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On the solutions to weakly coupled system of $\boldsymbol{k_i}$-Hessian equations ☆

Jingwen Ji [a], Feida Jiang [b,*], Baohua Dong [a]

[a] *College of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210094, PR China*

[b] *School of Mathematics and Shing-Tung Yau Center of Southeast University, Southeast University, Nanjing 211189, PR China*

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ABSTRACT

In this paper, the existence and multiplicity of nontrivial radial convex solutions to general coupled system of k_i-Hessian equations in a unit ball are studied via a fixed-point theorem. In particular, we obtain the uniqueness of nontrivial radial convex solution and nonexistence of nontrivial radial $\boldsymbol{k}$-admissible solution to a power-type system coupled by k_i-Hessian equations in a unit ball. Moreover, using a generalized Krein-Rutman theorem, the existence of $\boldsymbol{k}$-admissible solutions to an eigenvalue problem in a general strictly $(k-1)$-convex domain is also obtained.

1. Introduction

In this paper, we consider the existence and multiplicity of nontrivial radial $\boldsymbol{k}$-admissible solutions to the coupled system of the following k_i-Hessian equations:

$$\begin{cases} S_{k_1}\left(D^2 u_1\right) = f_1\left(|x|, -u_2\right), & \text{in } B, \\ S_{k_2}\left(D^2 u_2\right) = f_2\left(|x|, -u_3\right), & \text{in } B, \\ \qquad \vdots & \\ S_{k_{n-1}}\left(D^2 u_{n-1}\right) = f_{n-1}\left(|x|, -u_n\right), & \text{in } B, \\ S_{k_n}\left(D^2 u_n\right) = f_n\left(|x|, -u_1\right), & \text{in } B, \\ u_i = 0, i = 1, \ldots, n, & \text{on } \partial B, \end{cases} \tag{1.1}$$

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* Corresponding author.
E-mail addresses: 20201215013@nuist.edu.cn (J. Ji), jfd2001@163.com (F. Jiang), baohuadong@nuist.edu.cn (B. Dong).

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where $\boldsymbol{k} = (k_1, \ldots, k_n)$, $k_i \in \{1, \ldots, N\}$, $i \in \{1, \ldots, n\}$, $B = \{x \in \mathbb{R}^N : |x| < 1\}$ is a unit ball, $n \geq 2$ and $N \geq 2$ are integers. The nonlinearities f_i $(i = 1, \ldots, n)$ satisfy

$$\text{(F)} : f_i \in C\left([0,1] \times [0, +\infty), [0, +\infty)\right), \quad i = 1, \ldots, n$$

and each f_i is not identical to zero.

The k-Hessian operator S_k is defined by the k-th elementary symmetric function of eigenvalues of D^2u, i.e.

$$S_k\left(D^2u\right) := S_k\left(\lambda(D^2u)\right) = \sum_{1 \leq i_1 < \cdots < i_k \leq N} \lambda_{i_1} \cdots \lambda_{i_k}, \quad k = 1, \ldots, N,$$

where $\lambda(D^2u) = (\lambda_1, \ldots, \lambda_N)$ is the vector of eigenvalues of $D^2u = \left[\frac{\partial^2 u}{\partial x_i \partial x_j}\right]_{n \times n}$, (see [21,25] for instance). Notice that when $k = 1$, the Hessian operator reduces to the classical Laplace operator $S_1(D^2u) = \sum_{i=1}^N \lambda_i = \Delta u$. When $k = N$, the Hessian operator is the Monge-Ampère operator $S_N(D^2u) = \prod_{i=1}^N \lambda_i = \det(D^2u)$. In fact, the k-Hessian operator can be regarded as an extension of the Laplace operator and the Monge-Ampère operator. When $k \geq 2$, the k-Hessian operator is a fully nonlinear operator.

Let $u \in C^2(\Omega)$ and $\sigma_k = \{\lambda \in \mathbb{R}^N : S_l(\lambda) > 0, \forall l = 1, \ldots, k\}$ be a convex cone and its vertex be the origin. If $\lambda(D^2u) \in \overline{\sigma}_k$ (σ_k), u is said to be k-convex (uniformly k-convex) in Ω. Equivalently, if $\lambda(-D^2u) \in \overline{\sigma}_k$ (σ_k), u is k-concave (uniformly k-concave) in Ω. We say $u \in C^2(\Omega) \cup C^0(\overline{\Omega})$ is k-admissible if $\lambda(D^2u) \in \overline{\sigma}_k$. In particular, an N-admissible function u satisfying $\lambda(D^2u) \in \overline{\sigma}_N$ is said to be convex. It is clear that $\sigma_N \subset \cdots \subset \sigma_k \subset \cdots \subset \sigma_1$, which implies that convex functions are contained in k-admissible functions. Actually, we know from [2] that for a k-Hessian equation, it is elliptic when restricted to k-admissible functions. For $\boldsymbol{k} = (k_1, \ldots, k_n)$, $\boldsymbol{u} = (u_1, \ldots, u_n)$, if u_i is k_i-admissible and satisfies (1.1) for all $i = 1, \ldots, n$, we say $\boldsymbol{u}$ is a $\boldsymbol{k}$-admissible solution of (1.1).

Recalling that $f_i \in [0, +\infty)$ $(i = 1, \ldots, n)$ and $\boldsymbol{u} \in C^2(B)$ is $\boldsymbol{k}$-admissible solution of (1.1) vanishing on the boundary, we can achieve that $\boldsymbol{u}$ is sub-harmonic in B from [25]. Hence, we apply the maximum principle to conclude that $\boldsymbol{u}$ is negative in B.

The study of k-Hessian equations plays an important role in differential geometry, fluid mechanics and other applied disciplines. In the past years, many authors show great interest in solutions of k-Hessian equations and many excellent results on k-Hessian equations have been obtained, for instance, see [1–3,9,15–22,25]. However, there are few studies that consider the fully nonlinear coupled systems except [4–7,23,24,29] based on our cognition. For example, by using fixed point theorem, Wang [24] established the existence, multiplicity and nonexistence of convex radial solutions to a coupled system of Monge-Ampère equations in superlinear and sublinear cases. In [7], the authors studied the existence and multiplicity of nontrivial radial solutions for system coupled by multiparameter k-Hessian equations and obtained sufficient conditions for the existence of nontrivial radial solutions to power-type coupled k-Hessian system based on a eigenvalue theory in cones. In particular, Cui considered a Hessian type system coupled by different k-Hessian equations and obtained the existence of entire k-convex radial solutions, see [4].

Inspired by the above works, we are interested in a system coupled by different k-Hessian equations with general nonlinearities which satisfy α_i or β_i-asymptotic growth conditions. In this paper, we shall establish the existence and multiplicity of nontrivial radial $\boldsymbol{k}$-admissible solutions of the weakly coupled degenerated system (1.1). It is worth to notice that the system (1.1) contains a variety of different k-Hessian equations which is significantly different from that in [5,7,23,29] such that the problem we considered can contain Laplace equations and Monge-Ampère equations at the same time. This kind of system can represent the coupling of different types of elliptic equations, which makes our problem more comprehensive and more applicable.

If $\alpha_i, \beta_i > 0$, we let

$$\underline{f}_i^0 = \liminf_{c\to 0^+} \min_{0\le t\le 1} \frac{f_i(t,c)}{c^{\alpha_i}}, \qquad \underline{f}_i^\infty = \liminf_{c\to \infty} \min_{0\le t\le 1} \frac{f_i(t,c)}{c^{\beta_i}},$$
$$\overline{f}_i^0 = \limsup_{c\to 0^+} \max_{0\le t\le 1} \frac{f_i(t,c)}{c^{\alpha_i}}, \qquad \overline{f}_i^\infty = \limsup_{c\to \infty} \max_{0\le t\le 1} \frac{f_i(t,c)}{c^{\beta_i}}.$$

Here, we call them α_i or β_i-asymptotic growth condition, super-α_i or β_i-asymptotic growth condition and sub-α_i or β_i-asymptotic growth condition. Compared with some N-asymptotic growth (see, for instance [6,8,24] where the constants $\alpha_i = \beta_i = N$) in studying Monge-Ampère equations and some k-asymptotic growth (see, for instance [7,26,27] where the constants $\alpha_i = \beta_i = k$) in studying k-Hessian equations, our conditions are more flexible. By imposing suitable conditions on $\underline{f}_i^0$, $\underline{f}_i^\infty$, $\overline{f}_i^0$, $\overline{f}_i^\infty$ and coordinating inequality relations between α_i, β_i and k_i, we obtain existence and multiplicity results in general cases as follows.

