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Radford $[n, (n, l)]$-biproduct theorem for generalized Hom-crossed coproducts

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ABSTRACT

In this paper, we provide a new approach to construct monoidal Hom-Hopf algebras. We investigate monoidal Hom-Hopf algebra structure on a left (n, l)-Hom-crossed coproduct structure with a left n-Hom-smash product structure, obtaining Radford $[n, (n, l)]$-biproduct structure theorem. Then, we study a Hom-coaction admissible mapping system to characterize this Radford $[n, (n, l)]$-biproduct structure. Finally, we study the cosemisimplicity of a special Hom-smash coproduct and prove the related Maschke theorem.

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Hom-coaction admissible mapping system; Hom-Hopf algebra; Maschke theorem; n-Hom-smash product; (n, l)-Hom-crossed coproduct; Radford biproduct

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Introduction

In the classical Hopf algebraic theory, the one of the celebrated results is Radford' biproduct [21] which provided in particular an important approach to solve the classification of finite-dimensional pointed Hopf algebras (see [1, 2]). This biproduct says that if A is a braided Hopf algebra in the braided monoidal category of Yetter-Drinfeld modules ${}^H_H\mathcal{YD}$ over a Hopf algbera H, then a left smash product algebra structure and a left smash coproduct coalgebra structure afford a Hopf algebra structure on $A \otimes H$, see [16]. Radford's biproduct was generalized to many cases: replacing the smash product algebra structure by a left Hopf crossed product (see, [25]) and replacing the smash coproduct coalgebra structure by a left Hopf crossed coproduct [9] in the setting of Hopf algebras; replacing Hopf algebras by quasi-Hopf algebras [4], by multiplier Hopf algebras [8] and by monoidal Hom-Hopf algebras (see, [12, 15]).

As we know that the notion of a left Hopf crossed product was introduced in [3] and the dual Hopf crossed coproduct was introduced in [7] (see, [23, 24]). These notions have been studied in the setting of (monoidal) Hom-Hopf algebras (see, [13, 14]).

We recall from the papers [19] and [20] that the original notion of a Hom-Hopf algebra involved two different linear maps α and β for which α twists the associativity and β the coassociativity. At present, researchers have developed two directions of study: one considered the class such that $\beta = \alpha$, which are still called Hom-Hopf algebras (cf. [14, 17, 18]) and another one started by Caenepeel and Goyvaerts in [5], that the map α is assumed to be invertible and $\beta = \alpha^{-1}$, which are called monoidal Hom-Hopf algebras (cf. [6, 12, 13]). Therefore, Hom-Hopf algebras and monoidal Hom-Hopf algebras are different concepts. By the way, there are other developing of Hom-Hopf algebras combining with weak Hopf algebras (cf, [10, 11]) and linking with Hopf group-coalgebras (cf. [26]), and so on.

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The main object of this paper is to provide a new method to construct monoidal Hom-Hopf algebras by introducing the notion of a (n, l)-Hom-crossed coproduct with $n, l \in \mathbb{Z}$ and then building Radford $[n, (n, l)]$-biproduct theorem which is a generalization of the one both in [21] and in [9].

The organization of the paper is the following. In Section 1, some basic notations about monoidal Hom-Hopf algebras, Hom-(co)module algebras, Hom-(co)module coalgebras that we will need are recalled. In Sections 2 and 3, we will introduce the notion of a left (n, l)-Hom-crossed coproduct for a monoidal Hom-Hopf algebra and obtain Radford $[n, (n, l)]$-biproduct structure with $n, l \in \mathbb{Z}$ (see Theorems 2.3, 3.4, and 3.6). In Section 4, in order to characterize the Radford $[n, (n, l)]$-biproduct structure, we study Hom-coaction admissible mapping system (see, Theorem 4.10). In the final section, we study the cosemisimplicity of the Hom-smash coproduct and prove the related Maschke theorem (see Theorems 5.4 and 5.9).

Throughout, let k be a fixed field and everything is over k. We refer the readers to the book of Sweedler [22] for the relevant concepts on the general theory of Hopf algebras. Let (C, Δ) be a coalgebra, we use the Sweedler-Heyneman's notation for Δ as follows: $\Delta(c) = \sum c_1 \otimes c_2$, for all $c \in C$.

1. Preliminaries

In this section we will recall the notions of a monoidal Hom-category, a monoidal Hom-Hopf algebra, a (co)action of monoidal Hom-Hopf algebra and a Hom-smash (co)product.

1.1. A monoidal Hom-category $\widetilde{\mathcal{H}}(\mathcal{M}_k)$

Let $\mathcal{M}_k = (\mathcal{M}_k, \otimes, k, a, l, r)$ denote the usual monoidal category of k-vector spaces and linear maps between them. Recall from [6] that there is the *monoidal Hom-category* $\widetilde{\mathcal{H}}(\mathcal{M}_k) = (\mathcal{H}(\mathcal{M}_k), \otimes, (k, \mathrm{id}), \widetilde{a}, \widetilde{l}, \widetilde{r})$, a new monoidal category, associated with $\mathcal{M}_k$ as follows:

• $\mathcal{H}(\mathcal{M}_k)$ are couples (M, μ), where $M \in \mathcal{M}_k$ and $\mu \in Aut_k(M)$, the set of all k-linear automorphisms of M;

• $f : (M, \mu) \to (N, \nu)$ in $\mathcal{H}(\mathcal{M}_k)$ is a k-linear map $f : M \to N$ in $\mathcal{M}_k$ satisfying $\nu \circ f = f \circ \mu$ for any two objects $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$;

• The tensor product is given by

$$(M, \mu) \otimes (N, \nu) = (M \otimes N, \mu \otimes \nu)$$

for any $(M, \mu), (N, \nu) \in \mathcal{H}(\mathcal{M}_k)$;

• The tensor unit is given by (k, id);

• The associativity constraint $\widetilde{a}$ is given by the formula

$$\widetilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes \mathrm{id}) \otimes \varsigma^{-1}) = (\mu \otimes (\mathrm{id} \otimes \varsigma^{-1})) \circ a_{M,N,L},$$

for any objects $(M, \mu), (N, \nu), (L, \varsigma) \in \mathcal{H}(\mathcal{M}_k)$;

• The left and right unit constraint $\widetilde{l}$ and $\widetilde{r}$ are given by

$$\widetilde{l}_M = \mu \circ l_M = l_M \circ (\mathrm{id} \otimes \mu), \quad \widetilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes \mathrm{id})$$

for all $(M, \mu) \in \mathcal{H}(\mathcal{M}_k)$.

1.2. Monoidal Hom-associative algebras and monoidal Hom-coassociative coalgebras

A *unital monoidal Hom-associative algebra* is a vector space A together with an element $1_A \in A$ and linear maps

$$m : A \otimes A \longrightarrow A(a \otimes b \mapsto ab) \quad \text{and} \quad \alpha \in Aut_k(A)$$

such that

$$\alpha(a)(bc) = (ab)\alpha(c), \tag{1.1}$$

$$\alpha(ab) = \alpha(a)\alpha(b), \tag{1.2}$$

$$a1_A = 1_A a = \alpha(a), \tag{1.3}$$

$$\alpha(1_A) = 1_A \tag{1.4}$$

for all $a, b, c \in A$.

Remark. (1) In the language of algebras, m is called the Hom-multiplication, α is the twisting automorphism and 1_A is the unit. Note that Eq. (1.1) can be rewritten as $a(b\alpha^{-1}(c)) = (\alpha^{-1}(a)b)c$. The monoidal Hom-algebra A with a *structure map* α will be denoted by (A, α).

(2) A monoidal Hom-associative algebra is not the same as a Hom-associative algebra in which α is not necessary bijective, (see, [17, 19]).

(3) Let (A, α) and (A', α') be two monoidal Hom-algebras. A monoidal Hom-algebra map $f : (A, \alpha) \to (A', \alpha')$ is a linear map such that for any $a, b \in A, f \circ \alpha = \alpha' \circ f, f(ab) = f(a)f(b)$ and $f(1_A) = 1_{A'}$.

A counital monoidal Hom-coassociative coalgebra is a vector space C together with linear maps $\Delta : C \longrightarrow C \otimes C (\Delta(c) = c_1 \otimes c_2), \varepsilon : C \longrightarrow k$ and $\gamma \in Aut_k(C)$ so that

$$\gamma^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2), \tag{1.5}$$

$$\Delta(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \tag{1.6}$$

$$c_1\varepsilon(c_2) = \gamma^{-1}(c) = \varepsilon(c_1)c_2, \tag{1.7}$$

$$\varepsilon(\gamma(c)) = \varepsilon(c) \tag{1.8}$$

for all $c \in C$.

Remark. (1) Note that Eq. (1.5) is equivalent to $c_1 \otimes c_{21} \otimes \gamma(c_{22}) = \gamma(c_{11}) \otimes c_{12} \otimes c_2$. Similar to monoidal Hom-algebras, monoidal Hom-coalgebras will be short for counital monoidal Hom-coassociative coalgebras without any confusion. The monoidal Hom-coalgebra C with a *structure map* γ will be denoted by (C, γ).

(2) A monoidal Hom-coassociative coalgebra is not the same as a Hom-coassociative coalgebra in which Eqs. (1.5) and (1.7) are replaced by $\gamma(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma(c_2)$ and $c_1\varepsilon(c_2) = \gamma(c) = \varepsilon(c_1)c_2$ for any $c \in C$, respectively, and γ is not necessary bijective, (see, [17, 19]).

(3) Let (C, γ) and (C', γ') be two monoidal Hom-coalgebras. A monoidal Hom-coalgebra map $f : (C, \gamma) \to (C', \gamma')$ is a linear map such that $f \circ \gamma = \gamma' \circ f,\ \Delta \circ f = (f \otimes f) \circ \Delta$ and $\varepsilon' \circ f = \varepsilon$.

1.3. Monoidal Hom-Hopf algebras

A monoidal Hom-bialgebra $H = (H, \alpha, m, 1_H, \Delta, \varepsilon)$ is a bialgebra in the monoidal category $\tilde{\mathcal{H}}(\mathcal{M}_k)$. This means that $(H, \alpha, m, 1_H)$ is a monoidal Hom-algebra and $(H, \alpha, \Delta, \varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are morphisms of algebras.

A monoidal Hom-bialgebra (H, α) is called *a monoidal Hom-Hopf algebra* if there exists a morphism (called *antipode*) $S : H \to H$ in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ (i.e. $S \circ \alpha = \alpha \circ S$), which is the convolution inverse of the identity morphism id_H (i.e. $S * \mathrm{id} = 1_H \circ \varepsilon = \mathrm{id} * S$). Explicitly, for all $h \in H$,

$$S(h_1)h_2 = \varepsilon(h)1_H = h_1S(h_2).$$

Remark. (1) Note that a monoidal Hom-Hopf algebra is by definition a Hopf algebra in the category $\tilde{\mathcal{H}}(\mathcal{M}_k)$ (see, [5]).

(2) Furthermore, the antipode of monoidal Hom-Hopf algebras has almost all the properties of antipode of Hopf algebras such as: for any $g, h \in H$,

$$S(hg) = S(g)S(h), S(1_H) = 1_H, \Delta(S(h)) = S(h_2) \otimes S(h_1), \text{ and } \varepsilon \circ S = \varepsilon.$$

That is, S is a monoidal Hom-anti-(co)algebra homomorphism. Since α is bijective and commutes with S, we can also have that the inverse α^{-1} commutes with S, that is, $S \circ \alpha^{-1} = \alpha^{-1} \circ S$.

(3) A monoidal Hom-Hopf algebra (H, α) is not the same as a Hom-Hopf algebra (H, α) in which α is not necessary bijective, (see, [17, 19]).

1.4. m-Hom-smash products and m-Hom-smash coproducts

Let (A, α) be a monoidal Hom-algebra. *A left* (A, α)*-Hom-module* consists of an object (M, μ) in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a morphism $\psi : A \otimes M \longrightarrow M$, $\psi(a \otimes m) = a \cdot m$ such that

$$\alpha(a) \cdot (b \cdot m) = (ab) \cdot \mu(m), \quad \mu(a \cdot m) = \alpha(a) \cdot \mu(m), \text{ and } 1_A \cdot m = \mu(m),$$

for all $a, b \in A$ and $m \in M$.

Remark. (1) Monoidal Hom-algebra (A, α) can be considered as a Hom-module on itself by the Hom-multiplication.

(2) Let (M, μ) and (N, ν) be two left (A, α)-Hom-modules. A morphism $f : M \longrightarrow N$ is called a left (A, α)-linear if $f(a \cdot m) = a \cdot f(m)$, $f \circ \mu = \nu \circ f$ hold for any $a \in A, m \in M$. The category of left (A, α)-Hom-modules is denoted by $\tilde{\mathcal{H}}({}_A\mathcal{M})$.

Similarly, let (C, γ) be a monoidal Hom-coalgebra. *A right* (C, γ)*-Hom-comodule* is an object (M, μ) in $\tilde{\mathcal{H}}(\mathcal{M}_k)$ together with a k-linear map $\rho_M : M \longrightarrow M \otimes C$, $\rho_M(m) = m_{(0)} \otimes m_{(1)}$ such that

$$\mu^{-1}(m_{(0)}) \otimes \Delta_C(m_{(1)}) = (m_{(0)(0)} \otimes m_{(0)(1)}) \otimes \gamma^{-1}(m_{(1)}), \tag{1.9}$$

$$\rho_M(\mu(m)) = \mu(m_{(0)}) \otimes \gamma(m_{(1)}), \quad m_{(0)}\varepsilon(m_{(1)}) = \mu^{-1}(m), \tag{1.10}$$

for all $m \in M$.

Remark. (1) (C, γ) is a Hom-comodule on itself via the Hom-comultiplication.

(2) Let (M, μ) and (N, ν) be two right (C, γ)-Hom-comodules. A morphism $g : M \longrightarrow N$ is called right (C, γ)-colinear if $g \circ \mu = \nu \circ g$ and $g(m_{(0)}) \otimes m_{(1)} = g(m)_{(0)} \otimes g(m)_{(1)}$ hold for any $m \in M$. The category of right (C, γ)-Hom-comodules is denoted by $\tilde{\mathcal{H}}(\mathcal{M}^C)$.

Let (H, α) be a monoidal Hom-bialgebra. A monoidal Hom-algebra (B, β) is called a left *weak* (H, α)*-Hom-module algebra*, if (B, β) has a map: $\cdot : H \otimes B \longrightarrow B$, obeying the following axioms:

$$h \cdot (ab) = (h_1 \cdot a)(h_2 \cdot b), \qquad h \cdot 1_B = \varepsilon(h)1_B, \tag{1.11}$$

for all $a, b \in B, h \in H$. Furthermore, (B, β) is called *a left* (H, α)*-Hom-module algebra* if (B, β) is a left (H, α)-Hom-module with the action "$\cdot$".

Let (H, α) be a monoidal Hom-bialgebra. A monoidal Hom-coalgebra (B, β) is called *a left* (H, α)*-Hom-comodule coalgebra*, if (B, β) is a left (H, α)-Hom-comodule with coaction ρ obeying the following axioms:

$$b_{(-1)} \otimes b_{(0)1} \otimes b_{(0)2} = b_{1(-1)}b_{2(-1)} \otimes b_{1(0)} \otimes b_{2(0)}, \ \varepsilon(b_{(0)})b_{(-1)} = \varepsilon(b)1_H, \tag{1.12}$$

for all $b \in B$.

2. (n,l)-Hom-crossed coproducts

Let $n,l\in\mathbb{Z}$. In order to obtain Radford $[n,(n,l)]$-biproduct theorem, in this section, we introduce and study the notion of a left (n,l)-Hom-crossed coproduct for a monoidal Hom-Hopf algebra.

Definition 2.1. Let (H,α) be a monoidal Hom-bialgebra and (B,β) a left (H,α)-Hom-comodule coalgebra with coaction $\rho_B^H: B\to H\otimes B$. $\tau: B\to H\otimes H$ is a linear map and $\tau(b)=b^{'}\otimes b^{''}$ with $n,l\in\mathbb{Z}$. We say that the data $(B\rtimes_\tau H,\beta\otimes\alpha)$ is a *(n,l)-Hom-crossed coproduct* if

(1) $B\rtimes_\tau H=B\otimes H$, as a linear space;

(2) Hom-comultiplication is: $\forall b\rtimes_\tau h\in B\rtimes_\tau H$,

$$\triangle_{B\rtimes_\tau H}(b\rtimes_\tau h)=b_1\rtimes_\tau[\alpha^n(b_{21(-1)})\alpha^{l-1}(b_{22}^{'})]\alpha^{-1}(h_1)\otimes\beta^2(b_{21(0)})\rtimes_\tau\alpha^l(b_{22}^{''})\alpha^{-1}(h_2).$$

Remark 2.2. (1) When $\alpha=\mathrm{id}_H$ and $\beta=\mathrm{id}_A$, we will obtain the usual crossed coproduct (see, [23, 24]).

(2) When $n=0$ and $l=-1$, we obtain the monoidal Hom-crossed coproduct (see, [13]).

(3) When $\tau: B\to H\otimes H$ is trivial, that is $\tau(b)=\varepsilon(b)1_H\otimes 1_H$,

$$\begin{aligned}&\triangle_{B\rtimes_\tau H}(b\rtimes_\tau h)\\ =&b_1\rtimes_\tau[\alpha^n(b_{21(-1)})\alpha^{l-1}(b_{22}^{'})]\alpha^{-1}(h_1)\otimes\beta^2(b_{21(0)})\rtimes_\tau\alpha^l(b_{22}^{''})\alpha^{-1}(h_2)\\ =&b_1\rtimes_\tau[\alpha^n(b_{21(-1)})\varepsilon(b_{22})1_H]\alpha^{-1}(h_1)\otimes\beta^2(b_{21(0)})\rtimes_\tau 1_H\alpha^{-1}(h_2)\\ =&b_1\rtimes_\tau\alpha^n(b_{2(-1)})\alpha^{-1}(h_1)\otimes\beta(b_{2(0)})\rtimes_\tau h_2\end{aligned}$$

which is the n-smash coproduct.

Theorem 2.3. *Let $(B\rtimes_\tau H,\beta\otimes\alpha)$ be a (n,l)-Hom-crossed coproduct. Then $(B\rtimes_\tau H,\beta\otimes\alpha)$ is a monoidal Hom-coalgebra with counit $\varepsilon_{B\rtimes_\tau H}=\varepsilon_B\otimes\varepsilon_H$ if and only if the following conditions hold:*

(1) $\varepsilon(b^{'})b^{''}=b^{'}\varepsilon(b^{''})=\varepsilon(b)1_H$.

*(2) $\alpha^{n-1}(b_{1(-1)})\alpha^{l-2}(b_2^{'})\otimes\alpha^n(b_{1(0)(-1)})\alpha^{l-2}(b_2^{''})\otimes\beta(b_{1(0)(0)})$
$=\alpha^{l-2}(b_1^{'})\alpha^n(b_{2(-1)1})\otimes\alpha^{l-2}(b_1^{''})\alpha^n(b_{2(-1)2})\otimes b_{2(0)}$.*

*(3) $\alpha^{n-1}(b_{1(-1)})\alpha^{l-2}(b_2^{'})\otimes\alpha^{l-1}(b_{1(0)}^{'})\alpha^{l-1}(b_{2\,1}^{''})\otimes\alpha^{l-1}(b_{1(0)}^{''})\alpha^{l-1}(b_{2\,2}^{''})$
$=\alpha^{l-2}(b_1^{'})\alpha^{l-1}(b_{2\,1}^{'})\otimes\alpha^{l-2}(b_1^{''})\alpha^{l-1}(b_{2\,2}^{'})\otimes\alpha^{l-1}(b_2^{''})$,
for any $b\in B$.*

Proof. If $(B\rtimes_\tau H,\beta\otimes\alpha)$ is monoidal Hom-coalgebra, then we have, for any $b\in B$,

$$[(\beta^{-1}\otimes\alpha^{-1})\otimes\triangle_{B\rtimes_\tau H}]\triangle_{B\rtimes_\tau H}(b\rtimes_\tau 1_H)=[\triangle_{B\rtimes_\tau H}\otimes(\beta^{-1}\otimes\alpha^{-1})]\triangle_{B\rtimes_\tau H}(b\rtimes_\tau 1_H).$$

Then we have:
$\beta^{-1}(b_1)\rtimes_\tau\alpha^n(b_{21(-1)})\alpha^{l-1}(b_{22}^{'})\otimes\beta^2(b_{21(0)1})\rtimes_\tau[\alpha^{n+2}(b_{21(0)21(-1)})\alpha^{l+1}(b_{21(0)22}^{'})]\alpha^l(b_{22\,1}^{'})\otimes\beta^4(b_{21(0)21(0)})\rtimes_\tau\alpha^{l+2}(b_{21(0)22}^{''})\alpha^l(b_{22\,2}^{''})=b_{11}\rtimes_\tau[\alpha^n(b_{121(-1)})\alpha^{l-1}(b_{122}^{'})](\alpha^n(b_{21(-1)1})\alpha^{l-1}(b_{221}^{'}))\otimes\beta^2(b_{121(0)})\rtimes_\tau\alpha^l(b_{122}^{''})(\alpha^n(b_{21(-1)2})\alpha^{l-1}(b_{222}^{'}))\otimes\beta(b_{21(0)})\rtimes_\tau\alpha^l(b_{22}^{''})$.

Applying $\varepsilon_B\otimes\mathrm{id}_H\otimes\varepsilon_B\otimes\mathrm{id}_H\otimes\mathrm{id}_B\otimes\varepsilon_H$ to the both sides of the above identity, we will obtain the item (2).

Applying $\varepsilon_B\otimes\mathrm{id}_H\otimes\varepsilon_B\otimes\mathrm{id}_H\otimes\varepsilon_B\otimes\mathrm{id}_H$ to the both sides of the above identity, we will obtain the item (3).

