Why Indices Count the Total Number of Black Hole Microstates (at large N)

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Timeline

- **1970s: Black hole mechanics $\Longleftrightarrow$ thermodynamics**
  - [Bekenstein,Hawking][1, 2]...
- **1990s: Microscopic interpretation from string theory**
  - [Strominger,Vafa] [3] ...
- **Gauge/Gravity Duality**
  - AdS/CFT [Maldacena][4, 5] ...
- **2000s, 2010s: Natural question**
  - Black hole thermodynamics from gauge theories? [6] ...

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- **2000s, 2010s: Natural question**
  - Black hole thermodynamics from gauge theories? [6] ...
- **2016–202?: Answer**
  - **Yes**, for supersymmetric black holes (**with certain gaps**) [7][8, 9, 10] ...

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- **Main gap: Index vs. entropy**
  - How can an index [11]—which counts *bosons minus fermions*—"reproduce" the **total number of black hole microstates**?
- **Secondary (apparently unrelated) gap: gravitational constraint among charges and angular momenta**
  - What is the **microscopic origin** of a non-linear gravitational constraint among charges that defines the absence of naked CTC in supersymmetric black holes? [12, 13, 14]
- Why is it important to understand these questions?

Outline of the talk

1. Review the gaps and reach an **educated guess for the answer to both questions** that **applies** to fairly **universal cases**.

- Then we will **test** the claim **in two independent computations within $U(N)$ 4D $\mathcal{N} = 4$ super Yang–Mills theory**.

2. **Using saddle-point approximation:** at both $N = 2$ and $N \gg 1$
3. **Cauchy residue evaluation:** at $N = 2$. This will allow us to explicitly test the conclusions obtained from the saddle-point approach **2.** .

Part 1: Reviewing the gaps and the proposed answer

Supersymmetric Black Holes: Charge Constraints

- A universal feature of $\mathcal{Q}$-supersymmetric black holes across dimensions and theories is the following:
- Given their conserved charges

$$(\mathfrak{j}, \mathfrak{q})\,,$$

- regular Lorentzian solutions (i.e. without naked CTCs) exist only on a **co-dimension one surface** in charge space.
- This surface imposes a relation between:
  - $\mathcal{Q}$**-commuting charges** $\mathfrak{j} = 2J + \mathfrak{q}$
  - $\mathcal{Q}$**-non-commuting charges** $\mathfrak{q}$ and J
- characterized by

$$[\mathcal{Q}, \mathfrak{j}] = 0\,, \qquad [\mathcal{Q}, \mathfrak{q}] \neq 0\,.$$

Quite remarkably, in this co-dimension 1 region of charges, the classical thermodynamic entropy

$$\frac{A[\mathfrak{j},\, \mathfrak{q}[\mathfrak{j}]]}{4G_N}$$

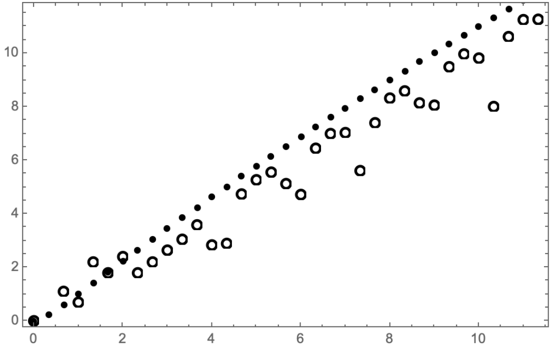
can be extracted* from a Witten index

$$\mathsf{Tr}_{\mathfrak{j}}(-1)^F$$

of the relevant theory of quantum gravity by performing a semiclassical expansion of the form

$$G_N \to 0\,, \quad \mathfrak{j}\, G_N = \text{fixed}\,.$$

For example, at least in the case of rotating black holes, the schematic picture is universally the following:



In those cases, when we say the index works*, we mean up to the removal of stationary oscillations (also in sign) that do not disappear in the thermodynamic expansion.

There are two natural questions that come to mind.
Firstly, **why does the Witten index work?**
Obviously,

$$\mathsf{Tr}_{\mathfrak{j}}(-1)^F \qquad (=: \widetilde{d}[\mathfrak{j}])$$

is not quite the same as the microcanonical entropy over $\mathcal{Q}$-supersymmetric states

$$\mathsf{Tr}_{\mathfrak{j},\,\mathfrak{q}} 1 \qquad (=: d[\mathfrak{j},\mathfrak{q}])$$

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(whose logarithm is the physical definition of the entropy of $\mathcal{Q}$-supersymmetric states).
The latter counts the total number of supersymmetric states at a given charge level. The former does not necessarily do so: it does so only if all such states are somehow either bosons (positive) or fermions (negative) [15].