We will assume $\boldsymbol{f} = \{f_1, \dots, f_n\}$ satisfies one of the following conditions:

(C1) $\underline{f}_i^0, \overline{f}_i^\infty \in (0, +\infty)$, $i = 1, \dots, n$ and $f_i(t, 0) = 0$, $i = 2, \dots, n$;
(C2) $\overline{f}_i^0, \underline{f}_i^\infty \in (0, +\infty)$, $i = 1, \dots, n$;
(C3) $\underline{f}_i^0, \underline{f}_i^\infty \in (0, +\infty)$, $i = 1, \dots, n$ and $f_i(t, 0) = 0$, $i = 2, \dots, n$;
(C4) $\overline{f}_i^0, \overline{f}_i^\infty \in (0, +\infty)$, $i = 1, \dots, n$.

Theorem 1.1. *(Existence theorem) Suppose that* (F) *and one of the following conditions hold:*
(a). (C1) *holds and positive constants* α_i, β_i $(i = 1, \dots, n)$ *satisfy*

$$\prod_{i=1}^n \alpha_i < \prod_{i=1}^n k_i, \quad \prod_{i=1}^n \beta_i < \prod_{i=1}^n k_i;$$

(b). (C2) *holds and positive constants* α_i, β_i $(i = 1, \dots, n)$ *satisfy*

$$\prod_{i=1}^n \alpha_i > \prod_{i=1}^n k_i, \quad \prod_{i=1}^n \beta_i > \prod_{i=1}^n k_i.$$

Then system (1.1) *has at least one nontrivial radial convex solution.*

Theorem 1.1 is concerning the existence of nontrivial radial convex solutions to the weakly coupled degenerate system (1.1) with general nonlinear terms. Furthermore, we can consider the result of multiplicity as well.

Let

$$G_i = \max\left\{ f_i(t, v_{i+1}(t)) : (t, v_{i+1}(t)) \in [0,1] \times [0, G_{i+1}^{\frac{1}{k_{i+1}}}] \right\}, \quad i = 1, \dots, n-1,$$
$$G_n = \max\left\{ f_n(t, v_1(t)) : (t, v_1(t)) \in [0,1] \times [0, \frac{r_0}{4}] \right\},$$
$$\tilde{G}_i = \max\left\{ f_i(t, v_{i+1}(t)) : (t, v_{i+1}(t)) \in [0,1] \times [0, \tilde{G}_{i+1}^{\frac{1}{k_{i+1}}}] \right\}, \quad i = 2, \dots, n-1,$$
$$\tilde{G}_n = \max\left\{ f_n(t, v_1(t)) : (t, v_1(t)) \in [0,1] \times [0, R_0] \right\},$$
$$E_i = \min\left\{ f_i(t, v_{i+1}(t)) : (t, v_{i+1}(t)) \in [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}\Gamma_{i+1} E_{i+1}^{\frac{1}{k_{i+1}}}, \tilde{G}_{i+1}^{\frac{1}{k_{i+1}}}] \right\}, \quad i = 1, \dots, n-1,$$
$$E_n = \min\left\{ f_n(t, v_1(t)) : (t, v_1(t)) \in [\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4} R_0, R_0] \right\}.$$

Theorem 1.2. *(Multiplicity theorem) Suppose that* (F) *and one of the following conditions hold:*
(c). (C3) *holds, positive constants* α_i, β_i $(i = 1, \ldots, n)$ *satisfy*

$$\prod_{i=1}^{n} \alpha_i < \prod_{i=1}^{n} k_i, \quad \prod_{i=1}^{n} \beta_i > \prod_{i=1}^{n} k_i,$$

and there exists a positive constant r_0 *such that* $r_0 > G_1^{\frac{1}{k_1}}$*;*
(d). (C4) *holds, positive constants* α_i, $\beta_i (i = 1, \ldots, n)$ *satisfy*

$$\prod_{i=1}^{n} \alpha_i > \prod_{i=1}^{n} k_i, \quad \prod_{i=1}^{n} \beta_i < \prod_{i=1}^{n} k_i,$$

and there exists a positive constant R_0 *such that* $R_0 < \Gamma_1 E_1^{\frac{1}{k_1}}$*. Then system* (1.1) *has at least two nontrivial radial convex solutions.*

Remark 1.1. Theorem 1.1 and Theorem 1.2 show that the existence and multiplicity of nontrivial radial convex solutions to system (1.1) respectively, see the penultimate paragraph in this section for more detailed explanations of convex solutions. Since the convex solutions are contained in the $\boldsymbol{k}$-admissible solutions, Theorems 1.1 and 1.2 show the existence and multiplicity of $\boldsymbol{k}$-admissible solutions as well.

Remark 1.2. It is worth to mention that the condition (C1) and (C3) can be replaced by $f_{m_k}(t, 0) = 0,\ k \in \{1, \ldots, n-1\}$, where $\{m_1, \ldots, m_{n-1}\} \subset \{1, \ldots, n\}$ and we describe as $f_i(t, 0) = 0$, $i = 2, \ldots, n$ for the sake of proof.

Specifically, we also study the uniqueness and nonexistence of nontrivial radial solutions to a power-type coupled system of $\boldsymbol{k}$-Hessian equations:

$$\begin{cases} S_{k_1}\left(D^2 u_1\right) = (-u_2)^{\gamma_1}, & \text{in } B, \\ S_{k_2}\left(D^2 u_2\right) = (-u_3)^{\gamma_2}, & \text{in } B, \\ \quad\vdots & \\ S_{k_{n-1}}\left(D^2 u_{n-1}\right) = (-u_n)^{\gamma_{n-1}}, & \text{in } B, \\ S_{k_n}\left(D^2 u_n\right) = (-u_1)^{\gamma_n}, & \text{in } B, \\ u_i = 0, i = 1, \ldots, n, & \text{on } \partial B, \end{cases} \tag{1.2}$$

where γ_i $(i = 1, \ldots, n)$ are positive constants.

It is obvious that system (1.2) is a special case of system (1.1). By the definitions of α_i or β_i-asymptotic growth condition, the growth of nonlinearities of the power-type system (1.2) satisfies $\alpha_i = \beta_i = \gamma_i$, which asserts the existence of nontrivial radial convex solutions to system (1.2) by Theorem 1.1 when $\prod_{i=1}^{n} \gamma_i \neq \prod_{i=1}^{n} k_i$. Next, we go further to study the uniqueness of nontrivial radial convex solution to system (1.2) in Theorem 1.3.

Theorem 1.3. *(Uniqueness theorem) Suppose that positive constant* $\prod_{i=1}^{n} \gamma_i$ *satisfies*

$$\prod_{i=1}^{n} \gamma_i < \prod_{i=1}^{n} k_i,$$

then system (1.2) *has a unique nontrivial radial convex solution.*

Here, we get the uniqueness result of nontrivial radial convex solution to system (1.2) in the assumption of $\prod_{i=1}^n \gamma_i < \prod_{i=1}^n k_i$. Besides, we obtain the nonexistence of nontrivial radial $\boldsymbol{k}$-admissible solution in B when $\prod_{i=1}^n \gamma_i = \prod_{i=1}^n k_i$.

Theorem 1.4. *(Nonexistence theorem) Suppose that positive constant $\prod_{i=1}^n \gamma_i$ satisfies*

$$\prod_{i=1}^n \gamma_i = \prod_{i=1}^n k_i,$$

then system (1.2) admits no nontrivial radial $\boldsymbol{k}$-admissible solution.

When $\prod_{i=1}^n \gamma_i = \prod_{i=1}^n k_i$, we are interested in the existence of nonzero $\boldsymbol{k}$-admissible solutions for the eigenvalue problem:

$$\begin{cases} S_{k_1}\left(D^2 u_1\right) = \lambda_1 \left(-u_2\right)^{\gamma_1}, & \text{in } \Omega, \\ S_{k_2}\left(D^2 u_2\right) = \lambda_2 \left(-u_3\right)^{\gamma_2}, & \text{in } \Omega, \\ \qquad \vdots & \\ S_{k_{n-1}}\left(D^2 u_{n-1}\right) = \lambda_{n-1} \left(-u_n\right)^{\gamma_{n-1}}, & \text{in } \Omega, \\ S_{k_n}\left(D^2 u_n\right) = \lambda_n \left(-u_1\right)^{\gamma_n}, & \text{in } \Omega, \\ u_i = 0, i = 1, \ldots, n, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

with positive parameters λ_i $(i = 1, \ldots, n)$, where $\Omega \in \mathbb{R}^N$ is a bounded, smooth and strictly $(k-1)$-convex domain, $N \geq 2$.