Meanwhile, we have

$$(\varepsilon_{B\rtimes_\tau H}\otimes\mathrm{id}_{B\rtimes_\tau H})\triangle_{B\rtimes_\tau H}(b\rtimes_\tau 1_H)=(\beta^{-1}\otimes\alpha^{-1})(b\rtimes_\tau 1_H),$$

$$(\mathrm{id}_{B\rtimes_\tau H}\otimes\varepsilon_{B\rtimes_\tau H})\triangle_{B\rtimes_\tau H}(b\rtimes_\tau 1_H)=(\beta^{-1}\otimes\alpha^{-1})(b\rtimes_\tau 1_H).$$

That is

$$b_1\rtimes_\tau\alpha^l(b_2^{'})\varepsilon_H(b_2^{''})=\beta^{-1}(b)\rtimes_\tau 1_H=b_1\rtimes_\tau\varepsilon_H(b_2^{'})\alpha^l(b_2^{''}).$$

Applying $\varepsilon_B \otimes \mathrm{id}_H$ to the both sides of the above identity, we will obtain the item (1).

Conversely, suppose (1)–(3) are established, it is obvious that $\varepsilon_{B\rtimes_\tau H}\circ(\beta\otimes\alpha)=\varepsilon_{B\rtimes_\tau H}$. Besides, for any $b\in B$, $h\in H$, it is not hard to verify the following equations:

$$(\varepsilon_{B\rtimes_\tau H}\otimes \mathrm{id}_{B\rtimes_\tau H})\,\triangle_{B\rtimes_\tau H}\,(b\rtimes_\tau h)=\beta^{-1}(b)\rtimes_\tau\alpha^{-1}(h),$$
$$(\mathrm{id}_{B\rtimes_\tau H}\otimes \varepsilon_{B\rtimes_\tau H})\,\triangle_{B\rtimes_\tau H}\,(b\rtimes_\tau h)=\beta^{-1}(b)\rtimes_\tau\alpha^{-1}(h),$$
$$[(\beta\otimes\alpha)\otimes(\beta\otimes\alpha)]\,\triangle_{B\rtimes_\tau H}\,(b\rtimes_\tau h)=\triangle_{B\rtimes_\tau H}[\beta(b)\rtimes_\tau\alpha(h)].$$

Finally, we check the Hom-coassociativity.

$$\begin{aligned}
&[(\beta^{-1}\otimes\alpha^{-1})\otimes\triangle_{B\rtimes_\tau H}]\,\triangle_{B\rtimes_\tau H}\,(b\rtimes_\tau h)\\
&=\beta^{-1}(b_1)\rtimes_\tau[\alpha^{n-1}(b_{21(-1)})\alpha^{l-2}(b_{22}^{'})]\alpha^{-2}(h_1)\otimes\beta^2(b_{21(0)1})\\
&\quad\rtimes_\tau[\alpha^{n+2}(b_{21(0)21(-1)})\alpha^{l+1}(b_{21(0)22}^{'})][\alpha^{l-1}(b_{22\,1}^{''})\alpha^{-2}(h_{21})]\otimes\beta^4(b_{21(0)21(0)})\\
&\quad\rtimes_\tau\alpha^{l+2}(b_{21(0)22}^{''})[\alpha^{l-1}(b_{22\,2}^{''})\alpha^{-2}(h_{22})]\\
&=\beta^{-1}(b_1)\rtimes_\tau[\alpha^{n-1}(b_{211(-1)}b_{212(-1)})\alpha^{l-2}(b_{22}^{'})]\alpha^{-2}(h_1)\otimes\beta^2(b_{211(0)})\rtimes_\tau[\alpha^{n+2}(b_{212(0)1(-1)})\\
&\quad\alpha^{l+1}(b_{212(0)2}^{'})][\alpha^{l-1}(b_{22\,1}^{''})\alpha^{-2}(h_{21})]\otimes\beta^4(b_{212(0)1(0)})\rtimes_\tau\alpha^{l+2}(b_{212(0)2}^{''})[\alpha^{l-1}(b_{22\,2}^{''})\alpha^{-2}(h_{22})]\\
&=\beta^{-1}(b_1)\rtimes_\tau[\alpha^{n-1}(b_{211(-1)}(b_{2121(-1)}b_{2122(-1)}))\alpha^{l-2}(b_{22}^{'})]\alpha^{-2}(h_1)\otimes\beta^2(b_{211(0)})\\
&\quad\rtimes_\tau[\alpha^{n+2}(b_{2121(0)(-1)})\alpha^{l+1}(b_{2122(0)}^{'})][\alpha^{l-1}(b_{22\,1}^{''})\alpha^{-2}(h_{21})]\otimes\beta^4(b_{2121(0)(0)})\\
&\quad\rtimes_\tau\alpha^{l+2}(b_{2122(0)}^{''})[\alpha^{l-1}(b_{22\,2}^{''})\alpha^{-2}(h_{22})]\\
&=b_{11}\rtimes_\tau[\alpha^{n-1}(b_{121(-1)})(\alpha^{n-2}(b_{122(-1)})\alpha^{n-3}(b_{21(-1)}))\alpha^{l-2}(b_{22}^{'})]\alpha^{-2}(h_1)\otimes\beta^2(b_{121(0)})\\
&\quad\rtimes_\tau[\alpha^{n+1}(b_{122(0)(-1)})\alpha^{l-1}(b_{21(0)}^{'})][\alpha^{l-1}(b_{22\,1}^{''})\alpha^{-2}(h_{21})]\otimes\beta^3(b_{122(0)(0)})\\
&\quad\rtimes_\tau\alpha^{l}(b_{21(0)}^{''})[\alpha^{l-1}(b_{22\,2}^{''})\alpha^{-2}(h_{22})]\\
&=b_{11}\rtimes_\tau[\alpha^{n-1}(b_{121(-1)})(\alpha^{n-2}(b_{122(-1)})\alpha^{n-3}(b_{21(-1)}))^{l-2}(b_{22}^{'})]\alpha^{-2}(h_1)\otimes\beta^2(b_{121(0)})\\
&\quad\rtimes_\tau[\alpha^{n+1}(b_{122(0)(-1)})\alpha^{l-1}(b_{21(0)}^{'})][\alpha^{l-1}(b_{22\,1}^{''})\alpha^{-2}(h_{21})]\otimes\beta^3(b_{122(0)(0)})\\
&\quad\rtimes_\tau\alpha^{l}(b_{21(0)}^{''})[\alpha^{l-1}(b_{22\,2}^{''})\alpha^{-2}(h_{22})]\\
&=b_{11}\rtimes_\tau[\alpha^{n-1}(b_{121(-1)}b_{122(-1)})[\alpha^{n-2}(b_{21(-1)})\alpha^{l-3}(b_{22}^{'})]]\alpha^{-2}(h_1)\otimes\beta^2(b_{121(0)})\rtimes_\tau(\alpha^{n+1}(b_{122(0)(-1)})\\
&\quad\alpha^{l-2}(b_{21(0)}^{'}b_{22\,1}^{''}))\alpha^{-1}(h_{21})\otimes\beta^3(b_{122(0)(0)})\rtimes_\tau[\alpha^{l-1}(b_{21(0)}^{''})\alpha^{l-1}(b_{22\,2}^{''})]\alpha^{-1}(h_{22})\\
&\overset{(3)}{=}b_{11}\rtimes_\tau[\alpha^{n-1}(b_{121(-1)}b_{122(-1)})[\alpha^{l-3}(b_{21}^{'})\alpha^{l-2}(b_{22\,1}^{'})]]\alpha^{-2}(h_1)\otimes\beta^2(b_{121(0)})\\
&\quad\rtimes_\tau[\alpha^{n+1}(b_{122(0)(-1)})[\alpha^{l-3}(b_{21}^{''})\alpha^{l-2}(b_{22\,2}^{'})]]\alpha^{-1}(h_{21})\otimes\beta^3(b_{122(0)(0)})\rtimes_\tau\alpha^{l-1}(b_{22}^{''})\alpha^{-1}(h_{22})\\
&=b_{11}\rtimes_\tau[\alpha^{n-1}(b_{121(-1)}b_{122(-1)})[\alpha^{l-3}(b_{21}^{'})\alpha^{l-2}(b_{22\,1}^{'})]]\alpha^{-1}(h_{11})\otimes\beta^2(b_{121(0)})\\
&\quad\rtimes_\tau[\alpha^{n+1}(b_{122(0)(-1)})[\alpha^{l-3}(b_{21}^{''})\alpha^{l-2}(b_{22\,2}^{'})]]\alpha^{-1}(h_{12})\otimes\beta^3(b_{122(0)(0)})\rtimes_\tau\alpha^{l-1}(b_{22}^{''})\alpha^{-2}(h_2)\\
&=\beta^{-1}(b_1)\rtimes_\tau[\alpha^{n-2}(b_{21(-1)})\alpha^{n}(b_{2211(-1)})[\alpha^{l-1}(b_{2212}^{'})\alpha^{l-1}(b_{222\,1}^{'})]]\alpha^{-1}(h_{11})\otimes\beta(b_{21(0)})\\
&\quad\rtimes_\tau[\alpha^{n+2}(b_{2211(0)(-1)})[\alpha^{l-1}(b_{2212}^{''})\alpha^{l-1}(b_{222\,2}^{'})]]\alpha^{-1}(h_{12})\otimes\beta^4(b_{2211(0)(0)})\rtimes_\tau\alpha^{l}(b_{222}^{''})\alpha^{-2}(h_2)\\
&=\beta^{-1}(b_1)\rtimes_\tau[[\alpha^{n-2}(b_{21(-1)})[\alpha^{n-1}(b_{2211(-1)})\alpha^{l-2}(b_{2212}^{'})]]\alpha^{l}(b_{222\,1}^{'})]\alpha^{-1}(h_{11})\otimes\beta(b_{21(0)})\\
&\quad\rtimes_\tau[[\alpha^{n+1}(b_{2211(0)(-1)})\alpha^{l-1}(b_{2212}^{''})]\alpha^{l}(b_{222\,2}^{'})]\alpha^{-1}(h_{12})\otimes\beta^4(b_{2211(0)(0)})\rtimes_\tau\alpha^{l}(b_{222}^{''})\alpha^{-2}(h_2)\\
&\overset{(2)}{=}\beta^{-1}(b_1)\rtimes_\tau[[\alpha^{n-2}(b_{21(-1)})[\alpha^{l-2}(b_{2211}^{'})\alpha^{n}(b_{2212(-1)1})]]\alpha^{l}(b_{222\,1}^{'})]\alpha^{-1}(h_{11})\otimes\beta(b_{21(0)})\\
&\quad\rtimes_\tau[[\alpha^{l-1}(b_{2211}^{''})\alpha^{n+1}(b_{2212(-1)2})]\alpha^{l}(b_{222\,2}^{'})]\alpha^{-1}(h_{12})\otimes\beta^3(b_{2212(0)})\rtimes_\tau\alpha^{l}(b_{222}^{''})\alpha^{-2}(h_2)\\
&=b_{11}\rtimes_\tau[[\alpha^{n-1}(b_{121(-1)})[\alpha^{l-3}(b_{122}^{'})\alpha^{n-2}(b_{21(-1)1})]]\alpha^{l-1}(b_{22\,1}^{'})]\alpha^{-1}(h_{11})\otimes\beta^2(b_{121(0)})\\
&\quad\rtimes_\tau[[\alpha^{l-2}(b_{122}^{''})\alpha^{n-1}(b_{21(-1)2})]\alpha^{l-1}(b_{22\,2}^{'})]\alpha^{-1}(h_{12})\otimes\beta(b_{21(0)})\rtimes_\tau\alpha^{l-1}(b_{22}^{''})\alpha^{-2}(h_2)\\
&=b_{11}\rtimes_\tau[\alpha^{n}(b_{121(-1)})\alpha^{l-1}(b_{122}^{'})][[\alpha^{n-1}(b_{21(-1)1})\alpha^{l-2}(b_{22\,1}^{'})]\alpha^{-2}(h_{11})]\otimes\beta^2(b_{121(0)})\\
&\quad\rtimes_\tau\alpha^{l}(b_{122}^{''})[(\alpha^{n-1}(b_{21(-1)2})\alpha^{l-2}(b_{22\,2}^{'}))\alpha^{-2}(h_{12})]\otimes\beta(b_{21(0)})\rtimes_\tau\alpha^{l-1}(b_{22}^{''})\alpha^{-2}(h_2)\\
&=[\triangle_{B\rtimes_\tau H}\otimes(\beta^{-1}\otimes\alpha^{-1})](b_1\rtimes_\tau[\alpha^{n}(b_{21(-1)})\alpha^{l-1}(b_{22}^{'})]\alpha^{-1}(h_1)\otimes\beta^2(b_{21(0)})\rtimes_\tau\alpha^{l}(b_{22}^{''})\alpha^{-1}(h_2))\\
&=[\triangle_{B\rtimes_\tau H}\otimes(\beta^{-1}\otimes\alpha^{-1})]\,\triangle_{B\rtimes_\tau H}\,(b\rtimes_\tau h).
\end{aligned}$$

In summary, $(B\rtimes_\tau H,\beta\otimes\alpha)$ is a monoidal Hom-coalgebra. □

Example 2.4. Let $H = \langle g, x | 1 \cdot g = g, 1 \cdot x = -x, g^2 = 1, x^2 = 0, xg = -gx \rangle$. Define the comultipication, counit, antipode and Hom-map by

$$\begin{aligned} \triangle(1) = 1 \otimes 1, \quad & \varepsilon(1) = 1, \quad S(1) = 1, \quad \alpha(1) = 1 \\ \triangle(g) = g \otimes g, \quad & \varepsilon(g) = 1, \quad S(g) = g, \quad \alpha(g) = g \\ \triangle(x) = -x \otimes 1 + 1 \otimes (-x), \quad & \varepsilon(x) = 0, \quad S(x) = -x, \quad \alpha(x) = -x \\ \triangle(gx) = -gx \otimes g + g \otimes (-gx), \quad & \varepsilon(gx) = 0, \quad S(gx) = -xg, \quad \alpha(gx) = -gx. \end{aligned}$$

Then (H, α) be a Hom-Hopf algebra. Let $G = \langle c \rangle$ be a infinite cyclic group generated by c and $A = K[G]$ be a group algebra. Define the comultipication and counit by

$$\triangle(c^k) = c^k \otimes c^k, \varepsilon(c^k) = 1.$$

Define the comodule map $\rho : A \to H \otimes A$ by $\rho(c^i) = 1_H \otimes c^i$ and $\tau : A \to H \otimes H$ by $\tau(c^k) = kx \otimes x + 1_H \otimes 1_H$. Easy to see that$(A, \mathrm{id})$ is a Hom-comodule coalgebra and τ satisfies the conditions in Theorem 2.3. Thus we have a (n, l)-Hom-crossed coproduct coalgebra $(A \rtimes_\tau H, \mathrm{id} \otimes \alpha)$.

We will always assume that τ is convolution inversible and its convolution inverse is $\tau^{-1} \in Hom(B, H \otimes H)$. For all $b \in B$, we write $\tau^{-1}(b) = b^s \otimes b^{ss} \in H \otimes H$. As the convolution inverse of τ, there is the following equation holds:

$$b_1^{'} b_2^s \otimes b_1^{''} b_2^{ss} = b_1^s b_2^{'} \otimes b_1^{ss} b_2^{''} = \varepsilon_B(b) 1_{H \otimes H}. \tag{2.1}$$

Theorem 2.5. *Let $(B \rtimes_\tau H, \beta \otimes \alpha)$ be a (n, l)-Hom-crossed coproduct and τ be convolution inversible. Then the following equations hold: for any $b \in B$,*

(1) $[\alpha^{l-1}(b_{11}^{'})\alpha^l(b_{12}^{'}{}_1)]\alpha^{l-1}(b_2{}^s) \otimes [\alpha^{l-2}(b_{11}^{''})\alpha^{l-1}(b_{12}^{'}{}_2)]\alpha^{l-1}(b_2^{ss}{}_1) \otimes \alpha^{l-1}(b_{12}^{''} b_2^{ss}{}_2)$
$= \alpha^n(b_{(-1)}) \otimes \alpha^{l-1}(b_{(0)}^{'}) \otimes \alpha^{l-1}(b_{(0)}^{''})$;

(2) $\alpha^{l-2}(b_1^{'})(\alpha^{l-1}(b_{21}{}^s{}_1)\alpha^{l-2}(b_{22}{}^s)) \otimes b_1^{''}{}_1(b_{21}{}^s{}_2\alpha^{-1}(b_{22}{}^{ss})) \otimes b_1^{''}{}_2 b_{21}{}^{ss}$
$= \alpha^{n-1}(b_{(-1)}) \otimes b_{(0)}^{s} \otimes b_{(0)}^{ss}$;

(3) $\alpha^{l-2}(b_1^s)(\alpha^{n-1}(b_{21(-1)})\alpha^{l-2}(b_{22}^{'})) \otimes \alpha^{l-2}(b_1^{ss})(\alpha^n(b_{21(0)(-1)})\alpha^{l-2}(b_{22}^{''})) \otimes \alpha^3(b_{21(0)(0)})$
$= \alpha^n(b_{(-1)1}) \otimes \alpha^n(b_{(-1)2}) \otimes b_{(0)}$;

(4) $(\alpha^{l-2}(b_{11}^{'})\alpha^n(b_{12(-1)1}))\alpha^{l-2}(b_2^s) \otimes (\alpha^{l-2}(b_{11}^{''})\alpha^n(b_{12(-1)2}))\alpha^{l-2}(b_2{}^{ss}) \otimes \beta(b_{12(0)})$
$= \alpha^{n-1}(b_{(-1)}) \otimes \alpha^n(b_{(0)(-1)}) \otimes b_{(0)(0)}$.

Proof. For any $b \in B$, we have

$$\begin{aligned} & \alpha^n(b_{(-1)}) \otimes \alpha^{l-1}(b_{(0)}^{'}) \otimes \alpha^{l-1}(b_{(0)}^{''}) \\ =& \alpha^n(b_{1(-1)})\alpha^{l-1}(b_{21}^{'} b_{22}^{s}) \otimes \alpha^{l-1}(b_{1(0)}^{'})\alpha^{l-1}(b_{21}^{''}{}_1 b_{22}^{ss}{}_1) \otimes \alpha^{l-1}(b_{1(0)}^{''})\alpha^{l-1}(b_{21}^{''}{}_2 b_{22}^{ss}{}_2) \\ =& [\alpha^n(b_{11(-1)})\alpha^{l-1}(b_{12}^{'})]\alpha^{l-1}(b_2{}^s) \otimes \alpha^{l-1}(b_{11(0)}^{'} b_{12}^{''}{}_1)\alpha^{l-1}(b_2^{ss}{}_1) \otimes \alpha^{l-1}(b_{11(0)}^{''} b_{12}^{''}{}_2)\alpha^{l-1}(b_2^{ss}{}_2) \\ =& [\alpha^{l-1}(b_{11}^{'})\alpha^l(b_{12}^{'}{}_1)]\alpha^{l-1}(b_2^s) \otimes [\alpha^{l-2}(b_{11}^{''})\alpha^{l-1}(b_{12}^{'}{}_2)]\alpha^{l-1}(b_2^{ss}{}_1) \otimes \alpha^{l-1}(b_{12}^{''} b_2^{ss}{}_2). \end{aligned}$$

So (1) holds. In a similar way, we can also prove (3) and (4):

$$\begin{aligned} & \alpha^n(b_{(-1)1}) \otimes \alpha^n(b_{(-1)2}) \otimes b_{(0)} \\ =& \alpha^{l-2}(b_1^s)[\alpha^{l-2}(b_{21}^{'})\alpha^n(b_{22(-1)1})] \otimes \alpha^{l-2}(b_1^{ss})[\alpha^{l-2}(b_{21}^{''})\alpha^n(b_{22(-1)2})] \otimes \alpha^2(b_{22(0)}) \\ =& \alpha^{l-2}(b_1^s)(\alpha^{n-1}(b_{21(-1)})\alpha^{l-2}(b_{22}^{'})) \otimes \alpha^{l-2}(b_1^{ss})(\alpha^n(b_{21(0)(-1)})\alpha^{l-2}(b_{22}^{''})) \otimes \alpha^3(b_{21(0)(0)}), \end{aligned}$$

and

$$
\begin{aligned}
&\alpha^{n-1}(b_{(-1)}) \otimes \alpha^{n}(b_{(0)(-1)}) \otimes b_{(0)(0)}\\
=&[\alpha^{n-1}(b_{11(-1)})\alpha^{l-2}(b_{12}{}^{'})]\alpha^{l-2}(b_2^{s}) \otimes [\alpha^{n}(b_{11(0)(-1)})\alpha^{l-2}(b_{12}{}^{''})]\alpha^{l-2}(b_2^{ss}) \otimes \beta^{2}(b_{11(0)(0)})\\
=&(\alpha^{l-2}(b_{11}{}^{'})\alpha^{n}(b_{12(-1)1}))\alpha^{l-2}(b_2^{s}) \otimes (\alpha^{l-2}(b_{11}{}^{''})\alpha^{n}(b_{12(-1)2}))\alpha^{l-2}(b_2^{ss}) \otimes \beta(b_{12(0)}).
\end{aligned}
$$

Besides, according to the Theorem 2.3(3), we will obtain the following equation:

$$
\begin{aligned}
&\alpha^{l-1}(b_1^{s}{}_1)\alpha^{l-2}(b_2^{s}) \otimes \alpha^{l-1}(b_1^{s}{}_2)\alpha^{l-2}(b_2^{ss}) \otimes \alpha^{l-1}(b_1^{ss})\\
=&\alpha^{l-2}(b_1^{s})\alpha^{n-1}(b_{2(-1)}) \otimes \alpha^{l-1}(b_1^{ss}{}_1)\alpha^{l-1}(b_{2(0)}{}^{s}) \otimes \alpha^{l-1}(b_1^{ss}{}_2)\alpha^{l-1}(b_{2(0)}{}^{ss}).
\end{aligned}
$$

According to the above equation, we can prove that

$$
\begin{aligned}
&\alpha^{n-1}(b_{(-1)}) \otimes b_{(0)}{}^{s} \otimes b_{(0)}{}^{ss}\\
=&\alpha^{l-2}(b_1{}^{'})[\alpha^{l-2}(b_{21}{}^{s})\alpha^{n-1}(b_{22(-1)})] \otimes b_1{}^{''}{}_1(b_{21}^{ss}{}_1 b_{22(0)}{}^{s}) \otimes b_1{}^{''}{}_2(b_{21}^{ss}{}_2 b_{22(0)}{}^{ss})\\
=&\alpha^{l-2}(b_1{}^{'})[\alpha^{l-1}(b_{21}^{s}{}_1)\alpha^{l-2}(b_{22}{}^{s})] \otimes b_1{}^{''}{}_1(b_{21}{}^{s}{}_2\alpha^{-1}(b_{22}{}^{ss})) \otimes b_1{}^{''}{}_2 b_{21}^{ss}.
\end{aligned}
$$

So (2) holds.

