Direct Sampling Methods for General Nonlinear Inverse Problems

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Joint work with

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OUTLINE

- Motivation of sampling-type methods
- General framework of direct sampling methods
- DSMs for Inverse acoustic/EM scattering, EIT, DOT
- **DSMs for moving inhomogeneous media**
- Optimal control approach for Sobolev scale

Most Popular Approach for Inverse Problems

- Most IPs: parameter identifications in PDEs, e.g.,

  EIT, DOT, Inverse Scattering, Seismic Tomography, …

Stationary PDE:

$$L_q(u) = 0$$

or time-dependent PDE:

$$D_t^{\alpha} u - L_q(u) = 0$$

Inverse problem is to solve

$$u(q) = u^{\delta} \quad \text{on} \quad \Gamma$$

Mostly the solution parameter *q* tells us information

geometric shape/location & distributional values

Least-squares formulation with regularization

- **IPs are mostly ill-posed:** $u(q) = u^\delta \quad \text{on} \quad \Gamma$

transform to a nearby "well-posed" problem:

$$\min_{q \in K} J(q) = \text{data fitting}\left(u(q) - u^\delta\right) \quad + \beta\, \Phi(q)$$

- **A most crucial mathematical issue :**

$$\text{choose } K,\ \Phi,\ \beta \text{ s.t. it is stable wrt data}$$

Solution of Nonlinear Optim Systems

- **Output LS Tikhonov regularization :**

$$\min_{q \in K} J(q) = \text{data fitting}\ \big(u(q) - u^\delta\big) + \beta\ \Phi(q)$$

- 1^{st} approach: coupled optimality PDE system

$$\left\{ \begin{array}{l} \text{Forward PDE ;} \\ \qquad\qquad \text{Adjoint PDE ;} \\ \qquad\qquad\qquad\qquad \text{Variational Inequality} \end{array} \right.$$

⇨ Singularities: parameters mostly disconts, unknown

e.g., conductivity in EIT, refractive index in inverse medium

Iterative Solvers for Nonlinear Optim Systems

- Least-squares minimization :

$$\min_{q\in K} J(q) = \text{data fitting } (u(q) - u^{\delta}) + \beta\ \Phi(q)$$

highly nonlinear, nonconvex, nonsmooth

- **2^{nd} approach: iterative**

Most popular iterative, e.g., Newton type:

need to choose β, h, Δt,

need good initial guess of q,
repeated forward solutions,
need the derivatives of u(q) wrt changes of q
often very sensitive to noise

⇒ Often very expensive & challenge to solve

- **Is it always worthwhile or necessary to do so?**

Alternative Solvers

- Indeed not worthwhile or necessary to do

$$\min_{q \in K} J(q) = \text{data fitting}\left(u(q) - u^{\delta}\right) + \beta\, \Phi(q) \qquad (1)$$

1st : if noise not small, we can see from

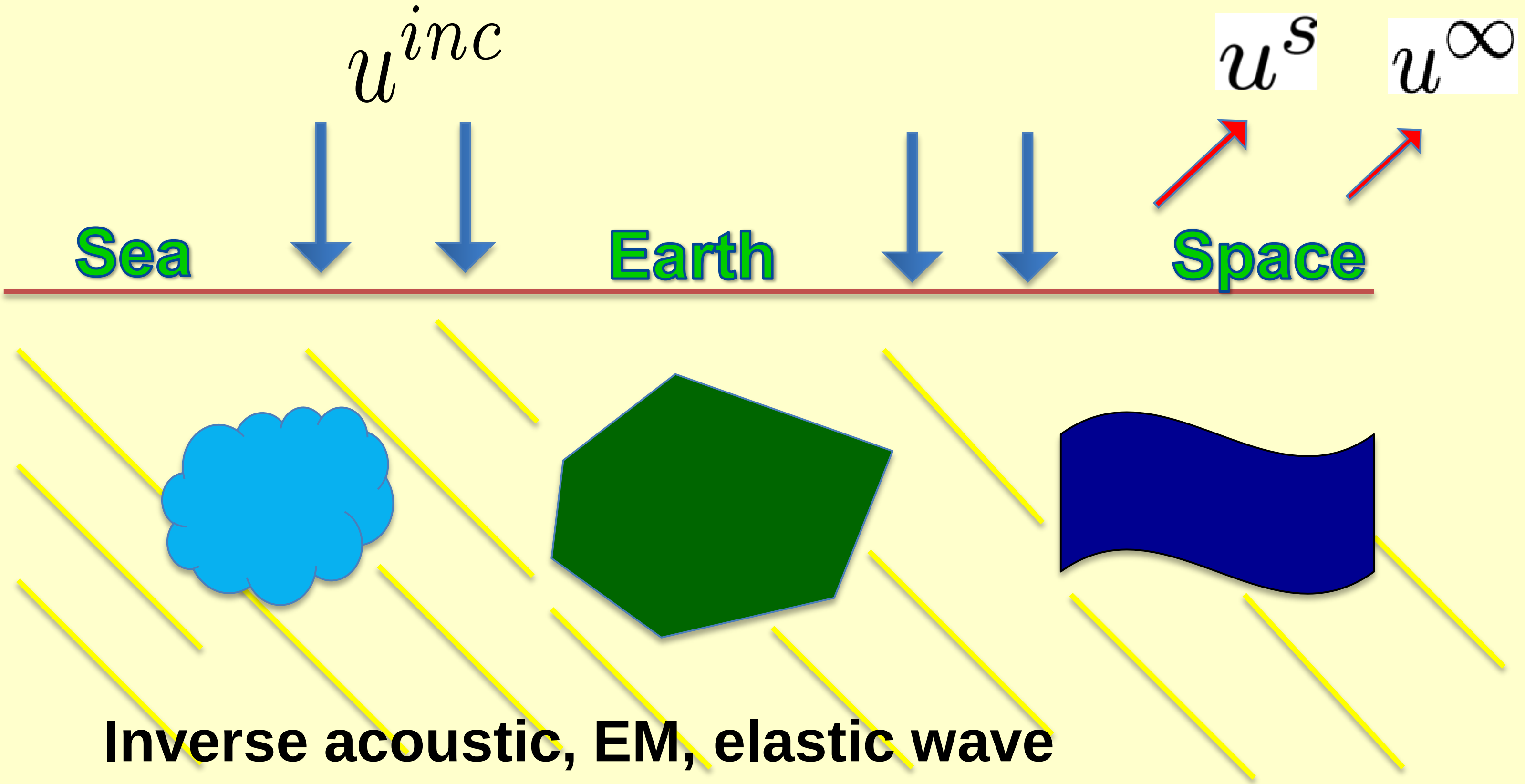
$$q - q^{\delta,h} = (q - q^{\delta}) + (q^{\delta} - q^{\delta,h}) = O(\delta^{\gamma}) + O(h^{\alpha})$$

2nd: no good accuracy needed for the concerned applications

- **Alternative solvers, overcoming technical barriers:**

no need good initial guess of q,
no repeated forward solutions,
no need the derivatives of u(q) wrt changes of q

Can we reconstruct
Shape & Location, without Physics?



Inverse acoustic, EM, elastic wave

EIT, DOT, MRI, … …

Linear Sampling Method

- Colton-Kirsch 96 : a truly revolutionary algorithm!

  Inverse acoustic scattering: $\Delta u + k^2 n^2(x) u = 0$

- Consider the far-field operator $F : L^2(S^{N-1}) \mapsto L^2(S^{N-1})$

$$(Fg)(\hat{x}) = \int u_\infty(\hat{x}, d)\, g(d)\, ds(d)\,, \quad \hat{x} \in S^{N-1}$$

  and the far-field equation for g:

$$Fg = \Phi_\infty(\cdot, z) \qquad \forall\, z \in R^N$$

- Solve for g at each z, and look at its energy

$$\|g(\cdot, z)\|_{L^2(S)}$$

Algorithm of LSM

- Turns inverse scattering into solving integral equations

- Algorithm of LSM : select a numerical cut-off value ***C***

1. Select a grid T_h of sampling points, covering ***D***
2. At each z , solve the far-field equation for $g(\cdot,z)$
3. Determine

$$z \in D \text{ if } \|g(\cdot,z)\| \le c\,;$$

$$z \notin D \text{ if } \|g(\cdot,z)\| > c$$

Drawbacks of LSM

- No effective strategies to choose numerical cut-off values.

- Huge computational efforts:

  need to solve the far-field equation for each sampling point, e.g.,

  for an $n \times n \times n$ grid, need to solve $\boldsymbol{n^3}$ ill-posed equations

  The grid should be very fine to get a fine reconstruction

New Variants of LSM

- Li-Liu-Zou, SISC 09:

  Multilevel Linear Sampling Method,

  reduce computational complexity from $O(n^3)$ to $O(n^2)$

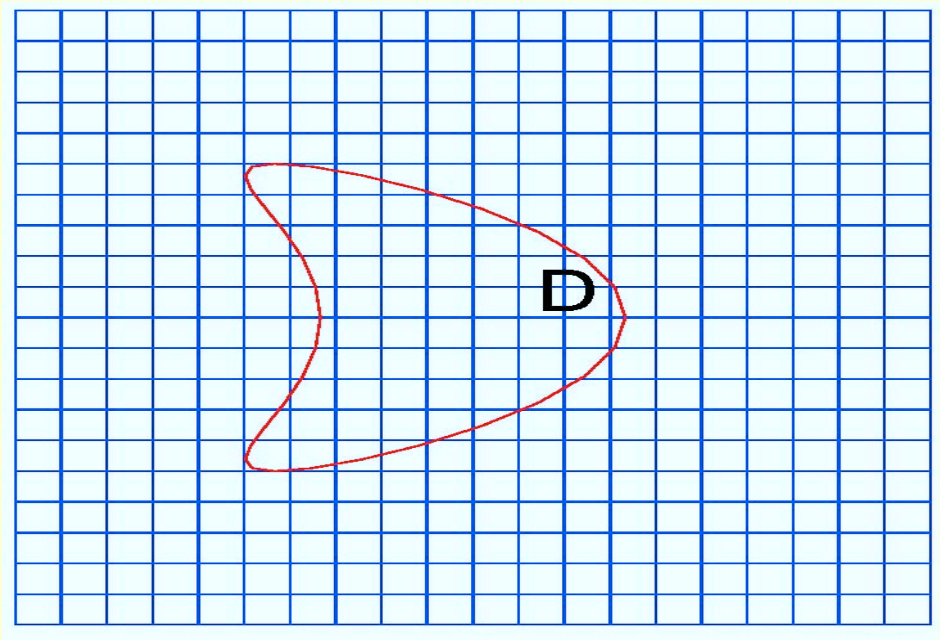
- Li-Liu-Zou, SISC 10:

  Strengthened LSM with a Reference Obstacle,

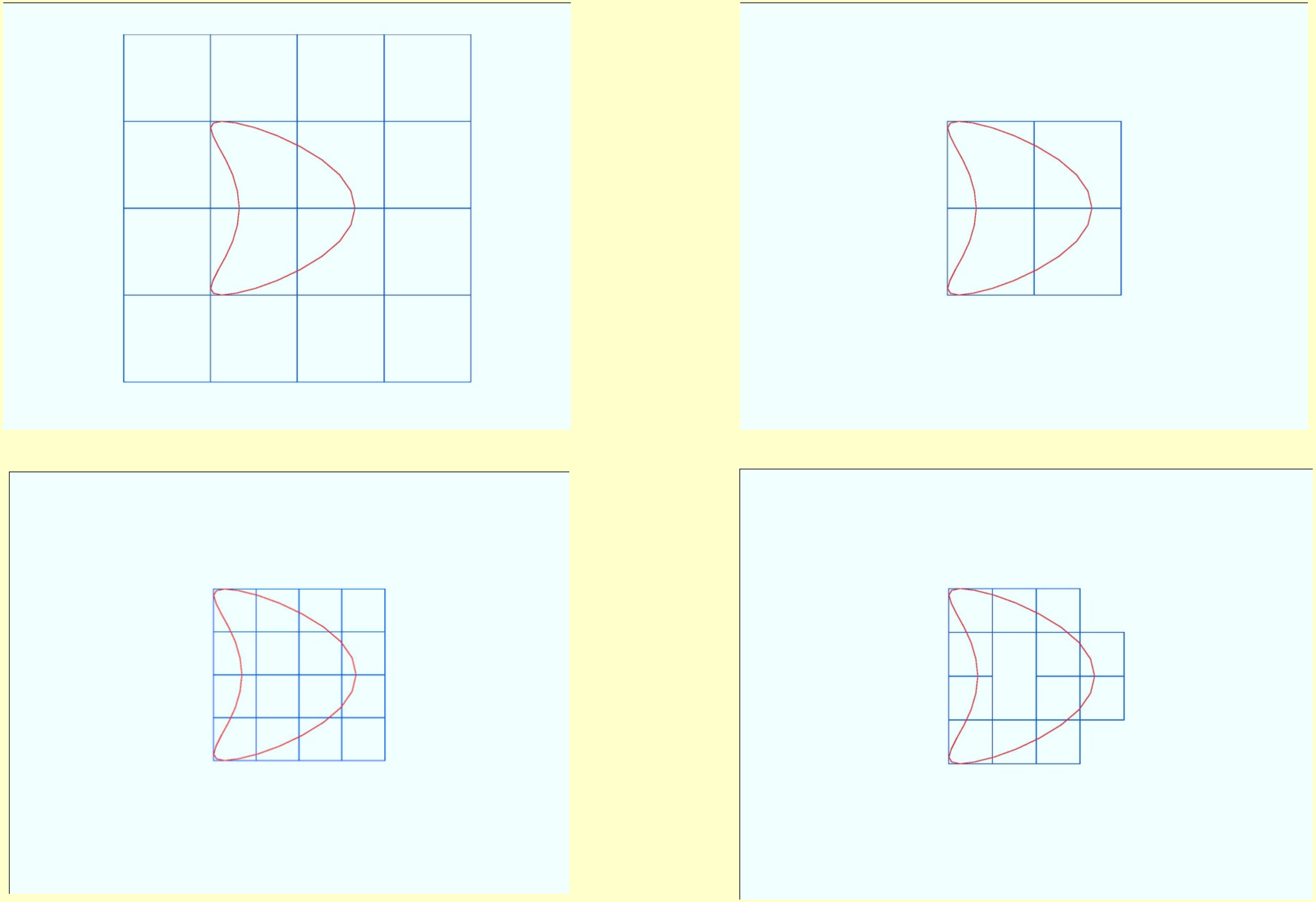
  provide a deterministic technique to select feasible numerical cut-off values