Secondly, **what is the microscopic meaning of the (non-linear) constraint**

$$\mathfrak{q} = \mathfrak{q}(\mathfrak{j}) \ ?$$

These are the two conceptual questions that we will address today using the canonical example of AdS/CFT duality.

In the microscopic theory, the $\mathcal{Q}$-non-commuting charge is linearly related to spin. Using this relation and the spin-statistics theorem,

$$\underbrace{(-1)^F = e^{\mp 2\pi \mathrm{i} J}}_{\text{Spin-statistics theorem}} \underset{e.g.}{=} e^{\pm\pi\mathrm{i}\mathfrak{q}}\, e^{\mp\pi\mathrm{i}\mathfrak{j}} \quad \text{or} \quad e^{\pm 3\pi\mathrm{i}\mathfrak{q}}\,.$$

This gives the **sum-rules**

$$e^{\pm\pi\mathrm{i}\mathfrak{j}}\, \mathsf{Tr}_{\mathfrak{j}}(-1)^F = \sum_{\mathfrak{q}} e^{\pm\pi\mathrm{i}\mathfrak{q}} \left(\mathsf{Tr}_{\mathfrak{j},\mathfrak{q}}(1)\right),$$
$$\mathsf{Tr}_{\mathfrak{j}}(-1)^F = \sum_{\mathfrak{q}} e^{\pm 3\pi\mathrm{i}\mathfrak{q}} \left(\mathsf{Tr}_{\mathfrak{j},\mathfrak{q}}(1)\right),$$

which require an average over the discrete eigenvalues of $\mathfrak{q}$ in the microscopic theory, rather than a restriction to a co-dimension one locus, as gravity predicts.

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A saddle point that gravity predicts to dominate the $\mathfrak{q}$-averaging in a thermodynamic expansion

$$G_N \to 0\,, \qquad \mathfrak{j}\, G_N = \text{fixed}\,.$$

Indeed, our computations will be consistent with such prediction e.g.

$$e^{\pm\pi \mathrm{i}\mathfrak{j}}\mathrm{Tr}_{\mathfrak{j}}(-1)^F \sim e^{\pm\pi \mathrm{i}\mathfrak{q}^\star(\mathfrak{j})}\left(\mathrm{Tr}_{\mathfrak{j},\mathfrak{q}^\star(\mathfrak{j})}(1)\right).$$

We will explain below how this happens in a concrete example.

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On the other hand **a natural expectation is that $\mathbf{Tr}_{\mathrm{j,q}}(1)$ is not protected**, and therefore computing it at strong coupling is generically considered a no-go task.

The protectedness of the index, together with the sum-rules

$$e^{\pm\pi \mathrm{i}\mathfrak{j}}\,\mathsf{Tr}_{\mathfrak{j}}(-1)^F \;=\; \sum_{\mathfrak{q}} e^{\pm\pi\mathrm{i}\mathfrak{q}}\left(\mathsf{Tr}_{\mathfrak{j},\mathfrak{q}}(1)\right),$$

$$\mathsf{Tr}_{\mathfrak{j}}(-1)^F \;=\; \sum_{\mathfrak{q}} e^{\pm 3\pi\mathrm{i}\mathfrak{q}}\left(\mathsf{Tr}_{\mathfrak{j},\mathfrak{q}}(1)\right),$$

imply that there is protected information in the BPS ($\mathcal{Q}$-supersymmetric) partition function

$$\mathsf{Tr}_{\mathfrak{j},\mathfrak{q}}(1).$$

Assuming the horizon area (which is captured by such protected data) defines thermodynamic entropy, then only such protected data must somehow be relevant at leading order in the corresponding thermodynamic expansion.
If this last prediction is correct, then the following expectations follow...

The computation of

$$d[\mathfrak{j}, \mathfrak{q}]$$

at zero coupling (without the need to go to strong coupling) is enough to recover both

1. The Witten index

$$\widetilde{d}[\mathfrak{j}]$$

2. The horizon area law

when restricted to the a co-dimension 1 locus such that

$$\mathfrak{q} = \mathfrak{q}(\mathfrak{j}) \,\mathsf{mod}\, 2k \,, \qquad k \in \mathbb{Z}$$

at leading order in the relevant thermodynamic expansion.
The plan is to explain how this happens in a concrete example.