The proof is completed. □

3. Radford $[n,(n,l)]$-biproduct

Let $n, l \in \mathbb{Z}$. In this section, we will prove our Radford $[n,(n,l)]$-biproduct theorem.

Let (H,α) be a monoidal Hom-bialgebra and (B,β) a left (H,α)-Hom-module algebra. Then $(B\natural H, \beta\natural\alpha)$ is called n-Hom-smash product if

(1) $B\natural H = B \otimes H$, as a linear space;

(2) Hom-multiplication is given by:

$$(a\natural h)(b\natural g) = a(\alpha^{n}(h_1)\cdot\beta^{-1}(b))\natural\alpha(h_2)g$$

for all $a\natural h, b\natural g \in B\natural H$.

Proposition 3.1. *With notations above. Then $(B\natural H, \beta\natural\alpha)$ is a monoidal Hom-algebra with unit $1_B\natural 1_H$.*

Proof. It is obvious that $(\beta\otimes\alpha)(1_B\natural 1_H) = 1_B\natural 1_H$. For any $a\natural h, b\natural g, c\natural l \in B\natural H$, we have

$$(\beta\otimes\alpha)[(a\natural h)(b\natural g)] = [(\beta\otimes\alpha)(a\natural h)][(\beta\otimes\alpha)(b\natural g)].$$

Now we prove Hom-associativity, on the one hand,

$$
\begin{aligned}
&[\beta(a)\natural\alpha(h)][(b\natural g)(c\natural l)]\\
=&[\beta(a)\natural\alpha(h)][b(\alpha^{n}(g_1)\cdot\beta^{-1}(c))\natural\alpha(g_2)l]\\
=&\beta(a)[\alpha^{n+1}(h_1)\cdot\beta^{-1}(b)(\alpha^{n-1}(g_1)\cdot\beta^{-2}(c))]\natural\alpha^{2}(h_2)(\alpha(g_2)l)\\
=&\beta(a)[(\alpha^{n+1}(h_{11})\cdot\beta^{-1}(b))(\alpha^{n+1}(h_{12})\cdot(\alpha^{n-1}(g_1)\cdot\beta^{-2}(c)))]\natural\alpha^{2}(h_2)(\alpha(g_2)l)\\
=&\beta(a)[(\alpha^{n+1}(h_{11})\cdot\beta^{-1}(b))((\alpha^{n}(h_{12})\alpha^{n-1}(g_1))\cdot\beta^{-2}(c))]\natural\alpha(h_2g_2)\alpha(l).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&[(a\natural h)(b\natural g)][\beta(c)\natural\alpha(l)]\\
=&[a(\alpha^{n}(h_1)\cdot\beta^{-1}(b))\natural\alpha(h_2)g][\beta(c)\natural\alpha(l)]\\
=&[a(\alpha^{n}(h_1)\cdot\beta^{-1}(b))](\alpha^{n+1}(h_{21})\alpha^{n}(g_1)\cdot c)\natural[\alpha^{2}(h_{22})\alpha(g_2)]\alpha(l)\\
=&\beta(a)[(\alpha^{n}(h_1)\cdot\beta^{-1}(b))(\alpha^{n}(h_{21})\alpha^{n-1}(g_1)\cdot\beta^{-1}(c))]\natural[\alpha^{2}(h_{22})\alpha(g_2)]\alpha(l)\\
=&\beta(a)[(\alpha^{n+1}(h_{11})\cdot\beta^{-1}(b))((\alpha^{n}(h_{12})\alpha^{n-1}(g_1))\cdot\beta^{-2}(c))]\natural\alpha(h_2g_2)\alpha(l).
\end{aligned}
$$

Thus,

$$[\beta(a)\sharp\alpha(h)][(b\sharp g)(c\sharp l)] = [(a\sharp h)(b\sharp g)][\beta(c)\sharp\alpha(l)].$$

Therefore, $(B\sharp H, \beta\otimes\alpha)$ is monoidal Hom-algebra. □

Definition 3.2. Let $(B\rtimes_\tau H, \beta\otimes\alpha)$ be (n,l)-Hom-crossed coproduct. Then τ is called a *twisted module cycle* if for any $a, b\in B$,

$$\alpha^n(a^{'})\cdot b_1\otimes(\alpha^n(a^{''})\cdot b_2)_{(-1)}\otimes(\alpha^n(a^{''})\cdot b_2)_{(0)} = \alpha^n(a^{'})\cdot b_1\otimes\alpha^n(a^{''}{}_1)b_{2(-1)}\otimes\alpha^n(a^{''}{}_2)\cdot b_{2(0)}.$$

Lemma 3.3. *Let $(B\rtimes_\tau H, \beta\otimes\alpha)$ be (n,l)-Hom-crossed coproduct with a twisted module cycle τ, then the following equations holds: for any $a, b\in B$,*

(1) $\alpha^n(a^{'})\cdot\beta^{-1}(b)\otimes\alpha^n(a^{''}) = \varepsilon_B(a)b\otimes 1_H$;

(2) $\alpha(a_1)(\alpha^n(a_2^{'})\cdot\beta^{-1}(b))\otimes\alpha^n(a_2^{''}) = ab\otimes 1_H$.

Proof. The proof is straightforward. □

Theorem 3.4. *Let (H,α) be a monoidal Hom-bialgebra and (B,β) a monoidal Hom-coalgebra. Suppose that (H,α) weakly coact on (B,β) and (B,β) is a left (H,α)-Hom-module algebra with the module structure map $\cdot : H\otimes B\to B$. Let $(B\rtimes_\tau H, \beta\otimes\alpha)$ be a (n,l)-Hom-crossed coproduct with a twisted module cycle τ and $(B\sharp H, \beta\otimes\alpha)$ a n-Hom-smash product. We use notation $(B^{\#}_{\rtimes_\tau}H, \beta\otimes\alpha)$ to denote the tensor product $B\otimes H$ with both the coalgebra structure $B\rtimes_\tau H$ and the algebra structure $B\sharp H$. Then the following conditions are equivalent:*

(1) $(B^{\#}_{\rtimes_\tau}H, \beta\otimes\alpha)$ is a monoidal Hom-bialgebra.

(2) The conditions:

(B1) τ and τ^{-1} are Hom-algebra map,

(B2) $\rho(1_B) = 1_H\otimes 1_B$,

(B3) ε_B is Hom-algebra map,

(B4) $\triangle_B(1_B) = 1_B\otimes 1_B$

(B5) $\triangle_B(h\cdot b) = h_1\cdot b_1\otimes h_2\cdot b_2$, $\varepsilon_B(h\cdot b) = \varepsilon_H(h)\varepsilon_B(b)$,

(B6) $\triangle_A(ab) = a_1[(\alpha^{2n}(a_{21(-1)})\alpha^{n+l-1}(a_{22}^{'}))\cdot\beta^{-1}(b_1)]\otimes\beta^2(a_{21(0)})[\alpha^{n+l}(a_{22}^{''})\cdot\beta^{-1}(b_2)]$,

(B7) $\alpha^{n-1}[(\alpha^{n+1}(h_1)\cdot b)_{(-1)}]h_2\otimes(\alpha^{n+1}(h_1)\cdot b)_{(0)} = h_1\alpha^n(b_{(-1)})\otimes(\alpha^n(h_2)\cdot b_{(0)})$,

(B8) $[\alpha^n(a_{1(-1)})\alpha^{l-1}(a_2^{'})]\alpha^n(b_{(-1)})\otimes\beta(a_{1(0)})(\alpha^{n+l-1}(a_2^{''})\cdot\beta^{-1}(b_{(0)})) = \alpha^n[(ab)_{(-1)}]\otimes(ab)_{(0)}$,

(B9) $\alpha^{l-1}[(\alpha^n(h_1)\cdot\beta^{-1}(b))^{'}]\alpha(h_{21})\otimes\alpha^{l-1}[(\alpha^n(h_1)\cdot\beta^{-1}(b))^{''}]\alpha(h_{22}) = h_1\alpha^{l-1}(b^{'})\otimes h_2\alpha^{l-1}(b^{''})$.

Proof. (1)⇒(2) follows from the similar calculations to those of [21, Theorem 1]. So we only need to show (2)⇒(1). Assume (2) holds, then $\varepsilon_{B\rtimes_\tau H}$ is Hom-algebra map by (B2) and (B3). By (B4) and (B5), we have $\triangle_{B\rtimes_\tau H}(1_B\sharp 1_H) = 1_B\sharp 1_H\otimes 1_B\sharp 1_H$. Next we proof for any $a\sharp h, b\sharp g\in B^{\#}_{\rtimes_\tau}H$,

$$\triangle_{B\rtimes_\tau H}[(a\sharp h)(b\sharp g)] = \triangle_{B\rtimes_\tau H}(a\sharp h)\ \triangle_{B\rtimes_\tau H}\ (b\sharp g).$$

It is enough to verify the following equations:

$$\triangle_{B\rtimes_\tau H}\ [(a\sharp 1_H)(b\sharp 1_H)] = \triangle_{B\rtimes_\tau H}(a\sharp 1_H)\ \triangle_{B\rtimes_\tau H}\ (b\sharp 1_H); \tag{3.1}$$

$$\triangle_{B\rtimes_\tau H}\ [(a\sharp 1_H)(1_B\sharp g)] = \triangle_{B\rtimes_\tau H}(a\sharp 1_H)\ \triangle_{B\rtimes_\tau H}\ (1_B\sharp g); \tag{3.2}$$

$$\triangle_{B\rtimes_\tau H}\ [(1_B\sharp h)(b\sharp 1_H)] = \triangle_{B\rtimes_\tau H}(1_B\sharp h)\ \triangle_{B\rtimes_\tau H}\ (b\sharp 1_H); \tag{3.3}$$

$$\triangle_{B\rtimes_\tau H}\ [(1_B\sharp h)(1_B\sharp g)] = \triangle_{B\rtimes_\tau H}(1_B\sharp h)\ \triangle_{B\rtimes_\tau H}\ (1_B\sharp g). \tag{3.4}$$

In fact, if (3.1)–(3.4) holds, then

$$\triangle_{B\rtimes_\tau H}[(a\sharp h)(b\sharp g)] = \triangle_{B\rtimes_\tau H}(a\sharp h)\ \triangle_{B\rtimes_\tau H}\ (b\sharp g).$$

Next we proof (3.1)–(3.4) holds, that is

$$
\begin{aligned}
&\triangle_{B\rtimes_\tau H}[(a\sharp 1_H)(b\sharp 1_H)] = \triangle_{B\rtimes_\tau H}(ab\sharp 1_H)\\
&= (ab)_1 \rtimes_\tau \alpha^{n+1}[(ab)_{21(-1)}]\alpha^l[(ab)_{22}{}'] \otimes \beta^2[(ab)_{21(0)}] \rtimes_\tau \alpha^{l+1}[(ab)_{22}{}'']\\
&\overset{(B6)}{=} a_1[\alpha^{2n}(a_{21(-1)})\alpha^{n+l-1}(a_{22}{}') \cdot \beta^{-1}(b_1)] \rtimes_\tau [\beta^{n+3}(a_{21(0)1})[(\alpha^{3n+2}(a_{21(0)21(-1)})\\
&\quad \alpha^{2n+l+1}(a_{21(0)22}{}'))\alpha^{2n+l}(a_{22}{}''{}_1) \cdot \beta^n(b_{21})]]_{(-1)}[\beta^{l+4}(a_{21(0)21(0)})[\alpha^{n+2l+1}(a_{21(0)22}{}'')\\
&\quad \alpha^{n+2l-1}(a_{22}{}''{}_2) \cdot \beta^{l-1}(b_{22})]]' \otimes [\beta^4(a_{21(0)1})[(\alpha^{2n+3}(a_{21(0)21(-1)})\alpha^{n+l+2}(a_{21(0)22}{}'))\\
&\quad \alpha^{n+l+1}(a_{22}{}''{}_1) \cdot \beta(b_{21})]]_{(0)} \rtimes_\tau [\beta^{l+5}(a_{21(0)21(0)})[\alpha^{n+2l+2}(a_{21(0)22}{}'')\alpha^{n+2l}(a_{22}{}''{}_2) \cdot \beta^l(b_{22})]]''\\
&= a_1[\alpha^{2n}(a_{21(-1)})\alpha^{n+l-1}(a_{22}{}') \cdot \beta^{-1}(b_1)] \rtimes_\tau \alpha^{n-1}[[\beta^4(a_{21(0)1})[(\alpha^{2n+3}(a_{21(0)21(-1)})\\
&\quad \alpha^{n+l+2}(a_{21(0)22}{}'))\alpha^{n+l+1}(a_{22}{}''{}_1) \cdot \beta(b_{21})]]_{(-1)}][\alpha^{l+4}(a_{21(0)21(0)}{}')[\alpha^{n+2l+1}(a_{21(0)22}{}'')\\
&\quad \alpha^{n+2l-1}(a_{22}{}''{}_2) \cdot \beta^{l-1}(b_{22})]'] \otimes [\beta^4(a_{21(0)1})[(\alpha^{2n+3}(a_{21(0)21(-1)})\alpha^{n+l+2}(a_{21(0)22}{}'))\\
&\quad \alpha^{n+l+1}(a_{22}{}''{}_1) \cdot \beta(b_{21})]]_{(0)} \rtimes_\tau \alpha[\alpha^{l+4}(a_{21(0)21(0)}{}'')[\alpha^{n+2l+1}(a_{21(0)22}{}'')\alpha^{n+2l-1}(a_{22}{}''{}_2) \cdot \beta^{l-1}(b_{22})]'']\\
&\overset{(B8)}{=} a_1[\alpha^{2n}(a_{21(-1)})\alpha^{n+l-1}(a_{22}{}') \cdot \beta^{-1}(b_1)] \rtimes_\tau \alpha^{-1}[[\alpha^{n+4}(a_{21(0)11(-1)})\alpha^{l+3}(a_{21(0)12}{}')]\\
&\quad \alpha^{n+1}[((\alpha^{2n+2}(a_{21(0)21(-1)})\alpha^{n+l+1}(a_{21(0)22}{}'))\alpha^{n+l}(a_{22}{}''{}_1) \cdot b_{21})_{(-1)}]][\alpha^{l+4}(a_{21(0)21(0)}{}')\\
&\quad (\alpha^{n+2l+1}(a_{21(0)22}{}'')\alpha^{n+2l-1}(a_{22}{}''{}_2) \cdot \beta^{l-1}(b_{22}))'] \otimes \beta^5(a_{21(0)11(0)})[\alpha^{n+l+3}(a_{21(0)12}{}'') \cdot\\
&\quad [[(\alpha^{2n+2}(a_{21(0)21(-1)})\alpha^{n+l+1}(a_{21(0)22}{}'))\alpha^{n+l}(a_{22}{}''{}_1) \cdot b_{21}]_{(0)}]] \rtimes_\tau \alpha[\alpha^{l+4}(a_{21(0)21(0)}{}'')\\
&\quad (\alpha^{n+2l+1}(a_{21(0)22}{}'')\alpha^{n+2l-1}(a_{22}{}''{}_2) \cdot \beta^{l-1}(b_{22}))'']\\
&= a_1[\alpha^{2n}(a_{211(-1)}a_{212(-1)})\alpha^{n+l-1}(a_{22}{}') \cdot \beta^{-1}(b_1)] \rtimes_\tau \alpha^{-1}[[\alpha^{n+4}(a_{211(0)1(-1)})\alpha^{l+3}(a_{211(0)2}{}')]\\
&\quad \alpha^{n+1}[((\alpha^{2n+2}(a_{212(0)1(-1)})\alpha^{n+l+1}(a_{212(0)2}{}'))\alpha^{n+l}(a_{22}{}''{}_1) \cdot b_{21})_{(-1)}]][\alpha^{l+4}(a_{212(0)1(0)}{}')\\
&\quad (\alpha^{n+2l+1}(a_{212(0)2}{}'')\alpha^{n+2l-1}(a_{22}{}''{}_2) \cdot \beta^{l-1}(b_{22}))'] \otimes \beta^5(a_{211(0)1(0)})[\alpha^{n+l+3}(a_{211(0)2}{}'') \cdot\\
&\quad [[(\alpha^{2n+2}(a_{212(0)1(-1)})\alpha^{n+l+1}(a_{212(0)2}{}'))\alpha^{n+l}(a_{22}{}''{}_1) \cdot b_{21}]_{(0)}]] \rtimes_\tau \alpha[\alpha^{l+4}(a_{212(0)1(0)}{}'')\\
&\quad (\alpha^{n+2l+1}(a_{212(0)2}{}'')\alpha^{n+2l-1}(a_{22}{}''{}_2) \cdot \beta^{l-1}(b_{22}))'']\\
&= a_1[\alpha^{2n}((a_{2111(-1)}a_{2112(-1)})(a_{2121(-1)}a_{2122(-1)}))\alpha^{n+l-1}(a_{22}{}') \cdot \beta^{-1}(b_1)] \rtimes_\tau\\
&\quad \alpha^{-1}[[\alpha^{n+4}(a_{2111(0)(-1)})\alpha^{l+3}(a_{2112(0)}{}')]\alpha^{n+1}[((\alpha^{2n+2}(a_{2121(0)(-1)})\alpha^{n+l+1}(a_{2122(0)}{}'))\\
&\quad \alpha^{n+l}(a_{22}{}''{}_1) \cdot b_{21})_{(-1)}]][\alpha^{l+4}(a_{2121(0)(0)}{}')(\alpha^{n+2l+1}(a_{2122(0)}{}'')\alpha^{n+2l-1}(a_{22}{}''{}_2) \cdot \beta^{l-1}(b_{22}))'] \otimes\\
&\quad \beta^5(a_{2111(0)(0)})[\alpha^{n+l+3}(a_{2112(0)}{}'') \cdot [[(\alpha^{2n+2}(a_{2121(0)(-1)})\alpha^{n+l+1}(a_{2122(0)}{}'))\alpha^{n+l}(a_{22}{}''{}_1) \cdot b_{21}]_{(0)}]]\\
&\quad \rtimes_\tau \alpha[\alpha^{l+4}(a_{2121(0)(0)}{}'')[\alpha^{n+2l+1}(a_{2122(0)}{}'')\alpha^{n+2l-1}(a_{22}{}''{}_2) \cdot \beta^{l-1}(b_{22})]'']\\
&= a_1[\alpha^{2n}(a_{211(-1)}a_{212(-1)})[\alpha^{2n}(a_{221(-1)})(\alpha^{2n}(a_{2221(-1)})\alpha^{n+l-1}(a_{2222}{}'))] \cdot \beta^{-1}(b_1)] \rtimes_\tau\\
&\quad \alpha^{-1}[(\alpha^{n+3}(a_{211(0)(-1)})\alpha^{l+2}(a_{212(0)}{}''))\alpha^{n+1}[(\alpha^{2n+2}(a_{221(0)(-1)})\alpha^{n+l+1}(a_{2221(0)}{}'a_{2222}{}''{}_1) \cdot b_{21})_{(-1)}]]\\
&\quad (\alpha^{l+3}(a_{221(0)(0)}{}')(\alpha^{n+2l+1}(a_{2221(0)}{}'')\alpha^{n+2l+1}(a_{2222}{}''{}_2) \cdot \beta^{l-1}(b_{22}))') \otimes \beta^4(a_{211(0)(0)})\alpha^{n+l+2}(a_{212(0)}{}'') \cdot\\
&\quad (\alpha^{2n+2}(a_{221(0)(-1)})\alpha^{n+l+1}(a_{2221(0)}{}'a_{2222}{}''{}_1) \cdot b_{21})_{(0)} \rtimes_\tau \alpha[\alpha^{l+3}(a_{221(0)(0)}{}'')(\alpha^{n+2l+1}(a_{2221(0)}{}''a_{2222}{}''{}_2) \cdot\\
&\quad \beta^{l-1}(b))'']\\
&= a_1[\alpha^{2n}(a_{211(-1)}a_{212(-1)})[\alpha^{2n}(a_{221(-1)})(\alpha^{n+l-1}(a_{2221}{}')\alpha^{n+l}(a_{2222}{}'{}_1))] \cdot \beta^{-1}(b_1)] \rtimes_\tau\\
&\quad \alpha^{-1}[(\alpha^{n+3}(a_{211(0)(-1)})\alpha^{l+2}(a_{212(0)}{}''))\alpha^{n+1}[(\alpha^{2n+2}(a_{221(0)(-1)})(\alpha^{n+l}(a_{2221}{}'')\alpha^{n+l+1}(a_{2222}{}'{}_2)) \cdot\\
&\quad b_{21})_{(-1)}]][\alpha^{l+3}(a_{221(0)(0)}{}')(\alpha^{n+2l+1}(a_{2222}{}'') \cdot \beta^{l-1}(b_{22}))'] \otimes \beta^4(a_{211(0)(0)})\alpha^{n+l+2}(a_{212(0)}{}'') \cdot\\
&\quad (\alpha^{2n+2}(a_{221(0)(-1)})(\alpha^{n+l}(a_{2221}{}'')\alpha^{n+l+1}(a_{2222}{}'{}_2)) \cdot b_{21})_{(0)} \rtimes_\tau \alpha[\alpha^{l+3}(a_{221(0)(0)}{}'')(\alpha^{n+2l+1}(a_{2222}{}'') \cdot\\
&\quad \beta^{l-1}(b))'']\\
&= a_1[\alpha^{2n}(a_{211(-1)}a_{212(-1)})[(\alpha^{2n}(a_{2211(-1)})\alpha^{n+l-1}(a_{2212}{}'))\alpha^{n+l}(a_{222}{}'{}_1)] \cdot \beta^{-1}(b_1)] \rtimes_\tau\\
&\quad \alpha^{-1}[[\alpha^{n+3}(a_{211(0)(-1)})\alpha^{l+2}(a_{212(0)}{}')]\alpha^{n+1}[((\alpha^{2n+2}(a_{2211(0)(-1)})\alpha^{n+l}(a_{2212}{}''))\alpha^{n+l+1}(a_{222}{}'{}_2) \cdot\\
&\quad b_{21})_{(-1)}]][\alpha^{l+4}(a_{2211(0)(0)}{}')(\alpha^{n+2l}(a_{222}{}'') \cdot \beta^{l-1}(b_{22}))'] \otimes \beta^4(a_{211(0)(0)})[\alpha^{n+l+2}(a_{212(0)}{}'') \cdot\\
&\quad [((\alpha^{2n+2}(a_{2211(0)(-1)})\alpha^{n+l}(a_{2212}{}''))\alpha^{n+l+1}(a_{222}{}'{}_2) \cdot b_{21})_{(0)}]] \rtimes_\tau \alpha[\alpha^{l+4}(a_{2211(0)(0)}{}'')(\alpha^{n+2l}(a_{222}{}'') \cdot\\
&\quad \beta^{l-1}(b_{22}))'']
\end{aligned}
$$