Multilevel Linear Sampling Method

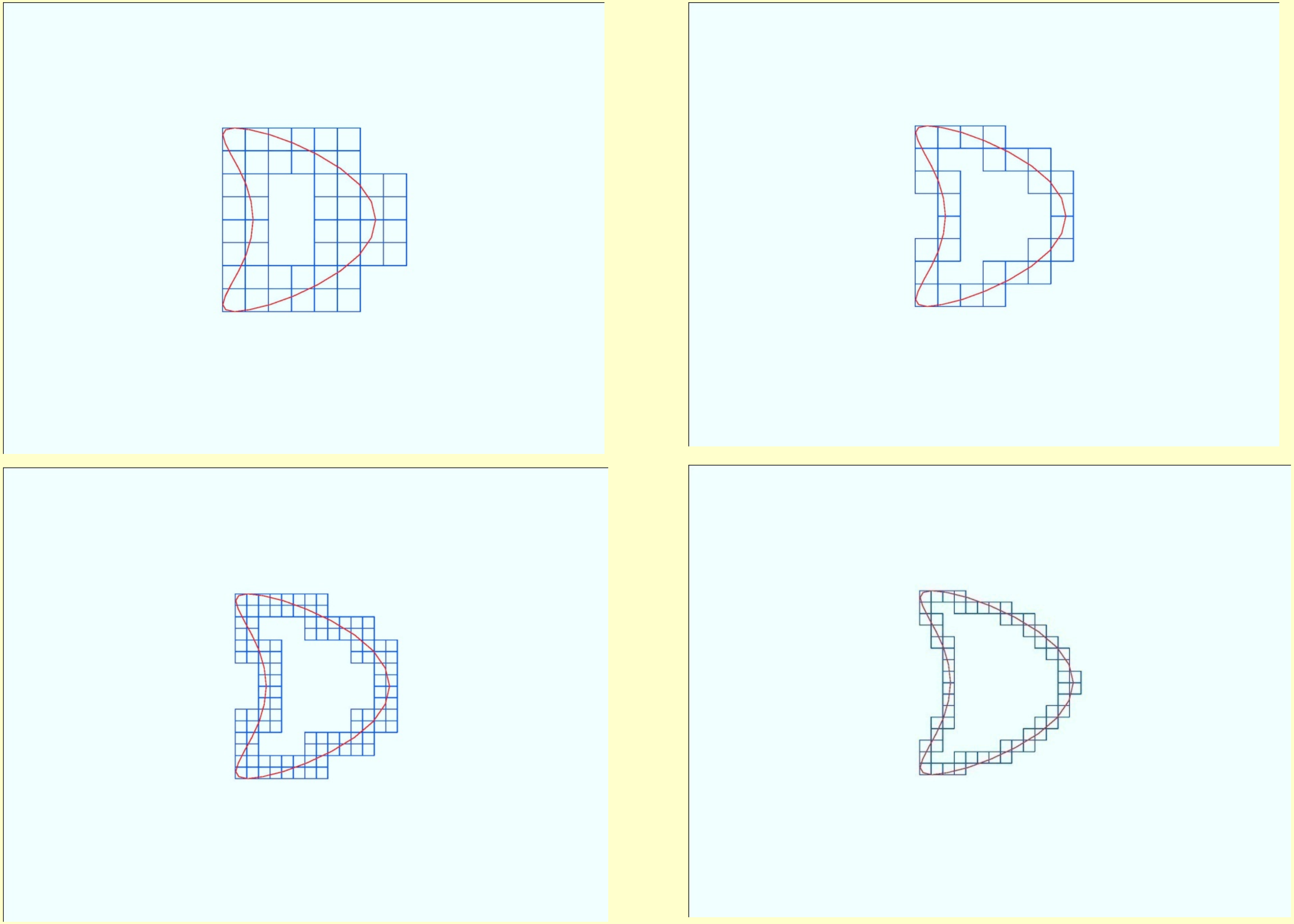
- MLSM : get rid of remote and inner cells