Part 2: Saddle-point analysis

The setup

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We focus on a pair of (Odd) symmetry generators $\mathcal{Q}$ and $\mathcal{S}$ such that

$$2\{\mathcal{Q},\mathcal{S}\} = E-2J-\frac{3\mathfrak{q}}{2} = E - \mathfrak{j} - \frac{1}{2}\mathfrak{q} \geq 0$$

and commute with $\mathfrak{j}=2J+\mathfrak{q}$

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The charges $\mathfrak{j}$ and $\mathfrak{q}$ are quantized in units of $\frac{1}{3}$

$$3\mathfrak{j}\,,\;3\mathfrak{q}\;\in\;\mathbb{Z}\,.$$

To perform the saddle-point approximation, it is convenient to work in the grand-canonical ensemble (at fixed chemical potentials).

Starting from the partition function

$$Z = \mathsf{Tr}_{\mathcal{H}}\, e^{-2\beta\{\mathcal{Q},\mathcal{S}\}} e^{\omega \mathfrak{j}}\, e^{2\pi \mathrm{i} \alpha \mathfrak{q}}\,,$$

the BPS states are counted by the Taylor coefficients of

$$Z_{\mathsf{BPS}} = \lim_{\beta\to\infty} Z = \sum_{\mathfrak{j},\,\mathfrak{q}} d[\mathfrak{j},\mathfrak{q}]\, e^{\omega \mathfrak{j}}\, e^{2\pi \mathrm{i}\alpha \mathfrak{q}}$$

and the (superconformal) index takes the form

$$\mathcal{I}[\omega \pm \pi \mathrm{i}] = \mathsf{Tr}(-1)^F e^{(\omega \pm {\pi \mathrm{i} \atop \text{or } 0})\mathfrak{j}} = \sum_{\mathfrak{j}} \widetilde{d}[\mathfrak{j}]\, e^{(\omega \pm {\pi \mathrm{i} \atop \text{or } 0})\mathfrak{j}} = Z_{\mathsf{BPS}}\Big|_{\alpha \bmod 3 = \pm\frac{1 \text{ or } 3}{2}}\,.$$

We will focus on Z_{BPS}; however, it will be useful to recall ...

Relations to recall

The grand-canonical version of the microcanonical sun-rules given before: the following two quantities coincide

$$\mathcal{I}[\omega \pm \pi \mathsf{i}] \quad \text{resp.} \quad \mathcal{I}[\omega] \; = \; Z_{\mathsf{BPS}}$$

when the latter is restricted to one of the infinitely many co-dimension 1 loci (α semi-integer)

$$\alpha \operatorname{mod} 3 \; = \; \pm \frac{1 \, \text{resp.} \, 3}{2}.$$

(These loci will be important below.)

The observable to study

Laplace transform (average or period) integral:

$$d[\mathfrak{j}, \mathfrak{q}] \,=\, \int_0^{6\pi \mathrm{i}} \frac{d\omega}{6\pi \mathrm{i}} \int_0^3 \frac{d\alpha}{3}\, Z_{\mathsf{BPS}}\, e^{-\omega \mathfrak{j} - 2\pi \mathrm{i} \alpha \mathfrak{q}}\,.$$

The explicit computation of this integral becomes intractable at sufficiently large N. This is due to the implicit dependence on N in Z_{BPS}. At $N = \infty$ we will study this observable in convenient expansions...

The expansions to study

Thermodynamic expansions = saddle-point expansion:

1. Gravitational:
$$N^2 \gg 1\,,\ \frac{\mathfrak{j}}{N^2}\,,\ \frac{\mathfrak{q}}{N^2}\ =\ \text{fixed}$$
2. Large charge (tested by Cauchy residues at $N = 2$):
$$N\ =\ \text{fixed}\,, \qquad \mathfrak{j} \gg 1\,,$$

In our case, it turns out that the dominant saddle point in 2. will just be the analytic continuation in N from $N \approx \infty$ to $N = 2$ of the one in 1. (This will become clear in due course.)