$$
\begin{aligned}
&= a_1[\alpha^{2n}(a_{211(-1)}a_{212(-1)})[(\alpha^{n+l-1}(a_{2211}{}')\alpha^{n}(a_{2212(-1)1}))\alpha^{n+l}(a_{222\,1}')]\cdot\beta^{-1}(b_1)]\rtimes_\tau\\
&\quad \alpha^{-1}[[\alpha^{n+3}(a_{211(0)(-1)})\alpha^{l+2}(a_{212(0)}{}')]\alpha^{n+1}[((\alpha^{n+l}(a_{2211}{}'')\alpha^{2n+2}(a_{2212(-1)2}))\\
&\quad \alpha^{n+l+1}(a_{222\,2}')\cdot b_{21})_{(-1)}]][\alpha^{l+3}(a_{2212(0)}{}')(\alpha^{n+2l}(a_{222}{}'')\cdot\beta^{l-1}(b_{22}))']\otimes\beta^4(a_{211(0)(0)})\\
&\quad [\alpha^{n+l+2}(a_{212(0)}{}'')\cdot[((\alpha^{n+l}(a_{2211}{}'')\alpha^{2n+2}(a_{2212(-1)2}))\alpha^{n+l+1}(a_{222\,2}')\cdot b_{21})_{(0)}]]\rtimes_\tau\\
&\quad \alpha[\alpha^{l+3}(a_{2212(0)}{}'')(\alpha^{n+2l}(a_{222}{}'')\cdot\beta^{l-1}(b_{22}))'']\\
&= a_1[\alpha^{2n}(a_{211(-1)}a_{212(-1)})[(\alpha^{n+l-1}(a_{2211}{}')\alpha^{n}(a_{2212(-1)1}))\alpha^{n+l}(a_{222\,1}')]\cdot b_{11}]\rtimes_\tau\\
&\quad \alpha^{-1}[[\alpha^{n+3}(a_{211(0)(-1)})\alpha^{l+2}(a_{212(0)}{}')]\alpha^{n+1}[((\alpha^{n+l}(a_{2211}{}'')\alpha^{2n+2}(a_{2212(-1)2}))\\
&\quad \alpha^{n+l+1}(a_{222\,2}')\cdot b_{12})_{(-1)}]][\alpha^{l+3}(a_{2212(0)}{}')(\alpha^{n+2l}(a_{222}{}'')\cdot\beta^{l-2}(b_{2}))']\otimes\beta^4(a_{211(0)(0)})\\
&\quad [\alpha^{n+l+2}(a_{212(0)}{}'')\cdot[((\alpha^{n+l}(a_{2211}{}'')\alpha^{2n+2}(a_{2212(-1)2}))\alpha^{n+l+1}(a_{222\,2}')\cdot b_{21})_{(0)}]]\rtimes_\tau\\
&\quad \alpha[\alpha^{l+3}(a_{2212(0)}{}'')(\alpha^{n+2l}(a_{222}{}'')\cdot\beta^{l-2}(b_{2}))'']\\
&= a_1[\alpha^{2n+1}(a_{211(-1)}a_{212(-1)})\cdot\beta^{-1}[\alpha^{n+l+2}(a_{2211}{}')\cdot(\alpha^{2n+2}(a_{2212(-1)1})\alpha^{n+l}(a_{222\,1}')\cdot\beta^{-1}(b_{11}))]]\\
&\quad \rtimes_\tau \alpha^{-1}[(\alpha^{n+3}(a_{211(0)(-1)})\alpha^{l+2}(a_{212(0)}{}'))\alpha^{n+1}[\alpha^{n+l+2}(a_{2211}{}'')\cdot(\alpha^{2n+2}(a_{2212(-1)2})\alpha^{n+l}(a_{222\,2}')\cdot\\
&\quad \beta^{-1}(b_{12}))_{(-1)}]][\alpha^{l+3}(a_{2212(0)}{}')(\alpha^{n+2l}(a_{222}{}'')\cdot\beta^{l-2}(b_2))']\otimes\beta^4(a_{211(0)(0)})[\alpha^{n+l+2}(a_{212(0)}{}'')\cdot\\
&\quad [\alpha^{n+l+2}(a_{2211}{}'')\cdot(\alpha^{2n+2}(a_{2212(-1)2})\alpha^{n+l}(a_{222\,2}')\cdot\beta^{-1}(b_{12}))]_{(0)}]\rtimes_\tau\alpha[\alpha^{l+3}(a_{2212(0)}{}'')\\
&\quad (\alpha^{n+2l}(a_{222}{}'')\cdot\beta^{l-2}(b_2))'']\\
&\overset{(Def\,3.2)}{=} a_1[\alpha^{2n+1}(a_{211(-1)}a_{212(-1)})\cdot\beta^{-1}[\alpha^{n+l+2}(a_{2211}{}')\cdot(\alpha^{2n+2}(a_{2212(-1)1})\alpha^{n+l}(a_{222\,1}')\cdot\beta^{-1}(b_{11}))]]\\
&\quad \rtimes_\tau \alpha^{-1}[(\alpha^{n+3}(a_{211(0)(-1)})\alpha^{l+2}(a_{212(0)}{}'))\alpha^{n+1}[\alpha^{n+l+2}(a_{2211\,1}'')(\alpha^{2n+2}(a_{2212(-1)2})\alpha^{n+l}(a_{222\,2}')\cdot\\
&\quad \beta^{-1}(b_{12}))_{(-1)}]][\alpha^{l+3}(a_{2212(0)}{}')(\alpha^{n+2l}(a_{222}{}'')\cdot\beta^{l-2}(b_2))']\otimes\beta^4(a_{211(0)(0)})[\alpha^{n+l+2}(a_{212(0)}{}'')\cdot\\
&\quad \alpha^{n+l+2}(a_{2211\,2}'')\cdot(\alpha^{2n+2}(a_{2212(-1)2})\alpha^{n+l}(a_{222\,2}')\cdot\beta^{-1}(b_{12}))_{(0)}]\rtimes_\tau\alpha[\alpha^{l+3}(a_{2212(0)}{}'')(\alpha^{n+2l}(a_{222}{}'')\cdot\\
&\quad \beta^{l-2}(b_2))'']\\
&= a_1[\alpha^{2n+1}(a_{211(-1)}((\alpha^{-n+l-1}(a_{21211}{}')\alpha^{-n+l}(a_{21212\,1}'))\alpha^{-n+l-1}(a_{2122}^{s})))\cdot\beta^{-1}[\alpha^{n+l+2}(a_{2211}{}')\cdot\\
&\quad (\alpha^{2n+2}(a_{2212(-1)1})\alpha^{n+l}(a_{222\,1}')\cdot\beta^{-1}(b_{11}))]]\rtimes_\tau\alpha^{-1}[(\alpha^{n+3}(a_{211(0)(-1)})((\alpha^{l+1}(a_{21211}{}'')\alpha^{l+2}(a_{21212\,2}'))\\
&\quad \alpha^{l+2}(a_{2122\,1}^{s})))\alpha^{n+1}[\alpha^{n+l+2}(a_{2211\,1}'')(\alpha^{2n+2}(a_{2212(-1)2})\alpha^{n+l}(a_{222\,2}')\cdot\beta^{-1}(b_{12}))_{(-1)}]]\\
&\quad (\alpha^{l+3}(a_{2212(0)}{}')(\alpha^{n+2l}(a_{222}{}'')\cdot\beta^{l-2}(b_2))')\otimes\beta^4(a_{211(0)(0)})[\alpha^{n+l+2}(a_{21212}{}''a_{2122\,2}^{ss})\cdot\alpha^{n+l+2}(a_{2211\,2}'')\cdot\\
&\quad (\alpha^{2n+2}(a_{2212(-1)2})\alpha^{n+l}(a_{222\,2}')\cdot\beta^{-1}(b_{12}))_{(0)}]\rtimes_\tau\alpha[\alpha^{l+3}(a_{2212(0)}{}'')(\alpha^{n+2l}(a_{222}{}'')\cdot\beta^{l-2}(b_2))'']\\
&= a_1[\alpha^{2n+1}(((\alpha^{-n+l-1}(a_{21111}{}')\alpha(a_{21112(-1)1}))\alpha^{-n+l-1}(a_{2112}^{s}))((\alpha^{-n+l-1}(a_{21211}{}')\alpha^{-n+l}(a_{21212\,1}'))\\
&\quad \alpha^{-n+l-1}(a_{2122}^{s})))\cdot\beta^{-1}[\alpha^{n+l+2}(a_{2211}{}')\cdot(\alpha^{2n+2}(a_{2212(-1)1})\alpha^{n+l}(a_{222\,1}')\cdot\beta^{-1}(b_{11}))]]\rtimes_\tau\\
&\quad \alpha^{-1}[(((\alpha^{l+1}(a_{21111}{}'')\alpha^{n+3}(a_{21112(-1)2}))\alpha^{l+1}(a_{2112}^{ss}))((\alpha^{l+1}(a_{21211}{}'')\alpha^{l+2}(a_{21212\,2}'))\\
&\quad \alpha^{l+2}(a_{2122\,1}^{s})))\alpha^{n+1}[\alpha^{n+l+2}(a_{2211\,1}'')(\alpha^{2n+2}(a_{2212(-1)2})\alpha^{n+l}(a_{222\,2}')\cdot\beta^{-1}(b_{12}))_{(-1)}]]\\
&\quad (\alpha^{l+3}(a_{2212(0)}{}')(\alpha^{n+2l}(a_{222}{}'')\cdot\beta^{l-2}(b_2))')\otimes\beta^5(a_{21112(0)})[\alpha^{n+l+2}(a_{21212}{}''a_{2122\,2}^{ss})\cdot\alpha^{n+l+2}(a_{2211\,2}'')\cdot\\
&\quad (\alpha^{2n+2}(a_{2212(-1)2})\alpha^{n+l}(a_{222\,2}')\cdot\beta^{-1}(b_{12}))_{(0)}]\rtimes_\tau\alpha[\alpha^{l+3}(a_{2212(0)}{}'')(\alpha^{n+2l}(a_{222}{}'')\cdot\beta^{l-2}(b_2))'']\\
&= a_1[[[(\alpha^{n+l-3}(a_{211}{}')\alpha^{2n}(a_{2121(-1)1}))\alpha^{n+l-1}(a_{2122}^{s})][(\alpha^{n+l-2}(a_{2211}{}')\alpha^{n+l-1}(a_{2212\,1}'))\\
&\quad \alpha^{n+l+1}(a_{222111}^{s})]]\alpha^{n+l+3}(a_{222112}{}')\cdot(\alpha^{2n+3}(a_{22212(-1)1})\alpha^{n+l+1}(a_{2222\,1}')\cdot\beta^{-1}(b_{11}))]\rtimes_\tau\\
&\quad \alpha^{-1}[[[(\alpha^{l-1}(a_{211}{}'')\alpha^{n+2}(a_{2121(-1)2}))\alpha^{l+1}(a_{2122}^{ss})][(\alpha^{l}(a_{2211}{}'')\alpha^{l+1}(a_{2212\,2}'))\alpha^{l+4}(a_{222111\,1}^{ss})]]\\
&\quad (\alpha^{l+5}(a_{222112\,1}'')\alpha^{n}((\alpha^{2n+4}(a_{22212(-1)2})\alpha^{n+l+2}(a_{2222\,2}')\cdot b_{12})_{(-1)}))][\alpha^{l+4}(a_{22212(0)}{}')\\
&\quad (\alpha^{n+2l+1}(a_{2222}{}'')\cdot\beta^{l-2}(b_2))']\otimes\beta^4(a_{2121(0)})[\alpha^{n+l+1}(a_{2212}{}'')\alpha^{n+l+3}(a_{222111\,2}^{ss}a_{222112\,2}'')\cdot\\
&\quad (\alpha^{2n+4}(a_{22212(-1)2})\alpha^{n+l+2}(a_{2222\,2}')\cdot b_{12})_{(0)}]\rtimes_\tau\alpha[\alpha^{l+4}(a_{22212(0)}{}'')(\alpha^{n+2l+1}(a_{2222}{}'')\cdot\beta^{l-2}(b_2))'']\\
&= a_1[[(\alpha^{n+l-2}(a_{211}{}')\alpha^{2n+1}(a_{2121(-1)1}))\alpha^{n+l}(a_{2122}^{s})][(\alpha^{n+l+1}(a_{2211}{}')\alpha^{n+l}(a_{2212\,1}'))\\
&\quad \alpha^{n+l+1}(a_{222111}^{s}a_{222112}{}')]\cdot[\alpha^{2n+3}(a_{22212(-1)1})\alpha^{n+l+1}(a_{2222\,1}')\cdot\beta^{-1}(b_{11})]]\rtimes_\tau\\
&\quad \alpha^{-1}[[(\alpha^{l}(a_{211}{}'')\alpha^{n+3}(a_{2121(-1)2}))\alpha^{l+2}(a_{2122}^{ss})][(\alpha^{l+1}(a_{2211}{}'')\alpha^{l+2}(a_{2212\,2}'))\\
&\quad (\alpha^{l+3}(a_{222111\,1}^{ss}a_{222112\,1}'')\alpha^{n-1}((\alpha^{2n+4}(a_{22212(-1)2})\alpha^{n+l+2}(a_{2222\,2}')\cdot b_{12})_{(-1)}))]][\alpha^{l+4}(a_{22212(0)}{}')\\
&\quad (\alpha^{n+2l+1}(a_{2222}{}'')\cdot\beta^{l-2}(b_2))']\otimes\beta^4(a_{2121(0)})[\alpha^{n+l+1}(a_{2212}{}'')\alpha^{n+l+3}(a_{222111\,2}^{ss}a_{222112\,2}'')\cdot
\end{aligned}
$$

$$(\alpha^{2n+4}(a_{22212(-1)2})\alpha^{n+l+2}(a_{2222'2})\cdot b_{12})_{(0)}]\rtimes_\tau \alpha[\alpha^{l+4}(a_{22212(0)}'')(\alpha^{n+2l+1}(a_{2222}'')\cdot\beta^{l-2}(b_2))'']$$

$$\overset{(2.1)}{=} a_1[[(\alpha^{n+l-2}(a_{211}')\alpha^{2n+1}(a_{2121(-1)1}))\alpha^{n+l}(a_{2122}^{s})](\alpha^{n+l}(a_{2211}')\alpha^{n+l+1}(a_{2212'1}))\cdot$$
$$(\alpha^{2n+2}(a_{2221(-1)1})\alpha^{n+l+1}(a_{2222'1})\cdot\beta^{-1}(b_{11}))]\rtimes_\tau \alpha^{-1}[[(\alpha^{l}(a_{211}'')\alpha^{n+3}(a_{2121(-1)2}))\alpha^{l+2}(a_{2122}^{ss})]$$
$$((\alpha^{l+1}(a_{2211}'')\alpha^{l+2}(a_{2212'2}))(\alpha^{n}((\alpha^{2n+3}(a_{2221(-1)2})\alpha^{n+l+2}(a_{2222'2})\cdot b_{12})_{(-1)})))]$$
$$(\alpha^{l+3}(a_{2221(0)}')(\alpha^{n+2l+1}(a_{2222}'')\cdot\beta^{l-2}(b_2))')\otimes\beta^4(a_{2121(0)})[\alpha^{n+l+2}(a_{2212}'')\cdot(\alpha^{2n+3}(a_{2221(-1)2})$$
$$\alpha^{n+l+2}(a_{2222'2})\cdot b_{12})_{(0)}]\rtimes_\tau \alpha[\alpha^{l+3}(a_{2221(0)}'')(\alpha^{n+2l+1}(a_{2222}'')\cdot\beta^{l-2}(b_2))'']$$

$$= a_1[[(\alpha^{n+l-2}(a_{211}')\alpha^{2n+1}(a_{2121(-1)1}))\alpha^{n+l}(a_{2122}^{s})](\alpha^{n+l}(a_{2211}')\alpha^{n+l+1}(a_{2212'1}))\cdot$$
$$(((\alpha^{n+l+1}(a_{222111'1})\alpha^{n+l+2}(a_{222112'11}))\alpha^{n+l+1}(a_{22212^{s}1}))\alpha^{n+l+1}(a_{2222'1})\cdot\beta^{-1}(b_{11}))]\rtimes_\tau$$
$$\alpha^{-1}[[(\alpha^{l}(a_{211}'')\alpha^{n+3}(a_{2121(-1)2}))\alpha^{l+2}(a_{2122}^{ss})][(\alpha^{l+1}(a_{2211}'')\alpha^{l+2}(a_{2212'2}))$$
$$(\alpha^{n}((((\alpha^{n+l+2}(a_{222111'2})\alpha^{n+l+3}(a_{222112'12}))\alpha^{n+l+2}(a_{22212^{s}2}))\alpha^{n+l+2}(a_{2222'2})\cdot b_{12})_{(-1)}))]]$$
$$(((\alpha^{l+2}(a_{222111}'')\alpha^{l+3}(a_{222112'2}))\alpha^{l+3}(a_{22212^{ss}1}))(\alpha^{n+2l+1}(a_{2222}'')\cdot\beta^{l-2}(b_2))')\otimes$$
$$\beta^4(a_{2121(0)})[\alpha^{n+l+2}(a_{2212}'')\cdot(((\alpha^{n+l+2}(a_{222111'2})\alpha^{n+l+3}(a_{222112'12}))\alpha^{n+l+2}(a_{22212^{s}2}))$$
$$\alpha^{n+l+2}(a_{2222'2})\cdot b_{12})_{(0)}]\rtimes_\tau \alpha[\alpha^{l+3}(a_{222112}''a_{22212^{ss}2})(\alpha^{n+2l+1}(a_{2222}'')\cdot\beta^{l-2}(b_2))'']$$

$$= a_1[(\alpha^{n+l+2}(a_{21}')\alpha^{2n+1}(a_{221(-1)1}))(\alpha^{n+l+1}(a_{222111}^{s}a_{222112}')\alpha^{n+l+2}(a_{22212}'))]\cdot$$
$$(((\alpha^{n+l+2}(a_{2222111'1})\alpha^{n+l+3}(a_{2222112'11}))\alpha^{n+l+2}(a_{222212^{s}1}))\alpha^{n+l+2}(a_{22222'1})\cdot\beta^{-1}(b_{11}))\rtimes_\tau$$
$$\alpha^{-1}[[(\alpha^{l-1}(a_{21}'')\alpha^{n+2}(a_{221(-1)2}))(\alpha^{l+2}(a_{222111}^{ss}a_{222112}'')\alpha^{l+3}(a_{22212'2}))]$$
$$\alpha^{n+1}[[[[(\alpha^{n+l+3}(a_{2222111'2})\alpha^{n+l+4}(a_{2222112'12}))\alpha^{n+l+3}(a_{222212^{s}2})]\alpha^{n+l+3}(a_{22222'2})]\cdot b_{12}]_{(-1)}]]$$
$$(((\alpha^{l+3}(a_{2222111}'')\alpha^{l+4}(a_{2222112'2}))\alpha^{l+4}(a_{222212^{ss}1}))(\alpha^{n+2l+2}(a_{22222}'')\cdot\beta^{l-2}(b_2))')\otimes$$
$$\beta^3(a_{221(0)})[\alpha^{n+l+3}(a_{22212}'')\cdot[[(\alpha^{n+l+3}(a_{2222111'2})\alpha^{n+l+4}(a_{2222112'12}))\alpha^{n+l+3}(a_{222212^{s}2})]$$
$$\alpha^{n+l+3}(a_{22222'2})\cdot b_{12}]_{(0)}]\rtimes_\tau \alpha[\alpha^{l+4}(a_{2222112}''a_{222212^{ss}2})(\alpha^{n+2l+2}(a_{22222}'')\cdot\beta^{l-2}(b_2))'']$$

$$\overset{(2.1)}{=} a_1[(\alpha^{n+l+2}(a_{21}')\alpha^{2n+1}(a_{221(-1)1}))(\alpha^{n+l+2}(a_{2221'1}))]\cdot[[(\alpha^{n+l+2}(a_{2222111'1})\alpha^{n+l+3}(a_{2222112'11}))$$
$$\alpha^{n+l+2}(a_{222212^{s}1})]\alpha^{n+l+2}(a_{22222'1})\cdot\beta^{-1}(b_{11})]\rtimes_\tau \alpha^{-1}[[(\alpha^{l-1}(a_{21}'')\alpha^{n+2}(a_{221(-1)2}))\alpha^{l+3}(a_{2221'2})]$$
$$\alpha^{n+1}[[[[(\alpha^{n+l+3}(a_{2222111'2})\alpha^{n+l+4}(a_{2222112'12}))\alpha^{n+l+3}(a_{222212^{s}2})]\alpha^{n+l+3}(a_{22222'2})]\cdot b_{12}]_{(-1)}]]$$
$$(((\alpha^{l+3}(a_{2222111}'')\alpha^{l+4}(a_{2222112'2}))\alpha^{l+4}(a_{222212^{ss}1}))(\alpha^{n+2l+2}(a_{22222}'')\cdot\beta^{l-2}(b_2))')\otimes$$
$$\beta^3(a_{221(0)})[\alpha^{n+l+2}(a_{2221}'')\cdot[[(\alpha^{n+l+3}(a_{2222111'2})\alpha^{n+l+4}(a_{2222112'12}))\alpha^{n+l+3}(a_{222212^{s}2})]$$
$$\alpha^{n+l+3}(a_{22222'2})\cdot b_{12}]_{(0)}]\rtimes_\tau \alpha[\alpha^{l+4}(a_{2222112}''a_{222212^{ss}2})(\alpha^{n+2l+2}(a_{22222}'')\cdot\beta^{l-2}(b_2))'']$$