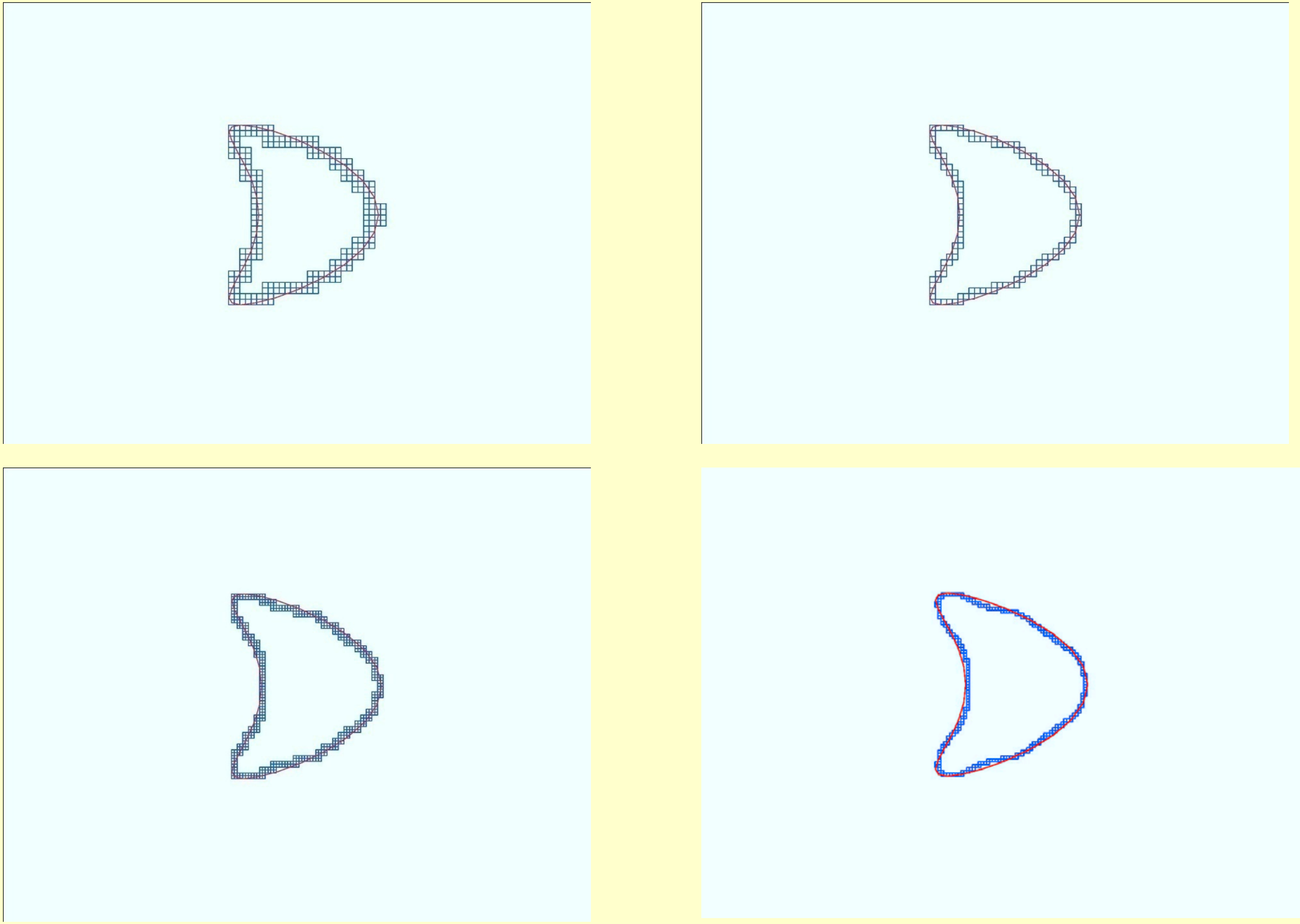
Numerical Example I



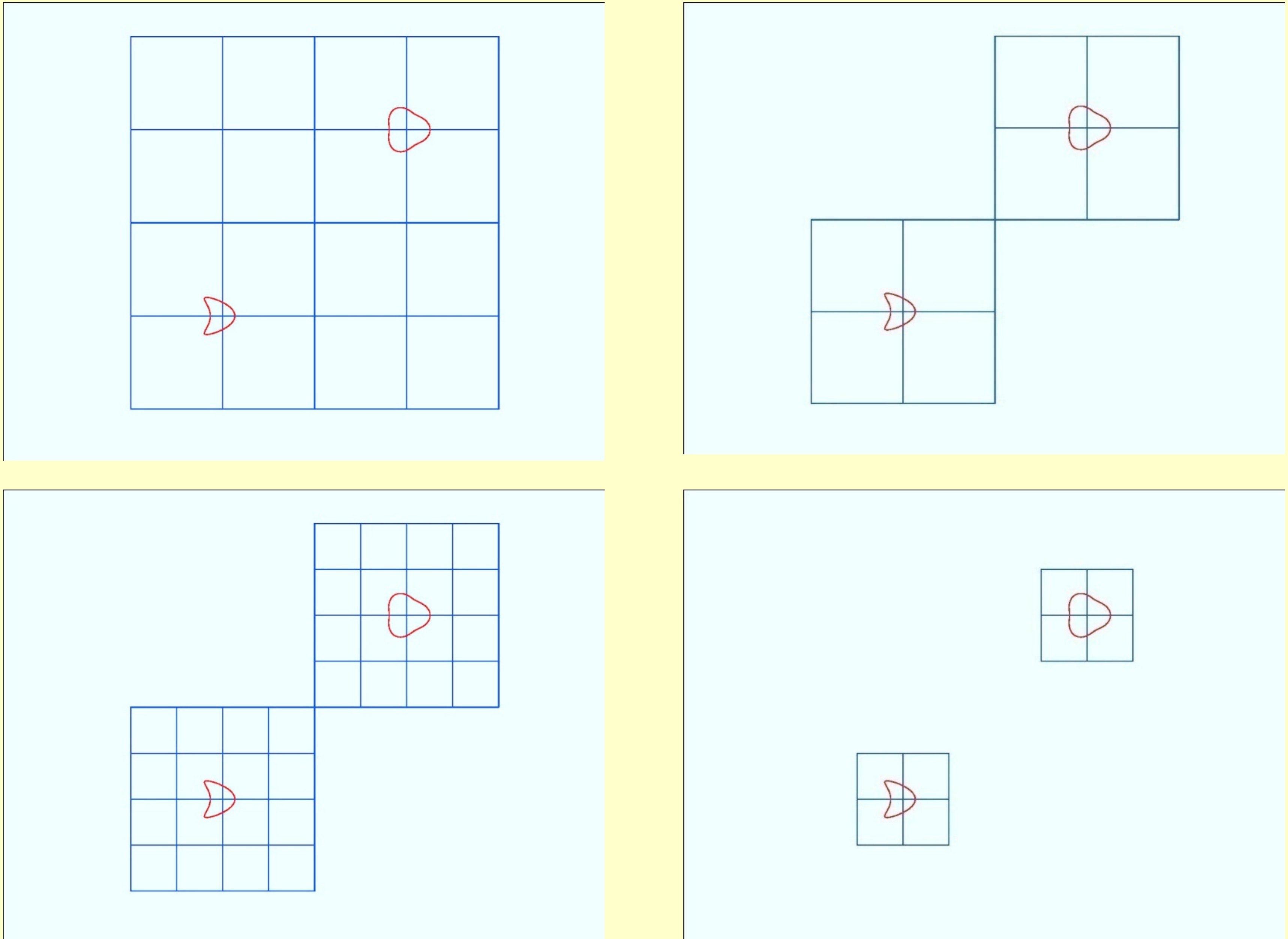
Numerical Example I



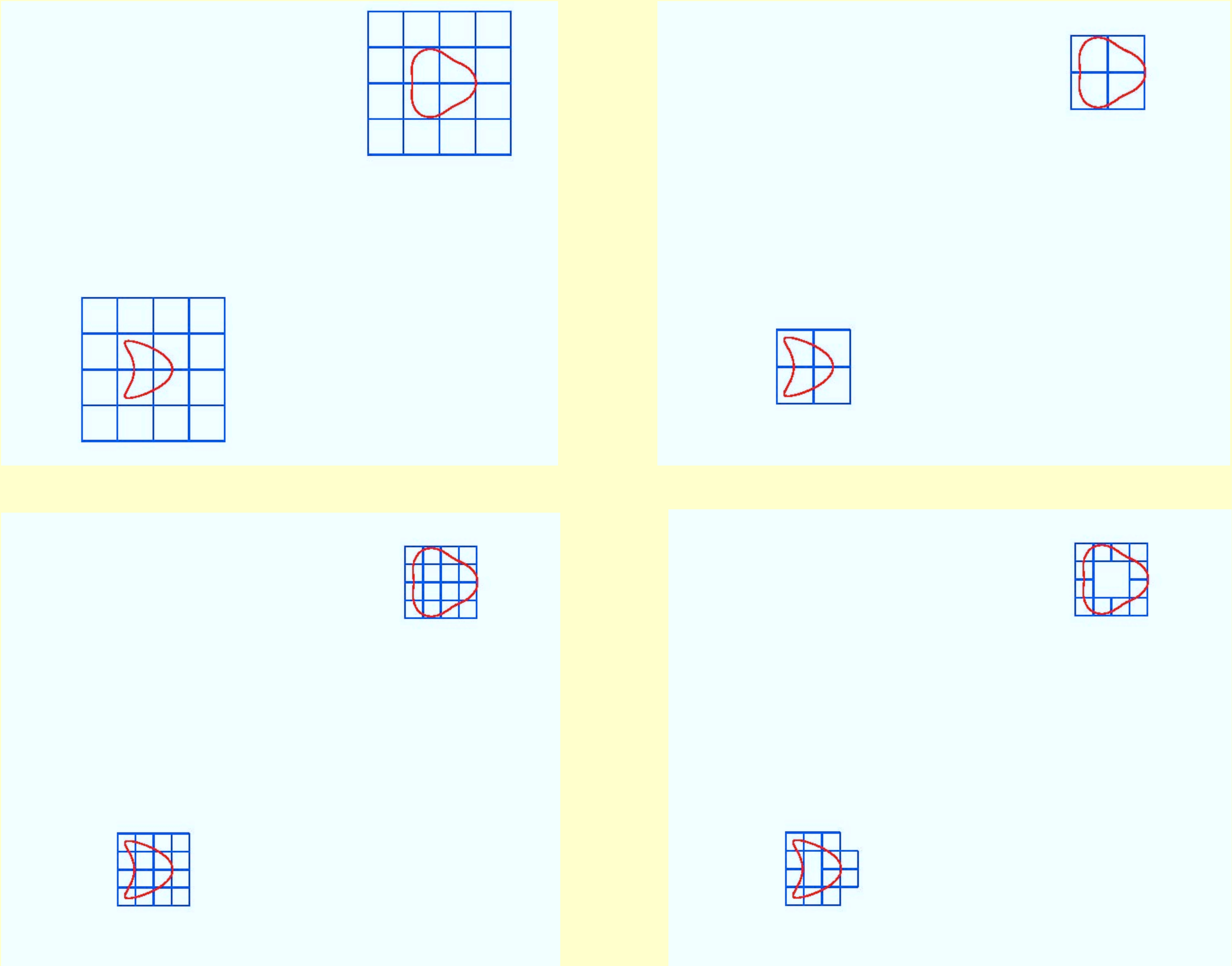
Numerical Example I

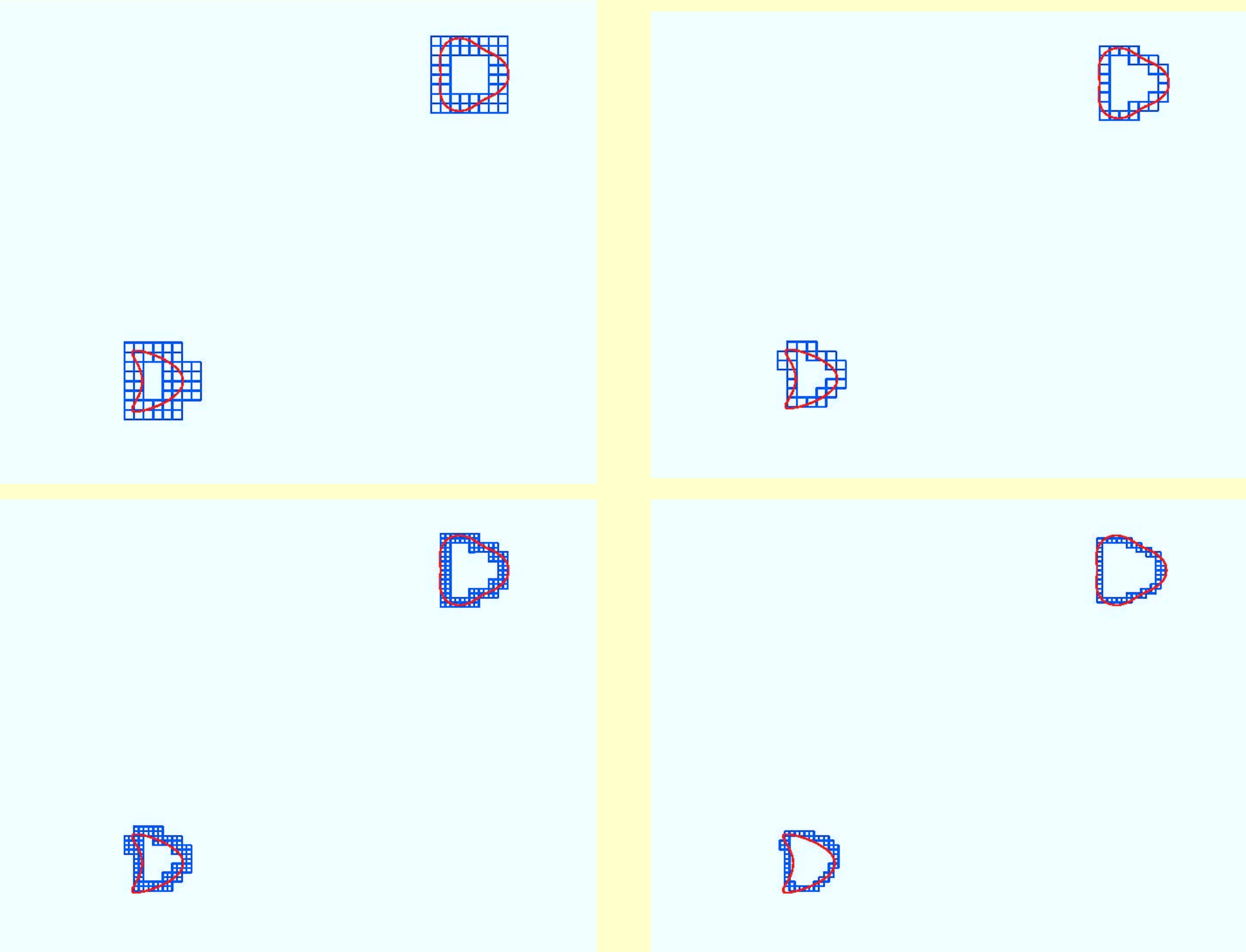


Numerical Example II



Numerical Example II





Sampling-type Methods

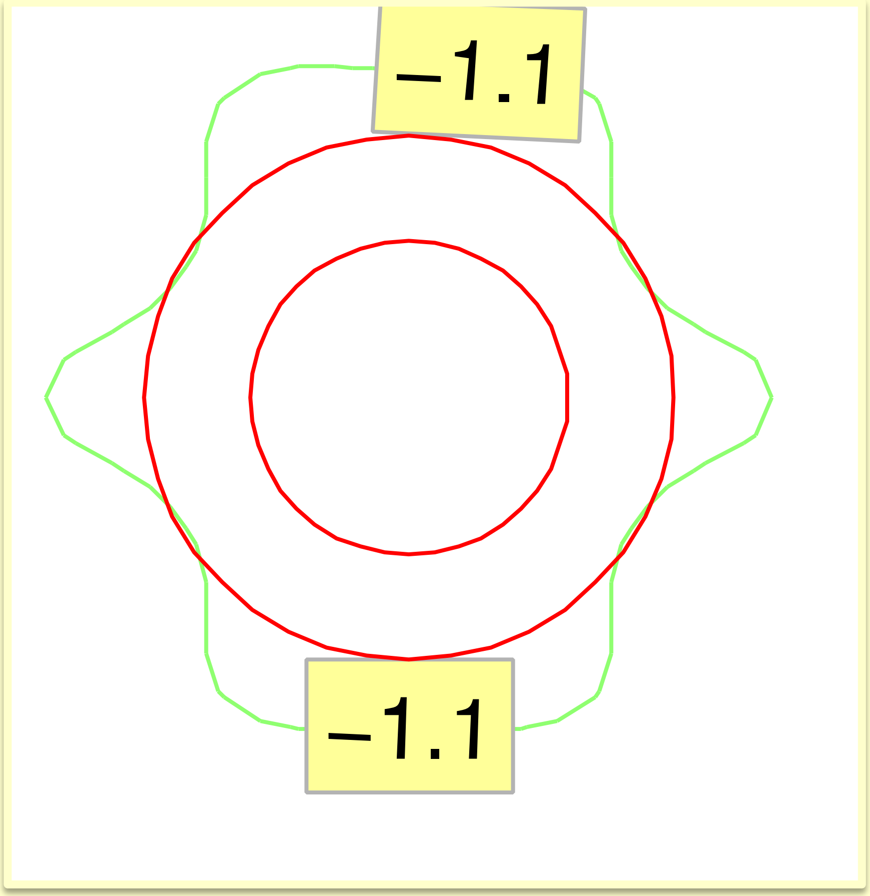
- **Linear sampling method (Colton-Kirsch 96);**
  Factorization method (Kirsch 98);
  Point source & multipole method (Potthast 98);
  Probe method (Potthast 01);
  Reciprocity Gap Sampling Method (Colton-Haddar, 05)
  Subspace-based optimization method (Chen 08)
  … …

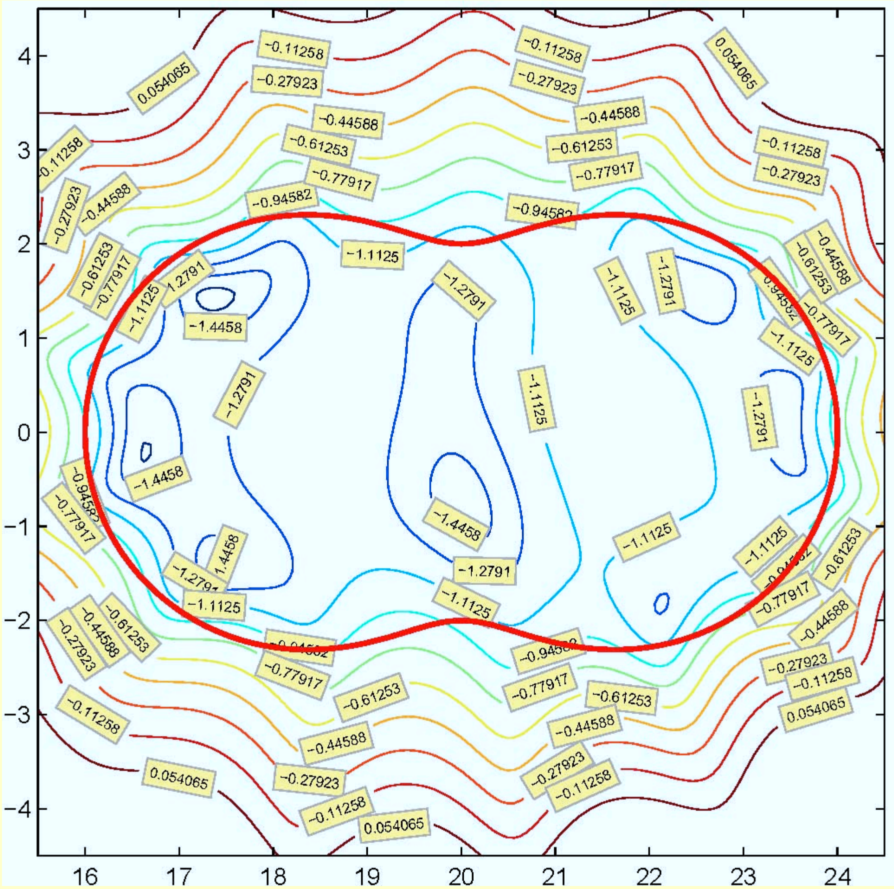
- **Monographs:**
  Potthast, Chapman & Hall, 01;
  Kirsch, Grinberg, Oxford 07;
  Cakoni, Colton, Monk: SIAM 11;
  Cakoni, Colton, Springer 14;
  Nakamura, Potthast, IOP, 15;
  X Chen, Wiley, 2018; … …

But

**when we apply these methods,
we may still encounter
several common difficulties**

(I) Cut-off Values & Noise





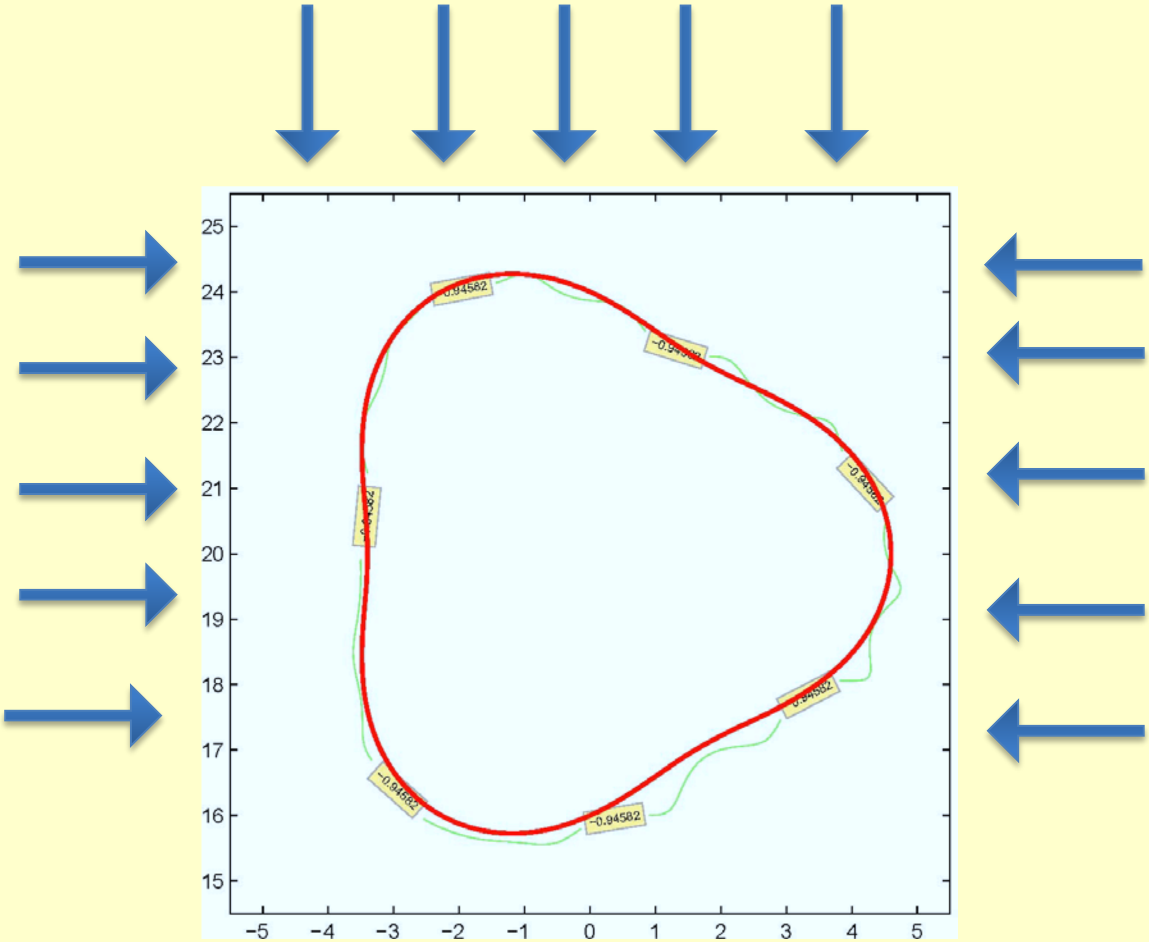
6 incidents & 30 receivers

inaccurate cut-off values

(II) Large Data for LSMs

u^{inc}

u^{inc}



And LSMs

- **derived only for wave-type inverse problems**

Find methods for more realistic cases

- Apply even with data from a single incident field
  or a single set of Cauchy data