The BPS partition function

Let us test our proposal by computing Z_{BPS} at $g_{\mathsf{YM}} = 0$ [16].
The answer can be obtained in two ways: via the path integral or via gauge-covariant letter counting. It is a unitary matrix integral ($u_{ij} := u_i - u_j$):

$$Z_{\mathsf{BPS}} = \frac{Z_0^N}{N!} \int_0^1 \prod_{i=1}^{N} du_i \cdot \prod_{i \neq j} \frac{(1 - e^{2\pi \mathsf{i} u_{ij}})}{(1 + e^{2\pi \mathsf{i}(u_{ij}+\alpha)})} \times$$
$$\frac{(1 + e^{2\pi \mathsf{i}(u_{ij}+\alpha)}; e^{\omega}, e^{\omega})_\infty}{(1 - e^{2\pi \mathsf{i}(u_{ij}+2\frac{\omega}{2\pi \mathsf{i}})}; e^{\omega}, e^{\omega})_\infty} \frac{(1 + e^{2\pi \mathsf{i}(u_{ij}+\frac{\alpha+4\frac{\omega}{2\pi \mathsf{i}}}{3})}; e^{\omega}, e^{\omega})_\infty^3}{(1 - e^{2\pi \mathsf{i}(u_{ij}+\frac{2\alpha+2\frac{\omega}{2\pi \mathsf{i}}}{3})}; e^{\omega}, e^{\omega})_\infty^3}$$

Indeed, at $\alpha \bmod 3 = \pm\frac{1 \text{ or } 3}{2}$ this integral reduces to the well-known expression for the superconformal index $\mathcal{I}[\omega \pm \pi \mathsf{i}]$ or $\mathcal{I}[\omega]$ of the $U(N)$ theory.

The effective potential

We define the effective potential V_{eff} as follows:

$$(\text{Integrand}) = \frac{(Z_0)^N}{N!}\, e^{\sum_{\substack{i,j=1 \\ i\neq j}}^{N} V_{\text{eff}}[U_{ij}]}\,, \qquad U_{ij} = e^{2\pi \mathrm{i} u_{ij}}\,.$$

We are interested in studying its asymptotic form near its essential singularities

$$\delta\omega \;=\; \omega - 2\pi\mathrm{i}\mathfrak{n} \;=\; 0\,, \qquad \mathfrak{n}\in\mathbb{Z}\,.$$

$$V_{\text{eff}}[U] \;=\; \frac{\sum_{n,m=0} V_{\text{eff}}^{(m,n)}[U;\chi]\,\delta\omega^{n+m}}{\delta\omega^2} + (\text{non-pert. terms suppressed as } \delta\omega$$

The truncated effective potential

$$(z;p,q)_\infty \;:=\; \exp\!\left(-\sum_{n=1}^{\infty}\frac{1}{n}\,\frac{z^n}{(1-p^n)(1-q^n)}\right).$$

For technical convenience, in intermediate computations we introduce a cutoff Λ in the sum

$$\sum_{n=1}^{\infty} \to \sum_{n=1}^{\Lambda}\,. \tag{1}$$

This introduces a cutoff Λ (in the polylogarithms that define $V_{\text{eff}}^{(m,n)}$):

$$V_{\text{eff}}^{(m,n)} \;=\; V_{\text{eff},\Lambda}^{(m,n)}\Big|_{\Lambda\to\infty}\,.$$

Some examples of the truncated $V_{\text{eff},\Lambda}^{(m,n)}$ are ...