$$= a_1[(\alpha^{n+l-2}(a_{21}')\alpha^{2n+1}(a_{221(-1)1}))\alpha^{n+l+2}(a_{2221'1})\cdot[(\alpha^{n+l+1}(a_{22221'1})\alpha^{n+l+3}(a_{222221'11}))$$
$$\alpha^{n+l+3}(a_{2222221^{s}1}a_{2222222'1})\cdot\beta^{-1}(b_{11})]]\rtimes_\tau \alpha^{-1}[[(\alpha^{l-1}(a_{21}'')\alpha^{n+2}(a_{221(-1)2}))\alpha^{l+3}(a_{2221'2})]$$
$$\alpha^{n+1}[[(\alpha^{n+l+2}(a_{22221'2})\alpha^{n+l+4}(a_{222221'12}))\alpha^{n+l+4}(a_{2222221^{s}2}a_{2222222'2})\cdot b_{12}]_{(-1)}]]$$
$$(((\alpha^{l+1}(a_{22221}'')\alpha^{l+3}(a_{222221'2}))\alpha^{l+5}(a_{2222221^{ss}1}))(\alpha^{n+2l+4}(a_{2222222}'')\cdot\beta^{l-2}(b_2))')\otimes$$
$$\beta^3(a_{221(0)})[\alpha^{n+l+2}(a_{2221}'')\cdot[(\alpha^{n+l+2}(a_{22221'2})\alpha^{n+l+4}(a_{222221'12}))\alpha^{n+l+4}(a_{2222221^{s}2}a_{2222222'2})\cdot$$
$$b_{12}]_{(0)}]\rtimes_\tau \alpha[(\alpha^{l+3}(a_{222221}'')\alpha^{l+5}(a_{2222221^{ss}2}))(\alpha^{n+2l+4}(a_{2222222}'')\cdot\beta^{l-2}(b_2))'']$$

$$= a_1[(\alpha^{n+l-2}(a_{21}')\alpha^{2n+1}(a_{221(-1)1}))\alpha^{n+l+2}(a_{2221'1})\cdot[(\alpha^{n+l+1}(a_{22221'1})\alpha^{n+l+3}(a_{222221'11}))$$
$$\alpha^{n+l+3}(a_{2222221^{s}1}a_{2222222'1})\cdot\beta^{-1}(b_{11})]]\rtimes_\tau \alpha^{-1}[[(\alpha^{l-1}(a_{21}'')\alpha^{n+2}(a_{221(-1)2}))\alpha^{l+3}(a_{2221'2})]$$
$$\alpha^{n+1}[[(\alpha^{n+l+2}(a_{22221'2})\alpha^{n+l+4}(a_{222221'12}))\alpha^{n+l+4}(a_{2222221^{s}2}a_{2222222'2})\cdot b_{12}]_{(-1)}]]$$
$$((\alpha^{l+2}(a_{22221}'')\alpha^{l+4}(a_{222221'2}))(\alpha^{l+4}(a_{2222221^{ss}1}a_{2222222''1})\alpha^{l-2}(b_2')))\otimes\beta^3(a_{221(0)})[\alpha^{n+l+2}(a_{2221}'')\cdot$$
$$[(\alpha^{n+l+2}(a_{22221'2})\alpha^{n+l+4}(a_{222221'12}))\alpha^{n+l+4}(a_{2222221^{s}2}a_{2222222'2})\cdot b_{12}]_{(0)}]\rtimes_\tau$$
$$\alpha[\alpha^{l+4}(a_{222221}'')(\alpha^{l+4}(a_{2222221^{ss}2}a_{2222222'2})\alpha^{l-2}(b_2'')))]$$

$$\overset{(2.1)}{=} a_1[(\alpha^{n+l-2}(a_{21}')\alpha^{2n+1}(a_{221(-1)1}))\alpha^{n+l+2}(a_{2221'1})\cdot[(\alpha^{n+l+2}(a_{22221'1})\alpha^{n+l+3}(a_{22222'11}))\cdot$$
$$\beta^{-1}(b_{11})]]\rtimes_\tau \alpha^{-1}[[(\alpha^{l-1}(a_{21}'')\alpha^{n+2}(a_{221(-1)2}))\alpha^{l+3}(a_{2221'2})]\alpha^{n+1}[[(\alpha^{n+l+3}(a_{22221'2})$$
$$\alpha^{n+l+4}(a_{22222'12}))\cdot b_{12}]_{(-1)}]][(\alpha^{l+2}(a_{22221}'')\alpha^{l+3}(a_{22222'2}))\alpha^{l-1}(b_2')]\otimes\beta^3(a_{221(0)})$$
$$[\alpha^{n+l+2}(a_{2221}'')\cdot[(\alpha^{n+l+3}(a_{22221'2})\alpha^{n+l+4}(a_{22222'12}))\cdot b_{12}]_{(0)}]\rtimes_\tau \alpha[\alpha^{l+3}(a_{22222}'')\alpha^{l-2}(b_2'')]$$

$$= a_1[(\alpha^{n+l-2}(a_{21}')\alpha^{2n+1}(a_{221(-1)1}))\alpha^{n+l+2}(a_{2221'1})\cdot(\alpha^{n+l+3}(a_{2222''11})\cdot\beta^{-1}(b_{11}))]\rtimes_\tau$$
$$[[(\alpha^{l-2}(a_{21}'')\alpha^{n+1}(a_{221(-1)2}))\alpha^{l+2}(a_{2221'2})]\alpha^{n}[(\alpha^{n+l+4}(a_{2222'12})\cdot b_{12})_{(-1)}]]$$

$$
\begin{aligned}
&(\alpha^{l+3}(a_{2222\ 2}{}^{'})\alpha^{l-1}(b_2^{'})) \otimes \beta^3(a_{221(0)})[\alpha^{n+l+2}(a_{2221}{}^{''}) \cdot [\alpha^{n+l+4}(a_{2222\ 12}{}^{'}) \cdot b_{12}]_{(0)}] \rtimes_\tau \\
&\alpha^{l+3}(a_{2222}{}^{''})\alpha^l(b_2^{''}) \\
&= a_1[(\alpha^{n+l-2}(a_{21}{}^{'})\alpha^{2n+1}(a_{221(-1)1}))\alpha^{n+l+2}(a_{2221\ 1}{}^{'}) \cdot (\alpha^{n+l+2}(a_{2222\ 1}{}^{'}) \cdot \beta^{-1}(b_{11}))] \rtimes_\tau \\
&[(\alpha^{l-1}(a_{21}{}^{''})\alpha^{n+2}(a_{221(-1)2}))\alpha^{l+3}(a_{2221\ 2}{}^{'})][[\alpha^{n-1}[(\alpha^{n+l+4}(a_{2222\ 21}{}^{'}) \cdot b_{12})_{(-1)}] \\
&\alpha^{l+3}(a_{2222\ 22}{}^{'})]\alpha^{l-1}(b_2^{'})] \otimes \beta^3(a_{221(0)})[\alpha^{n+l+2}(a_{2221}{}^{''}) \cdot (\alpha^{n+l+4}(a_{2222\ 21}{}^{'}) \cdot b_{12})_{(0)}] \rtimes_\tau \\
&\alpha^{l+3}(a_{2222}{}^{''})\alpha^l(b_2^{''}) \\
&\overset{(B7)}{=} a_1[(\alpha^{n+l-2}(a_{21}{}^{'})\alpha^{2n+1}(a_{221(-1)1}))\alpha^{n+l+2}(a_{2221\ 1}{}^{'}) \cdot (\alpha^{n+l+2}(a_{2222\ 1}{}^{'}) \cdot \beta^{-1}(b_{11}))] \rtimes_\tau \\
&[(\alpha^{l-1}(a_{21}{}^{''})\alpha^{n+2}(a_{221(-1)2}))\alpha^{l+3}(a_{2221\ 2}{}^{'})][(\alpha^{l+3}(a_{2222\ 21}{}^{'})\alpha^n(b_{12(-1)}))\alpha^{l-1}(b_2^{'})] \otimes \\
&\beta^3(a_{221(0)})[\alpha^{n+l+2}(a_{2221}{}^{''}) \cdot (\alpha^{n-l+3}(a_{2222\ 22}{}^{'}) \cdot b_{12(0)})] \rtimes_\tau \alpha^{l+3}(a_{2222}{}^{''})\alpha^l(b_2^{''}) \\
&= \beta(a_{11})[\alpha^{n+l}(a_{12}{}^{'}) \cdot [\alpha^{2n}(a_{21(-1)1})\alpha^{n+l}(a_{221\ 1}{}^{'}) \cdot (\alpha^{n+l}(a_{222\ 1}{}^{'}) \cdot \beta^{-2}(b_{11}))]] \rtimes_\tau \\
&[(\alpha^{l-1}(a_{12}{}^{''})\alpha^{n+1}(a_{21(-1)2}))\alpha^{l+2}(a_{221\ 2}{}^{'})][(\alpha^{l+2}(a_{222\ 21}{}^{'})\alpha^n(b_{12(-1)}))\alpha^{l-1}(b_2^{'})] \otimes \\
&\beta^2(a_{21(0)})[\alpha^{n+l+1}(a_{221}{}^{''}) \cdot (\alpha^{n+l+2}(a_{222\ 22}{}^{'}) \cdot b_{12(0)})] \rtimes_\tau \alpha^{l+2}(a_{222}{}^{''})\alpha^l(b_2^{''}) \\
&= a_1[\alpha^{2n+1}(a_{21(-1)1})\alpha^{n+l+1}(a_{221\ 1}{}^{'}) \cdot (\alpha^{n+l+1}(a_{222\ 1}{}^{'}) \cdot \beta^{-1}(b_{11}))] \rtimes_\tau \\
&[\alpha^{n+2}(a_{21(-1)2})\alpha^{l+2}(a_{221\ 2}{}^{'})][(\alpha^{l+2}(a_{222\ 21}{}^{'})\alpha^n(b_{12(-1)}))\alpha^{l-1}(b_2^{'})] \otimes \beta^2(a_{21(0)}) \\
&[\alpha^{n+l+1}(a_{221}{}^{''}) \cdot (\alpha^{n+l+2}(a_{222\ 22}{}^{'}) \cdot b_{12(0)})] \rtimes_\tau \alpha^{l+2}(a_{222}{}^{''})\alpha^l(b_2^{''}) \\
&= a_1[\alpha^{2n+1}(a_{21(-1)1})\alpha^{n+l+1}(a_{221\ 1}{}^{'}) \cdot (\alpha^{n+l+2}(a_{222\ 11}{}^{'}) \cdot \beta^{-1}(b_{11}))] \rtimes_\tau \\
&[\alpha^{n+2}(a_{21(-1)2})\alpha^{l+2}(a_{221\ 2}{}^{'})][(\alpha^{l+2}(a_{222\ 12}{}^{'})\alpha^n(b_{12(-1)}))\alpha^{l-1}(b_2^{'})] \otimes \beta^2(a_{21(0)}) \\
&[\alpha^{n+l+1}(a_{221}{}^{''}) \cdot (\alpha^{n+l+1}(a_{222\ 2}{}^{'}) \cdot b_{12(0)})] \rtimes_\tau \alpha^{l+2}(a_{222}{}^{''})\alpha^l(b_2^{''}) \\
&= a_1[(\alpha^{2n}(a_{21(-1)1})\alpha^{n+l}(a_{221\ 1}{}^{'}))\alpha^{n+l+2}(a_{222\ 11}{}^{'}) \cdot b_{11}] \rtimes_\tau \\
&[\alpha^{n+2}(a_{21(-1)2})\alpha^{l+2}(a_{221\ 2}{}^{'})][(\alpha^{l+2}(a_{222\ 12}{}^{'})\alpha^n(b_{12(-1)}))\alpha^{l-1}(b_2^{'})] \otimes \beta^2(a_{21(0)}) \\
&[\alpha^{n+l}(a_{221}{}^{''})\alpha^{n+l+1}(a_{222\ 2}{}^{'}) \cdot \beta(b_{12(0)})] \rtimes_\tau \alpha^{l+2}(a_{222}{}^{''})\alpha^l(b_2^{''}) \\
&= a_1[\alpha^{2n+1}(a_{21(-1)})\alpha^{n+l}(a_{221\ 1}{}^{'}\alpha(a_{222\ 11}{}^{'})) \cdot b_{11}] \rtimes_\tau \\
&[\alpha^{n+2}(a_{21(-1)2})[(\alpha^l(a_{221\ 2}{}^{'})\alpha^{l+1}(a_{222\ 12}{}^{'}))\alpha^n(b_{12(-1)})]]\alpha^l(b_2^{'}) \otimes \beta^2(a_{21(0)}) \\
&[\alpha^{n+l}(a_{221}{}^{''}\alpha(a_{222\ 2}{}^{'})) \cdot \beta(b_{12(0)})] \rtimes_\tau \alpha^{l+2}(a_{222}{}^{''})\alpha^l(b_2^{''}) \\
&= a_1[\alpha^{2n+1}(a_{21(-1)})(\alpha^{2n+1}(a_{221(-1)1})\alpha^{n+l}(a_{222\ 1}{}^{'})) \cdot b_{11}] \rtimes_\tau \\
&[\alpha^{n+2}(a_{21(-1)2})[(\alpha^{n+1}(a_{221(-1)2})\alpha^l(a_{222\ 2}{}^{'}))\alpha^n(b_{12(-1)})]]\alpha^l(b_2^{'}) \otimes \beta^2(a_{21(0)}) \\
&[\alpha^{n+l+1}(a_{221(0)}{}^{'}a_{222\ 1}{}^{''}) \cdot \beta(b_{12(0)})] \rtimes_\tau \alpha^{l+2}(a_{221(0)}{}^{''}a_{222\ 2}{}^{''})\alpha^l(b_2^{''}) \\
&= a_1[\alpha^{2n+1}(a_{211(-1)1}a_{212(-1)1})\alpha^{n+l}(a_{22\ 1}{}^{'}) \cdot b_{11}] \rtimes_\tau \\
&[(\alpha^{n+1}(a_{211(-1)2}a_{212(-1)2})\alpha^l(a_{22\ 1}{}^{'}))\alpha^{n+1}(b_{12(-1)})]\alpha^l(b_2^{'}) \otimes \beta^3(a_{211(0)}) \\
&[\alpha^{n+l+1}(a_{212(0)}{}^{'})\alpha^{n+l}(a_{22\ 1}{}^{''}) \cdot \beta(b_{12(0)})] \rtimes_\tau [\alpha^{l+2}(a_{212(0)}{}^{''})\alpha^{l+1}(a_{22\ 2}{}^{''})]\alpha^l(b_2^{''}) \\
&= a_1[\alpha^{2n+1}(a_{21(-1)1})\alpha^{n+l}(a_{22\ 1}{}^{'}) \cdot \beta^{-1}(b_1)] \rtimes_\tau [(\alpha^{n+1}(a_{21(-1)2})\alpha^l(a_{22\ 1}{}^{'}))\alpha^{n+1}(b_{21(-1)})]\alpha^{l+1}(b_{22}^{'}) \\
&\otimes \beta^3(a_{21(0)1})[\alpha^{n+l+1}(a_{21(0)2}{}^{'})\alpha^{n+l}(a_{22\ 1}{}^{''}) \cdot \beta(b_{21(0)})] \rtimes_\tau [\alpha^{l+2}(a_{21(0)2}{}^{''})\alpha^{l+1}(a_{22\ 2}{}^{''})]\alpha^{l+1}(b_{22}^{''}) \\
&= a_1[\alpha^{2n+1}(a_{21(-1)1})\alpha^{n+l}(a_{22\ 1}{}^{'}) \cdot \beta^{-1}(b_1)] \rtimes_\tau [(\alpha^{n+1}(a_{21(-1)2})\alpha^l(a_{22\ 1}{}^{'}))\alpha^{n+1}(b_{21(-1)})]\alpha^{l+1}(b_{22}^{'}) \\
&\otimes \beta^3(a_{21(0)1})[\alpha^{n+l+2}(a_{21(0)2}{}^{'}) \cdot (\alpha^{n+l}(a_{22\ 1}{}^{''}) \cdot b_{21(0)})] \rtimes_\tau [\alpha^{l+2}(a_{21(0)2}{}^{''})\alpha^{l+1}(a_{22\ 2}{}^{''})]\alpha^{l+1}(b_{22}^{''}) \\
&= a_1[\alpha^{2n+1}(a_{21(-1)1})\alpha^{n+l}(a_{22\ 1}{}^{'}) \cdot \beta^{-1}(b_1)] \rtimes_\tau [(\alpha^{n+1}(a_{21(-1)2})\alpha^l(a_{22\ 1}{}^{'}))\alpha^{n+1}(b_{21(-1)})]\alpha^{l+1}(b_{22}^{'}) \\
&\otimes \beta^2(a_{21(0)1})(\alpha^{n+l+1}(a_{22\ 1}{}^{''}) \cdot \beta(b_{21(0)})) \rtimes_\tau \alpha^{l+2}(a_{22\ 2}{}^{''})\alpha^{l+1}(a_{22}{}^{''}) \\
&= \triangle_{B\rtimes_\tau H}(a\sharp 1_H)\ \triangle_{B\rtimes_\tau H}\ (b\sharp 1_H)
\end{aligned}
$$

and (3.1) is proved.

$$
\begin{aligned}
&\triangle_{B\rtimes_\tau H}(a\sharp 1_H)\ \triangle_{B\rtimes_\tau H}\ (1_B\sharp g) \\
&= [a_1 \rtimes_\tau \alpha^{n+1}(a_{21(-1)})\alpha^l(a_{22}{}^{'}) \otimes \beta^2(a_{21(0)}) \rtimes_\tau \alpha^{l+1}(a_{22}{}^{''})][1_B \rtimes_\tau g_1 \otimes 1_B \rtimes_\tau g_2] \\
&= a_1[\alpha^{2n+1}(a_{21(-1)1})\alpha^{n+l}(a_{22\ 1}{}^{'}) \cdot 1_B] \rtimes_\tau [\alpha^{n+2}(a_{21(-1)2})\alpha^{l+1}(a_{22\ 2}{}^{'})]g_1 \\
&\otimes \beta^2(a_{21(0)})(\alpha^{n+l+1}(a_{22\ 1}{}^{''}) \cdot 1_B) \rtimes_\tau \alpha^{l+2}(a_{22\ 2}{}^{''})g_2 \\
&= \beta(a_1) \rtimes_\tau [\alpha^{n+1}(a_{21(-1)})\alpha^l(a_{22}{}^{'})]g_1 \otimes \beta^3(a_{21(0)}) \rtimes_\tau \alpha^{l+1}(a_{22}{}^{''})g_2 \\
&= \triangle_{B\rtimes_\tau H}[\beta(a)\sharp\alpha(g)] \\
&= \triangle_{B\rtimes_\tau H}[(a\sharp 1_H)(1_B\sharp g)].
\end{aligned}
$$

and (3.2) is proved.

$$
\begin{aligned}
&\triangle_{B\rtimes_\tau H}[(1_B\sharp h)(b\sharp 1_H)]\\
&= \triangle_{B\rtimes_\tau H}[(\alpha^{n+1}(h_1)\cdot b)\sharp\alpha^2(h_2)]\\
&= (\alpha^{n+1}(h_1)\cdot b)_1 \rtimes_\tau [\alpha^n[(\alpha^{n+1}(h_1)\cdot b)_{21(-1)}]\alpha^{l-1}[(\alpha^{n+1}(h_1)\cdot b)_{22}{}^{'}]]\alpha(h_{21})\\
&\quad \otimes \beta^2[(\alpha^{n+1}(h_1)\cdot b)_{21(0)}] \rtimes_\tau \alpha^l[(\alpha^{n+1}(h_1)\cdot b)_{22}{}^{''}]\alpha(h_{22})\\
&\overset{(B5)}{=} (\alpha^{n+1}(h_{11})\cdot b_1) \rtimes_\tau [\alpha^n[(\alpha^{n+1}(h_{121})\cdot b_{21})_{(-1)}]\alpha^{l-1}[(\alpha^{n+1}(h_{122})\cdot b_{22})^{'}]]\alpha(h_{21})\\
&\quad \otimes \beta^2[(\alpha^{n+1}(h_{121})\cdot b_{21})_{(0)}] \rtimes_\tau \alpha^l[(\alpha^{n+1}(h_{122})\cdot b_{22})^{''}]\alpha(h_{22})\\
&\overset{(B9)}{=} (\alpha^{n+1}(h_{11})\cdot b_1) \rtimes_\tau \alpha^{n+1}[(\alpha^{n+1}(h_{121})\cdot b_{21})_{(-1)}][\alpha^{l-1}[(\alpha^{n+1}(h_{122})\cdot b_{22})^{'}]h_{21}]\\
&\quad \otimes \beta^2[(\alpha^{n+1}(h_{121})\cdot b_{21})_{(0)}] \rtimes_\tau \alpha[\alpha^{l-1}[(\alpha^{n+1}(h_{122})\cdot b_{22})^{''}]h_{22}]\\
&= (\alpha^{n+1}(h_{11})\cdot b_1) \rtimes_\tau \alpha^{n+1}[(\alpha^{n+1}(h_{121})\cdot b_{21})_{(-1)}][\alpha(h_{122})\alpha^l(b_{22}{}^{'})] \otimes \beta^2[(\alpha^{n+1}(h_{121})\cdot b_{21})_{(0)}]\\
&\quad \rtimes_\tau \alpha[\alpha(h_2)\alpha^l(b_{22}{}^{''})]\\
&= (\alpha^{n+1}(h_{11})\cdot b_1) \rtimes_\tau [\alpha^n[(\alpha^{n+1}(h_{121})\cdot b_{21})_{(-1)}]\alpha(h_{122})]\alpha^{l+1}(b_{22}{}^{'}) \otimes \beta^2[(\alpha^{n+1}(h_{121})\cdot b_{21})_{(0)}]\\
&\quad \rtimes_\tau \alpha^2(h_2)\alpha^{l+1}(b_{22}{}^{''})\\
&\overset{(B7)}{=} (\alpha^{n+1}(h_{11})\cdot b_1) \rtimes_\tau \alpha(h_{12})[\alpha^{n+1}(b_{21(-1)})\alpha^l(b_{22}{}^{'})] \otimes [\alpha^{n+1}(h_{21})\cdot\beta^2(b_{21(0)})] \rtimes_\tau \alpha(h_{22})\alpha^{l+1}(b_{22}{}^{''})\\
&= \triangle_{B\rtimes_\tau H}(1_B\sharp h)\ \triangle_{B\rtimes_\tau H}\ (b\sharp 1_H).
\end{aligned}
$$

and (3.3) is proved.

$$
\begin{aligned}
&\triangle_{B\rtimes_\tau H}(1_B\sharp h)\ \triangle_{B\rtimes_\tau H}\ (1_B\sharp g)\\
&= 1_B[\alpha^n(h_{11})\cdot 1_B] \rtimes_\tau \alpha(h_{12})g_1 \otimes 1_B[\alpha^n(h_{21})\cdot 1_B] \rtimes_\tau \alpha(h_{22})g_2\\
&= 1_B \rtimes_\tau h_1g_1 \otimes 1_B \rtimes_\tau h_2g_2\\
&= \triangle_{B\rtimes_\tau H}(1_B\sharp hg)\\
&= \triangle_{B\rtimes_\tau H}[(1_B\sharp h)(1_B\sharp g)].
\end{aligned}
$$

and (3.4) is proved.

