On the free boundary hard phase fluid in Minkowski spacetime

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Outline

Part 1 **Introducing the model**

- Ideal fluids in relativistic framework
- The hard phase model with free boundary

Part 2 **Main result I–Well-posedness**

- Comparison with Newtonian problem
- A priori estimates

Part 3 **Main result II–Newtonian limit**

- Re-scaled equations
- Uniform estimates

Relativistic ideal fluids in Minkowski background

- Let $(\mathbb{R}^{1+3}, m)$ be the standard Minkowski spacetime with

$$m := \begin{pmatrix} -1 & 0 \\ \mathbf{0} & \mathbf{I}_{3\times 3} \end{pmatrix}.$$

- We denote by $m_{\alpha\beta}$ and $m^{\alpha\beta}$ the components for m and m^{-1} respectively.
- All the indices are raised and lowered with respect to m and m^{-1}.
- The Greek letters are all from 0 to 3.

Relativistic fluids in Minkowski background

- The motion of the fluid is described by the *fluid velocity* and several *thermodynamical quantities*:
- The fluid velocity is denoted by

$$u = u^\mu \frac{\partial}{\partial x^\mu},$$

and satisfies

$$u^0 > 0, \quad u^\mu u_\mu = -1.$$

Relativistic fluids in Minkowski background

- There are five thermodynamic quantities:

$$\begin{aligned} n &: \quad \text{number density of particles} \\ p &: \quad \text{pressure} \\ \rho &: \quad \text{energy density} \\ s &: \quad \text{entropy per particle} \\ \theta &: \quad \text{temperature} \end{aligned}$$

- They satisfy the following relation

$$p = n\frac{\partial \rho}{\partial n} - \rho, \quad \theta = \frac{1}{n}\frac{\partial \rho}{\partial s}.$$

- The *sound speed* η is given by

$$\eta := \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}, \quad 0 \leq \eta \leq 1.$$

- Here by choosing appropriate units, we assume the speed of light is 1.

Relativistic fluids in Minkowski background

- We also need the *energy-momentum tensor* $T^{\mu\nu}$ and the *particle current* I^{μ} which are given by

$$T^{\mu\nu} := (p+\rho)u^{\mu}u^{\nu} + pm^{\mu\nu}, \quad I^{\mu} = nu^{\mu}.$$

- The equation of motion is given by

$$\nabla_{\mu}T^{\mu\nu} = 0, \quad \nabla_{\mu}I^{\mu} = 0. \tag{1}$$

- Here ∇ is the canonical Levi-Civita connection of the Minkowski metric m.

Barotropic fluids

In this work we consider *barotropic fluids*, namely, the pressure p is a function of the energy density ρ *only*:

$$p = f(\rho), \quad f' > 0.$$

Define

$$F(p) := \int_0^p \frac{dp'}{\rho(p') + p'}, \quad V := e^F u,$$

and

$$\|V\| := e^F, \quad \|V\|^2 := -V^\mu V_\mu.$$

Equation of motion-Alternative

The equation of motion (1) becomes

$$V^\nu \nabla_\nu V^\mu + \frac{1}{2}\nabla^\mu \left(\|V\|^2\right) = 0, \quad \nabla_\mu \left(G(\|V\|)V^\mu\right) = 0, \tag{2}$$

where the function G is defined by

$$G(\|V\|) := \frac{\rho + p}{\|V\|^2}.$$

Note that p and ρ are functions of $\|V\|$.

The hard phase model-Assumptions

- We assume the fluid is *irrotational*:

$$\nabla_\mu V_\nu - \nabla_\nu V_\mu = 0, \quad \Rightarrow \quad V^\mu = \nabla^\mu \phi$$

  for a scalar function ϕ.

- p and ρ are given by

$$p = \frac{1}{2}\left(\|V\|^2 - 1\right), \quad \rho = \frac{1}{2}\left(\|V\|^2 + 1\right),$$
$$\Rightarrow \quad \eta \equiv 1, \quad G \equiv 1.$$

- We denote $\sigma^2 := \|V\|^2$. σ^2 is the *enthalpy*.