In fact, Wang has proved the existence of a positive eigenvalue λ^* for a single k-Hessian equation with $f(u) = \lambda|u|^k (k < N)$ in [25]. When $\lambda = \lambda^*$, the corresponding eigenfunction φ^* is nonzero k-admissible and that any other eigenfunction would be a positive constant multiple of φ^*. Since λ^* acts like a bifurcation point for system (1.3), we can be reminiscent of the generalized Krein-Rutman theorem in [13] to obtain the existence of $\boldsymbol{k}$-admissible solutions to eigenvalue problem (1.3).

Theorem 1.5. *(Eigenvalue problem) Suppose that $\Omega \in \mathbb{R}^N$ is a bounded, smooth and strictly $(k-1)$-convex domain, positive constant $\prod_{i=1}^n \gamma_i$ satisfies*

$$\prod_{i=1}^n \gamma_i = \prod_{i=1}^n k_i,$$

then system (1.3) admits a nonzero $\boldsymbol{k}$-admissible solution if and only if $\lambda_1 \lambda_2^{\frac{\gamma_1}{k_2}} \cdots \lambda_n^{\frac{\prod_{i=1}^{n-1}\gamma_i}{\prod_{i=2}^{n} k_i}} = \lambda_0^{k_1}$, where $\lambda_0 \neq 1$ is a positive constant, such that the system

$$\begin{cases} S_{k_1}\left(D^2 (\dfrac{u_1}{\lambda_0})\right) = \left(-u_2\right)^{\gamma_1}, & \text{in } \Omega, \\ S_{k_2}\left(D^2 u_2\right) = \left(-u_3\right)^{\gamma_2}, & \text{in } \Omega, \\ \qquad \vdots & \\ S_{k_{n-1}}\left(D^2 u_{n-1}\right) = \left(-u_n\right)^{\gamma_{n-1}}, & \text{in } \Omega, \\ S_{k_n}\left(D^2 u_n\right) = \left(-u_1\right)^{\gamma_n}, & \text{in } \Omega, \\ u_i = 0, \quad i = 1, \ldots, n, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

has a nonzero $\boldsymbol{k}$*-admissible solution.*

Note that the existence of nonzero $\boldsymbol{k}$-admissible solution of (1.4) is guaranteed by a generalized Krein-Rutman theorem, see Section 5 for details.

In this article, we study the existence and multiplicity of radial convex solutions to system (1.1), the uniqueness of radial convex solution and nonexistence of radial $\boldsymbol{k}$-admissible solution to system (1.2), and the existence of radial $\boldsymbol{k}$-admissible solutions to the related eigenvalue problem (1.3). The reasons why Theorems 1.1, 1.2 and 1.3 are only restricted to the convex solutions will be further explained in Remarks 2.1 and 4.1. The improvement from convex solutions to $\boldsymbol{k}$-admissible solutions in Theorem 1.1, Theorem 1.2 and Theorem 1.3 is still an interesting problem, which attracts us to find another way or technique to solve this problem in a sequel.

The rest of the paper is organized as follows. In Section 2, we make some preliminary calculations of C^2 radial solutions and present a fixed point theorem in Theorem 2.1. In Section 3, we give the proof of existence and multiplicity results for system (1.1) with general nonlinearities by using the fixed point theorem. In Section 4, the uniqueness and nonexistence results for power-type coupled system (1.2) which is a special case of (1.1) are considered. In Section 5, by overcoming the difficulties caused by verifying the condition of generalized Krein-Rutman theorem which to prove the operator is strong, we obtain the existence of nonzero $\boldsymbol{k}$-admissible solutions to the eigenvalue problem (1.3) in a general strictly $(k-1)$-convex domain.

2. Preliminaries

To study radial classical solutions of system (1.1), we assume $u(|x|) = u(t)$ be the radial function with $t = \sqrt{\sum_{i=1}^N x_i^2}$, then it follows from Lemma 2.1 in [14] that the k-Hessian operator becomes

$$S_k(D^2u) = C_{N-1}^{k-1} u''(t) (\frac{u'(t)}{t})^{k-1} + C_{N-1}^{k} (\frac{u'(t)}{t})^{k}, \quad t \in (0,1).$$

Then we can convert (1.1) to the following system of ordinary differential equations:

$$\begin{cases} C_{N-1}^{k_1-1} u_1''(t)(\frac{u_1'(t)}{t})^{k_1-1} + C_{N-1}^{k_1}(\frac{u_1'(t)}{t})^{k_1} = f_1(t,-u_2), & 0<t<1, \\ C_{N-1}^{k_2-1} u_2''(t)(\frac{u_2'(t)}{t})^{k_2-1} + C_{N-1}^{k_2}(\frac{u_2'(t)}{t})^{k_2} = f_2(t,-u_3), & 0<t<1, \\ \qquad \vdots \\ C_{N-1}^{k_{n-1}-1} u_{n-1}''(t)(\frac{u_{n-1}'(t)}{t})^{k_{n-1}-1} + C_{N-1}^{k_{n-1}}(\frac{u_{n-1}'(t)}{t})^{k_{n-1}} = f_{n-1}(t,-u_n), & 0<t<1, \\ C_{N-1}^{k_n-1} u_n''(t)(\frac{u_n'(t)}{t})^{k_n-1} + C_{N-1}^{k_n}(\frac{u_n'(t)}{t})^{k_n} = f_n(t,-u_1), & 0<t<1, \\ u_i(1) = u_i'(0) = 0, \quad i = 1,\dots,n. \end{cases} \tag{2.1}$$

Equivalently, we seek nonnegative k-concave solutions for convenience by making a simple transformation $v_i = -u_i$ $(i = 1,\dots,n)$ in (2.1), which leads to the following system:

$$\begin{cases} C_{N-1}^{k_1-1} (-v_1)''(t)(\frac{(-v_1)'(t)}{t})^{k_1-1} + C_{N-1}^{k_1}(\frac{(-v_1)'(t)}{t})^{k_1} = f_1(t,v_2), & 0<t<1, \\ C_{N-1}^{k_2-1} (-v_2)''(t)(\frac{(-v_2)'(t)}{t})^{k_2-1} + C_{N-1}^{k_2}(\frac{(-v_2)'(t)}{t})^{k_2} = f_2(t,v_3), & 0<t<1, \\ \qquad \vdots \\ C_{N-1}^{k_{n-1}-1} (-v_{n-1})''(t)(\frac{(-v_{n-1})'(t)}{t})^{k_{n-1}-1} + C_{N-1}^{k_{n-1}}(\frac{(-v_{n-1})'(t)}{t})^{k_{n-1}} = f_{n-1}(t,v_n), & 0<t<1, \\ C_{N-1}^{k_n-1} (-v_n)''(t)(\frac{(-v_n)'(t)}{t})^{k_n-1} + C_{N-1}^{k_n}(\frac{(-v_n)'(t)}{t})^{k_n} = f_n(t,v_1), & 0<t<1, \\ v_i(1) = v_i'(0) = 0, \quad i = 1,\dots,n. \end{cases} \tag{2.2}$$

By integration, we get from (2.2) that

$$\begin{cases} v_1(t) = \int_t^1 \left(\frac{k_1}{\tau^{N-k_1}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}} f_1\left(s, v_2(s)\right) \mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau, & 0 \le t \le 1, \\ v_2(t) = \int_t^1 \left(\frac{k_2}{\tau^{N-k_2}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_2-1}} f_2\left(s, v_3(s)\right) \mathrm{d}s \right)^{\frac{1}{k_2}} \mathrm{d}\tau, & 0 \le t \le 1, \\ \qquad \vdots \\ v_{n-1}(t) = \int_t^1 \left(\frac{k_{n-1}}{\tau^{N-k_{n-1}}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_{n-1}-1}} f_{n-1}\left(s, v_n(s)\right) \mathrm{d}s \right)^{\frac{1}{k_{n-1}}} \mathrm{d}\tau, & 0 \le t \le 1, \\ v_n(t) = \int_t^1 \left(\frac{k_n}{\tau^{N-k_n}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_n-1}} f_n\left(s, v_1(s)\right) \mathrm{d}s \right)^{\frac{1}{k_n}} \mathrm{d}\tau, & 0 \le t \le 1. \end{cases}$$