Therefore, $(B^{\sharp}_{\rtimes_\tau}H, m_{B\sharp H}, 1_B\otimes 1_H, \triangle_{B\rtimes_\tau H}, \varepsilon_{B\rtimes_\tau H}, \beta\otimes\alpha)$ is monoidal Hom-bialgebra. □

Definition 3.5. Let (H,α) be a monoial Hom-bialgebra and (C,β) a monoidal Hom-coalgebra. $\tau : C \to H\otimes H$ and $S : H \to H$ are linear maps. S is called a τ-antipode if for all $c\in C, h\in H$,

(1) $\alpha\circ S = S\circ\alpha$,

(2) $m(\mathrm{id}_H\otimes S)m_{H\otimes H}(\tau\otimes\triangle_H)(c\otimes h) = \varepsilon_C(c)\varepsilon_H(h)1_H$,

(3) $m(S\otimes\mathrm{id}_H)m_{H\otimes H}(\tau\otimes\triangle_H)(c\otimes h) = \varepsilon_C(c)\varepsilon_H(h)1_H$.

In this case (H,α) is called a τ-monoidal Hom-Hopf algebra.

Theorem 3.6. *Let $(B\rtimes_\tau H, \beta\otimes\alpha)$ be a monoidal Hom-bialgebra. If (H,α) is a τ-monoidal Hom-Hopf algebra with τ-antipode S_H, $S_B\in Hom(B,B)$ is a convolution invertible element of* id_B *with $\beta\circ S_B = S_B\circ\beta$, then $((B^{\#}_{\rtimes_\tau}H, \beta\otimes\alpha)$ is a monoidal Hom-Hopf algebra with the antipode given by*

$$S_{B^{\sharp}_{\rtimes_\tau}H}(b\otimes h) = [1_B\otimes S_H[(\alpha^{l-3}(b_1^s)\alpha^{n-2}(b_{2(-1)}))\alpha^{-3}(h)]\alpha^{l-1}(b_1^{ss})][S_B(\beta(b_{2(0)}))\otimes 1_H].$$

Proof. For any $b\otimes h\in B^{\sharp}_{\rtimes_\tau}H$, it is easy to proof that

$$S_{B^{\sharp}_{\rtimes_\tau}H}(\beta(b)\otimes\alpha(h)) = (\beta\otimes\alpha)S_{B^{\sharp}_{\rtimes_\tau}H}(a\otimes h)$$

and we have

$$
\begin{aligned}
&(\mathrm{id} * S)\circ\triangle_{B\rtimes_\tau H}(b\otimes h)\\
&= [b_1\otimes[\alpha^n(b_{21(-1)})\alpha^{l-1}(b_{22}{}^{'})]\alpha^{-1}(h_1)]S_{B^{\sharp}_{\rtimes_\tau}H}[\beta^2(b_{21(0)})\otimes\alpha^l(b_{22}{}^{''})\alpha^{-1}(h_2)]\\
&= [b_1\otimes[\alpha^n(b_{21(-1)})\alpha^{l-1}(b_{22}{}^{'})]\alpha^{-1}(h_1)][[1_B\otimes S_H[(\alpha^{l-3}(\beta^2(b_{21(0)})_1{}^s)\alpha^{n-2}(\beta^2(b_{21(0)})_{2(-1)}))\\
&\quad \alpha^{-3}(\alpha^l(b_{22}{}^{''})\alpha^{-1}(h_2))]\alpha^{l-1}(\beta^2(b_{21(0)})_1{}^{ss})][S_B(\beta(\beta^2(b_{21(0)})_{2(0)}))\otimes 1_H]]\\
&= [[\beta^{-1}(b_1)\otimes[\alpha^{n-1}(b_{21(-1)})\alpha^{l-2}(b_{22}{}^{'})]\alpha^{-2}(h_1)][1_B\otimes S_H[(\alpha^{l-3}(\beta^2(b_{21(0)})_1{}^s)\\
&\quad \alpha^{n-2}(\beta^2(b_{21(0)})_{2(-1)}))\alpha^{-3}(\alpha^l(b_{22}{}^{''})\alpha^{-1}(h_2))]\alpha^{l-1}(\beta^2(b_{21(0)})_1{}^{ss})]][S_B(\beta^4(b_{21(0)})_{2(0)})\otimes 1_H]
\end{aligned}
$$

$$
\begin{aligned}
&= [b_1 \otimes [[\alpha^{n-1}(b_{21(-1)})\alpha^{l-2}(b_{22}{}')]\alpha^{-2}(h_1)][S_H[(\alpha^{l-1}(b_{21(0)1}{}^{s})\alpha^{n}(b_{21(0)2(-1)}))\\
&\quad (\alpha^{l-3}(b_{22}{}'')\alpha^{-4}(h_2))]\alpha^{l+1}(b_{21(0)1}{}^{ss})]][S_B(\beta^{4}(b_{21(0)2(0)})) \otimes 1_H]\\
&= [b_1 \otimes [[\alpha^{n-1}(b_{211(-1)}b_{212(-1)})\alpha^{l-2}(b_{22}{}')]\alpha^{-2}(h_1)][S_H[(\alpha^{l-1}(b_{211(0)}{}^{s})\alpha^{n}(b_{212(0)(-1)}))\\
&\quad (\alpha^{l-3}(b_{22}{}'')\alpha^{-4}(h_2))]\alpha^{l+1}(b_{211(0)}{}^{ss})]][S_B(\beta^{4}(b_{212(0)(0)})) \otimes 1_H]\\
&= [b_1 \otimes [[\alpha^{n-1}(b_{21(-1)})(\alpha^{-n-1}(b_{221(-1)})\alpha^{l-2}(b_{222}{}'))]\alpha^{-2}(h_1)][S_H[\alpha^{l-1}(b_{21(0)}{}^{s})\\
&\quad ((\alpha^{n-1}(b_{221(0)(-1)})\alpha^{l-3}(b_{222}{}''))\alpha^{-4}(h_2))]\alpha^{l}(b_{21(0)}{}^{ss})]][S_B(\beta^{4}(b_{221(0)(0)})) \otimes 1_H]\\
&= [b_1 \otimes [[\alpha^{n-1}(b_{21(-1)})(\alpha^{l-2}(b_{221}{}')\alpha^{n}(b_{222(-1)1}))]\alpha^{-2}(h_1)][S_H[\alpha^{l-1}(b_{21(0)}{}^{s})[(\alpha^{l-3}(b_{221}{}'')\\
&\quad \alpha^{-n-1}(b_{222(-1)2}))\alpha^{-4}(h_2)]]\alpha^{l}(b_{21(0)}{}^{ss})]][S_B(\beta^{3}(b_{222(0)})) \otimes 1_H]\\
&= [b_1 \otimes [[(\alpha^{l-2}(b_{211}{}')(\alpha^{l-1}(b_{2121}{}^{s}{}_{1})\alpha^{l-2}(b_{2122}{}^{s})))(\alpha^{l-2}(b_{221}{}')\alpha^{n}(b_{222(-1)1}))]\alpha^{-2}(h_1)]\\
&\quad S_H[\alpha^{l-1}(b_{211}{}''{}_{1}(b_{2121}{}^{s}{}_{2}\alpha^{-1}(b_{2122}{}^{ss})))[\alpha^{-1}(\alpha^{l-2}(b_{221}{}'')\alpha^{n}(b_{222(-1)2}))\alpha^{-4}(h_2)]]\alpha^{l}(b_{211}{}''{}_{2}b_{2121}{}^{ss})]\\
&\quad (S_B(\beta^{3}(b_{222(0)})) \otimes 1_H)\\
&= [b_1 \otimes [[\alpha^{l-2}(b_{21}{}')[\alpha^{l-1}(b_{221}{}^{s}{}_{1})(\alpha^{l-2}(b_{22211}{}^{s}b_{22212}{}')\alpha^{n}(b_{2222(-1)1}))]]\alpha^{-2}(h_1)](S_H[\alpha^{l-1}(b_{21}{}''{}_{1})\\
&\quad ((\alpha^{l-2}(b_{221}{}^{s}{}_{2})(\alpha^{l-3}(b_{22211}{}^{ss}b_{22212}{}'')\alpha^{n-1}(b_{2222(-1)2})))\alpha^{-4}(h_2))]\alpha^{l-1}(b_{21}{}''{}_{2}b_{221}{}^{ss}))]\\
&\quad (S_B(\beta^{4}(b_{2222(0)})) \otimes 1_H)\\
&\overset{(2.1)}{=} [b_1 \otimes [[\alpha^{l-2}(b_{21}{}')[\alpha^{l-1}(b_{221}{}^{s}{}_{1})\alpha^{n}(b_{222(-1)1})]]\alpha^{-2}(h_1)][S_H[\alpha^{l-1}(b_{21}{}''{}_{1})[(\alpha^{l-2}(b_{221}{}^{s}{}_{2})\\
&\quad \alpha^{n-1}(b_{222(-1)2}))\alpha^{-4}(h_2)]]\alpha^{l-1}(b_{21}{}''{}_{2}b_{221}{}^{ss})]][S_B(\beta^{3}(b_{222(0)})) \otimes 1_H]\\
&= [b_1 \otimes [[\alpha^{l-2}(b_{21}{}')[\alpha^{l-1}(b_{221}{}^{s}{}_{1})\alpha^{n}(b_{222(-1)1})]]\alpha^{-2}(h_1)][S_H[[(\alpha^{l-2}(b_{221}{}^{s}{}_{2})\alpha^{n-1}(b_{222(-1)2}))\\
&\quad \alpha^{-4}(h_2)]S_H(\alpha^{l-1}(b_{21}{}''{}_{1}))]\alpha^{l-1}(b_{21}{}''{}_{2}b_{221}{}^{ss})]][S_B(\beta^{3}(b_{222(0)})) \otimes 1_H]\\
&= [b_1 \otimes \alpha^{l}(b_{21}{}')[[((\alpha^{l-2}(b_{221}{}^{s}{}_{1})\alpha^{n-1}(b_{222(-1)1}))\alpha^{-4}(h_1))S_H[(\alpha^{l-2}(b_{221}{}^{s}{}_{2})\alpha^{n-1}(b_{222(-1)2}))\\
&\quad \alpha^{-4}(h_2)]][(S_H[\alpha^{l-2}(b_{21}{}''{}_{1})]\alpha^{l-2}(b_{21}{}''{}_{2}))\alpha^{l-1}(b_{221}{}^{ss})]]][S_B(\beta^{3}(b_{222(0)})) \otimes 1_H]\\
&= [b_1 \otimes \alpha^{l+1}(b_{21}{}')\varepsilon(b_{221}{}^{s}b_{222(-1)}hb_{21}{}'')\alpha^{l}(b_{221}{}^{ss})][S_B(\beta^{3}(b_{222(0)})) \otimes 1_H]\\
&= [b_1 \otimes \varepsilon_H(h)1_H][S_B(b_2) \otimes 1_H]\\
&= \varepsilon_H(h)b_1S_B(b_2) \otimes 1_H\\
&= \varepsilon_H(h)\varepsilon_B(b)1_B \otimes 1_H.
\end{aligned}
$$

Similarly, we also have $(S * \mathrm{id}) \circ \triangle_{B\rtimes_\tau H}(b \otimes h) = \varepsilon_H(h)\varepsilon_B(b)1_B \otimes 1_H$. Therefore, $S_{B^\sharp_{\rtimes_\tau}H}$ is the convolution inverse of $\mathrm{id}_{B^\sharp_{\rtimes_\tau}H}$ and $((B^\#_{\rtimes_\tau}H, \beta \otimes \alpha)$ is a monoidal Hom-Hopf algebra.

This completes the proof. □

Remark 3.7. In the case of Hopf algebras, it follows from Theorem 3.6 that [9, Theorem 2.5] by taking $\alpha = \mathrm{id}_H$ and $\beta = \mathrm{id}_A$, and furthermore, it follows that [21, Theorem 1] when τ is trivial.

4. Hom-coaction admissible mapping system

In this section, we study a Hom-coaction admissible mapping system to characterize this Radford $[n, (n, l)]$-biproduct structure $(B^\#_{\rtimes_\tau}H, \beta \otimes \alpha)$ established in Theorem 3.4.

Definition 4.1. Let (H, α) be a monoidal Hom-Hopf algebra and (C, β) a monoidal Hom-coalgebra. Suppose $\bar{\tau} : C \longrightarrow H \otimes H$, $\bar{\tau}(c) \triangleq c' \otimes c''$ is bilinear and $\bar{\omega} : C \otimes H \otimes H \longrightarrow H \otimes H$ is a bilinear map defined by

$$\bar{\omega}(c \otimes h \otimes g) = c'h \otimes c''g,$$

for any $c \in C, h, g \in H$.

Then (C, β) is called left $(H, \alpha, \bar{\tau})$-*Hom-comodule* if there is a map $\rho^l : C \longrightarrow H \otimes C$ such that the following two conditions hold:

(1) $(\alpha^{-1} \otimes \rho^l)\rho^l = (\bar{\omega} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta_H \otimes \mathrm{id})(\beta^{-1} \otimes \rho^l)\Delta_C$,

(2) $(\varepsilon \otimes \mathrm{id})\rho^l = \beta^{-1}$.

Remark 4.2. If there is a map $\rho^r : C \to C\otimes H$ such that
(1) $(\rho^r\otimes\alpha^{-1})\rho^r = (\mathrm{id}\otimes\bar{\omega})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta_H)(\beta^{-1}\otimes\rho^r)\Delta_C$,
(2) $(\mathrm{id}\otimes\varepsilon)\rho^r = \beta^{-1}$
hold, then (C,β) is called right $(H,\alpha,\bar{\sigma})$-Hom-module.

Definition 4.3. Let (A,α), (C,β) be monoidal Hom-coalgebras. Then $f : A \longrightarrow C$ is called weak Hom-coalgebra map if $\varepsilon_C\circ f = \varepsilon_A, f\circ\alpha = \beta\circ f$.

Definition 4.4. Let (H,α) be a monoidal Hom-Hopf algebra and (A,β) a monoidal Hom-coalgebra. Let $\rho^+ : A \longrightarrow H\otimes A$ and $\rho^- : A \longrightarrow A\otimes H$ are left and right coactions, respectively. (A,β) is called weak (H,α) Hom-bicomodule if for any $a\in A$,
(1) $(\varepsilon_H\otimes\mathrm{id})\rho^+(a) = \beta^{-1}(a) = (\mathrm{id}\otimes\varepsilon_H)\rho^-(a)$,
(2) $(\alpha^{-1}\otimes\rho^-)\rho^+ = (\rho^+\otimes\alpha^{-1})\rho^-$.

Definition 4.5. Let $(B^{\#}_{\rtimes_\tau}H,\beta\otimes\alpha)$ be monoidal Hom-bialgebra and (A,γ) a monoidal Hom-bialgebra. Then $B \overset{p}{\underset{j}{\leftrightarrows}} A \overset{\pi}{\underset{i}{\rightleftarrows}} H$ is called a *n-Hom-coaction admissible mapping system* if
(1) $p\circ j = \mathrm{id}_B, \pi\circ i = \mathrm{id}_H$,
(2) i is a Hom-bialgebra map, π is a weak Hom-coalgebra map and Hom-algebra map, p is a Hom-coalgebra map, j is a Hom-algebra map,
(3) p is a (H,α) Hom-bimodule map [(A,γ) is given a left action $h\to a \triangleq i(\alpha^{-n}(h))a$, a right action $a\leftarrow h\triangleq ai(\alpha^{-n}(h))$ by (H,α) and (B,β) is given a right action $b\leftarrow h\triangleq\varepsilon_H(h)\beta(b)$ by (H,α)],
(4) $j(B)$ is a (H,α) Hom sub-bicomodule of (A,γ) and $p|_{j(B)}$ is a weak (H,α) Hom-bicomodule map. [(A,γ) is given the weak (H,α) Hom-bicomodule defined by $\rho^l_A(a) = a_{(-1)}\otimes a_{(0)} \triangleq \alpha^{-n}\circ\pi(a_1)\otimes a_2$ and $\rho^r_A(a) = a_{[0]}\otimes a_{[1]} \triangleq a_1\otimes\alpha^{-n}\circ\pi(a_2)$; (B,β) is given the (H,α) Hom-bicomodule defined by $\rho^r_B(b) = b_{[0]}\otimes b_{[1]}\triangleq\beta^{-1}(b)\otimes 1_H$]. And there is $\bar{\tau} : A\to H\otimes H$ such that (A,γ,ρ^l_A), (A,γ,ρ^r_A) are left, right $(H,\alpha,\bar{\tau})$-Hom comodules,
(5) $(j\circ p)*(i\circ\pi) = \mathrm{id}_A$.

Now let $(B^{\#}_{\rtimes_\tau}H,\beta\otimes\alpha)$ be a monoidal Hom-bialgebra built in Theorem 3.4. Then there is some natural maps:

$$\bar{p} : B^{\#}_{\rtimes_\tau}H \longrightarrow B, b\otimes h\mapsto\varepsilon_H(h)b; \qquad \bar{\pi} : B^{\#}_{\rtimes_\tau}H \longrightarrow H, b\otimes h\mapsto\varepsilon_B(b)h;$$

$$\bar{j} : B \longrightarrow B^{\#}_{\rtimes_\tau}H, b\mapsto b\otimes 1_H; \qquad \bar{i} : H \longrightarrow B^{\#}_{\rtimes_\tau}H, h\mapsto 1_B\otimes h.$$

A left, right (H,α) Hom-module action on $B^{\#}_{\rtimes_\tau}H$ defined by

$$\varphi^l : k\otimes(b\otimes h)\mapsto\alpha(k_1)\cdot b\otimes\alpha^{1-n}(k_2)h$$

and

$$\varphi^r : (b\otimes h)\otimes k\mapsto\beta(b)\otimes h\alpha^{-n}(k).$$

A left,right (H,α) Hom-comodule structure on $B^{\#}_{\rtimes_\tau}H$ defined by

$$\rho_l : b\otimes h\mapsto[\alpha^{-1}(b_{1(-1)})\alpha^{l-2-n}(b_2^{'})]\alpha^{-n-1}(h_1)\otimes\beta(b_{1(0)})\otimes\alpha^{l-1}(b_2^{''})\alpha^{-1}(h_2)$$

and

$$\rho_r : b\otimes h\mapsto b_1\otimes\alpha^{l-1}(b_2^{'})\alpha^{-1}(h_1)\otimes\alpha^{l-1-n}(b_2^{''})\alpha^{-n-1}(h_2).$$

With the above notion, we have

Lemma 4.6. *If $(B^{\#}_{\rtimes_\tau}H, \beta\otimes\alpha)$ is a monoidal Hom-bialgebra, then $(B^{\#}_{\rtimes_\tau}H, \beta\otimes\alpha, \rho_l)$ is a left $(H,\alpha,\bar{\tau})$ Hom comodule, where $\bar{\tau}: b\otimes h\mapsto \varepsilon(h)\alpha^{l-1-n}(b^{'})\otimes\alpha^{l-1-n}(b^{''})$.*

Proof. For any $b\otimes h\in B^{\#}_{\rtimes_\tau}H$, we have

$$
\begin{aligned}
&(\alpha^{-1}\otimes\rho_l)\rho_l(b\otimes h)\\
=&[\alpha^{-2}(b_{1(-1)})\alpha^{l-n-3}(b_2^{'})]\alpha^{-n-2}(h_1)\otimes[b_{1(0)1(-1)}\alpha^{l-n-1}(b_{1(0)2}^{'})][\alpha^{l-n-2}(b_{2\,1}^{''})\alpha^{-n-2}(h_{21})]\\
&\otimes\beta^2(b_{1(0)1(0)})\otimes\alpha^l(b_{1(0)2}^{''})[\alpha^{l-2}(b_{2\,2}^{''})\alpha^{-2}(h_{22})]\\
=&[\alpha^{-2}(b_{11(-1)}b_{12(-1)})\alpha^{l-n-3}(b_2^{'})]\alpha^{-n-2}(h_1)\otimes[b_{11(0)(-1)}\alpha^{l-n-1}(b_{12(0)}^{'})][\alpha^{l-n-2}(b_{2\,1}^{''})\alpha^{-n-2}(h_{21})]\\
&\otimes\beta^2(b_{11(0)(0)})\otimes\alpha^l(b_{12(0)}^{''})[\alpha^{l-2}(b_{2\,2}^{''})\alpha^{-2}(h_{22})]\\
=&[\alpha^{-2}(b_{1(-1)})[\alpha^{l-n-3}(b_{21}^{'})\alpha^{l-n-2}(b_{22\,1}^{'})]]\alpha^{-n-2}(h_1)\otimes[\alpha^{-1}(b_{1(0)(-1)})[\alpha^{l-n-3}(b_{21}^{''})\alpha^{l-n-2}(b_{22\,2}^{'})]]\\
&\alpha^{-n-1}(h_{21})\otimes\beta(b_{1(0)(0)})\otimes\alpha^{l-1}(b_{22}^{''})\alpha^{-1}(h_{22})\\
=&[[\alpha^{l-n-3}(b_{11}^{'})\alpha^{-1}(b_{12(-1)1})]\alpha^{-l-n-2}(b_{2\,1}^{'})]\alpha^{-n-2}(h_1)\otimes[[\alpha^{l-n-3}(b_{11}^{''})\alpha^{-1}(b_{12(-1)2})]\alpha^{l-n-2}(b_{2\,2}^{'})]\\
&\alpha^{-n-1}(h_{21})\otimes\beta(b_{12(0)})\otimes\alpha^{l-2}(b_2^{''})\alpha^{-1}(h_{22})\\
=&(\bar{\omega}\otimes\mathrm{id})(\mathrm{id}\otimes\Delta_H\otimes\mathrm{id})(\beta^{-1}\otimes\alpha^{-1}\otimes\rho_l)\Delta(b\otimes h)
\end{aligned}
$$

the second equality holds by (1.12), the third equality holds by Theorem 2.3(3) and the forth equality holds by Theorem 2.3(2). This completes the proof. □