The hard phase model with free boundary

We are interested in the following free boundary problem for hard phase model:

- Let Ω be a spacetime domain in $(\mathbb{R}^{1+3}, m)$. Ω will be part of the unknown of our problem.
- The free boundary problem is

$$\begin{aligned} &\nabla_\mu V^\mu = 0, \quad dV = 0, \quad \text{in} \quad \Omega \\ &\sigma^2 = -V^\mu V_\mu \equiv 1 \quad \text{on} \quad \partial\Omega \\ &V \quad \text{tangential to} \quad \partial\Omega. \end{aligned} \tag{3}$$

- The initial data satisfies

$$\begin{aligned} &\nabla_\mu \sigma^2 \nabla^\mu \sigma^2 > 0 \quad \text{on} \quad \partial\Omega_0 \\ &\sigma_0^2 > 1 \quad \text{in} \quad \Omega_0. \end{aligned} \tag{4}$$

Main result I: Well-posedness

Theorem

Any sufficiently regular data satisfying (4) *and certain compatibility conditions leads to a unique local-in-time solution to* (3)*.*

- The conditions (4) on initial data is the *relativistic Taylor sign condition*.
- Since we are solving an initial-boundary value problem for a hyperbolic PDE system, the initial data should satisfy certain compatibility conditions
- Seeking the optimal regularity is not our concern in this work.

Remarks on the model

- The hard phase model has independent physical interest: It is an idealized model for the physical situation when the mass-energy density exceeds the nuclear saturation density during the gravitational collapse of the degenerate core of a massive star. In this situation, the sound speed is thought to approach the speed of light (Christodoulou, Friedman-Pandharipande, Lichnerowicz, Rezzolla-Zanotti, Walecka, and Zel'dovich, etc.)
- The hard phase model captures main mathematical features of a class of free boundary problems. Our approach in this work can be applied to general barotropic fluids with non-zero vorticity.

Historical results on related models

- Gaseous models: Makino, Rendall (Existence for a class of solutions), Hadzić-Shkoller-Speck, Jang-LeFloch-Masmoudi (A priori estimates), Trakhinin (Existence using Nash-Moser)
- Liquid models: Oliynyk (Existence for a similar liquid model using different methods), Ginsberg (A priori estimates for the same model with smallness assumption on initial data).

Comparison with Newtonian problem

- The Newtonian free boundary problem for incompressible irrotational fluid is

$$\begin{aligned}
&\nabla \cdot \tilde{V} = 0, \quad \nabla \times \tilde{V} = 0 \quad \text{in} \quad \tilde{\Omega}_t \\
&\tilde{V}_t + (\tilde{V} \cdot \nabla)\tilde{V} = -\nabla \tilde{P} \quad \text{in} \quad \tilde{\Omega}_t \\
&\tilde{P} \equiv 0 \quad \text{on} \quad \partial\tilde{\Omega}_t \\
&(1, \tilde{V}) \quad \text{tangential to} \quad \cup_{t>0}(t, \partial\tilde{\Omega}_t).
\end{aligned} \tag{5}$$

- Hopf Lemma implies the *Taylor sign condition*

$$-\frac{\partial \tilde{P}}{\partial \tilde{n}} \geq c_t > 0 \quad \text{on} \quad \partial\tilde{\Omega}_t \tag{6}$$

- Here $\tilde{P}$ is the pressure. $\tilde{V}$ is the fluid velocity. $\tilde{\Omega}_t$ is the unknown domain occupied by fluid at time t. $\tilde{n}$ is the outward unit normal to $\partial\tilde{\Omega}_t$.