Considering the Banach space $X := C[0,1]$, for $\boldsymbol{v} = (v_1, \dots, v_n) \in \underbrace{X \times \cdots \times X}_{n}$, we define $||\boldsymbol{v}|| = \sum_{i=1}^n ||\boldsymbol{v_i}(t)|| = \sum_{i=1}^n \sup_{t\in[0,1]} |v_i(t)|$. Let K be a cone in X defined as

$$K := \left\{ v \in X : v(t) \ge 0, t \in [0,1], \min_{\frac{1}{4} \le t \le \frac{3}{4}} v(t) \ge \frac{1}{4} ||v|| \right\}. \tag{2.3}$$

We define the operators $T_i : K \to X$ $(i = 1, \dots, n)$ to be

$$\begin{aligned} T_1(v_2)(t) &= \int_t^1 \left(\frac{k_1}{\tau^{N-k_1}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}} f_1\left(s, v_2(s)\right) \mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau, \\ T_2(v_3)(t) &= \int_t^1 \left(\frac{k_2}{\tau^{N-k_2}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_2-1}} f_2\left(s, v_3(s)\right) \mathrm{d}s \right)^{\frac{1}{k_2}} \mathrm{d}\tau, \\ &\ \ \vdots \\ T_{n-1}(v_n)(t) &= \int_t^1 \left(\frac{k_{n-1}}{\tau^{N-k_{n-1}}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_{n-1}-1}} f_{n-1}\left(s, v_n(s)\right) \mathrm{d}s \right)^{\frac{1}{k_{n-1}}} \mathrm{d}\tau, \\ T_n(v_1)(t) &= \int_t^1 \left(\frac{k_n}{\tau^{N-k_n}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_n-1}} f_n\left(s, v_1(s)\right) \mathrm{d}s \right)^{\frac{1}{k_n}} \mathrm{d}\tau. \end{aligned}$$

Note that each image of operator is a nonnegative k-concave function on $[0,1]$ and we define $T_1(v_2) = v_1, T_2(v_3) = v_2, \cdots, T_n(v_1) = v_n$ in K. Thus, by the concavity of v_i $(i = 1, \dots, n)$, it is easy to see that T_i $(i = 1, \dots, n)$ maps K into itself. Besides, by standard arguments, we know that every operator is completely continuous.

Next, we define a composite operator $Tv_1 = T_1 T_2 \cdots T_n(v_1)$, which is also completely continuous from K to K. We can see that positive solutions of (2.2) are equivalent to nonzero fixed points of operator T in cone K. If $\boldsymbol{v} = (v_1, \dots, v_n) \in \underbrace{C[0,1] \times \cdots \times C[0,1]}_{n}$ is a positive solution of (2.2), then v_1 must be a nonzero fixed point of T in K; conversely if $v_1 \in K \setminus \{0\}$ is a fixed point of T, we can define $v_n = T_n(v_1), v_{n-1} = T_{n-1}(v_n), \cdots, v_2 = T_2(v_3)$ such that $(v_1, \dots, v_n) \in \underbrace{C[0,1] \times \cdots \times C[0,1]}_{n}$ solves (2.2).

Remark 2.1. As we shall see in the last two paragraphs, we let each T_i $(i = 1\dots,n)$ maps K to itself which implies that $v'(t) = (-u)'(t)$ is nonincreasing from Lemma 2.2 in [24]. On the other hand, the eigenvalues of the second derivative of radial classical function in a unit ball can be represented by $\lambda(D^2u) = (u''(t), \frac{u'(t)}{t}, \dots, \frac{u'(t)}{t}), t \in [0,1]$, we combine this with the definition of $\boldsymbol{k}$-admissible function, an immediate consequence is that we essentially achieve the $(N-1)$-admissible function in $\mathbb{R}^N$, that is, all $\frac{u'(t)}{t} \geq 0$. To sum up, all eigenvalues of the Hessian matrix of nontrivial radial $\boldsymbol{k}$-admissible solutions of system (1.1) and system (1.2) are nonnegative and exist in its closure of convex cone, which can draw our conclusion.

The proofs of our existence and multiplicity results are based on the following well-known fixed point theorem of cone, (see Theorem 2.3.4 in Guo and Lakshmikantham [10]).

Theorem 2.1. *Let X be a Banach space and K is a cone in X. Assume that Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$ and let*

$$T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$$

be completely continuous such that either

(i) $||Tu|| \geq ||u||$, $u \in K \cap \partial\Omega_1$ and $||Tu|| \leq ||u||$, $u \in K \cap \partial\Omega_2$; or
(ii) $||Tu|| \leq ||u||$, $u \in K \cap \partial\Omega_1$ and $||Tu|| \geq ||u||$, $u \in K \cap \partial\Omega_2$

holds, where $||\cdot||$ is a norm in X, $\Omega_R = \{u \in K : ||u|| < R\}$ and $\partial\Omega_R = \{u \in K : ||u|| = R\}$. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence and multiplicity

In this section, we apply the fixed point theorem of cone in Theorem 2.1 to prove the existence and multiplicity results in Theorem 1.1 and Theorem 1.2. To simplify notation, we denote v_1 by v_{n+1}.

3.1. Existence

In order to prove the Theorem 1.1, we first introduce two useful lemmas.

Lemma 3.1. *Assume* (F) *holds. Let $\eta, m > 0$ and $v_i \in K$, $i = 1, \dots, n$. If for any $t \in [\frac{1}{4}, \frac{3}{4}]$ and $i = 1, \dots, n$, we have*

$$f_i\left(t, v_{i+1}(t)\right) \geq \eta v_{i+1}^m(t),$$

then

$$T_i(v_{i+1})(\frac{1}{4}) \geq \Gamma_i \eta^{\frac{1}{k_i}} (\frac{1}{4})^{\frac{m}{k_i}} ||v_{i+1}||^{\frac{m}{k_i}}, \quad i = 1, \dots, n,$$

where Γ_i are positive constants given by $\Gamma_i = \int_{\frac{1}{4}}^{\frac{3}{4}} \left(\frac{k_i}{\tau^{N-k_i}} \int_{\frac{1}{4}}^{\tau} \frac{s^{N-1}}{C_{N-1}^{k_i-1}} \,\mathrm{d}s \right)^{\frac{1}{k_i}} \mathrm{d}\tau, i = 1, \dots, n.$

Proof. For $v_1 \in K$, we have

$$
\begin{aligned}
T_n(v_1)(\frac{1}{4}) =& \int_{\frac{1}{4}}^{1} \left(\frac{k_n}{\tau^{N-k_n}} \int_0^{\tau} \frac{s^{N-1}}{C_{N-1}^{k_n-1}} f_n\left(s, v_1(s)\right) \mathrm{d}s \right)^{\frac{1}{k_n}} \mathrm{d}\tau \\
\geq& \int_{\frac{1}{4}}^{\frac{3}{4}} \left(\frac{k_n}{\tau^{N-k_n}} \int_{\frac{1}{4}}^{\tau} \frac{s^{N-1}}{C_{N-1}^{k_n-1}} \eta v_1^m(s) \, \mathrm{d}s \right)^{\frac{1}{k_n}} \mathrm{d}\tau \\
\geq& \int_{\frac{1}{4}}^{\frac{3}{4}} \left(\frac{k_n}{\tau^{N-k_n}} \int_{\frac{1}{4}}^{\tau} \frac{s^{N-1}}{C_{N-1}^{k_n-1}} \eta \left(\frac{1}{4}||v_1||\right)^m \mathrm{d}s \right)^{\frac{1}{k_n}} \mathrm{d}\tau \\
=& \Gamma_n \eta^{\frac{1}{k_n}} \left(\frac{1}{4}||v_1||\right)^{\frac{m}{k_n}}.
\end{aligned}
$$

For $v_i \in K$ $(i = 2, \dots, n)$, we have similar calculations. Here we omit them for simplicity. □

Lemma 3.2. *Assume* (F) *holds. Let $\varepsilon, d > 0$ and $v_i \in K, i = 1, \dots, n$. If for any $t \in [0, 1]$ and $i = 1, \dots, n$, we have*

$$
f_i\left(t, v_{i+1}(t)\right) \leq \varepsilon v_{i+1}^d(t),
$$

then

$$
T_i(v_{i+1})(t) < \left(\varepsilon||v_{i+1}||^d\right)^{\frac{1}{k_i}}, \quad i = 1, \dots, n.
$$