Lemma 4.7. *If $(B^{\#}_{\rtimes_\tau}H, \beta\otimes\alpha)$ is a monoidal Hom-bialgebra, then $(B^{\#}_{\rtimes_\tau}H, \beta\otimes\alpha, \rho_r)$ is a right $(H,\alpha,\bar{\tau})$ Hom-comodule, where $\bar{\tau}: b\otimes h\mapsto \varepsilon(h)\alpha^{l-1-n}(b^{'})\otimes\alpha^{l-1-n}(b^{''})$.*

Proof. For any $b\otimes h\in B^{\#}_{\rtimes_\tau}H$, we have

$$
\begin{aligned}
&(\rho_r\otimes\alpha^{-1})\rho_r(b\otimes h)\\
=&b_{11}\otimes\alpha^{l-1}(b_{12}^{'})[\alpha^{l-2}(b_{2\,1}^{'})\alpha^{-2}(h_{11})]\otimes\alpha^{l-n-1}(b_{12}^{''})[\alpha^{l-n-2}(b_{2\,2}^{'})\alpha^{-n-2}(h_{12})]\\
&\otimes\alpha^{l-n-2}(b_2^{''})\alpha^{-n-2}(h_2)\\
=&\beta^{-1}(b_1)\otimes[\alpha^{n-1}(b_{21(-1)})\alpha^{l-2}(b_{22}^{'})]\alpha^{-1}(h_{11})\otimes[\alpha^{l-n-1}(b_{21(0)}^{'})\alpha^{l-n-1}(b_{22\,1}^{''})]\alpha^{-n-1}(h_{12})\\
&\otimes\alpha^{l-n-1}(b_{21(0)}^{''}b_{22\,2}^{''})\alpha^{-n-2}(h_2)\\
=&(\mathrm{id}\otimes\bar{\omega})(\mathrm{id}\otimes\mathrm{id}\otimes\Delta_H)(\beta^{-1}\otimes\alpha^{-1}\otimes\rho_r)\Delta(b\otimes h)
\end{aligned}
$$

the second equality holds by Theorem 2.3(3). This completes the proof. □

Lemma 4.8. *If $(B^{\#}_{\rtimes_\tau}H, \beta\otimes\alpha)$ is a monoidal Hom-bialgebra, then $(B^{\#}_{\rtimes_\tau}H, \beta\otimes\alpha, \rho_l, \rho_r)$ is a weak (H,α) Hom-bicomodule.*

Proof. For any $b\otimes h\in B^{\#}_{\rtimes_\tau}H$, we have

$$
\begin{aligned}
&(\alpha^{-1}\otimes\rho_r)\rho_l(b\otimes h)\\
=&[\alpha^{-2}(b_{1(-1)})\alpha^{l-n-3}(b_2^{'})]\alpha^{-n-2}(h_1)\otimes\beta(b_{1(0)1})\otimes\alpha^l(b_{1(0)2}^{'})[\alpha^{l-2}(b_{2\,1}^{''})\alpha^{-2}(h_{21})]\\
&\otimes\alpha^{l-n}(b_{1(0)2}^{''})[\alpha^{l-n-2}(b_{2\,2}^{''})\alpha^{-n-2}(h_{22})]\\
=&[\alpha^{-2}(b_{11(-1)}b_{12(-1)})\alpha^{l-n-3}(b_2^{'})]\alpha^{-n-2}(h_1)\otimes\beta(b_{11(0)})\otimes\alpha^l(b_{12(0)}^{'})[\alpha^{l-2}(b_{2\,1}^{''})\alpha^{-2}(h_{21})]\\
&\otimes\alpha^{l-n}(b_{12(0)}^{''})[\alpha^{l-n-2}(b_{2\,2}^{''})\alpha^{-n-2}(h_{22})]\\
=&[\alpha^{-2}(b_{1(-1)})[\alpha^{l-n-3}(b_{21}^{'})\alpha^{l-n-2}(b_{22\,1}^{'})]]\alpha^{-n-2}(h_1)\otimes b_{1(0)}\otimes[\alpha^{l-2}(b_{21}^{''})\alpha^{l-1}(b_{22\,2}^{'})]\alpha^{-1}(h_{21})
\end{aligned}
$$

$$\otimes\,\alpha^{l-n-1}(b_{22}^{''})\alpha^{-n-1}(h_{22})$$
$$=(\rho_l\otimes\alpha^{-1})\rho_r(b\otimes h)$$

the second equality holds by (1.12) and the third equality holds by Theorem 2.3(3). This completes the proof. □

Proposition 4.9. *If $(B^{\#}_{\rtimes_\tau}H,\beta\otimes\alpha)$ is a monoidal Hom-bialgebra built in Theorem 3.4, then $B\overset{\bar{p}}{\underset{\bar{j}}{\leftrightarrows}}B^{\#}_{\rtimes_\tau}H\overset{\bar{\pi}}{\underset{\bar{i}}{\rightleftarrows}}H$ is a n-Hom-coaction admissible mapping system.*

Theorem 4.10. *Let $B\overset{p}{\underset{j}{\leftrightarrows}}A\overset{\pi}{\underset{i}{\rightleftarrows}}H$ be a n-Hom-coaction admissible mapping system. Then $B^{\#}_{\rtimes_\tau}H\cong A$ as monoidal Hom-bialgebra.*

Proof. Define the following maps:

$$f:B^{\#}_{\rtimes_\tau}H\to A,b\otimes h\mapsto\gamma^{-1}(j(b)i(h))$$
$$g:A\to B^{\#}_{\rtimes_\tau}H,a\mapsto(\beta\otimes\alpha)(p(a_1)\otimes\pi(a_2)).$$

We need to prove the following aspects:

- $f\circ g=\mathrm{id}_A,g\circ f=\mathrm{id}_{B^{\#}_{\rtimes_\tau}H}$;
- f is a Hom algebra homomorphism;
- g is a Hom coalgebra homomorphism.

Firstly, it is easy to verify $f\circ(\beta\otimes\alpha)=\gamma\circ f,g\circ\gamma=(\beta\otimes\alpha)\circ g$, then for any $a\in A$,

$$\begin{aligned}f(g(a))=&f[(\beta\otimes\alpha)(p(a_1)\otimes\pi(a_2))]\\=&\gamma\circ f[(p(a_1)\otimes\pi(a_2))]\\=&j(p(a_1))i(\pi(a_2))=(j\circ p)*(i\circ\pi)(a)=a.\end{aligned}$$

And

$$\begin{aligned}g\circ f(b\otimes h)=&g[\gamma^{-1}(j(b)i(h))]\\=&(\beta^{-1}\otimes\alpha^{-1})g[(j(b)i(h))]\\=&p((j(b)_1i(h)_1))\otimes\pi((j(b)_2i(h)_2))\\=&p((j(b)_1i(h_1)))\otimes\pi((j(b)_2i(h_2)))\\=&\varepsilon(h_1)\beta[p(j(b)_1)]\otimes\pi[j(b)_2]h_2\\=&\beta[\beta^{-1}(b)]\otimes 1_H\alpha^{-1}(h_2)=b\otimes h.\end{aligned}$$

Next, we prove f is a Hom-algebra homomorphism, we only need to prove $f[(a\otimes h)(b\otimes g)]=f(a\otimes h)f(b\otimes g)$. First we compute for any $h\in H,b\in B$

$$\begin{aligned}i(h)j(b)=&j\circ p[i(h_1)j(b)_1]i\circ\pi[i(h_2)j(b)_2]\\=&j\circ p[\alpha^n(h_1)\to j(b)_1]i[h_2\pi(j(b)_2)]\\=&j[\alpha^n(h_1)\cdot p(j(b)_1)]i[h_2\pi(j(b)_2)]\\=&j[\alpha^n(h_1)\cdot\beta^{-1}(b)]i[\alpha(h_2)].\end{aligned}$$

Therefore, it is easy to prove $f[(a\otimes h)(b\otimes g)]=f(a\otimes h)f(b\otimes g)$ by j is a Hom algebra homomorphism and i is a Hom-bialgebra homomorphism.

Finally, we prove g is a Hom-coalgebra homomorphism, we only need to prove $\Delta(g(a))=(g\otimes g)\Delta(a)$. First by Definition 4.5(3)and(4), it is not hard to verify for any $a\in A$,

$$\alpha^{n+1}(p(a_1)_{(-1)})\pi(a_2)\otimes\beta(p(a_1)_{(0)})=\alpha\circ\pi(a_1)\otimes p(a_2).\tag{4.1}$$

Therefore,

$$\begin{aligned}\Delta(g(a)) =& p[\gamma(a_{11})]\otimes[\alpha^{n+1}(p(a_{121})_{(-1)})\alpha^{l}(p(a_{122})^{'})]\pi(a_2)_1\otimes\beta^3[p(a_{121})_{(0)}]\otimes\alpha^{l+1}[p(a_{122})^{''}]\pi(a_2)_2\\ =& p(a_1)\otimes\alpha^{n+1}(p(a_{21})_{(-1)})[\alpha^{l}(p(a_{221})^{'})\alpha(\pi(a_{222})_1)]\otimes\beta^2[p(a_{21})_{(0)}]\otimes\alpha^{l+1}[p(a_{221})^{''}]\alpha^2[\pi(a_{222})_2]\\ =& p(a_1)\otimes\alpha^{n+1}(p(a_{21})_{(-1)})\alpha(\pi(a_{221}))\otimes\beta^2[p(a_{21})_{(0)}]\otimes\alpha^2[\pi(a_{222})]\\ \overset{(4.1)}{=}& p(a_1)\otimes\alpha^2(\pi(a_{211}))\otimes\beta^2[p(a_{212})]\otimes\alpha[\pi(a_{22})]\\ =& \beta[p(a_{11})]\otimes\alpha(\pi(a_{22}))\otimes\beta[p(a_{21})]\otimes\alpha[\pi(a_{22})]\\ =& (g\otimes g)\Delta(a)\end{aligned}$$

the third equality holds since $(A,\gamma,\rho_A^l))$ is left $(H,\alpha,\bar{\tau})$-Hom comodule.

In fact, it is easy to prove f is a Hom coalgebra homomorphism and g is a Hom coalgebra homomorphism by the relation of f and g. Thus, f and g are Hom bialgebra homomorphisms and $B^{\#}_{\rtimes_\tau}H\cong A$.

This completes the proof. □

5. The Maschke theorem of monoidal Hom-smash coproduct

In this section, we study the cosemisimplicity of a special Hom-smash coproduct and prove the related Maschke theorem.

Let (H,α) be a monoidal Hom-Hopf algebra and (C,β) a right (H,α)-Hom-comodule coalgebra. Then $(C\ltimes H,\beta\otimes\alpha)$ is called a monoidal Hom-smash coproduct (cf. [13, 15]) if

(1) As a linear space, $C\ltimes H = C\otimes H$;

(2) Hom-comultiplication is: $\forall c\ltimes h\in C\ltimes H$,

$$\triangle_{C\ltimes H}(c\ltimes h) = c_1\ltimes c_{2(1)}\alpha^{-1}(h_2)\otimes\beta(c_{2(0)})\ltimes h_1.$$

We have the fact that $(C\ltimes H,\beta\otimes\alpha)$ is a monoidal Hom-coalgebra with counit $\varepsilon_{C\ltimes H}=\varepsilon_C\otimes\varepsilon_H$ under the Hom-comultiplication.

Remark 5.1. We define two maps:

$$\pi_C: C\ltimes H\to C,\ c\ltimes h\mapsto\beta(c)\varepsilon_H(h);$$
$$\pi_H: C\ltimes H\to H,\ c\ltimes h\mapsto\varepsilon_C(c)S^{-1}(\alpha(h)),$$

where $S^{-1}\in Hom(H,H)$ is the inverse of antipode S. And it is easy to verify the following equation:

$$c\ltimes h = (\mathrm{id}_C\otimes S)(\pi_C\otimes\pi_H)\triangle_{C\ltimes H}(c\ltimes h). \tag{5.1}$$

Lemma 5.2. *π_C and π_H are Hom-coalgebra maps.*

Proof. It is not hard to verify the following equations according to the definition of $\varepsilon_{C\ltimes H}$, π_C and π_H.

$$\pi_C\circ(\beta\otimes\alpha)=\beta\circ\pi_C,\ \pi_H\circ(\beta\otimes\alpha)=\alpha\circ\pi_H,$$
$$\varepsilon_C\circ\pi_C=\varepsilon_C\otimes\varepsilon_H=\varepsilon_{C\ltimes H},\ \varepsilon_H\circ\pi_H=\varepsilon_C\otimes\varepsilon_H=\varepsilon_{C\ltimes H}.$$

And for any $c\ltimes h\in C\ltimes H$,

$$\begin{aligned}(\pi_C\otimes\pi_C)\circ\triangle_{C\ltimes H}(c\ltimes h) =& (\pi_C\otimes\pi_C)(c_1\ltimes c_{2(1)}\alpha^{-1}(h_2)\otimes\beta(c_{2(0)})\ltimes h_1)\\ =& \varepsilon_H(h)\beta(c_1)\otimes\beta(c_2)=\triangle_C(\varepsilon_H(h)\beta(c))=\triangle_C\circ\pi_C(c\ltimes h)\end{aligned}$$

and

$$\begin{aligned}(\pi_H\otimes\pi_H)\circ\triangle_{C\ltimes H}(c\ltimes h)=&\varepsilon_C(c)S^{-1}(\alpha(h_2))\otimes S^{-1}(\alpha(h_1))\\ =&\varepsilon_C(c)S^{-1}(\alpha(h))_1\otimes S^{-1}(\alpha(h))_2\\ =&\triangle_H\left(\varepsilon_C(c)S^{-1}(\alpha(h))\right)=\triangle_H\circ\pi_H(c\ltimes h).\end{aligned}$$

Thus, π_C and π_H are Hom-coalgebra maps. □

Definition 5.3. Let (H,α) be a monoidal Hom-Hopf algebra and (C,β) a right (H,α)-Hom-comodule coalgebra with comodule map $\rho_C^H: C\to C\otimes H, c\mapsto c_{(0)}\otimes c_{(1)}$. We say that a pair (M,μ) is called a right (C,H)-Hom-comodule if

(1) (M,μ,ρ_M^C) is a right (C,β)-Hom-comodule with $\rho_M^C: M\to M\otimes C, m\mapsto m_{\{0\}}\otimes m_{\{1\}}$,
(2) (M,μ,ρ_M^H) is a right (H,α)-Hom-comodule with $\rho_M^H: M\to M\otimes H, m\mapsto m_{[0]}\otimes m_{[1]}$,
(3) For all $m\in M$,

$$m_{[0]\{0\}}\otimes m_{[0]\{1\}}\otimes m_{[1]}=m_{\{0\}[0]}\otimes m_{\{1\}(0)}\otimes m_{\{0\}[1]}m_{\{1\}(1)}. \tag{5.2}$$

Let C and H be as above. We denote the category of right (C,H) Hom-comodule by $\widetilde{\mathcal{H}}(\mathcal{M}^{C,H})$. Similarly, we denote the category of right $(C\ltimes H,\beta\otimes\alpha)$ Hom-comodule by $\widetilde{\mathcal{H}}(\mathcal{M}^{C\ltimes H})$. Then we have the following theorem.

Theorem 5.4. *$\widetilde{\mathcal{H}}(\mathcal{M}^{C,H})$ and $\widetilde{\mathcal{H}}(\mathcal{M}^{C\ltimes H})$ are isomorphic.*

Proof. Let $F:\widetilde{\mathcal{H}}(\mathcal{M}^{C\ltimes H})\to\widetilde{\mathcal{H}}(\mathcal{M}^{C,H})$ be a functor and $(M,\mu,\rho_M^{C\ltimes H})\in\widetilde{\mathcal{H}}(\mathcal{M}^{C\ltimes H})$ a right $(C\ltimes H,\beta\otimes\alpha)$-Hom- comodule, $\rho_M^{C\ltimes H}: M\to M\otimes(C\ltimes H), m\mapsto m_{\langle 0\rangle}\otimes m_{\langle 1\rangle}$. We define

$$m_{\{0\}}\otimes m_{\{1\}}=m_{\langle 0\rangle}\otimes\pi_C(m_{\langle 1\rangle})\in M\otimes C; \tag{5.3}$$
$$m_{[0]}\otimes m_{[1]}=m_{\langle 0\rangle}\otimes\pi_H(m_{\langle 1\rangle})\in M\otimes H. \tag{5.4}$$

Let $\gamma=\beta\otimes\alpha$. Then we have

$$\begin{aligned}\rho_M^{C\ltimes H}(m)&=m_{\langle 0\rangle}\otimes m_{\langle 1\rangle}\\ &\overset{(5.1)}{=}m_{\langle 0\rangle}\otimes(\pi_C(m_{\langle 1\rangle 1})\otimes S(\pi_H(m_{\langle 1\rangle 2})))\\ &=\mu(m_{\langle 0\rangle\langle 0\rangle})\otimes(\pi_C(m_{\langle 0\rangle\langle 1\rangle})\otimes S(\pi_H(\gamma^{-1}(m_{\langle 1\rangle}))))\\ &=\mu(m_{\langle 0\rangle\langle 0\rangle})\otimes(\pi_C(m_{\langle 0\rangle\langle 1\rangle})\otimes S(\alpha^{-1}\circ\pi_H(m_{\langle 1\rangle})))\\ &\overset{(5.3)}{=}\mu(m_{\langle 0\rangle\{0\}})\otimes(m_{\langle 0\rangle\{1\}}\otimes S(\alpha^{-1}\circ\pi_H(m_{\langle 1\rangle})))\\ &\overset{(5.4)}{=}\mu(m_{[0]\{0\}})\otimes(m_{[0]\{1\}}\otimes S(\alpha^{-1}(m_{[1]}))).\end{aligned}$$

We define two maps:

$$\rho_M^C: M\to M\otimes C, m\mapsto m_{\{0\}}\otimes m_{\{1\}},$$
$$\rho_M^H: M\to M\otimes H, m\mapsto m_{[0]}\otimes m_{[1]}.$$

Next we proof (M,μ,ρ_M^C) is a right (C,β)-Hom-comodule and (M,μ,ρ_M^H) a right (H,α)-Hom-comodule. It follows from Lemma 5.2 that

$$\varepsilon_C(m_{\{1\}})m_{\{0\}}=\mu^{-1}(m),\ \mu(m)_{\{0\}}\otimes\mu(m)_{\{1\}}=\mu(m_{\{0\}})\otimes\beta(m_{\{1\}}).$$

And we have

$$\begin{aligned}m_{\{0\}\{0\}}\otimes m_{\{0\}\{1\}}\otimes\beta^{-1}(m_{\{1\}})&=m_{\{0\}\langle 0\rangle}\otimes\pi_C(m_{\{0\}\langle 1\rangle})\otimes\beta^{-1}(m_{\{1\}})\\ &=m_{\langle 0\rangle\langle 0\rangle}\otimes\pi_C(m_{\langle 0\rangle\langle 1\rangle})\otimes\beta^{-1}\circ\pi_C(m_{\langle 1\rangle})\\ &=\mu^{-1}(m_{\langle 0\rangle})\otimes\pi_C(m_{\langle 1\rangle 1})\otimes\beta^{-1}\circ\pi_C(\gamma(m_{\langle 1\rangle 2}))\end{aligned}$$

$$
\begin{aligned}
&= \mu^{-1}(m_{\langle 0\rangle}) \otimes \pi_C(m_{\langle 1\rangle 1}) \otimes \pi_C(m_{\langle 1\rangle 2})\\
&= \mu^{-1}(m_{\langle 0\rangle}) \otimes \pi_C(m_{\langle 1\rangle})_1 \otimes \pi_C(m_{\langle 1\rangle})_2\\
&= \mu^{-1}(m_{\{0\}}) \otimes m_{\{1\}1} \otimes m_{\{1\}2}.
\end{aligned}
$$

Similarly,

$$\varepsilon_H(m_{[1]})m_{[0]} = \mu^{-1}(m),\ \mu(m)_{[0]} \otimes \mu(m)_{[1]} = \mu(m_{[0]}) \otimes \alpha(m_{[1]}).$$

And

$$
\begin{aligned}
m_{[0][0]} \otimes m_{[0][1]} \otimes \alpha^{-1}(m_{[1]}) &= m_{[0]\langle 0\rangle} \otimes \pi_H(m_{[0]\langle 1\rangle}) \otimes \alpha^{-1}(m_{[1]})\\
&= m_{[0]\langle 0\rangle} \otimes \pi_H(m_{[0]\langle 1\rangle}) \otimes \alpha^{-1}(m_{[1]})\\
&= m_{\langle 0\rangle\langle 0\rangle} \otimes \pi_H(m_{\langle 0\rangle\langle 1\rangle}) \otimes \alpha^{-1}\pi_H(m_{\langle 1\rangle})\\
&= \mu^{-1}(m_{\langle 0\rangle}) \otimes \pi_H(m_{\langle 0\rangle 1}) \otimes \alpha^{-1}\pi_H(\gamma(m_{\langle 1\rangle 2}))\\
&= \mu^{-1}(m_{\langle 0\rangle}) \otimes \pi_H(m_{\langle 0\rangle 1}) \otimes \pi_H(m_{\langle 1\rangle 2})\\
&= \mu^{-1}(m_{\langle 0\rangle}) \otimes \pi_H(m_{\langle 0\rangle})_1 \otimes \pi_H(m_{\langle 1\rangle})_2\\
&= \mu^{-1}(m_{\{0\}}) \otimes m_{[1]1} \otimes m_{[1]2}
\end{aligned}
$$

which shows that (M, μ, ρ_M^C) is a right (C, β)-Hom-comodule and (M, μ, ρ_M^H) is a right (H, α)-Hom-comodule.