(m,n)	$V^{(m,n)}_{\text{eff},\Lambda}[U,\chi]$
$(0,0)$	$\sum_{\substack{i,j=1\\ i\neq j}}^{N}\Big(-\sum_{a=1}^{3}\mathrm{Li}_{3\Lambda}(X_aU_{ij})+\sum_{a=1}^{3}\mathrm{Li}_{3\Lambda}(Y_aU_{ij})-\ldots$ $-\mathrm{Li}_{3\Lambda}(ZU_{ij})+\mathrm{Li}_{3\Lambda}(U_{ij})\Big)$
$(1,0)=(0,1)$	$\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{2}\Big(-\sum_{a=1}^{3}\mathrm{Li}_{2\Lambda}(X_aU_{ij})-\sum_{a=1}^{3}\mathrm{Li}_{2\Lambda}(Y_aU_{ij})+\ldots$ $+\mathrm{Li}_{2\Lambda}(ZU_{ij})+\mathrm{Li}_{2\Lambda}(U_{ij})\Big)$
$(1,1)$	$\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{4}\Big(-\sum_{a=1}^{3}\mathrm{Li}_{1\Lambda}(X_aU_{ij})+\sum_{a=1}^{3}\mathrm{Li}_{1\Lambda}(Y_aU_{ij})+\ldots$ $+3\mathrm{Li}_{1\Lambda}(ZU_{ij})-3\mathrm{Li}_{1\Lambda}(U_{ij})\Big)$
$(2,1)=(1,2)$	$\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{24}\Big(-\sum_{a=1}^{3}\mathrm{Li}_{0\Lambda}(X_aU_{ij})-\sum_{a=1}^{3}\mathrm{Li}_{0\Lambda}(Y_aU_{ij})+\ldots$ $+\mathrm{Li}_{0\Lambda}(ZU_{ij})+\mathrm{Li}_{0\Lambda}(U_{ij})\Big)$
$(2,2)$	$\sum_{\substack{i,j=1\\ i\neq j}}^{N}\frac{1}{144}\Big(-\sum_{a=1}^{3}\mathrm{Li}_{-1\Lambda}(X_aU_{ij})+\sum_{a=1}^{3}\mathrm{Li}_{-1\Lambda}(Y_aU_{ij})-\ldots$ $-\mathrm{Li}_{-1\Lambda}(ZU_{ij})+\mathrm{Li}_{-1\Lambda}(U_{ij})\Big)$
(X_a,Y_a,Z)	$(\frac{1}{\nu_a}e^{\frac{2\pi\mathrm{i}(\alpha+k/2+\mathfrak{n})}{3}},\nu_a e^{\frac{4\pi\mathrm{i}(\alpha+k/2+\mathfrak{n})}{3}},e^{2\pi\mathrm{i}(\alpha+k/2)})$

Saddle points at finite cut-off

In terms of eigenvalues of the unitaries,

$$U_{ij} = e^{2\pi \mathsf{i} u_{ij}} = e^{2\pi \mathsf{i}(u_i - u_j)}\,, \qquad |U_{ij}| = 1\,.$$

At any finite cut-off Λ, the ansatz

$$U_{ij} = 1$$

is a saddle of all the function coefficients $V^{(m,n)}_{\text{eff},\Lambda}$, for all values of $\chi = e^{4\pi \mathsf{i}(\alpha + \frac{k}{2})}$.

If we assume the usual domain of convergence of the q-series expansion of Z_{BPS} as a function of $q = e^{\omega}$:

$$\delta\omega < 0$$

then, remarkably,

$$e^{\sum_{m,n=0} \frac{V_{\text{eff, }\Lambda}^{(m,n)}[1+\mathrm{i}0\,,\,\chi]\,\delta\omega^{m+n}}{\delta\omega^2}}\Bigg|_{\Lambda\to+\infty} = e^{-\infty} = 0\,.$$

if

$$\boxed{\alpha + \frac{k}{2} + \mathfrak{n} \mod 3 \neq \pm 1}\,.$$

(Asymptotic localization mechanism)

Sketch of the computation

Let us focus on the coefficients

$$V^{(m,n)}_{\mathsf{eff},\,\Lambda}[U_{ij}\,,\,\chi] \qquad m+n\,\geq\,4\,,$$

which include combinations of the form

$$(-1)^{3-m-n}\,\mathsf{Li}_{3-m-n,\Lambda}\,(ZU_{ij}) + \mathsf{Li}_{3-m-n,\Lambda}\left(\frac{1}{U_{ij}}\right).$$

For the series representation of each of these two individual contributions to make sense at $\Lambda=\infty$, it is necessary that

$$|U_{ij}|\,=\,1\,, \qquad U_{ij}\,\neq\,1$$

hold.

Sketch of the computation

The pole at $U_{ij} = 1$ of the second contribution can be cancelled by the pole of the first iff the following balancing condition holds:

$$Z := e^{2\pi \mathrm{i}(\alpha + \frac{k}{2})} = 1 \,.$$

Analogously, from the other polylogs it follows that the additional balancing condition

$$X \times Y = e^{\frac{2\pi \mathrm{i}(\alpha + k/2 + \mathfrak{n})}{3}} \times e^{\frac{4\pi \mathrm{i}(\alpha + k/2 + \mathfrak{n})}{3}} = 1$$

must hold, excluding the case

$$X = Y = 1$$

which gives zero on-shell potential. **It then follows that the asymptotic localization mechanism mentioned above holds...**