Proof. Since $v_1(t) \in K$, $\forall t \in [0, 1]$, we have

$$
\begin{aligned}
T_n(v_1)(t) \leq& \int_0^1 \left(\frac{k_n}{\tau^{N-k_n}} \int_0^{\tau} \frac{s^{N-1}}{C_{N-1}^{k_n-1}} f_n\left(s, v_1(s)\right) \mathrm{d}s \right)^{\frac{1}{k_n}} \mathrm{d}\tau \\
\leq& \int_0^1 \left(\frac{k_n}{\tau^{N-k_n}} \int_0^{\tau} \frac{s^{N-1}}{C_{N-1}^{k_n-1}} \varepsilon v_1^d(s) \, \mathrm{d}s \right)^{\frac{1}{k_n}} \mathrm{d}\tau \\
\leq& \frac{1}{2} \left(\frac{k_n}{N C_{N-1}^{k_n-1}} \right)^{\frac{1}{k_n}} \left(\varepsilon||v_1||^d\right)^{\frac{1}{k_n}} \\
<& \left(\varepsilon||v_1||^d\right)^{\frac{1}{k_n}},
\end{aligned}
\tag{3.1}
$$

where the fact $\frac{1}{2}\left(\frac{k_n}{NC_{N-1}^{k_n-1}}\right)^{\frac{1}{k_n}} < 1$ is used in the last inequality, which is easily checked. For $v_i \in K$ $(i = 2, \dots, n)$, we also have similar conclusions. □

On the basis of the above preparations, we give the proof for the existence result in Theorem 1.1 with the aid of the fixed point theorem of cone.

Proof of Theorem 1.1. (a). It follows from $\underline{f}_i^0 \in (0,+\infty)$ $(i = 1,\dots,n)$ that for any given $\varepsilon_1 \in (0, \min\{\underline{f}_i^0, i = 1,\dots,n\})$, there exists a constant $r_1 \in (0,1)$ such that

$$f_i(t, v_{i+1}(t)) \geq (\underline{f}_i^0 - \varepsilon_1) v_{i+1}^{\alpha_i}, \quad 0 \leq v_{i+1} \leq r_1, \tag{3.2}$$

for any $t \in [0,1]$ and $i = 1,\dots,n$. Let

$$L_1 := \Gamma_1 \cdots \Gamma_n^{\frac{\prod_{i=1}^{n-1}\alpha_i}{\prod_{i=1}^{n-1}k_i}} (\underline{f}_1^0 - \varepsilon_1)^{\frac{1}{k_1}} \cdots (\underline{f}_n^0 - \varepsilon_1)^{\frac{\prod_{i=1}^{n-1}\alpha_i}{\prod_{i=1}^{n}k_i}} (\frac{1}{4})^{\frac{\alpha_1}{k_1}+\cdots+\frac{\prod_{i=1}^{n}\alpha_i}{\prod_{i=1}^{n}k_i}}$$

be a positive constant. Since $f_i(t,0) = 0$ for $i = 2,\dots,n$, there exists another constant r_2:

$$0 < r_2 < \min\left\{r_1, L_1^{\frac{\prod_{i=1}^{n}k_i}{\prod_{i=1}^{n}k_i - \prod_{i=1}^{n}\alpha_i}}\right\}$$

such that

$$f_i(t, v_{i+1}(t)) \leq r_1^{k_i}, \quad 0 \leq v_{i+1} \leq r_2, \tag{3.3}$$

for any $t \in [0,1]$ and $i = 2,\dots,n$. For $v_1 \in K \cap \partial\Omega_{r_2}$, it follows from Lemma 3.2 and (3.3) that

$$v_i(t) = T_i(v_{i+1})(t) < r_1, \quad i = 2,\dots,n$$

which shows that for any $v_1 \in K \cap \partial\Omega_{r_2}$, we have $v_i \in (0, r_1)$, for all $i = 1,\dots,n$. Then by Lemma 3.1 and (3.2), we get

$$T_i(v_{i+1})(\frac{1}{4}) \geq \Gamma_i (\underline{f}_i^0 - \varepsilon_1)^{\frac{1}{k_i}} (\frac{1}{4})^{\frac{\alpha_i}{k_i}} ||v_{i+1}||^{\frac{\alpha_i}{k_i}}, \quad i = 1,\dots,n.$$

This suggests that for any $v_1 \in K \cap \partial\Omega_{r_2}$, we have

$$\begin{aligned}
||Tv_1|| = &\sup_{t\in[0,1]} |T_1T_2\cdots T_n(v_1)(t)| \\
\geq &T_1T_2\cdots T_n(v_1)(\frac{1}{4}) \\
\geq &\Gamma_1(\underline{f}_1^0 - \varepsilon_1)^{\frac{1}{k_1}} (\frac{1}{4})^{\frac{\alpha_1}{k_1}} ||T_2(v_3)||^{\frac{\alpha_1}{k_1}} \\
\geq &\Gamma_1(\underline{f}_1^0 - \varepsilon_1)^{\frac{1}{k_1}} (\frac{1}{4})^{\frac{\alpha_1}{k_1}} |T_2(v_3)(\frac{1}{4})|^{\frac{\alpha_1}{k_1}} \\
\geq &\Gamma_1\Gamma_2^{\frac{\alpha_1}{k_1}} (\underline{f}_1^0 - \varepsilon_1)^{\frac{1}{k_1}} (\underline{f}_2^0 - \varepsilon_1)^{\frac{\alpha_1}{k_1k_2}} (\frac{1}{4})^{\frac{\alpha_1}{k_1}+\frac{\alpha_2\alpha_2}{k_1k_2}} ||T_3(v_4)||^{\frac{\alpha_1\alpha_2}{k_1k_2}} \\
&\vdots \\
\geq &L_1||v_1||^{\frac{\prod_{i=1}^{n}\alpha_i}{\prod_{i=1}^{n}k_i}}.
\end{aligned}$$

Notice that $||v_1|| = r_2 < L_1^{\frac{\prod_{i=1}^{n}k_i}{\prod_{i=1}^{n}k_i - \prod_{i=1}^{n}\alpha_i}}$ and $\prod_{i=1}^{n}\alpha_i < \prod_{i=1}^{n}k_i$, then

$$\frac{L_1||v_1||^{\frac{\prod_{i=1}^{n}\alpha_i}{\prod_{i=1}^{n}k_i}}}{||v_1||} = \frac{L_1}{||v_1||^{\frac{\prod_{i=1}^{n}k_i - \prod_{i=1}^{n}\alpha_i}{\prod_{i=1}^{n}k_i}}} > 1,$$

which implies that

$$||Tv_1|| > ||v_1||, \quad v_1 \in K \cap \partial\Omega_{r_2}. \tag{3.4}$$

On the other hand, it can be obtained from $\overline{f}_i^{\infty} \in (0, +\infty)$ that for any given $\varepsilon_2 > 0$, there exists a constant $R_1 > 1$ such that for any $t \in [0, 1]$ and $i = 1, \ldots, n$,

$$f_i\,(t, v_{i+1}(t)) \leq (\overline{f}_i^{\infty} + \varepsilon_2) v_{i+1}^{\beta_i}, \quad v_{i+1} \geq R_1. \tag{3.5}$$

Furthermore, by the continuity of f_i $(i = 1, \ldots, n)$, there exist constants $M_i(R_1) > 0$ $(i = 1, \ldots, n)$ such that for any $(t, v_{i+1}(t)) \in [0, 1] \times [0, R_1]$,

$$f_i\,(t, v_{i+1}(t)) \leq M_i(R_1), \quad i = 1, \ldots, n. \tag{3.6}$$

Combining (3.5) with (3.6), we have for any $(t, v_{i+1}(t)) \in [0, 1] \times [0, +\infty)$,

$$f_i\,(t, v_{i+1}(t)) \leq M_i(R_1) + (\overline{f}_i^{\infty} + \varepsilon_2) v_{i+1}^{\beta_i}, \quad i = 1, \ldots, n. \tag{3.7}$$

Then from the Lemma 3.2 and (3.7), we have for any $t \in [0, 1]$,

$$T_i(v_{i+1})(t) \leq \left[M_i(R_1) + (\overline{f}_i^{\infty} + \varepsilon_2)||v_{i+1}||^{\beta_i}\right]^{\frac{1}{k_i}}, \quad i = 1, \ldots, n.$$