- Insensitive to data noise

- Involve no solutions of ill-posed & well-posed linear
  or nonlinear systems

- Apply to general inverse problems

  ⇒ Clearly, hard to have efficient methods for all these

  ⇒ Let us try what we can do

DSMs for General Inverse Problems

- Inverse acoustic medium scattering, Ito-Jin-Zou 12;

  Inverse EM medium scattering, Ito-Jin-Zou 13;

- Non-wave type IPs:

  Electric impedance tomography, Chow-Ito-Zou 14;

  Diffusive optical tomography, Chow-Ito-Liu-Zou 14 ;

  Moving objects, Chow-Ito-Zou 16;

  Several other important applications, Chow-Han-Zou 20

General Framework of DSM

(Chow-Ito-Zou 2019)

- Define a Sobolev dual product on Γ with index γ :

$$\langle \chi, \phi \rangle_{\gamma,\Gamma} \quad \forall\, \chi \in Y \,, \phi \in Z$$

- Select probing & testing funcs $\{\eta_x\}$, $\{\mu_x\}$ based on PDEs

(1) nearly orthogonal wrt $\langle \cdot \,, \cdot \rangle_\gamma$, i.e., $\forall\, x \in \Omega, y \in D$,

kernel $$K(x,y) = \frac{\langle \eta_x \,, \mu_y \rangle_\gamma}{|\eta_x|_Y}$$ like a Gaussian

(2) family of testing funcs is fundamental over testing points:

$$u - u_0 \approx \sum_k a_k\, \mu_{x_k} \quad \text{on} \quad \Gamma$$

General Index functions for DSMs

- **We define the index function**

$$I(x) := \frac{\langle \eta_x , u - u_0 \rangle_{\gamma,\Gamma}}{|\eta_x|_Y} \qquad \forall\, x \in \Omega$$

- Then the index provides a probability :

$$I(x) \approx \sum_k a_k \frac{\langle \eta_x , \mu_{x_k} \rangle_{\gamma,\Gamma}}{|\eta_x|_Y}$$

DSM for Acoustic Media
(Ito-Jin-Zou 2011)

- Acoustic wave, TM or TE mode :

$$\Delta u + k^2 n^2(x) u = 0$$

- Fundamental solution *G* :

$$\Delta G + k^2 G = \delta(x - y)$$

- By Lippmann-Schwinger representation:

$$u^s(x) = \int_{\widetilde{\Omega}} G(x,y) I(y) dy \approx \sum w_j G(x, y_j)$$

- From the above:

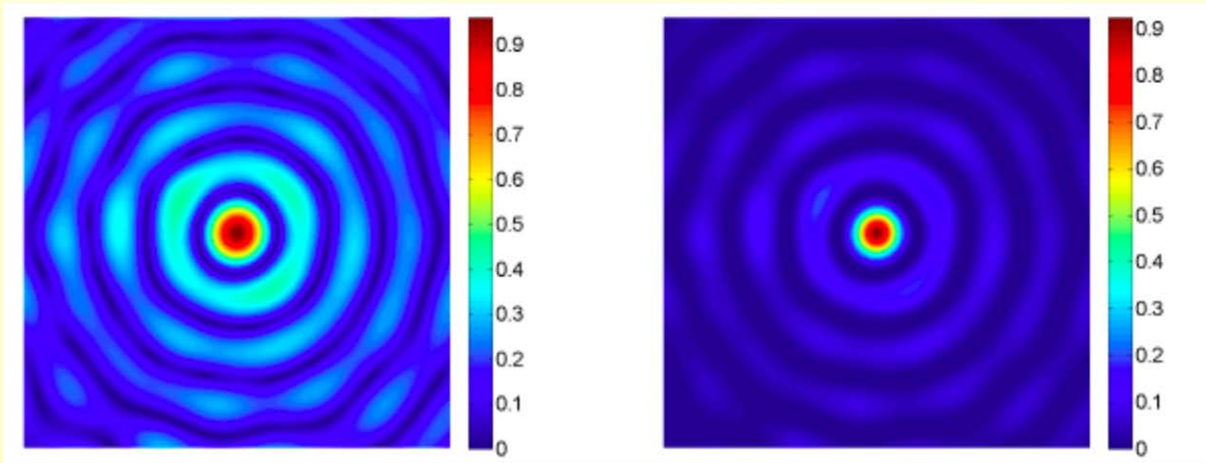
$$\int_\Gamma u^s(x)\, \overline{G}(x, x_p)\, ds \approx k^{-1} \sum w_j \,\mathrm{Im}(G(y_j, x_p))$$

Direct Sampling Algorithm
(Ito-Jin-Zou 2011)

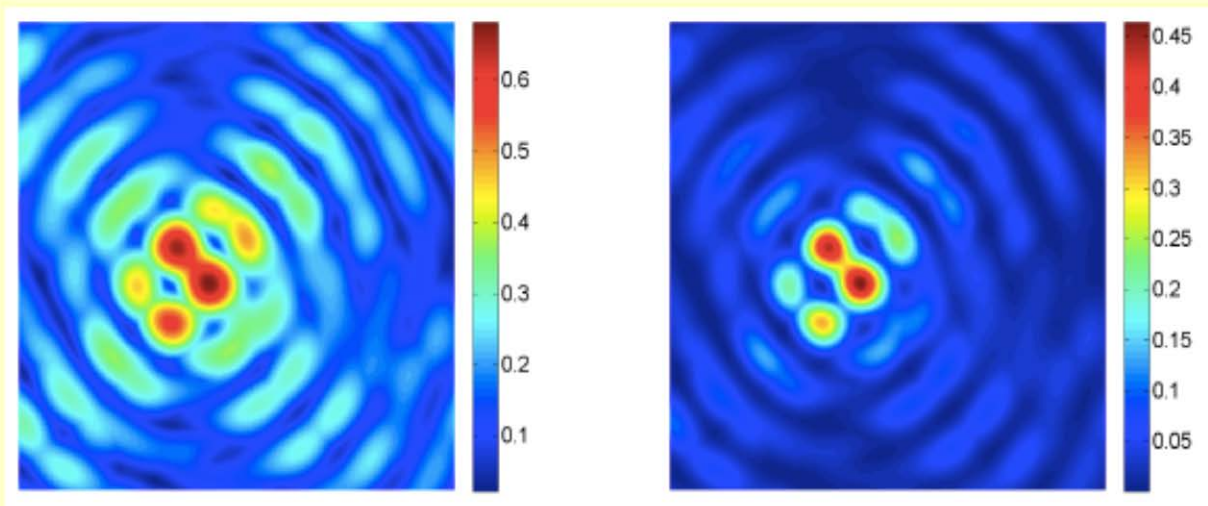
- Index func for probability of sampling point:

$$\Phi(x_p) = \frac{|\langle u^s, G(\cdot, x_p)\rangle_\Gamma|}{\|u^s\| \, \|G(\cdot, x_p)\|}$$

Numerical Examples I



Φ Φ^2



Two incidents: 20% noise

DSM for Inverse EM Media Scattering
(Ito-Jin-Zou 2013)

- Time harmonic EM system :

$$
\begin{aligned}
\mathrm{i}\omega\epsilon\, E + \nabla \times H &= 0 \quad \text{in} \quad \mathbb{R}^{\mathrm{d}} \\
-\mathrm{i}\omega\mu\, H + \nabla \times E &= 0 \quad \text{in} \quad \mathbb{R}^{\mathrm{d}}
\end{aligned}
$$

$\Rightarrow$

$$\nabla \times (\nabla \times E) - k^2 n^2(x)\, E = 0$$

- Fundamental solution *G* : $(-\Delta - k^2)G(x,y) = \delta(x-y)$

$\Rightarrow$ Maxwell fundamental soln:

$$\Phi(x,y) = k^2 G(x,y)\, I + D^2 G(x,y)$$

Direct Sampling Algorithm
(Ito-Jin-Zou 2013)

- Nearly orthogonality:

$$\int_\Gamma (\Phi(x,x_p)p, \overline{\Phi}(x,x_q)q)ds \approx k^{-1}(p, \Im(\Phi(x_p,x_q))q) \quad \forall\, p \in \mathbb{C}^{\mathrm{d}},\ q \in \mathbb{R}^{\mathrm{d}}$$

- By Lippmann-Schwinger representation:

$$E^s(x) = \int_\Omega \Phi(x,y)J(y)dy \approx \sum_j \Phi(x,y_j)J(y_j)\,|\tau_j|,$$

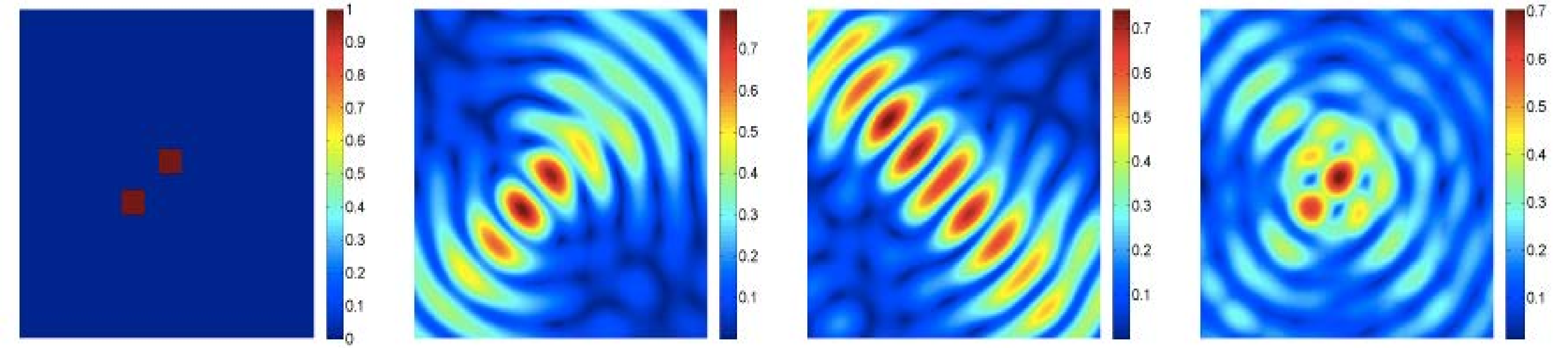
$$\langle E^s, \Phi(\cdot,x_p)q\rangle_{L^2(\Gamma)} \approx k^{-1}\textstyle\sum_j |\tau_j|(J(y_j), \Im(\Phi(x_p,y_j))q)$$

- Index func for probability of sampling point:

$$\Psi(x_p;q) = \frac{|\langle E^s, \Phi(\cdot,x_p)q\rangle_\Gamma|}{\|E^s\|_\Gamma \|\Phi(\cdot,x_p)q\|_\Gamma}$$

polarization q: basically quite arbitrary; incident polarization works well

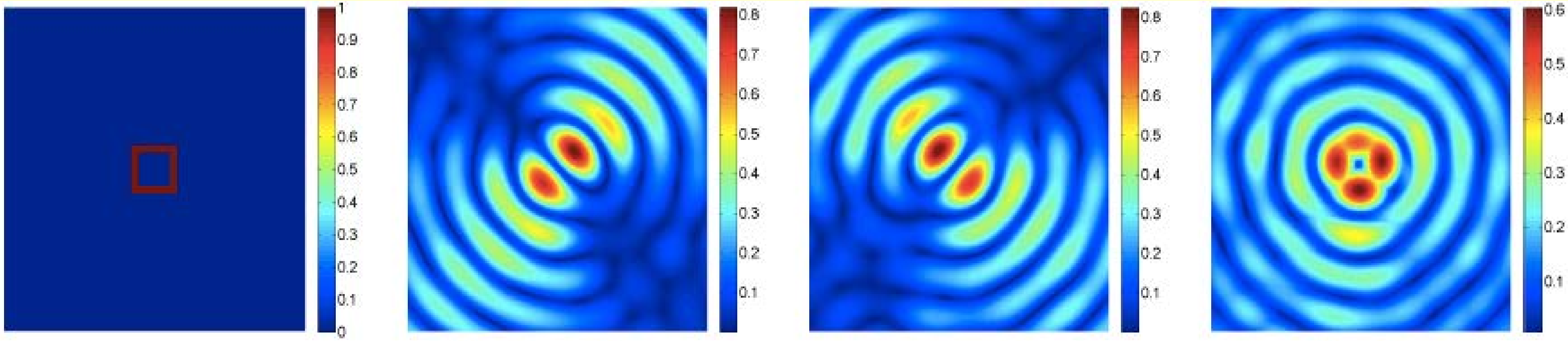
Numerical Examples I



(a) true scatterers (b) index $\Psi(x_p; p_1)$ (c) index $\Psi(x_p; p_2)$ (d) index Ψ_c

Two incidents, same polarizations p & q: 20% noise

Numerical Examples II



(a) true scatterers (b) index $\Psi(x_p; p_1)$ (c) index $\Psi(x_p; p_2)$ (d) index Ψ_c

Two incidents, same polarizations p & q: 20% noise

DSM for EIT

- Electrical Impedance Tomography:

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{in} \quad \Omega$$

Inject current on Γ: $\qquad g = \sigma \frac{\partial u}{\partial \nu}$

Measure potential on Γ: $\qquad f = u$

- EIT :
  given (f, g), recover electrical conductivity $\sigma(x)$

Choice of Probing & Testing Spaces/Funcs

- **Define on the measurement surface Γ:**

$$\langle \chi, \phi \rangle_{\gamma,\Gamma} := \langle (-\Delta_\Gamma)^\gamma \chi, \phi \rangle \quad \forall\, \chi \in H^{2\gamma}(\Gamma)\,, \phi \in L^2(\Gamma)$$

- Select probing & testing funcs $\{\eta_x\}$, $\{\mu_y\}$ s.t.