Implications for Z_{BPS}

Near its essential singularities

$$\delta\omega = \omega - 2\pi \mathrm{i}\mathfrak{n} \to 0$$

the BPS partition function reduces asymptotically to an ensemble of well-known saddle-point contributions to superconformal indices. Formally,

$$Z_{\mathsf{BPS}} \underset{\delta\omega\to 0}{\sim} \sum_{\ell\in\mathbb{Z}} \mathcal{I}_+^{sp}\, \delta(\alpha + \frac{k}{2} + \mathfrak{n} - 1 + 3\ell) + \sum_{\ell\in\mathbb{Z}} \mathcal{I}_-^{sp}\, \delta(\alpha + \frac{k}{2} + \mathfrak{n} + 1 + 3\ell) + \ldots$$

where

$$\mathcal{I}_\pm^{sp} = e^{-\frac{4N^2(\mp\mathrm{i}\pi + \delta\omega)^3}{27\delta\omega^2}} .$$

Other saddles

$$\ldots = (\text{contribution from other saddles})$$

$$U_{ij} \sim 1 + (\mathsf{sub})_{i,j}$$

But all of them are localized at indices

$$\boxed{\alpha + \frac{k}{2} + \mathfrak{n} \mod 3 = \pm 1},$$

which means that they are saddle points of superconformal indices (e.g. eigenvalue-instanton saddles of the index).

Rational essential singularities

Expansions of Z_{BPS} around rational essential singularities $\delta\omega = \omega - 2\pi \mathrm{i} \frac{\mathfrak{n}}{M}$ localize to superconformal indices as well:

$$\boxed{M(\alpha + \frac{k}{2} + \mathfrak{n}) \mod 3 = \pm 1}.$$

Superconformal indices counting only subsets of protected states with charges $\mathfrak{q}$

$$(\text{integer}) \times \frac{M}{3}$$

Types of saddles: orbifold saddles, eigenvalue instantons.

Coming back to the Laplace transform

In thermodynamic expansions such that

$$\mathfrak{j} \gg \frac{1}{3}$$

essential singularities with $M = 1$ **dominate the Laplace transform** (Cardy-like expansion)

$$d[\mathfrak{j}, \mathfrak{q}] \, = \int_0^{6\pi \mathrm{i}} \frac{d\omega}{6\pi \mathrm{i}} \int_0^3 \frac{d\alpha}{3} \, Z_{\mathsf{BPS}} \, e^{-\omega \mathfrak{j} - 2\pi \mathrm{i} \alpha \mathfrak{q}} \, .$$

The integration contour intersects 9 pairs of complex-conjugate contributions with orbifold number $M = 1 \ldots$

Oscillations from c.c. saddles

One of these two pairs ($M = 1$) contributes as follows:

$$e^{\mathcal{S}_+ - \pi \mathrm{i} \mathfrak{q}} + e^{\mathcal{S}_- + \pi \mathrm{i} \mathfrak{q}}$$

where

$$\begin{aligned} \mathcal{S}_\pm[\mathfrak{j}] &= \underset{\omega}{\mathrm{ext}}\left(-\frac{4N^2}{27}\,\frac{(\pm\pi\mathrm{i}+\omega)^3}{\omega^2} - \omega\,\mathfrak{j}\right) \\ &= \mathcal{S}[\mathfrak{j}] \pm \pi \mathrm{i} \mathfrak{q}_{\mathrm{ctr}}[\mathfrak{j}]\,, \end{aligned}$$

whenever these two saddles dominate, there are sign oscillations defined by

$$\mathfrak{q}_{\mathrm{ctr}}[\mathfrak{j}] - \mathfrak{q}.$$

Non-linear constraint + positivity

Sign oscillations disappear only in the **co-dimension 1 region of charges**

$$\mathfrak{q}_{\mathsf{ctr}}[\mathfrak{j}] - \mathfrak{q} \mod 2 = 0.$$

Thus, the **positivity condition**

$$d[\mathfrak{j}, \mathfrak{q}] \geq 0$$

implies that **these saddles can dominate the Laplace transform** (the total number of BPS states) **only in such a co-dimension 1 region of charges**

$$\boxed{d[\mathfrak{j}, \mathfrak{q}_{\mathsf{ctr}}[\mathfrak{j}]] \sim e^{\mathcal{S}[\mathfrak{j}]}}.$$

Index vs. partition function (relation to BHs)

As these saddles dominate the index at $\mathfrak{j} \gg \frac{1}{3}$ as well (up to oscillations), we arrive at the triad of relations

$$d[\mathfrak{j}, \mathfrak{q}_{\mathsf{ctr}}[\mathfrak{j}]] \sim e^{\mathcal{S}[\mathfrak{j}]} \underbrace{\sim}_{\text{up to oscillations}} |\widetilde{d}[\mathfrak{j}]|.$$

Furthermore, identifying $G_N \equiv \frac{\pi}{2N^2}$ and the charges $\mathfrak{j}$ and $\mathfrak{q}$ on the dual sides,

$$\mathcal{S}[\mathfrak{j}] = \frac{A_H[\mathfrak{j}, \mathfrak{q}_{\mathsf{ctr}}[\mathfrak{j}]]}{4G_N}$$

(even at finite N, e.g. $N = 2$ (!)).