Let

$$\begin{aligned}
H &:= M_1(R_1)^{\frac{1}{k_1}} + \cdots + (\overline{f}_1^{\infty} + \varepsilon_2)^{\frac{1}{k_1}} \cdots (\overline{f}_{n-1}^{\infty} + \varepsilon_2)^{\frac{\prod_{i=1}^{n-2}\beta_i}{\prod_{i=1}^{n-1}k_i}} M_n(R_1)^{\frac{\prod_{i=1}^{n-1}\beta_i}{\prod_{i=1}^{n}k_i}}, \\
L_2 &:= (\overline{f}_1^{\infty} + \varepsilon_2)^{\frac{1}{k_1}} \cdots (\overline{f}_n^{\infty} + \varepsilon_2)^{\frac{\prod_{i=1}^{n-1}\beta_i}{\prod_{i=1}^{n}k_i}}.
\end{aligned}$$

Thus, there exists a large constant R_2:

$$R_2 > \max\left\{R_1, 2H, (2L_2)^{\frac{\prod_{i=1}^{n}k_i}{\prod_{i=1}^{n}k_i - \prod_{i=1}^{n}\beta_i}}\right\}$$

such that for any $v_1 \in K \cap \partial\Omega_{R_2}$ and $t \in [0, 1]$,

$$\begin{aligned}
Tv_1(t) =& T_1 T_2 \cdots T_n(v_1)(t) \\
\leq& \left[M_1(R_1) + (\overline{f}_1^{\infty} + \varepsilon_2)||T_2(v_3)||^{\beta_1}\right]^{\frac{1}{k_1}} \\
\leq& M_1(R_1)^{\frac{1}{k_1}} + \left[(\overline{f}_1^{\infty} + \varepsilon_2)||T_2(v_3)||^{\beta_1}\right]^{\frac{1}{k_1}} \\
\leq& M_1(R_1)^{\frac{1}{k_1}} + (\overline{f}_1^{\infty} + \varepsilon_2)^{\frac{1}{k_1}} M_2(R_1)^{\frac{\beta_1}{k_1 k_2}} + (\overline{f}_1^{\infty} + \varepsilon_2)^{\frac{1}{k_1}} (\overline{f}_2^{\infty} + \varepsilon_2)^{\frac{\beta_1}{k_1 k_2}} ||v_3||^{\frac{\beta_1\beta_2}{k_1 k_2}} \\
&\vdots \\
\leq& H + L_2||v_1||^{\frac{\prod_{i=1}^{n}\beta_i}{\prod_{i=1}^{n}k_i}}.
\end{aligned}$$

Since $\prod_{i=1}^{n}\beta_i < \prod_{i=1}^{n}k_i$, we get that for any $v_1 \in K \cap \partial\Omega_{R_2}$,

$$\frac{H + L_2||v_1||^{\frac{\prod_{i=1}^n \beta_i}{\prod_{i=1}^n k_i}}}{||v_1||} = \frac{H}{||v_1||} + \frac{L_2}{||v_1||^{\frac{\prod_{i=1}^n k_i - \prod_{i=1}^n \beta_i}{\prod_{i=1}^n k_i}}} < 1,$$

which implies that

$$||Tv_1|| < ||v_1||, \quad v_1 \in K \cap \partial\Omega_{R_2}. \tag{3.8}$$

Therefore, combining with (3.4) and (3.8), it follows from Theorem 2.1 that T has at least one fixed point in $K \cap (\overline{\Omega}_{R_2} \setminus \Omega_{r_2})$.

(b). By the assumption of $\overline{f}_i^0 \in (0, +\infty)$, for any given $\eta_1 > 0$, there exists a positive constant $r_3 < 1$ such that for any $t \in [0, 1]$ and $i = 1, \ldots, n$,

$$f_i(t, v_{i+1}(t)) \leq (\overline{f}_i^0 + \eta_1) v_{i+1}^{\alpha_i}, \quad v_{i+1} \in [0, r_3]. \tag{3.9}$$

Let

$$L_3 := (\overline{f}_1^0 + \eta_1)^{\frac{1}{k_1}} (\overline{f}_2^0 + \eta_1)^{\frac{\alpha_1}{k_1 k_2}} \cdots (\overline{f}_n^0 + \eta_1)^{\frac{\prod_{i=1}^{n-1} \alpha_i}{\prod_{i=1}^n k_i}}.$$

Since $\overline{f}_i^0 \in (0, +\infty)$, there exists another constant r_4:

$$0 < r_4 < \min\left\{ r_3, L_3^{\frac{\prod_{i=1}^n k_i}{\prod_{i=1}^n k_i - \prod_{i=1}^n \alpha_i}} \right\}$$

such that for $i = 2, \ldots, n$,

$$f_i(t, v_{i+1}(t)) \leq r_3^{k_i}, \quad (t, v_{i+1}(t)) \in [0, 1] \times [0, r_4]. \tag{3.10}$$

Then for any $v_1 \in K \cap \partial\Omega_{r_4}$, it follows from Lemma 3.2 and (3.10) that

$$v_i(t) = T_i(v_{i+1})(t) \leq r_3, \quad (t, v_{i+1}(t)) \in [0, 1] \times [0, r_4], \quad i = 2, \ldots, n.$$

Thus, by Lemma 3.2 and (3.9), we get

$$T_i(v_{i+1})(t) \leq \left[(\overline{f}_i^0 + \eta_1) ||v_{i+1}||^{\alpha_i} \right]^{\frac{1}{k_i}}, \quad i = 1, \ldots, n,$$

for any $t \in [0, 1]$. For $v_1 \in K \cap \partial\Omega_{r_4}$, we have

$$\begin{aligned}
||Tv_1|| &= \sup_{t \in [0,1]} |T_1 T_2 \cdots T_n(v_1)(t)| \\
&\leq \left[(\overline{f}_1^0 + \eta_1) ||T_2(v_3)||^{\alpha_1} \right]^{\frac{1}{k_1}} \\
&\leq (\overline{f}_1^0 + \eta_1)^{\frac{1}{k_1}} \left[(\overline{f}_2^0 + \eta_1) ||T_3(v_4)||^{\alpha_2} \right]^{\frac{\alpha_1}{k_1 k_2}} \\
&\leq (\overline{f}_1^0 + \eta_1)^{\frac{1}{k_1}} (\overline{f}_2^0 + \eta_1)^{\frac{\alpha_1}{k_1 k_2}} \left[(\overline{f}_3^0 + \eta_1) ||T_4(v_5)||^{\alpha_3} \right]^{\frac{\alpha_1 \alpha_2}{k_1 k_2 k_3}} \\
&\quad \vdots \\
&\leq L_3 ||v_1||^{\frac{\prod_{i=1}^n \alpha_i}{\prod_{i=1}^n k_i}}.
\end{aligned}$$

Recalling that $\prod_{i=1}^n \alpha_i > \prod_{i=1}^n k_i$, then

$$\frac{L_3||v_1||^{\frac{\prod_{i=1}^n \alpha_i}{\prod_{i=1}^n k_i}}}{||v_1||} = \frac{L_3}{||v_1||^{\frac{\prod_{i=1}^n k_i - \prod_{i=1}^n \alpha_i}{\prod_{i=1}^n k_i}}} < 1,$$

which implies that

$$||Tv_1|| < ||v_1||, \quad v_1 \in K \cap \partial\Omega_{r_4}. \tag{3.11}$$

On the other hand, it follows from $\underline{f}_i^\infty \in (0, +\infty)$ that for any given $\eta_2 \in (0, \min\{\underline{f}_i^\infty, i = 1, \dots, n\})$, there exists a constant $R_3 > 1$ such that

$$f_i\left(t, v_{i+1}(t)\right) \geq (\underline{f}_i^\infty - \eta_2) v_{i+1}^{\beta_i}, \quad v_{i+1} \geq R_3, \tag{3.12}$$

for any $t \in [0, 1]$ and $i = 1, \dots, n$. Let

$$L_4 := \Gamma_1 \cdots \Gamma_n^{\frac{\prod_{i=1}^{n-1} \beta_i}{\prod_{i=1}^{n-1} k_i}} (\underline{f}_1^\infty - \eta_2)^{\frac{1}{k_1}} \cdots (\underline{f}_n^\infty - \eta_2)^{\frac{\prod_{i=1}^{n-1} \beta_i}{\prod_{i=1}^n k_i}} (\frac{1}{4})^{\frac{\beta_1}{k_1} + \cdots + \frac{\prod_{i=1}^n \beta_i}{\prod_{i=1}^n k_i}}.$$