(1) Nearly orthogonal wrt $\langle \cdot\,,\,\cdot \rangle_\gamma$, i.e., $\forall\, x \in \Omega, y \in D$,

$$K(x,y) = \frac{\langle \eta_x\,, \mu_y \rangle_{\gamma\,,\Gamma}}{|\eta_x|_Y}$$ like a Gaussian

(2) The testing family is fundamental:

$$u - u_0 \approx \sum_k a_k\, \mu_{y_k} \quad \text{on} \quad \Gamma$$

Choice of Probing Functions

- Define

$$-\Delta w_{x,d} = -d \cdot \nabla \delta_x \quad \text{in} \quad \Omega\,; \quad \frac{\partial w_{x,d}}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega$$

- Dipole potential :

$$D_{x,d}(\xi) := c_n \frac{(x-\xi)\cdot d}{|x-\xi|^n}\,, \quad \xi \in \mathbb{R}^n$$

$$\text{Set } \varphi_{x,d} = D_{x,d} - w_{x,d} \; :$$

$$-\Delta \varphi_{x,d} = 0 \;\; \text{in } \Omega\,; \quad \frac{\partial \varphi_{x,d}}{\partial \nu} = \frac{\partial D_{x,d}}{\partial \nu} \;\; \text{on } \partial\Omega$$

- Probing functions as dir. derivative of Green funcs :

$$\eta_{x,d}(\xi) := w_{x,d}(\xi) = -d \cdot \nabla G_x(\xi) \;\; \forall \xi \in \Gamma$$

Probing functions for special geometries

- For 3D spheric measurement surface :

$$\eta_{x,d}(\xi) = \frac{d\cdot\xi - \frac{(x-\xi)\cdot d}{|x-\xi|}}{\sqrt{4\pi}(|x-\xi| - x\cdot\xi + 1)}$$

- For 2D circular measurement curve :

$$\eta_{x,d}(\xi) = \frac{1}{\pi}\frac{(\xi - x)\cdot d}{|x-\xi|^2}$$

Verification of Fundamental Properties

- For p.w. constant inclusions $\Omega_1, \Omega_2, \cdots, \Omega_k$:

$$(u - u_0)(\xi) = -\sum_i \int_{\partial\Omega_i} [\eta] \frac{\partial G_\xi}{\partial \nu} \, ds \approx \sum_k a_k \eta_{x_k, d_k}(\xi)$$

so testing funcs take the same as probing, with $d_k = \nu(x_k)$

- Similarly for p.w. smooth inclusions

Verification of Othogonality

- For circular measurement curve :

$$\langle \eta_{x,d_x}, \mu_{y,d_y} \rangle_{\gamma,\mathbb{S}^1} = 2\mathrm{Re}\left(\frac{e^{i(\theta_{d_x}-\theta_{d_y}-\theta_x+\theta_y)}}{r_x r_y} G^{2\gamma}(r_x r_y e^{i(\theta_x-\theta_y)}) \right)$$

with a complex polynomial

$$G^\beta(z) := \left(z\frac{\partial}{\partial z} \right)^\beta \left(\frac{1}{1-z} \right) = \sum_{m=0}^{\infty} m^\beta z^m$$

so when $y \approx x$ and $d_y \approx d_x$, like a Gaussian

- Similarly for spherical measurement surface, but much more technical

Index Function

- Index function for EIT :

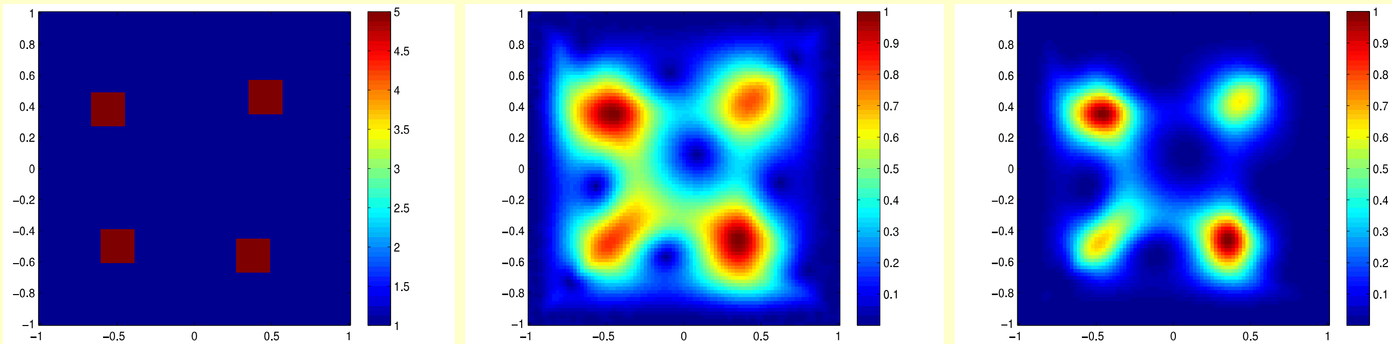
$$K(x,y) = \frac{\langle \eta_x, G_y \rangle_\gamma}{|\eta_x|_Y} = \frac{\langle (-\Delta_\Gamma)^\gamma \chi, \phi \rangle}{|\eta_x|_Y}$$

With the Sobolev index

$$\gamma = 2$$

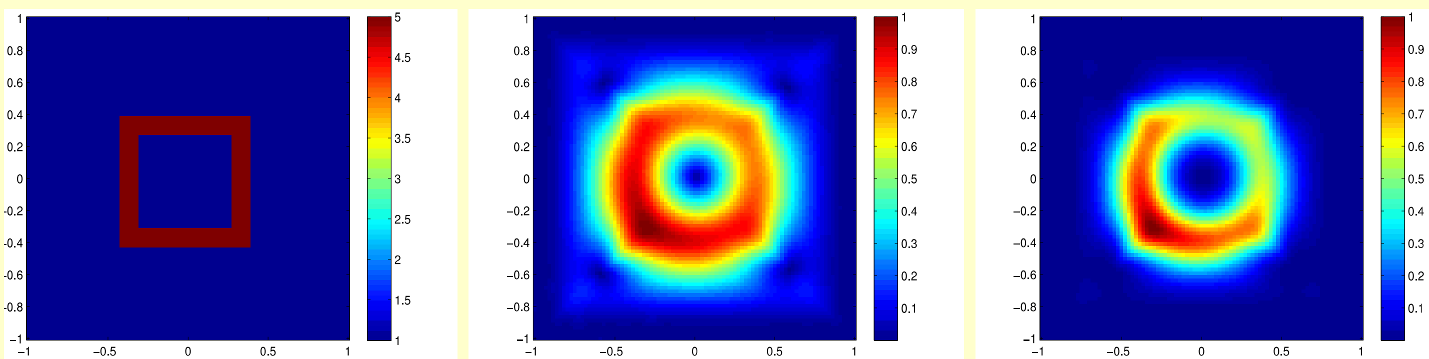
Numerical Experiments

- Four separated square objects



5% noise

- Thin square ring object:



DSM for DOT

- Diffusive optical tomography in absorption medium Ω with absorption coeff μ & photon density u :

$$-\Delta u + \mu u = 0 \quad \text{in} \;\; \Omega$$

Inject current on Γ: $\qquad g = \dfrac{\partial u}{\partial \nu}$

Measure density on Γ: $\qquad f = u$

- DOT :
given *(f, g),* *recover* the absorption coeff μ

General Principle of DSM

- **Define on the measurement surface Γ :**

$$\langle \chi, \phi \rangle_{\gamma,\Gamma} := \langle (-\Delta_\Gamma)^\gamma \chi, \phi \rangle \quad \forall\, \chi \in H^{2\gamma}(\Gamma)\,, \phi \in L^2(\Gamma)$$

- Select a set of probing & testing funcs $\{\eta_x\}$ & $\{\mu_x\}$:

(1) nearly orthogonal wrt $\langle \cdot\, , \, \cdot \rangle_\gamma$, i.e., $\forall\, x \in \Omega, y \in D$,

$$K(x,y) = \frac{\langle \eta_x \,, \mu_y \rangle_{\gamma,\Gamma}}{|\eta_x|_Y}$$ like a Gaussian

(2) family probing funcs is fundamental:

$$(u - u_0)(\xi) \approx \sum_k a_k \mu_{y_k}(\xi) \quad \forall\, \xi \in \Gamma$$

Choice of Testing Functions

- Green function :

$$-\Delta G_x + \mu_0 G_x = \delta_x \text{ in } \Omega ; \quad \frac{\partial G_x}{\partial \nu} = 0 \text{ on } \partial\Omega$$

- Scattered potential $u - u_0 \text{ on } \Gamma :$

$$(u - u_0)(\xi) = \int_D G_y(\xi)(\mu_0 - \mu(y))u(y)\,dy \quad \forall\, \xi \in \Gamma$$

- Fundamental representation :

$$(u - u_0)(\xi) \approx \sum_k a_k G_{y_k}(\xi) \quad \forall\, \xi \in \Gamma$$

- Green functions: good candidates for testing funcs

Choice of Probing Functions

- Green function :

$$-\Delta w_x + \mu_0 w_x = \delta_x \ \text{ in } \ \Omega; \quad w_x = 0 \ \text{ on } \ \Gamma; \quad \frac{\partial w_x}{\partial \nu} = 0 \ \text{ on } \ \partial\Omega \setminus \Gamma$$

- Fundamental solution in the whole space Φ_x

- Define ψ_x :

$$-\Delta \psi_x + \mu_0 \psi_x = 0 \ \text{ in } \ \Omega; \quad \psi_x = \Phi_x \ \text{ on } \ \Gamma; \quad \frac{\partial \psi_x}{\partial \nu} = \frac{\partial \Phi_x}{\partial \nu} \ \text{ on } \ \partial\Omega \setminus \Gamma$$

- Probing functions, $w_x = \Phi_x - \psi_x$:

$$\eta_x(\xi) := \frac{\partial w_x}{\partial \nu}(\xi) \quad \forall\, \xi \in \Gamma$$

Probing functions for special geometries

- For 2D circular measurement curve :

$$\eta_x(y) = \frac{1-|x|^2}{2\pi|x-y|^2} \quad \forall\, y \in \mathbb{S}^1$$

- Orthogonality or Gaussian like behaviour :

$$K(x,z) = \frac{\langle \eta_x, G_z \rangle_1}{|\eta_x|_1^{\frac{1}{2}}\,|\eta_x|_0^{\frac{3}{4}}} = C(x)\left\{\frac{r_z r_x \cos(\theta_x-\theta_z)(1+r_z^2 r_x^2)-2r_z^2 r_x^2}{(1-2r_z r_x \cos(\theta_x-\theta_z)+r_z^2 r_x^2)^2}\right\}$$

Index Function for DSM

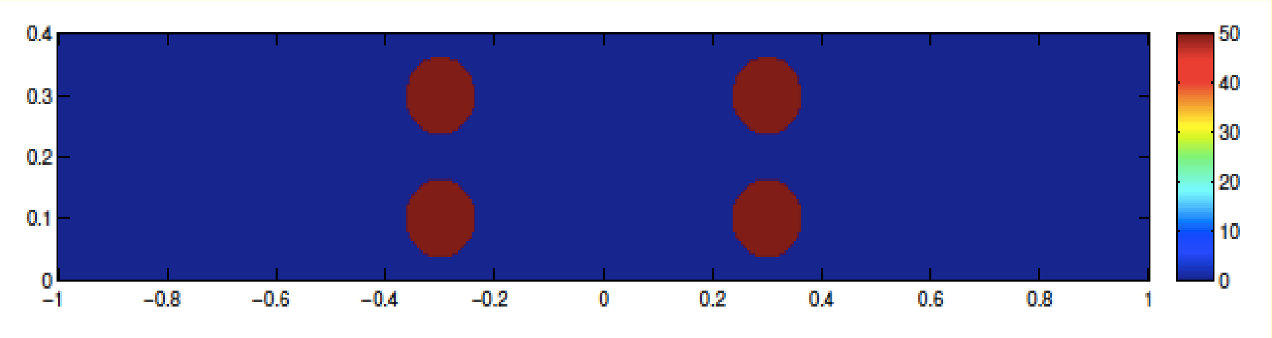
- Recall the kernel functions for DOT:

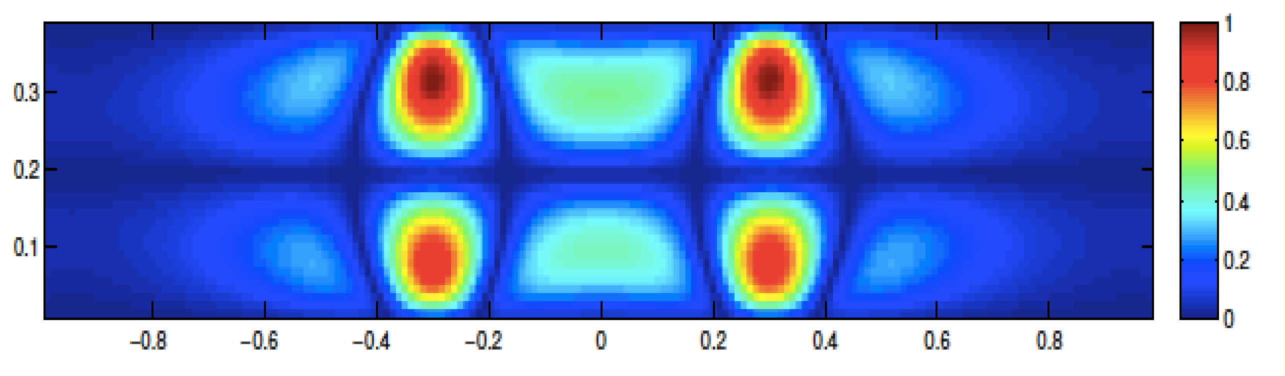
$$K(x,y) = \frac{\langle \eta_x, G_y \rangle_\gamma}{|\eta_x|_Y} = \frac{\langle (-\Delta_\Gamma)^\gamma \chi, \phi \rangle}{|\eta_x|_Y}$$

with the Sobolev index

$$\gamma = 1$$

Example I (5% noise)





- Severely ill-posed, 4 inclusions close to each other & to the boundary, but reconstructions quite satisfactory:
  5% noise, only one Cauchy data, data far away from inclusions

General Principle of time-dependent DSM

- **Define a Sobolev dual product on** $\Gamma \times (\tau_0, T)$ **:**

$$\langle \chi, \phi \rangle_{\gamma, \Gamma\times(\tau_0,T)} \quad \forall \chi \in Y \,, \phi \in Z$$

- Select probing & testing funcs $\{\eta_{x,t}\}$, $\{\mu_{y,s}\}$ based on PDEs

(1) nearly orthogonal wrt $\langle \cdot \,, \, \cdot \rangle_\gamma$, i.e., $\forall y \in D, \, s \in (\tau_0, T),$

kernel func: $$K(x,t;y,s) = \frac{\langle \eta_{x,t}, \mu_{y,s} \rangle_{\gamma, \Gamma\times(\tau_0,T)}}{|\eta_{x,t}|_Y}$$ Gaussian

(2) testing funcs are fundamental over set of testing points:

$$u - u_0 \approx \sum_{k,j} a_{k,j} \, \mu_{y_k, s_j} \quad \text{on} \quad \Gamma \times (\tau_0, T)$$

General Index functions for DSMs

- **We define the index function**

$$I(x,t) := \frac{\langle \eta_{x,t}, u - u_0 \rangle_{\gamma,\Gamma\times(\tau_0,T)}}{|\eta_{x,t}|_Y} \quad \forall\, x \in \Omega\,, t \in (0,T)$$

- Then the index provides a probability :

$$I(x,t) \approx \sum_k a_{k,j} \frac{\langle \eta_{x,t}, \mu_{y_k,s_j} \rangle_{\gamma,\Gamma\times(\tau_0,T)}}{|\eta_{x,t}|_Y}$$

DSM for Moving Potential

- Heat conduction/moving DOT:

$$\frac{\partial u}{\partial t} = a\Delta u - q(x,t)u$$

Measure heat intensity on Γ : u

corresponding to one single u_0

- **Inverse Problem:**

given u , recover $q(x,t)$

Testing Functions

- u_0 : heat intensity with background potential q_0

$$\frac{\partial(u-u_0)}{\partial t} - a\Delta(u-u_0) = -q_0(u-u_0) - (q-q_0)u$$

- we have

$$(u-u_0)(x,t) = -\int_0^T \int_{D(t)} \Phi(x-y,t-s)\, c(y,s)\, dyds$$

with fundamental solution

$$\Phi(x,t) = \frac{1}{4\pi at} \exp\left(-\frac{|x|^2}{4at}\right)$$

- Therefore

$$(u-u_0)(x,t) \approx \sum_{k,j} c_{kj} \Phi(x-y_k, t-s_j) \ \ \forall (x,t) \in \Gamma \times (0,T)$$

- Probing functions :

$$\eta_{x,t} := \Phi_{x,t}(y,s) \equiv \Phi(x-y,t-s)\,\chi_+(t-s-\delta)$$

Index Function

- **Define on the measurement surface** :

$$\langle \chi, \phi \rangle_{\alpha, \Gamma\times(0,t)} := \langle \Delta_x^\alpha \chi, \phi \rangle \quad \forall \chi \in H^{2\alpha}(\Gamma), \phi \in L^2(\Gamma)$$

not surface Laplacian this time

- DSM index functions :

$$I_0^\alpha(x,t) = \frac{\langle \eta_{x,t}, u - u_0 \rangle_{\alpha, \Gamma\times(0,t)}}{|w_{x,t}^\alpha|_Y}$$

- Normalization :

$$\hat{I}_0^\alpha(x,t) = \frac{|I_0^\alpha(x,t)|}{\max |I_0^\alpha(x,t)|}$$

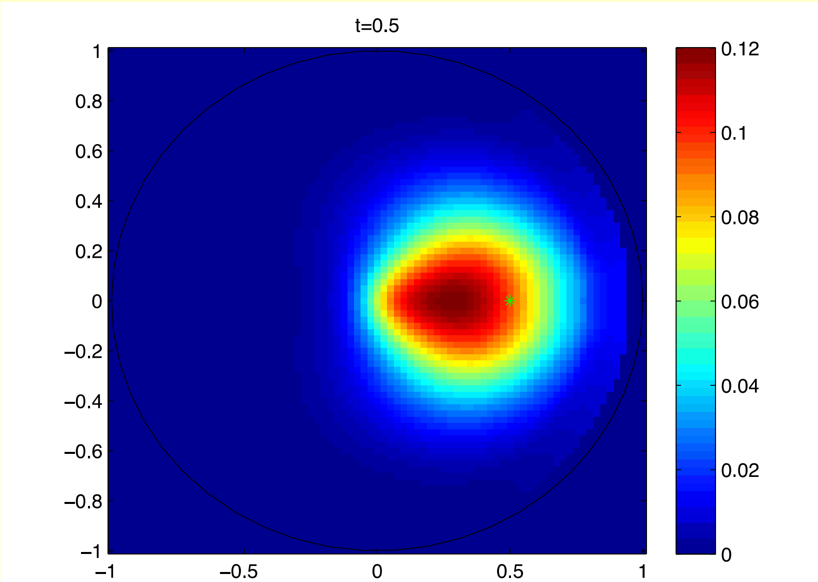
Index Function: real-time

- DSM index functions : real time reconstruction

$$I_0^{\alpha}(x,t) = \frac{\langle \eta_{x,t}, u-u_0 \rangle_{\alpha,\Gamma\times(0,t)}}{|w_{x,t}^{\alpha}|_Y}$$

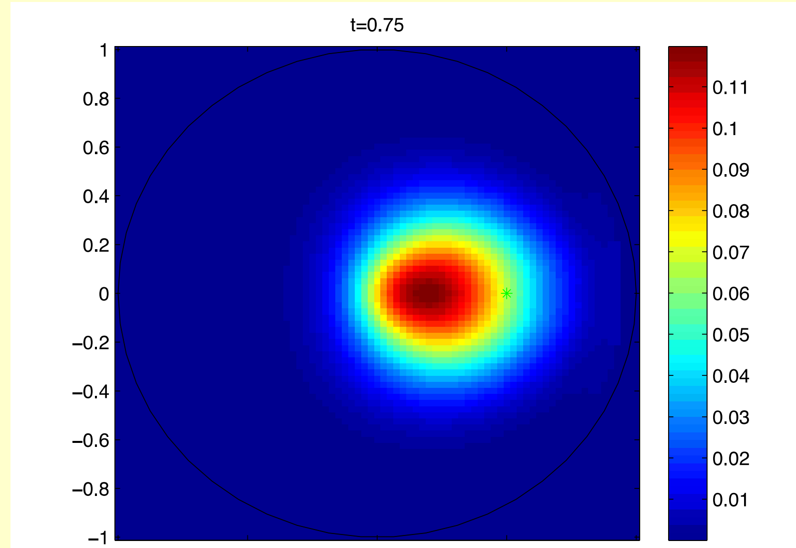
→ no any data after time t needed

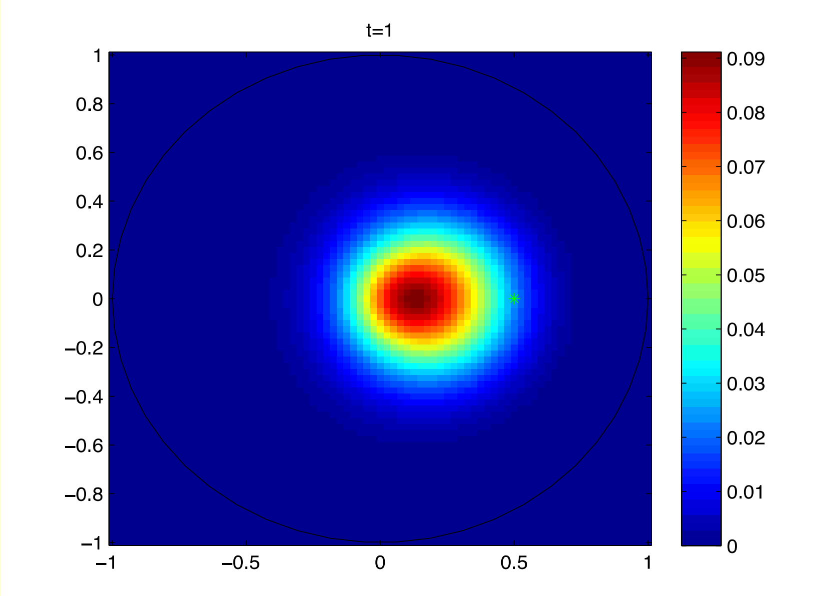
Behavior of index for point source $q = \delta_{(0.5,0)}(x)\delta_1(t)$



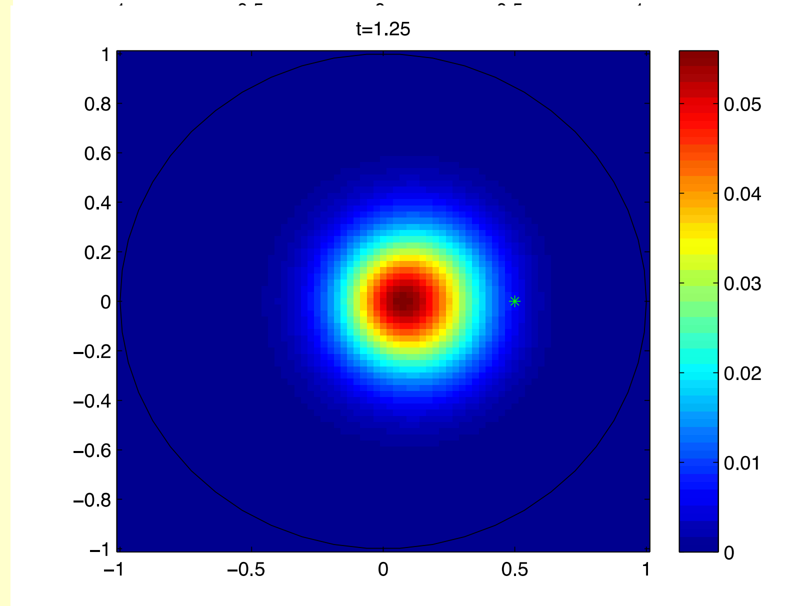
zeroth order index I_0^α

$\alpha = 2$





max at t=0.5, far from t=1, further away

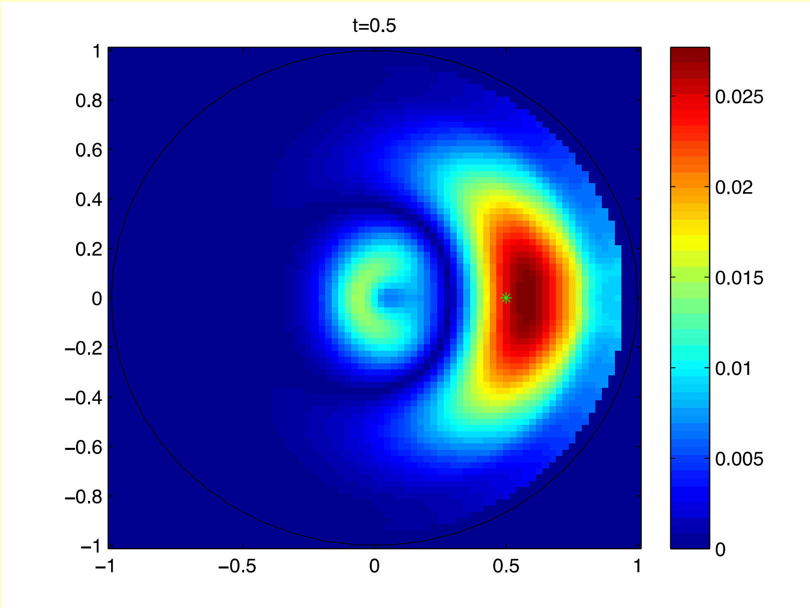


Temporal derivatives of zeroth order index I_0^α

- From the behaviour of I_0^α, we see big drop in spatial maximum with time as time goes on from t=1, so a rate of change of I_0^α may capture the inclusion more effectively:

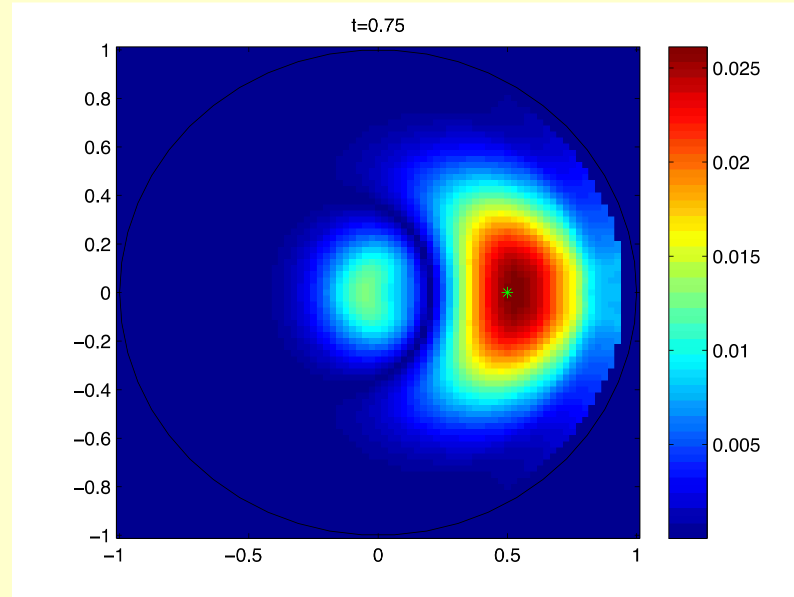
$$I_\gamma^\alpha = \frac{\partial^\gamma}{\partial t^\gamma} I_0^\alpha$$

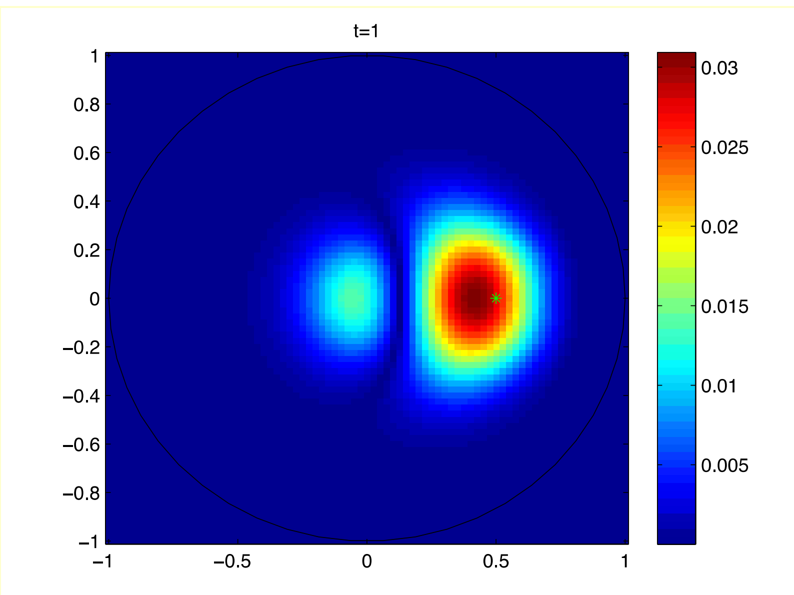
Behavior of index for point source $q = \delta_{(0.5,0)}(x)\delta_1(t)$



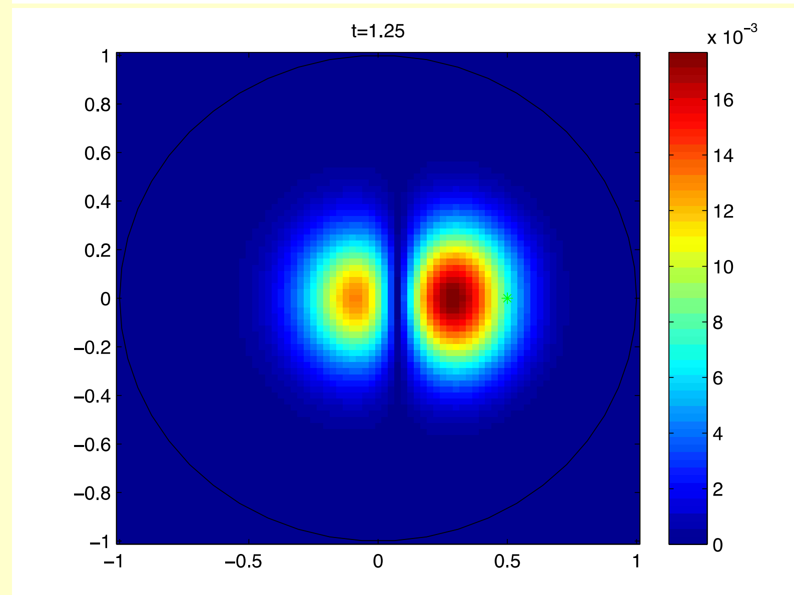
1st order index I_1^{α}

$\alpha = 2$





max well reached at t=1



Verification of Index Function

- Consider the kernel

$$K_0^\alpha(x,t;y,s) = \frac{\int_0^{t-\delta}\int_\Gamma \Phi(y-z,s-k)\Delta_z^\alpha\Phi(x-z,t-k)d\sigma_z dk}{\sqrt{\int_0^{t-\delta}\int_\Gamma(\Delta_z^\alpha\Phi(x-z,t-k))^2 d\sigma_z dk}}$$

and its derivatives

$$K_\gamma^\alpha(x,y,t,s) := \partial_t^\gamma K_0^\alpha(x,y,t,s)$$

For some $t > t_0$,

$$\partial_t\left(\sqrt{\int_0^{t-\delta}\int_\Gamma\left(\Delta_z^\alpha\Phi(x-z,t-k)\right)^2 d\sigma_z dk}\right) = O(t_0^{-2\alpha-2})$$

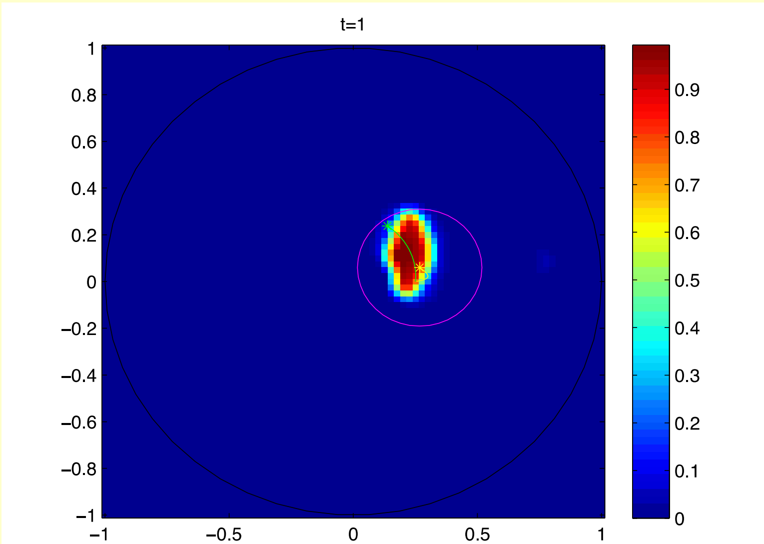
Explicit relation for $\alpha,\gamma\in\mathbb{N}$, $t > t_0$:

$$K_\gamma^\alpha(x,t;y,s) = \frac{\partial_t^\gamma\left(\int_0^{t-\delta}\int_\Gamma \Phi(y-z,s-k)\Delta_z^\alpha\Phi(x-z,t-k)d\sigma_z dk\right)}{\sqrt{\int_0^{t-\delta}\int_\Gamma(\Delta_z^\alpha\Phi(x-z,t-k))^2 d\sigma_z dk}} + O(t_0^{-2\alpha-2})$$

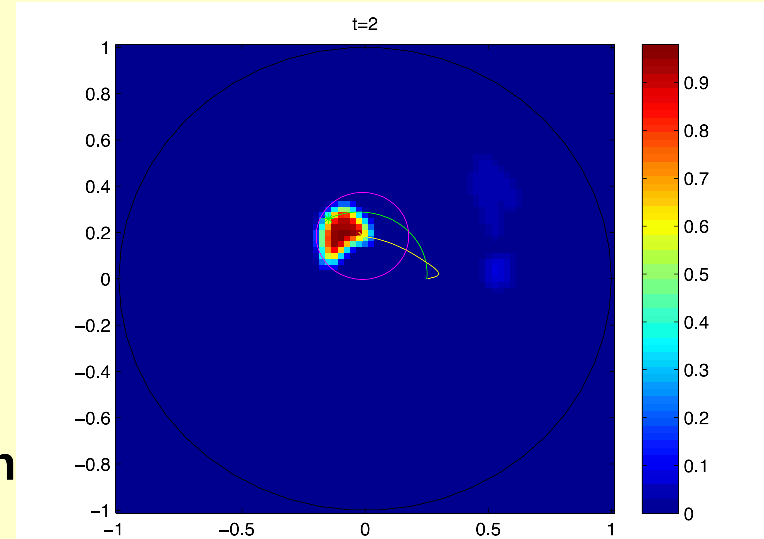
Max at $y = x$, $s = t$, but shift away more if $\alpha-\gamma$ bigger

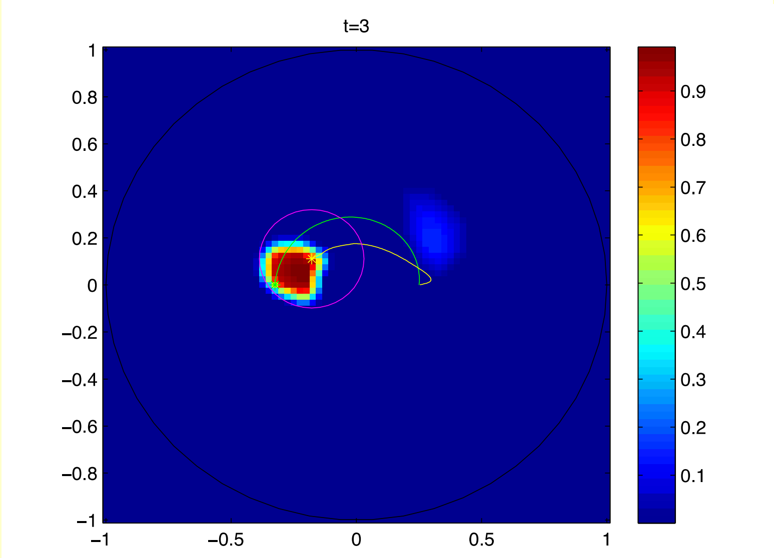
Numerical Experiments I

$$\Gamma(t) = \left(\frac{t}{8T} + 0.25\right)\left(\cos\left(\frac{t\pi}{3}\right),\ \sin\left(\frac{t\pi}{3}\right)\right), \quad t \in (0,5)$$



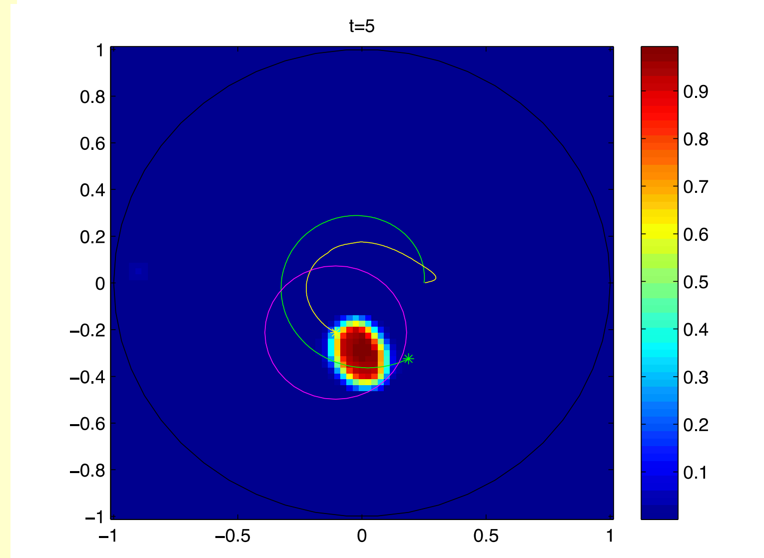
there is a time lag, reconstructed inclusion follows closely exact inclusion