Summary of saddle-point analysis

These relations explain why the saddle points of the index succeeded in reproducing the black hole entropy. However, they also show that the natural observable for counting states is really

$$d[\mathfrak{j}, \mathfrak{q}],$$

the total number of BPS operators.

The saddle-point analysis (via the asymptotic localization mechanism), together with a positivity constraint, predicts that in the co-dimension 1 region of charges

$$\mathfrak{q}_{\mathsf{ctr}}[\mathfrak{j}] - \mathfrak{q} \mod 2 = 0$$

The microcanonical partition function should match the absolute value of the index (after averaging out oscillations).

The asymptotic localization mechanism explains:

- Why the zero-coupling computation works (recombination effects are predicted to be negligible there)
- How the continuous charges of the saddle-point analysis (or equivalently of gravity) become the discrete charges of the microscopic theory (via Poisson resummation):

$$\sum_{\ell\in\mathbb{Z}} \delta(\alpha + \frac{k}{2} + \mathfrak{n} - 1 + 3\ell) \sim \sum_{\ell\in\mathbb{Z}} e^{2\pi\mathrm{i}(\alpha+\frac{k}{2}+\mathfrak{n}-1)\,\frac{\ell}{3}}$$

- In gravity, these infinitely many saddles $\mathfrak{n}$ correspond to different periodicity conditions at the tip of complex cigar geometries [8, 17].

Part 3: Saddle point vs. exact computation at $N = 2$

The goal is to compute

$$Z_{\mathsf{BPS}} = \frac{Z_0^2}{N!} \int_0^1 \prod_{i=1}^{2} du_i \cdot \prod_{i\neq j=1}^{2} \frac{(1 - e^{2\pi\mathsf{i}u_{ij}})}{(1 + e^{2\pi\mathsf{i}(u_{ij}+\alpha)})} \times$$

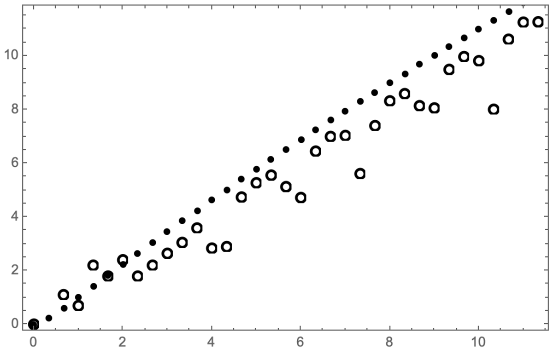
$$\frac{(1 + e^{2\pi\mathsf{i}(u_{ij}+\alpha)}; e^{\omega}, e^{\omega})_\infty}{(1 - e^{2\pi\mathsf{i}(u_{ij}+2\frac{\omega}{2\pi\mathsf{i}})}; e^{\omega}, e^{\omega})_\infty} \frac{(1 + e^{2\pi\mathsf{i}(u_{ij}+\frac{\alpha+4\frac{\omega}{2\pi\mathsf{i}}}{3})}; e^{\omega}, e^{\omega})_\infty^3}{(1 - e^{2\pi\mathsf{i}(u_{ij}+\frac{2\alpha+2\frac{\omega}{2\pi\mathsf{i}}}{3})}; e^{\omega}, e^{\omega})_\infty^3}$$

by Taylor-expanding the integrand in small rapidities $q = e^{\omega}$ and $t^{\frac{1}{3}} = e^{\frac{2\pi\mathsf{i}\alpha}{3}} q^{\frac{1}{3}}$.