Ideas to solve the Newtonian problem: Wu (97,99)

- Reducing the problem to the boundary.
- Differentiating the momentum equation in (5) with respect to $\tilde{D}_t := \partial_t + \tilde{V} \cdot \nabla$ to obtain the system:

$$\begin{aligned} \left(\tilde{D}_t^2 + \tilde{a}\nabla_{\tilde{n}}\right)\tilde{V} &= -\nabla \tilde{D}_t \tilde{p} \quad \text{on} \quad \partial\tilde{\Omega}_t \\ \Delta \tilde{V} &= 0 \quad \text{in} \quad \tilde{\Omega}_t. \end{aligned} \tag{7}$$

- Here $\nabla_{\tilde{n}}$ is the standard Dirichlet-Neumann operator, and $\tilde{a} := -\frac{\partial \tilde{P}}{\partial \tilde{n}}$.
- Using boundary integrals we express $\tilde{a}$ and $\nabla D_t \tilde{p}$ in terms of the boundary values of $\tilde{V}$ and its derivatives.
- It turns out that the first equation in (7) is a quasilinear equation of $\tilde{V}$.

Ideas to solve the Newtonian problem: Christodoulou-Lindblad (00)

- Instead of using boundary integrals, one considers the elliptic problems:

$$\begin{aligned} &\Delta \tilde{P} = -(\partial_i \tilde{V}^\ell)\partial_\ell \tilde{V}^i \quad \text{in} \quad \tilde{\Omega}_t, \quad \tilde{P} = 0 \quad \text{on} \quad \partial\tilde{\Omega}_t \\ &\Delta D_t \tilde{P} = G(\partial \tilde{V}, \partial^2 \tilde{P}) \quad \text{in} \quad \tilde{\Omega}_t, \quad D_t\tilde{P} = 0 \quad \text{on} \quad \partial\tilde{\Omega}_t. \end{aligned} \tag{8}$$

- Here $G(\partial \tilde{V}, \partial^2 \tilde{P})$ consists of the product between $\partial \tilde{V}$ and $\partial^2 \tilde{P}$, as well as a cubic expression of $\partial \tilde{V}$.
- The elliptic equations (8) recover the regularity of $\tilde{P}$ and $D_t \tilde{P}$.

Back to hard phase model

- Let $D_V := V^\mu \partial_\mu$, and n be the outward unit normal to $\partial\Omega$.
-
$$\sigma^2 \equiv 1 \quad \text{on} \quad \partial\Omega \quad \Rightarrow \quad \nabla\sigma^2 = -an \quad \text{on} \quad \partial\Omega$$
$$a = \sqrt{\nabla_\mu \sigma^2 \nabla^\mu \sigma^2} > 0.$$
- Differentiating the equation $D_V V^\mu + \frac{1}{2}\nabla^\mu \sigma^2 = 0$ by D_V on $\partial\Omega$, the original system (3) becomes
$$\left(D_V^2 + \frac{1}{2} a \nabla_n\right) V^\mu = -\frac{1}{2}\nabla^\mu D_V \sigma^2 \quad \text{on} \quad \partial\Omega$$
$$\Box V^\mu = 0 \quad \text{in} \quad \Omega. \tag{9}$$

Quasilinear system

- The operator ∇_n in (9) is the *hyperbolic Dirichlet-Neumann map*. It is not clear at all whether this operator is positive or not.
- σ^2 and $D_V\sigma^2$ satisfy the following wave equations with Dirichlet boundary data:

$$\Box\sigma^2 = -2(\nabla^\mu V^\nu)(\nabla_\mu V_\nu), \quad \sigma^2 \equiv 1 \quad \text{on} \quad \partial\Omega. \tag{10}$$

$$\Box D_V\sigma^2 = 4(\nabla^\mu V^\nu)(\nabla_\mu\nabla_\nu\sigma^2) + 4(\nabla^\lambda V^\nu)(\nabla_\lambda V^\mu)(\nabla_\nu V_\mu) \quad \text{in} \quad \Omega$$
$$D_V\sigma^2 \equiv 0 \quad \text{on} \quad \partial\Omega. \tag{11}$$

Well-posedness: Main ingredients of the proof

- A priori estimates for the system (9), (10) and (11).
- Higher order regularity: Commuting D_V^k. Note that D_V is defined globally both in the interior of Ω and $\partial\Omega$, and tangential to $\partial\Omega$. Using the equation we show that $D_V^2 \sim \partial_x$.
- Galerkin method to construct approximation sequences and prove the convergence of the sequences.