Simultaneously, for any $m \in M$, we have

$$(\rho_M^{C\ltimes H} \otimes \gamma^{-1})\rho_M^{C\ltimes H}(m) = (\mu^{-1} \otimes \triangle_{C\ltimes H})\rho_M^{C\ltimes H}(m).$$

That is

$$
\begin{aligned}
&\mu^2(m_{[0]\{0\}[0]\{0\}}) \otimes \beta(m_{[0]\{0\}[0]\{1\}}) \otimes S(m_{[0]\{0\}[1]}) \otimes \beta^{-1}(m_{[0]\{1\}}) \otimes S(\alpha^{-2}(m_{[1]}))\\
=&m_{[0]\{0\}} \otimes m_{[0]\{1\}1} \otimes m_{[0]\{1\}2(1)}S(\alpha^{-2}(m_{[1]1})) \otimes \beta(m_{[0]\{1\}2(0)}) \otimes S(\alpha^{-1}(m_{[1]2})).
\end{aligned}
$$

First, applying $\mathrm{id}_M \otimes \pi_H \otimes \pi_C$ to both sides, we have

$$m_{\{0\}[0]} \otimes m_{\{0\}[1]} \otimes \beta^{-1}(m_{\{1\}}) = m_{[0]\{0\}} \otimes \alpha^{-2}(m_{[1]})S^{-1}(m_{[0]\{1\}(1)}) \otimes \beta(m_{[0]\{1\}(0)}).$$

Second, applying $\mathrm{id}_M \otimes \mathrm{id}_H \otimes \gamma \circ \rho_C^H$ to both sides, we have

$$
\begin{aligned}
&m_{\{0\}[0]} \otimes m_{\{0\}[1]} \otimes m_{\{1\}(0)} \otimes m_{\{1\}(1)}\\
=&m_{[0]\{0\}} \otimes \alpha^{-2}(m_{[1]})S^{-1}(m_{[0]\{1\}(1)}) \otimes \beta^2(m_{[0]\{1\}(0)(0)}) \otimes \alpha^2(m_{[0]\{1\}(0)(1)}).
\end{aligned}
$$

Third, applying $(\mathrm{id}_M \otimes m_H \otimes \mathrm{id}_C) \circ (\mathrm{id}_M \otimes \mathrm{id}_H \otimes \tau)$ to both sides, we have

$$
\begin{aligned}
&m_{\{0\}[0]} \otimes m_{\{0\}[1]}m_{\{1\}(1)} \otimes m_{\{1\}(0)}\\
=&m_{[0]\{0\}} \otimes [\alpha^{-2}(m_{[1]})S^{-1}(m_{[0]\{1\}(1)})]\alpha^2(m_{[0]\{1\}(0)(1)}) \otimes \beta^2(m_{[0]\{1\}(0)(0)})\\
=&m_{[0]\{0\}} \otimes [\alpha^{-2}(m_{[1]})S^{-1}(\alpha(m_{[0]\{1\}(1)2}))]\alpha^2(m_{[0]\{1\}(1)1}) \otimes \beta(m_{[0]\{1\}(0)})\\
=&m_{[0]\{0\}} \otimes \alpha^{-1}(m_{[1]})[S^{-1}(\alpha(m_{[0]\{1\}(1)2}))\alpha(m_{[0]\{1\}(1)1})] \otimes \beta(m_{[0]\{1\}(0)})\\
=&m_{[0]\{0\}} \otimes m_{[1]} \otimes m_{[0]\{1\}}.
\end{aligned}
$$

Then we have proof (5.2) and $(M, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}^{C,H})$.

Conversely, let $G : \widetilde{\mathcal{H}}(\mathcal{M}^{C,H}) \to \widetilde{\mathcal{H}}(\mathcal{M}^{C\ltimes H})$ be a functor and $(N, \nu) \in \widetilde{\mathcal{H}}(\mathcal{M}^{C,H})$. Then (N, ν) is both a right (C, β)-Hom-comodule with comodule map $\rho_N^C : N \to N \otimes C$, $n \mapsto n_{\{0\}} \otimes n_{\{1\}}$ and a right (H, α)-Hom-comodule with comodule map $\rho_N^H : N \to N \otimes H$, $n \mapsto n_{[0]} \otimes n_{[1]}$. We define a map as follow:

$$\rho_N^{C\ltimes H} : N \to N \otimes (C \ltimes H), n \mapsto \nu(n_{[0]\{0\}}) \otimes n_{[0]\{1\}} \otimes S(\alpha^{-1}(n_{[1]})).$$

Let $n_{\langle 0\rangle}\otimes n_{\langle 1\rangle}=\nu(n_{[0]\{0\}})\otimes(n_{[0]\{1\}}\otimes S(\alpha^{-1}(n_{[1]})))$. We next prove that $(N,\nu,\rho_N^{C\ltimes H})$ is a right $(C\ltimes H,\beta\otimes\alpha)$-Hom-comodule. By the definitions of $\rho_N^{C\ltimes H}$, comodule and counit, we have

$$\varepsilon_{C\ltimes H}(n_{\langle 1\rangle})n_{\langle 0\rangle}=\nu^{-1}(n),\ \nu(n)_{\langle 0\rangle}\otimes\nu(n)_{\langle 1\rangle}=\nu(n_{\langle 0\rangle})\otimes\gamma(n_{\langle 1\rangle}).$$

And we also have

$$\begin{aligned}
&(\nu^{-1}\otimes\triangle_{C\ltimes H})\rho_N^{C\ltimes H}(n)\\
&\quad=n_{[0]\{0\}}\otimes n_{[0]\{1\}1}\otimes n_{[0]\{1\}2(1)}S(\alpha^{-2}(n_{[1]1}))\otimes\beta(n_{[0]\{1\}2(0)})\\
&\quad\otimes S(\alpha^{-1}(n_{[1]2}))\\
&=\nu(n_{[0]\{0\}\{0\}})\otimes n_{[0]\{0\}\{1\}}\otimes\alpha^{-1}(n_{[0]\{1\}(1)})S(\alpha^{-2}(n_{[1]1}))\\
&\quad\otimes n_{[0]\{1\}(0)}\otimes S(\alpha^{-1}(n_{[1]2}))\\
&=\nu^2(n_{[0][0]\{0\}\{0\}})\otimes\beta(n_{[0][0]\{0\}\{1\}})\otimes n_{[0][0]\{1\}(1)}S(\alpha^{-2}(n_{[0][1]}))\\
&\quad\otimes\nu(n_{[0][0]\{1\}(0)})\otimes S(\alpha^{-2}(n_{[1]}))\\
&\overset{(5.2)}{=}\nu^2(n_{[0]\{0\}[0]\{0\}})\otimes\beta(n_{[0]\{0\}[0]\{1\}})\otimes n_{[0]\{1\}(0)(1)}\\
&\quad S(\alpha^{-2}(n_{[0]\{0\}[1]}n_{[0]\{1\}(1)}))\otimes\nu(n_{[0]\{1\}(0)(0)})\otimes S(\alpha^{-2}(n_{[1]}))\\
&=\nu^2(n_{[0]\{0\}[0]\{0\}})\otimes\beta(n_{[0]\{0\}[0]\{1\}})\otimes n_{[0]\{1\}(1)1}S(\alpha^{-2}(n_{[0]\{0\}[1]})\\
&\quad\alpha^{-1}(n_{[0]\{1\}(1)2}))\otimes n_{[0]\{1\}(0)}\otimes S(\alpha^{-2}(n_{[1]}))\\
&=\nu^2(n_{[0]\{0\}[0]\{0\}})\otimes\beta(n_{[0]\{0\}[0]\{1\}})\otimes[\alpha^{-1}(n_{[0]\{1\}(1)1})\\
&\quad S(\alpha^{-1}(n_{[0]\{1\}(1)2}))]S(\alpha^{-1}(n_{[0]\{0\}[1]}))\otimes n_{[0]\{1\}(0)}\otimes S(\alpha^{-2}(n_{[1]}))\\
&=\nu^2(n_{[0]\{0\}[0]\{0\}})\otimes\beta(n_{[0]\{0\}[0]\{1\}})\otimes S(n_{[0]\{0\}[1]})\otimes\beta^{-1}(n_{[0]\{1\}})\\
&\quad\otimes S(\alpha^{-2}(n_{[1]}))\\
&=(\rho_N^{C\ltimes H}\otimes\gamma^{-1})\rho_N^{C\ltimes H}(n).
\end{aligned}$$

Then, $(N,\nu,\rho_N^{C\ltimes H})$ is a right $(C\ltimes H,\beta\otimes\alpha)$-Hom-comodule.

Thus, we have $GF(M)=M, FG(N)=N$, showing that $\widetilde{\mathcal{H}}(\mathcal{M}^{C\ltimes H})$ and $\widetilde{\mathcal{H}}(\mathcal{M}^{C,H})$ are isomorphic. □

Definition 5.5. Let (H,α) be finite dimensional monoidal Hom-Hopf algebra with its dual Hopf algebra (H^*,α^*). Then $\phi\in H^*$ is called the left integral of H^* if ϕ is α^*-invariable (i.e. $\alpha^*(\phi)=\phi$) and for all $\xi\in H^*$,

$$\xi\phi=\varepsilon_{H^*}(\xi)\phi.$$

A left integral is normalized if $\varepsilon_{H^*}(\phi)=1$.

Lemma 5.6. *Let (H,α) be finite dimensional monoidal Hom-Hopf algebra with its dual Hopf algebra (H^*,α^*). If α is involutive, then (H^*,α^*) is both a left (H,α)-Hom-module and a right (H,α)-Hom-module with left and right action: for all $x,y\in H, f\in H^*$*

$$(x\rightharpoonup f)(y)=f(yx)\quad\textit{and}\quad(f\leftharpoonup x)(y)=f(yS(x)).$$

Proof. This proof is straightforward. □

Lemma 5.7. *Let (H,α) be finite dimensional monoidal Hom-Hopf algebra and α an involution. If (H^*,α^*) is the dual Hopf algebra and $\phi\in H^*$ is a left integral of H^*, then for any $f,g\in H^*$, a, $b\in H$,*

(1) $(a_2\rightharpoonup f)g\leftharpoonup a_1=f(g\leftharpoonup a)$,

(2) $\phi(aS(b_1))\alpha(b_2)=\phi(a_2S(b))a_1$.

Proof. (1) For any $a, x \in H, f, g \in H^*$, we have

$$\begin{aligned}\langle (a_2 \rightharpoonup f)g \leftharpoonup a_1, x\rangle &= \langle (a_2 \rightharpoonup f)g, xS(a_1)\rangle \\ &= \langle a_2 \rightharpoonup f, x_1S(a_1)_1\rangle \langle g, x_2S(a_1)_2\rangle \\ &= \langle f, (x_1S(a_1)_1)a_2\rangle \langle g, x_2S(a_1)_2\rangle \\ &= \langle f, \alpha(x_1)(S(a_{12})\alpha^{-1}(a_2))\rangle \langle g, x_2S(a_{11})\rangle \\ &= \langle f, \alpha(x_1)(S(a_{21})a_{22})\rangle \langle g, x_2S(\alpha^{-1}(a_1))\rangle \\ &= \langle f, x_1\rangle \langle g, x_2S(a)\rangle \\ &= \langle f, x_1\rangle \langle g \leftharpoonup a, x_2\rangle \\ &= \langle f(g \leftharpoonup a), x\rangle .\end{aligned}$$

(2) ϕ is a left integral of H^*, then for all $a \in H, f \in H^*$, we have

$$(a_2 \rightharpoonup f)\phi = \varepsilon_{H^*}(a_2 \rightharpoonup f)\phi = \langle a_2 \rightharpoonup f, 1_H\rangle \phi = \langle f, \alpha(a_2)\rangle \phi.$$

Therefore,

$$f(\phi \leftharpoonup a) \overset{(1)}{=} (a_2 \rightharpoonup f)\phi \leftharpoonup a_1 = \langle f, \alpha(a_2)\rangle \phi \leftharpoonup a_1. \tag{5.5}$$

$$\begin{aligned}\langle \phi, aS(b_1)\rangle \langle f, \alpha(b_2)\rangle &= \langle f, \alpha(b_2)\rangle \langle \phi \leftharpoonup b_1, a\rangle \\ &\overset{(5.5)}{=} \langle f(\phi \leftharpoonup b), a\rangle \\ &= \langle f, a_1\rangle \langle \phi \leftharpoonup b, a_2\rangle \\ &= \langle f, a_1\rangle \langle \phi, a_2S(b)\rangle \\ &= \langle \phi, a_2S(b)\rangle \langle f, a_1\rangle .\end{aligned}$$

□

Lemma 5.8. *Let $(V, \mu) \in \widetilde{\mathcal{H}}(\mathcal{M}^{C\ltimes H})$ and (W, μ) be a right $(C \ltimes H, \beta \otimes \alpha)$-Hom subcomodule of (V, μ). If $\lambda : V \to W$ is right (C, β)-colinear and α is involutive, then $\widetilde{\lambda} : (V, \mu) \to (W, \mu)$,*

$$\widetilde{\lambda}(v) = \phi[v_{[1]}S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{[0]}))_{[1]}))]\lambda(\mu^2(v_{[0]}))_{[0]},$$

is right $(C \ltimes H, \beta \otimes \alpha)$-colinear.

Proof. By Theorem 5.4, we only prove $\widetilde{\lambda}$ is both right (C, β)-colinear and right (H, α)-colinear.

On the one hand, by Lemma 5.7, we have

$$\begin{aligned}\widetilde{\lambda}(v)_{[0]} \otimes \widetilde{\lambda}(v)_{[1]} &= \phi[v_{[1]}S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{[0]}))_{[1]}))]\lambda(\mu^2(v_{[0]}))_{[0][0]} \otimes \lambda(\mu^2(v_{[0]}))_{[0][1]} \\ &= \phi[v_{[1]}S^{-1}(\lambda(\mu^2(v_{[0]}))_{[1]2})]\mu^{-1}(\lambda(\mu^2(v_{[0]}))_{[0]}) \otimes \lambda(\mu^2(v_{[0]}))_{[1]1} \\ &= \phi[v_{[1]1}S^{-1}(\lambda(\mu^2(v_{[0]}))_{[1]})]\mu^{-1}(\lambda(\mu^2(v_{[0]}))_{[0]}) \otimes \alpha(v_{[1]2}) \\ &= \phi[v_{[1]1}S^{-1}(\alpha^{-1}(\lambda(\mu(v_{[0]}))_{[1]}))](\lambda(\mu(v_{[0]}))_{[0]}) \otimes \alpha(v_{[1]2}) \\ &= \phi[v_{[0][1]}S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{[0][0]}))_{[1]}))](\lambda(\mu^2(v_{[0][0]}))_{[0]}) \otimes v_{[1]} \\ &= \widetilde{\lambda}(v_{[0]}) \otimes v_{[1]}.\end{aligned}$$

On the other hand, since λ is right (C, β)-colinear, we have

$$\lambda(v)_{\{0\}} \otimes \lambda(v)_{\{1\}} = \lambda(v_{\{0\}}) \otimes v_{\{1\}}. \tag{5.6}$$

Then

$$\begin{aligned}\widetilde{\lambda}(v)_{\{0\}} \otimes \widetilde{\lambda}(v)_{\{1\}} =& \phi[v_{[1]}S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{[0]}))_{[1]}))]\lambda(\mu^2(v_{[0]}))_{[0]\{0\}} \otimes \lambda(\mu^2(v_{[0]}))_{[0]\{1\}} \\ \overset{(5.2)}{=}& \phi[v_{[1]}S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{[0]}))_{\{0\}[1]}\lambda(\mu^2(v_{[0]}))_{\{1\}(1)}))]\lambda(\mu^2(v_{[0]}))_{\{0\}[0]} \otimes\end{aligned}$$

$$
\begin{aligned}
&\lambda(\mu^2(v_{[0]}))_{\{1\}(0)}\\
&\overset{(5.6)}{=}\phi[v_{[1]}S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{[0]})_{\{0\}})_{[1]}\mu^2(v_{[0]})_{\{1\}(1)}))]\lambda(\mu^2(v_{[0]})_{\{0\}})_{[0]}\otimes\\
&\quad\mu^2(v_{[0]})_{\{1\}(0)}\\
&=\phi[v_{[1]}S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{[0]\{0\}}))_{[1]}\beta^2(v_{[0]\{1\}})_{(1)}))]\lambda(\mu^2(v_{[0]\{0\}}))_{[0]}\otimes\\
&\quad\beta^2(v_{[0]\{1\}})_{(0)}\\
&\overset{(5.2)}{=}\phi[(v_{\{0\}[1]}v_{\{1\}(1)})S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{\{0\}[0]}))_{[1]}\beta^2(v_{\{1\}(0)})_{(1)}))]\\
&\quad\lambda(\mu^2(v_{\{0\}[0]}))_{[0]}\otimes\beta^2(v_{\{1\}(0)})_{(0)}\\
&=\phi[(v_{\{0\}[1]}v_{\{1\}(1)})S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{\{0\}[0]}))_{[1]}\alpha^2(v_{\{1\}(0)(1)})))]\\
&\quad\lambda(\mu^2(v_{\{0\}[0]}))_{[0]}\otimes\beta^2(v_{\{1\}(0)(0)})\\
&=\phi[(v_{\{0\}[1]}\alpha(v_{\{1\}(1)2}))S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{\{0\}[0]}))_{[1]}\alpha^2(v_{\{1\}(1)1})))]\\
&\quad\lambda(\mu^2(v_{\{0\}[0]}))_{[0]}\otimes\beta(v_{\{1\}(0)})\\
&=\phi[(v_{\{0\}[1]}\alpha(v_{\{1\}(1)2}))(S^{-1}(\alpha(v_{\{1\}(1)1}))S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{\{0\}[0]}))_{[1]})))]\\
&\quad\lambda(\mu^2(v_{\{0\}[0]}))_{[0]}\otimes\beta(v_{\{1\}(0)})\\
&=\phi[\alpha(v_{\{0\}[1]})[(v_{\{1\}(1)2}S^{-1}(v_{\{1\}(1)1}))S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{\{0\}[0]}))_{[1]}))]]\\
&\quad\lambda(\mu^2(v_{\{0\}[0]}))_{[0]}\otimes\beta(v_{\{1\}(0)})\\
&=\phi[\alpha(v_{\{0\}[1]})S^{-1}(\lambda(\mu^2(v_{\{0\}[0]}))_{[1]})]\lambda(\mu^2(v_{\{0\}[0]}))_{[0]}\otimes v_{\{1\}}\\
&=\phi[\alpha(v_{\{0\}[1]}S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{\{0\}[0]}))_{[1]})))]\lambda(\mu^2(v_{\{0\}[0]}))_{[0]}\otimes v_{\{1\}}\\
&=\widetilde{\lambda}(v_{\{0\}})\otimes v_{\{1\}}.
\end{aligned}
$$

This completes the proof. □

Theorem 5.9. *Let (H,α) be finite dimensional cosemisimple monoidal Hom-Hopf algebra and (C,β) a right (H,α) Hom-comodule coalgebra. Then $(V,\mu)\in\widetilde{\mathcal{H}}(\mathcal{M}^{C\ltimes H})$ and (W,μ) is a right $(C\ltimes H,\beta\otimes\alpha)$-Hom subcomodule of (V,μ). If (W,μ) is a right (C,β) Hom-comodule direct summand of (V,μ), then (W,μ) is a right $(C\ltimes H,\beta\otimes\alpha)$ Hom-comodule direct summand of (V,μ).*

Proof. Since (H,α) is a finite dimensional cosemisimple monoidal Hom-Hopf algebra, then there is a normalized left integral $\phi\in H^*$ by [6, Theorem 4.6]. Let

$$\widetilde{\lambda}:(V,\mu)\to(W,\mu)\,,\widetilde{\lambda}(v)=\phi[v_{[1]}S^{-1}(\alpha^{-1}(\lambda(\mu^2(v_{[0]}))_{[1]}))]\lambda(\mu^2(v_{[0]}))_{[0]}$$

which $\lambda:V\to W$ is right (C,β) comodule projection. Then $\widetilde{\lambda}$ is right $(C\ltimes H,\beta\otimes\alpha)$-colinear. We just proof $\widetilde{\lambda}$ is projection. In fact, for all $w\in W$, we have

$$
\begin{aligned}
\widetilde{\lambda}(w)&=\phi[w_{[1]}S^{-1}(\alpha^{-1}(\lambda(\mu^2(w_{[0]}))_{[1]}))]\lambda(\mu^2(w_{[0]}))_{[0]}\\
&=\phi[w_{[1]}S^{-1}(\alpha^{-1}(\mu^2(w_{[0]})_{[1]}))]\mu^2(w_{[0]})_{[0]}\\
&=\phi[w_{[1]}S^{-1}(\alpha^{-1}(\alpha^2(w_{[0][1]})))]\mu^2(w_{[0][0]})\\
&=\phi[\alpha(w_{[1]2})S^{-1}(\alpha^{-1}(\alpha^2(w_{[1]1})))]\mu(w_{[0]})\\
&=\phi[\alpha(w_{[1]2})S^{-1}(\alpha(w_{[1]1}))]\mu(w_{[0]})\\
&=\phi(\varepsilon(w_{[1]})1_H)\mu(w_{[0]})\\
&=w.
\end{aligned}
$$

So $\widetilde{\lambda}$ is a right $(C\ltimes H,\beta\otimes\alpha)$ projection and (W,μ) is a right $(C\ltimes H,\beta\otimes\alpha)$ Hom-comodule direct summand of (V,μ). □

Corollary 5.10. *Let (H, α) be finite dimensional cosemisimple monoidal Hom-Hopf algebra and α be an involution. If (C, β) is a cosemisimple right (H, α) Hom-comodule coalgebra, then Hom-smash coproduct $(C \ltimes H, \beta \otimes \alpha)$ is also cosemisimple.*

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