There exists another constant R_4:

$$R_4 > \left\{ 4R_3, L_4^{\frac{\prod_{i=1}^n k_i}{\prod_{i=1}^n k_i - \prod_{i=1}^n \beta_i}}, L_5 \right\} \tag{3.13}$$

such that for any $v_1 \in K \cap \partial\Omega_{R_4}$, we have

$$\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} v_1(t) \geq \frac{1}{4}||v_1|| = \frac{1}{4} R_4 > R_3, \tag{3.14}$$

where

$$L_5 := \max_{l \in \{2, \dots, n\}} \left(\frac{4R_3}{\Gamma_l \cdots \Gamma_n^{\frac{\prod_{i=l}^{n-1} \beta_i}{\prod_{i=l}^{n-1} k_i}} (\underline{f}_l^\infty - \eta_2)^{\frac{1}{k_l}} \cdots (\underline{f}_n^\infty - \eta_2)^{\frac{\prod_{i=l}^{n-1} \beta_i}{\prod_{i=l}^n k_i}} (\frac{1}{4})^{\frac{\beta_l}{k_l} + \cdots + \frac{\prod_{i=l}^n \beta_l}{\prod_{i=l}^n k_l}}} \right)^{\frac{\prod_{i=l}^n k_i}{\prod_{i=l}^n \beta_i}}.$$

Here in L_5, when $l = n$ we set $\frac{\prod_{i=l}^{n-1} \beta_i}{\prod_{i=l}^{n-1} k_i} = 1$, so that the terms $\frac{\prod_{i=l}^{n-1} \beta_i}{\prod_{i=l}^n k_i}$ make sense for all $i = 2, \dots, n$. Combining (3.12) and (3.13), it follows from Lemma 3.1 that for any $v_1 \in K \cap \partial\Omega_{R_4}$,

$$\begin{aligned}
&||v_n|| \geq v_n(\frac{1}{4}) = T_n(v_1)(\frac{1}{4}) \geq \Gamma_n(\underline{f}_n^\infty - \eta_2)^{\frac{1}{k_n}} (\frac{1}{4})^{\frac{\beta_n}{k_n}} ||v_1||^{\frac{\beta_n}{k_n}} > 4R_3, \\
&||v_{n-1}|| \geq v_{n-1}(\frac{1}{4}) = T_{n-1}(v_n)(\frac{1}{4}) \geq \Gamma_{n-1}(\underline{f}_{n-1}^\infty - \eta_2)^{\frac{1}{k_{n-1}}} (\frac{1}{4})^{\frac{\beta_{n-1}}{k_{n-1}}} ||v_n||^{\frac{\beta_{n-1}}{k_{n-1}}} > 4R_3, \\
&\quad \vdots \\
&||v_2|| \geq v_2(\frac{1}{4}) = T_2(v_3)(\frac{1}{4}) \geq \Gamma_2(\underline{f}_2^\infty - \eta_2)^{\frac{1}{k_2}} (\frac{1}{4})^{\frac{\beta_2}{k_2}} ||v_3||^{\frac{\beta_2}{k_2}} > 4R_3.
\end{aligned} \tag{3.15}$$

From (3.14) and (3.15), we get that for any $v_1 \in K \cap \partial\Omega_{R_4}$,

$$\min_{\frac{1}{4}\le t\le\frac{3}{4}} v_i(t) \ge \frac{1}{4}||v_i|| \ge R_3, \quad i=1,\dots,n.$$

Then by Lemma 3.1, we deduce that

$$\begin{aligned}
Tv_1(\frac{1}{4}) =& T_1T_2\cdots T_n(v_1)(\frac{1}{4}) \\
\ge& \Gamma_1(\underline{f}_1^\infty - \eta_2)^{\frac{1}{k_1}}(\frac{1}{4})^{\frac{\beta_1}{k_1}}||T_2(v_3)||^{\frac{\beta_1}{k_1}} \\
\ge& \Gamma_1(\underline{f}_1^\infty - \eta_2)^{\frac{1}{k_1}}(\frac{1}{4})^{\frac{\beta_1}{k_1}}\left(\Gamma_2(\underline{f}_2^\infty - \eta_2)^{\frac{1}{k_2}}(\frac{1}{4})^{\frac{\beta_2}{k_2}}||T_3(v_4)||^{\frac{\beta_2}{k_2}}\right)^{\frac{\beta_1}{k_1}} \\
&\vdots \\
\ge& L_4||v_1||^{\frac{\prod_{i=1}^n \beta_i}{\prod_{i=1}^n k_i}}.
\end{aligned}$$

Since $\prod_{i=1}^n \beta_i > \prod_{i=1}^n k_i$, it follows from

$$\frac{L_4||v_1||^{\frac{\prod_{i=1}^n \beta_i}{\prod_{i=1}^n k_i}}}{||v_1||} = \frac{L_4}{||v_1||^{\frac{\prod_{i=1}^n k_i - \prod_{i=1}^n \beta_i}{\prod_{i=1}^n k_i}}} > 1$$

that

$$||Tv_1|| > ||v_1||, \quad v_1 \in K\cap\partial\Omega_{R_4}. \tag{3.16}$$

Therefore, from Theorem 2.1 combining (3.11) and (3.16), we obtain that T has at least one fixed point in $K\cap\left(\overline{\Omega}_{R_4}\setminus\Omega_{r_4}\right)$. □

3.2. Multiplicity

In Section 3.1, applying the fixed-point theorem in Theorem 2.1, we achieve the existence result in a cone with different combinations of asymptotic growth condition and relations of α_i, β_i and k_i. In order to obtain the multiplicity of nontrivial radial convex solutions of (1.1), we recombine the conditions in Theorem 1.1 and find two kinds of "intermediate state" as in (3.17) and (3.19).

Proof of Theorem 1.2. (c). As we assumed, for any $(t, v_1(t)) \in [0,1]\times[0,\frac{r_0}{4}]$, we have

$$||v_1|| \le 4\min_{\frac{1}{4}\le t\le\frac{3}{4}} v_1(t) \le r_0.$$

Then for $v_1 \in K\cap\partial\Omega_{r_0}$, by the definition of G_n, we have

$$\begin{aligned}
v_n(t) = T_n(v_1)(t) \le& \int_0^1\left(\frac{k_n}{\tau^{N-k_n}}\int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_n-1}} f_n(s, v_1(s))\,\mathrm{d}s\right)^{\frac{1}{k_n}}\mathrm{d}\tau \\
\le& \int_0^1\left(\frac{k_n}{\tau^{N-k_n}}\int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_n-1}} G_n\right)^{\frac{1}{k_n}}\mathrm{d}\tau \\
=& \frac{1}{2}\left(\frac{k_nG_n}{NC_{N-1}^{k_n-1}}\right)^{\frac{1}{k_n}}
\end{aligned}$$

$$<G_n^{\frac{1}{k_n}},$$

for any $t \in [0,1]$. Similarly, for $v_1 \in K \cap \partial\Omega_{r_0}$, we have $v_i(t) \le G_i^{\frac{1}{k_i}}$ $(i = 2, \dots, n-1)$, $\forall t \in [0,1]$. Therefore, by the definition of G_1, we have

$$\begin{aligned} Tv_1(t) =& T_1 T_2 \cdots T_n(v_1)(t) \\ =& \int_t^1 \left(\frac{k_1}{\tau^{N-k_1}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}} f_1\left(s, v_2(s)\right) \mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau \\ \le& \int_0^1 \left(\frac{k_1}{\tau^{N-k_1}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}} G_1 \, \mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau \\ =& \frac{1}{2} \left(\frac{k_1 G_1}{N C_{N-1}^{k_1-1}} \right)^{\frac{1}{k_1}} \\ <& G_1^{\frac{1}{k_1}}, \quad \forall t \in [0,1], \end{aligned}$$

where the fact $\frac{1}{2}(\frac{k_1}{NC_{N-1}^{k_1-1}})^{\frac{1}{k_1}} = \frac{1}{2C_N^{k_1}} < 1$ is used in the last inequality. For $v_1 \in K \cap \partial\Omega_{r_0}$, we have $||v_1|| = r_0$. Thanks to $G_1^{\frac{1}{k_1}} < r_0$, we have

$$||Tv_1|| < ||v_1||, \quad v_1 \in K \cap \partial\Omega_{r_0}. \tag{3.17}$$

Since $\prod_{i=1}^n \alpha_i < \prod_{i=1}^n k_i$, $\prod_{i=1}^n \beta_i > \prod_{i=1}^n k_i$, it follows from Theorem 1.1 that there exist sufficient small constant $r_2 \in (0, r_0)$ and sufficient large constant $R_4 > r_0$ such that

$$||Tv_1|| \ge ||v_1||, \quad v_1 \in K \cap \partial\Omega_{r_2} \quad \text{and} \quad ||Tv_1|| \ge ||v_1||, \quad v_1 \in K \cap \partial\Omega_{R_4}. \tag{3.18}$$

Due to (3.17), we have $Tv_1 \ne v_1$ for any $v_1 \in K \cap \partial\Omega_{r_0}$, which shows that T has no fixed point in $K \cap \partial\Omega_{r_0}$. Otherwise, if there exists a fixed point $v_1 \in K \cap \partial\Omega_{r_0}$ such that $Tv_1 = v_1$, then we have $||Tv_1|| = ||v_1||$, which contradicts with (3.17). Combining (3.17) and (3.18), it follows from Theorem 2.1 that there exist at least two fixed points of T in $K \cap \left(\overline{\Omega}_{r_0} \setminus \Omega_{r_2}\right)$ and $K \cap \left(\overline{\Omega}_{R_4} \setminus \Omega_{r_0}\right)$ respectively.