A priori estimates-Positivity of the hyperbolic DN map

- Main idea: Multiplying both the boundary equation $\left(D_V^2 + \frac{1}{2}a\nabla_n\right) V = \dots$ and the equation $\Box V = 0$ by $D_V V$, and integrate on Ω and $\partial\Omega$. We obtain the following positive energy

$$\int_{\Omega_t} |\partial_{t,x} V|^2 \, dx + \int_{\partial\Omega_t} \frac{1}{a} |D_V V|^2 \, dS. \tag{12}$$

  Here Ω_t and $\partial\Omega_t$ are the $x^0 = t$-slices of Ω and $\partial\Omega$ respectively.
- Let us illustrate the idea with a simpler model, where B is the unit ball:

$$\begin{aligned} &\Box u = F \quad \text{in} \quad [0, T] \times B \\ &\left(\partial_t^2 + \partial_r\right) u = f \quad \text{on} \quad [0, T] \times \partial B \end{aligned} \tag{13}$$

Positivity of the hyperbolic DN map

- Multiplying the system (13) by $\partial_t u$, we have

$$\begin{aligned} &\frac{1}{2}\partial_t(\partial_t u)^2 + (\partial_t u)(\partial_r u) = (\partial_t u)f \quad \text{on} \quad \partial B \\ &\frac{1}{2}\partial_t\left((\partial_t u)^2 + |\nabla u|^2\right) - \nabla\cdot(\partial_t u \nabla u) = -F\cdot\partial_t u \quad \text{in} \quad B. \end{aligned} \tag{14}$$

- Integrating the second equation in (14) on $[0, T]\times B$:

$$\begin{aligned} &\frac{1}{2}\int_B |\partial_{t,x} u(T)|^2\, dx - \frac{1}{2}\int_B |\partial_{t,x} u(0)|^2\, dx \\ &- \int_0^T \int_{\partial B} (\partial_t u)(\partial_r u)\, dS\, dt = -\int_0^T\int_B F\cdot\partial_t u\, dx\, dt \end{aligned} \tag{15}$$

Positivity of the hyperbolic DN map

- Integrating the first equation in (14) on $[0, T] \times \partial B$:

$$\frac{1}{2}\int_{\partial B} |\partial_t u(T)|^2 \, dS - \frac{1}{2}\int_{\partial B} |\partial_t u(0)|^2 \, dS + \int_0^T \int_{\partial B} (\partial_t u)(\partial_r u) \, dS \, dt = \int_0^T \int_{\partial B} (\partial_t u) f \, dS \, dt \tag{16}$$

- Adding (15) and (16), we obtain

$$\begin{aligned} &\frac{1}{2}\int_B |\partial_t u(T)|^2 \, dx + \frac{1}{2}\int_{\partial B} |\partial_t u(T)|^2 \, dS \\ =&\frac{1}{2}\int_B |\partial_t u(0)|^2 \, dx + \frac{1}{2}\int_{\partial B} |\partial_t u(0)|^2 \, dS \\ &- \int_0^T \int_B F \cdot \partial_t u \, dx \, dt + \int_0^T \int_{\partial B} (\partial_t u) f \, dS \, dt. \end{aligned} \tag{17}$$

Boundary equation-Source term

- To control the source term of the RHS of the equation $\left(D_V^2 + \frac{1}{2} a \nabla_n\right) V = \ldots$, we need to control

$$\int_0^T \int_{\partial\Omega_t} |\nabla D_V \sigma^2|^2 \, dS \, dt.$$

- We consider the following wave equation with Dirichlet data:

$$\Box D_V \sigma^2 = G \quad \text{in} \quad \Omega, \quad D_V \sigma^2 \equiv 0 \quad \text{on} \quad \partial\Omega. \tag{18}$$

- We use the multiplier vectorfield $V + \alpha n$ to derive an energy identity for (18). Here the constant $\alpha > 0$ is sufficiently small such that $V + \alpha n$ is *timelike* with respect to the Minkowski metric.