For example, the coefficient of $t^{\frac{4}{3}}$ $(q = e^{\omega})$ is

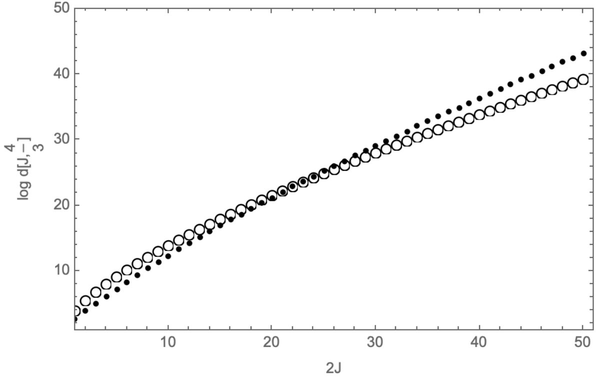
$$\begin{aligned} 12 &+ 48q + 225q^2 + 840q^3 + 2811q^4 + 8496q^5 + 24204q^6 \\ &+ 65112q^7 + 168081q^8 + 417528q^9 + 1005930q^{10} \\ &+ 2356152q^{11} + 5388702q^{12} + 12057384q^{13} \\ &+ 26464212q^{14} + 57064188q^{15} + 121096059q^{16} \\ &+ 253205850q^{17} + 522317775q^{18} + 1063943844q^{19} \\ &+ 2142034122q^{20} + 4265607348q^{21} + 8407975740q^{22} \\ &+ 16414170480q^{23} + 31754763042q^{24} + 60908083752q^{25} \\ &+ 115882142814q^{26} + 218781699552q^{27} + 410037270660q^{28} \\ &+ 763137119370q^{29} + 1410871015434q^{30} + \dots \end{aligned}$$

Saddle point and the index at $N = 2$



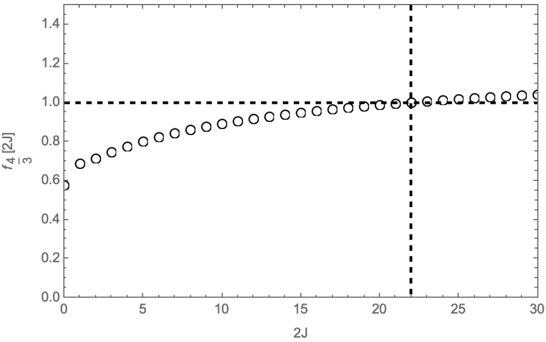
saddle point $\mathcal{S}[\mathfrak{j}]$ vs. $|\log \widetilde{d}[\mathfrak{j}]|$

Saddle points and the partition function at $N = 2$



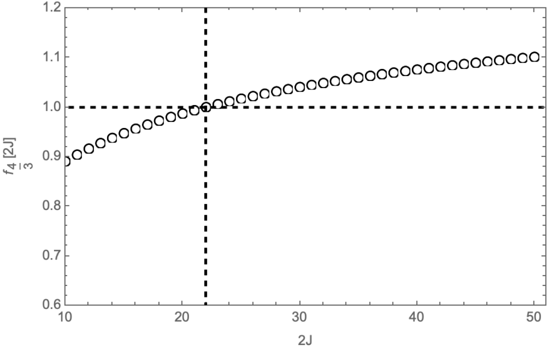
The $\log d[J, \frac{4}{3}]$ vs. J at fixed $Q = \frac{4}{3}$

The quotient $f_{\mathfrak{q}=Q}[J] := \frac{\mathcal{S}[\mathfrak{j}=2J+Q]}{\log d[\mathfrak{j},Q]}$ vs. J at fixed $Q = \frac{4}{3}$



The vertical dashed line denotes $J = J_{\mathsf{ctr}} = 21.9574 \approx 22$ where

$$3\,\mathfrak{q}_{\mathsf{ctr}}[J_{\mathsf{ctr}}] \bmod 6 \;=\; 4.00003 \;\approx\; 4.$$



The approximate triple intersection confirms our previous saddle-point result. The pattern repeats for the other values of charges $\mathfrak{q}$ explored.

Final comments/questions I

- At $N = \infty$ the orbifold saddles correspond to complex cigar geometries with labels $\mathfrak{n}$ and $\frac{\mathfrak{n}}{M}$. Infinitely many of them come from the same Lorentzian solution [17].
- What about other saddles beyond orbifold ones e.g. eigenvalue instantons? Gravitational interpretation
- Do they correspond to classical geometries, or to quantum-deformed geometries backreacted by branes?
- Is the horizon-area beyond the non-CTC region dominated by other saddle points? Does it have meaning of entropy? [18]
- Finite-N questions about counting of BPS states: giant brane expansions, trace relations, fortuity, etc. Z_{BPS} is the natural observable there not the index.
- A unitary matrix integral representation of Z_{BPS} accounting for recombinations?

Thanks for your attention.

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