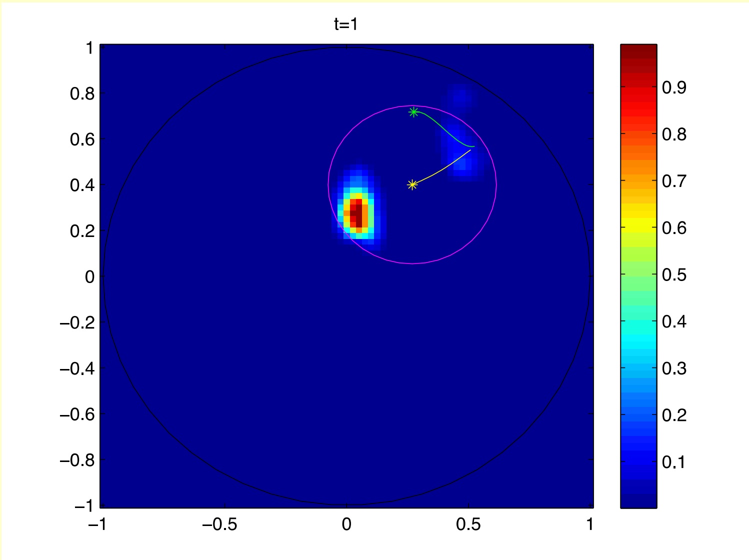
5% noise

green * yellow : centre of inclusion mass

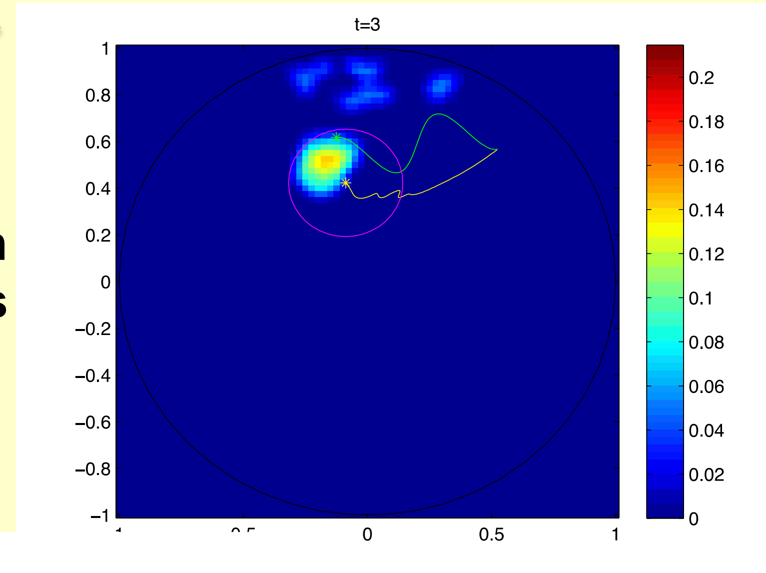


Numerical Experiments II

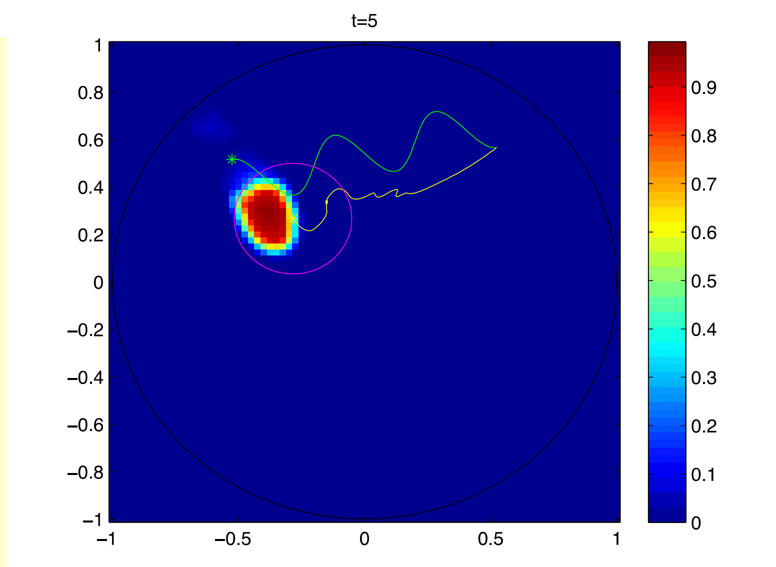
$$\Gamma(t) = \left(-\frac{t}{5} + \frac{1}{2} + \frac{1}{40}\cos(t\pi), -\frac{t}{20} + \frac{2}{3} - \frac{1}{0}\cos(t\pi) \right), \quad t \in (0,5)$$



once it succeeds to approach exact inclusion for t>2, it starts to follow exact path



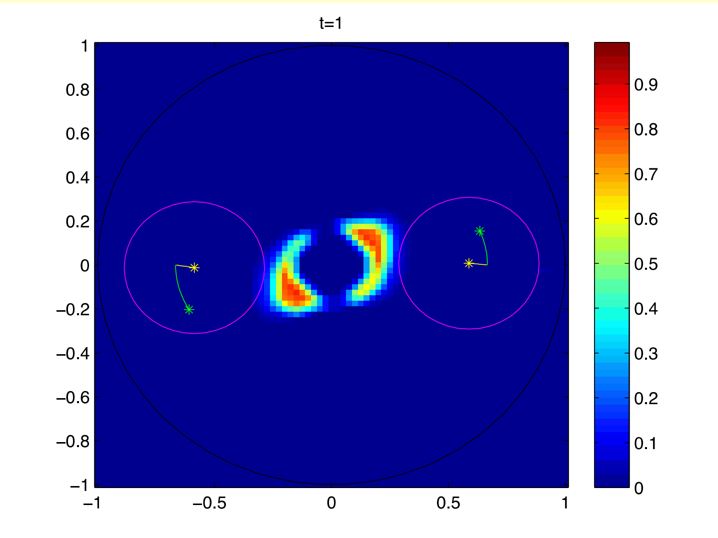
Initially, for t<2, the reconstructed inclusion tries to find the exact inclusion



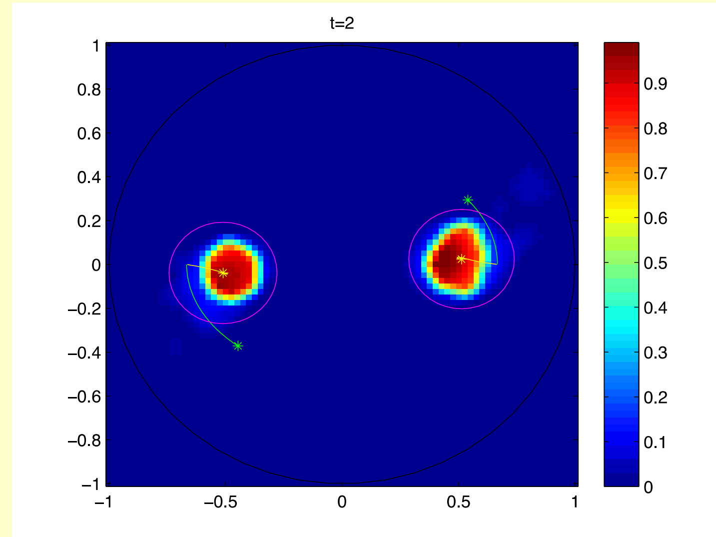
recovered trajectory can even follow very fine turnings as exact one from t >4 onwards

Numerical Experiments III

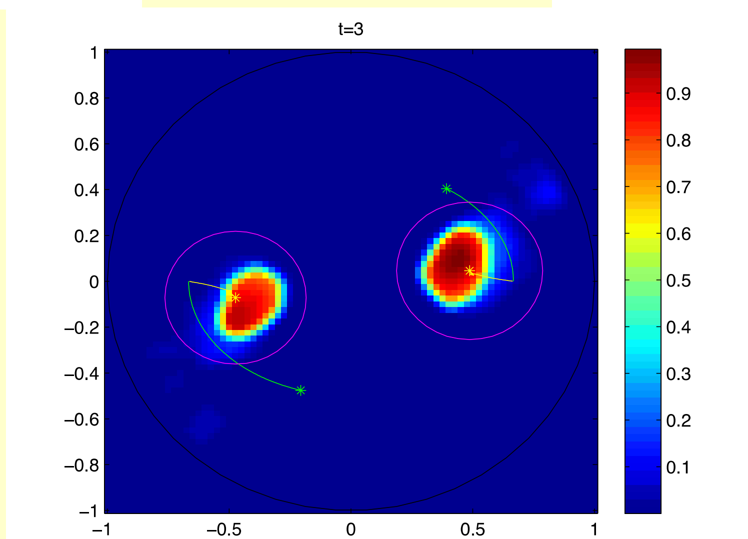
$$\Gamma_1(t) = \left(\tfrac{2}{3}\cos\left(\tfrac{t\pi}{10}\right),\ \tfrac{1}{2}\sin\left(\tfrac{t\pi}{10}\right)\right), \qquad \Gamma_2(t) = \left(-\tfrac{2}{3}\cos\left(\tfrac{2t\pi}{15}\right),\ -\tfrac{1}{2}\sin\left(\tfrac{2t\pi}{15}\right)\right)$$



2 objects, with different speeds, highly ill-posed



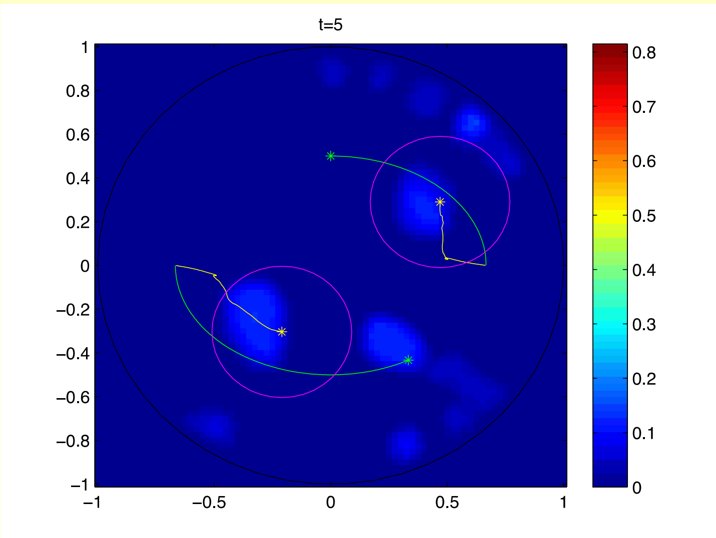
initially, strongly coupling,



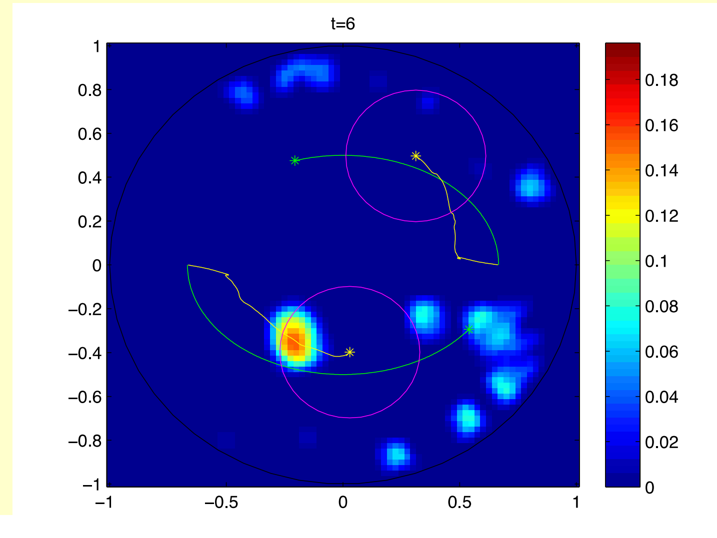
gradually, clearly seen 2 objects,

Numerical Experiments III

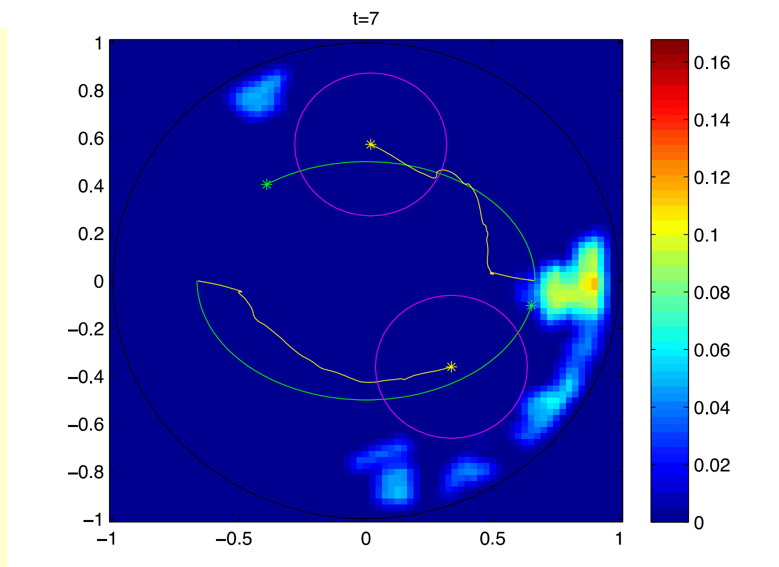
$$\Gamma_1(t) = \left(\tfrac{2}{3}\cos\left(\tfrac{t\pi}{10}\right),\ \tfrac{1}{2}\sin\left(\tfrac{t\pi}{10}\right)\right), \quad \Gamma_2(t) = \left(-\tfrac{2}{3}\cos\left(\tfrac{2t\pi}{15}\right),\ -\tfrac{1}{2}\sin\left(\tfrac{2t\pi}{15}\right)\right)$$



after long time, signal to noise weak, less stable, more oscillatory



2 objects, with one set of data, very challenging task



still tracing 2 objects reasonably well

An Optimal Control framework

- Forward equation:

$$L_q(u) = 0\,, \quad Bu = f$$

- Background equation:

$$L_{q_0}(u_0) = 0\,, \quad Bu_0 = f$$

- Parameter-to-solution:

$$L_{q_0}(u - u_0) = -(L_q - L_{q_0})(u) := J(q - q_0)$$

$$\Longrightarrow \quad u - u_0 = G\,[J(q - q_0)]$$

An Optimal Control Framework

- Parameter-to-solution:

$$u - u_0 = G\left[J(q-q_0)\right]$$

$$\Rightarrow \quad E : u - u_0 \to (u-u_0)|_{\Gamma}$$

- Index function :

$$I = \left(\Phi \circ W_{X,\gamma}(\eta) \circ E \circ G\right)\left[J(q-q_0)\right]$$

- Hope I provides an estimate of support of $J(c-c_0)$

$$\Phi \circ W_{X,\gamma}(\eta) \circ E \circ G \approx id : \ X \to X^*$$

$$\Rightarrow \quad \min_{\Phi,\gamma} \|\Phi \circ W_{X,\gamma}(\eta) \circ E \circ G - id\|^2_{X \to X^*}$$

Index Function for DSM

- Recall the kernel function :

$$K(x,y) = \frac{\langle \eta_x , G_y \rangle_\gamma}{|\eta_x|_Y} = \frac{\langle (-\Delta_\Gamma)^\gamma \chi, \phi \rangle}{|\eta_x|_Y}$$

- Sobolev index :

Wave-type: $\gamma = 0$

EIT: $\gamma = 2$

DOT: $\gamma = 1$

Features of DSMs

- Computationally very cheap, completely parallel
- Stability: straightforward
- Works for a single measurement data
- Robust against noise in data, due to orthogonality:

  high frequency components in data orthogonal to fundamental solutions on measurement surface

Other related sampling methods

- **R Potthast 2010, inverse obstacle scattering**

  ZM Chen et al. (since 2013):
  reverse time migration, inverse obstacle acoustic & EM scattering

  H Ammari, et al.

  WK Park, et al.

  HY Liu, XD Liu, JZ Li, YK Guo, … …

  … …

➡ **Mostly for inverse wave scattering**

THANK YOU!