IBVP for wave

- Again we illustrate the idea with the following simpler model for (18):

$$\Box u = G \quad \text{in} \quad [0, T] \times B, \quad u \equiv 0 \quad \text{on} \quad [0, T] \times \partial B. \quad (19)$$

- Denote the multiplier vectorfield $\partial_t + \alpha \partial_r$ by

$$\partial_t + \alpha \partial_r := Q = Q^\mu \partial_\mu.$$

- A direct calculation shows

$$\begin{aligned} \Box u \cdot Qu =& \nabla_\mu \left((Qu)(\nabla^\mu u) - \frac{1}{2} Q^\mu \left(\nabla_\nu u\right) \left(\nabla^\nu u\right) \right) \\ &+ \frac{1}{2} \left(\nabla_\mu Q^\mu\right) \left(\nabla_\nu u\right) \left(\nabla^\nu u\right) - \left(\nabla^\mu Q^\nu\right) \left(\nabla_\mu u\right) \left(\nabla_\nu u\right) \end{aligned}$$

- The last two terms on the RHS above are of lower order.

IBVP for wave -conti

- Q being timelike implies that the flux on $\{T\} \times B$ is positive and controls the standard energy $\int_B |\partial_{t,x} D_V \sigma^2(T)|^2$.
- On $[0, T] \times \partial B$, we obtain

$$\frac{\alpha}{2} \int_0^T \int_{\partial B} (\nabla_\nu u)(\nabla^\nu u)\, dS\, dt \simeq \frac{\alpha}{2} \int_0^T \int_{\partial B} |\partial_r u|^2\, dS\, dt,$$

since

$$\partial_t u = \partial_\theta u = 0 \quad \text{on} \quad [0, T] \times \partial B.$$

Here θ is the angular variable on ∂B.

$H^k(\Omega_t)$-bounds

- To obtain the L^∞-control in the a priori estimates, we need the control of $\partial_x^k V$ in $L^2(\Omega_t)$.
- The energy controls $D_V^k V \in H^1(\Omega_t)$ and $D_V^{k+1} V \in L^2(\partial\Omega_t)$.
- Using the boundary equation $\left(D_V^2 + \frac{1}{2} a \nabla_n\right) V = \ldots$ we have

$$\nabla_n V \simeq D_V^2 V + \mathrm{l.o.t.}$$

The Trace Theorem implies

$$\|\nabla_n V\|_{H^{\frac{1}{2}}(\partial\Omega_t)} \lesssim \left\|D_V^2 V\right\|_{H^1(\Omega_t)} \lesssim \text{“Energy for } D_V^2 V\text{”} \quad (20)$$

$H^k(\Omega_t)$-bounds -conti

- On the other hand, we have

$$0 = \Box V = \partial_{t,x} D_V V + AV,$$

  where A is an elliptic operator on Ω_t. This together with (20) gives control on $\|V\|_{H^2(\Omega_t)}$ in terms of the energy (i.e., the $H^1(\Omega_t)$-norm) for $D_V^2 V$.
- This finally shows $D_V^2 \sim \nabla_x$.

Newtonian limit-Rescaled quantities

- To study the Newtonian limit as the speed of light approaches infinity, we of course cannot set the speed of light $c = 1$ anymore.
- Now the pressure p and energy density ρ are given by

$$p = \frac{1}{2}c^{-2}\left(\sigma^2 - c^4\right), \quad \rho = \frac{1}{2}c^{-4}\left(\sigma^2 + c^6\right).$$

- The sound speed $\eta = c$ and on the boundary $\partial\Omega$ we have $\sigma^2 \equiv c^4$.
- The initial data satisfies

$$\begin{aligned} &\sigma_0^2 \geq c^4 \quad \text{in} \quad \Omega_0 \\ &\sigma_0^2 = c^4 \quad \text{on} \quad \partial\Omega_0 \\ &\nabla_\mu \sigma_0^2 \nabla^\mu \sigma_0^2 \geq c_0^2 c^4 > 0 \quad \text{on} \quad \partial\Omega_0. \end{aligned} \tag{21}$$

Rescaled quantities and time variable

- Instead of V, σ^2, we work with the rescaled quantities

$$\overline{V} := c^{-1} V, \quad \overline{\sigma}^2 := c^{-2}\sigma^2 - c^2 \tag{22}$$

- Here $\overline{V}, \overline{\sigma}$ are to be shown of order $O(1)$ as $c \to \infty$.
- In addition to the standard time variable t in the proof of the well-posedness, we also work with the rescaled time variable $t' := c^{-1} t$. Therefore we have

$$\frac{\partial}{\partial t} = c^{-1} \frac{\partial}{\partial t'} \quad m = -c^2 (dt')^2 + \sum_{i=1}^{3} (dx^i)^2$$

$$\Box = -\frac{1}{c^2} \partial_{t'}^2 + \sum_{i=1}^{3} \partial_i^2 .$$

- Note that $\overline{V}^0 \simeq c$ as $c \to \infty$

Rescaled energy

- We strive for an a priori estimate which is independent of c. Therefore the energy must be of order $O(1)$ as $c \to \infty$.
- Systematically, let $E[V](t)$ and $E[D_V \sigma^2](t)$ be the energies we bound in the above a priori estimate. A direct observation shows that

$$E[\overline{V}](t) \simeq c, \quad E[D_{\overline{V}} \overline{\sigma}^2](t) \simeq c, \quad \text{as} \quad c \to \infty.$$

  The reason for this is that $\overline{V}^0 \simeq c$, which appears in the definition of $E[\overline{V}]$ and $E[D_{\overline{V}} \overline{\sigma}^2]$.
- To get an order $O(1)$ energy, we need to consider the rescaled energies

$$c^{-1} E[\overline{V}](t), \quad c^{-1} E[D_{\overline{V}} \overline{\sigma}^2](t).$$

Sources in the energy estimates

- Systematically, the energy estimates have the following form

$$c^{-1}E[\overline{V}](T)+c^{-1}E[D_{\overline{V}}\overline{\sigma}^2](T)$$
$$\lesssim \text{“Initial data of order } O(1)\text{''}+c^{-1}\int_0^T \text{“Nonlinear sources''}\, dt$$

- The “Nonlinear sources” above is of order $O(1)$ as $c\to\infty$.
- This observation implies that in the time variable t, we can extend the solution given by the well-posedness theorem up to the scale $t\simeq c$, and in the time variable t' up to the scale $t'\simeq 1$.
- This is crucial because t' is the time variable for the Newtonian problem.

The discrepancy for energy hierarchy given by the a priori estimates

- Suppose as $c \to \infty$, Θ is a quantity of order $O(1)$. Then $\partial_t \Theta$ must be of order $O(c^{-1})$ and $\partial_i \Theta = O(1)$. However, the a priori estimate gives the same estimate for $\partial_t \Theta = O(1)$. In the Newtonian limit, we need the improved estimate $\partial_t \Theta = O(c^{-1})$.
- To overcome this difficulty, we look at $\overline{\sigma}^2$:

$$\overline{\sigma}^2 = (\overline{V}^0 - c)^2 - \sum_{i=1}^{3} (\overline{V}^i)^2 + 2c(\overline{V}^0 - c) \tag{23}$$

  The a priori estimate shows that $\overline{V}^0 - c, \overline{V}^i, \overline{\sigma}^2$ remains bounded as $c \to \infty$, which in turn shows

$$\overline{V}^0 - c = O(c^{-1}) \quad \text{as} \quad c \to \infty.$$

- Differentiating (23) in ∂_t, we get

$$\partial_t \overline{V}^0 = O(c^{-1}) \quad \text{as} \quad c \to \infty.$$

Main result II-Newtonian limit

Finally we have the result on Newtonian limit, which can be roughly stated as following:

Theorem

The rescaled solution $(\overline{V}, \overline{\sigma})$ to the free boundary problem (3)-(4) *converges to the solution to the free boundary problem* (5) *as* $c \to \infty$.

Thank you!