(d). For $v_1 \in K \cap \partial\Omega_{R_0}$, by the definition of $\tilde{G}_n$, we have

$$\begin{aligned} v_n(t) = T_n(v_1)(t) \le& \int_0^1 \left(\frac{k_n}{\tau^{N-k_n}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_n-1}} f_n\left(s, v_1(s)\right) \mathrm{d}s \right)^{\frac{1}{k_n}} \mathrm{d}\tau \\ \le& \int_0^1 \left(\frac{k_n}{\tau^{N-k_n}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_n-1}} \tilde{G}_n \, \mathrm{d}s \right)^{\frac{1}{k_n}} \mathrm{d}\tau \\ =& \frac{1}{2} \left(\frac{k_n \tilde{G}_n}{N C_{N-1}^{k_n-1}} \right)^{\frac{1}{k_n}} \\ <& \tilde{G}_n^{\frac{1}{k_n}}, \quad \forall t \in [0,1]. \end{aligned}$$

Besides, by the definition of E_n, we get that for any $v_1 \in K \cap \partial\Omega_{R_0}$,

$$
\begin{aligned}
v_n(\frac{1}{4}) = T_n(v_1)(\frac{1}{4}) = & \int_{\frac{1}{4}}^{1} \left(\frac{k_n}{\tau^{N-k_n}} \int_0^{\tau} \frac{s^{N-1}}{C_{N-1}^{k_n-1}} f_n\left(s, v_1(s)\right) \mathrm{d}s \right)^{\frac{1}{k_n}} \mathrm{d}\tau \\
\geq & \int_{\frac{1}{4}}^{\frac{3}{4}} \left(\frac{k_n}{\tau^{N-k_n}} \int_{\frac{1}{4}}^{\tau} \frac{s^{N-1}}{C_{N-1}^{k_n-1}} E_n \, \mathrm{d}s \right)^{\frac{1}{k_n}} \mathrm{d}\tau \\
= & \Gamma_n E_n^{\frac{1}{k_n}},
\end{aligned}
$$

then

$$
\min_{\frac{1}{4} \leq t \leq \frac{3}{4}} v_n(t) \geq \frac{1}{4} ||v_n|| \geq \frac{1}{4} v_n(\frac{1}{4}) \geq \frac{1}{4} \Gamma_n E_n^{\frac{1}{k_n}}.
$$

Thus for any $t \in [\frac{1}{4}, \frac{3}{4}]$, we have $\frac{1}{4}\Gamma_n E_n^{\frac{1}{k_n}} \leq v_n(t) \leq \tilde{G}_n^{\frac{1}{k_n}}$. Repeating the above steps, we have $\frac{1}{4}\Gamma_i E_i^{\frac{1}{k_i}} \leq v_i(t) \leq \tilde{G}_i^{\frac{1}{k_i}}$, for any $t \in [\frac{1}{4}, \frac{3}{4}]$, $(i = 2, \dots, n)$. It follows from the assumption of E_1 that for any $v_1 \in K \cap \partial\Omega_{R_0}$,

$$
\begin{aligned}
Tv_1(\frac{1}{4}) = & T_1 T_2 \cdots T_n(v_1)(\frac{1}{4}) \\
= & \int_{\frac{1}{4}}^{1} \left(\frac{k_1}{\tau^{N-k_1}} \int_0^{\tau} \frac{s^{N-1}}{C_{N-1}^{k_1-1}} f_1\left(s, T_2(v_3)(s)\right) \mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau \\
\geq & \int_{\frac{1}{4}}^{\frac{3}{4}} \left(\frac{k_1}{\tau^{N-k_1}} \int_0^{\tau} \frac{s^{N-1}}{C_{N-1}^{k_1-1}} E_1 \, \mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau \\
\geq & \Gamma_1 E_1^{\frac{1}{k_1}} > R_0,
\end{aligned}
$$

which deduce that

$$
||Tv_1|| > ||v_1||, \quad v_1 \in K \cap \partial\Omega_{R_0}. \tag{3.19}
$$

Moreover, since $\prod_{i=1}^{n} \alpha_i > \prod_{i=1}^{n} k_i, \prod_{i=1}^{n} \beta_i < \prod_{i=1}^{n} k_i$, we know from Theorem 1.1 that there exist sufficient small constant $r_4 \in (0, R_0)$ and sufficient large constant $R_2 > R_0$ such that

$$
||Tv_1|| \leq ||v_1||, \quad v_1 \in K \cap \partial\Omega_{R_2} \quad \text{and} \quad ||Tv_1|| \leq ||v_1||, \quad v_1 \in K \cap \partial\Omega_{r_4}. \tag{3.20}
$$

Due to (3.19), we have $Tv_1 \neq v_1$ for any $v_1 \in K \cap \partial\Omega_{R_0}$, which shows that fixed point of T can not exist on $K \cap \partial\Omega_{R_0}$. Otherwise, if there exists a fixed point $v_1 \in K \cap \partial\Omega_{R_0}$ such that $Tv_1 = v_1$, then we have $||Tv_1|| = ||v_1||$, which contradict with (3.19). Thus, basing on (3.19) and (3.20), it follows from Theorem 2.1 that there exist at least two fixed points of T in $K \cap \left(\overline{\Omega}_{R_0} \setminus \Omega_{r_4}\right)$ and $K \cap \left(\overline{\Omega}_{R_2} \setminus \Omega_{R_0}\right)$ respectively.

Remark 3.1. If we do not prove $Tv_1 \neq v_1$ for any $v_1 \in K \cap \partial\Omega_{r_0}$ in the proof of (c), by taking $0 < r_2 < r_0 < r' < R_4$, there also exist at least two fixed points of T: $v_1 \in K \cap \left(\overline{\Omega}_{r_0} \setminus \Omega_{r_2}\right)$ and $v_2 \in K \cap \left(\overline{\Omega}_{R_4} \setminus \Omega_{r'}\right)$. Similarly, if we do not prove $Tv_1 \neq v_1$ for any $v_1 \in K \cap \partial\Omega_{R_0}$, we can alternatively take $0 < r_4 < R' < R_0 < R_2$ such that $v_1 \in K \cap \left(\overline{\Omega}_{R'} \setminus \Omega_{r_4}\right)$ and $v_2 \in K \cap \left(\overline{\Omega}_{R_2} \setminus \Omega_{R_0}\right)$. □

4. Uniqueness and nonexistence

In this section, we study the uniqueness and nonexistence results for a special case of the system (1.1) where the nonlinearities are power functions with respect to u.

4.1. Uniqueness

In [12], the authors gave a proof of uniqueness and approximation by iterations of the solution to a general Dirichlet problem of Monge-Ampère equation. Here, we will use their method to prove Theorem 1.3. We first introduce the definition of u_0-sublinear operator and a corresponding existence result.

Definition 4.1. Let P be a cone from a Banach space Y. With some $u_0 \in P$ positive, $A : P \to P$ is called u_0-sublinear if

(i) for any $x > 0$, there exist positive constants θ_1 and θ_2 which depend on x, such that

$$\theta_1 u_0 \le Ax \le \theta_2 u_0;$$

(ii) for any $\theta_1 u_0 \le x \le \theta_2 u_0$ and $0 < \xi < 1$, there always exists some $\eta > 0$ such that

$$A(\xi x) \ge (1 + \eta)\xi Ax.$$

Lemma 4.1. *An increasing and u_0-sublinear operator A can have at most one positive fixed-point.*

The proof can be found in [12], we omit it here.

Proof of Theorem 1.3. Let $X := C[0, 1]$ and cone $P := \{v \in X : v(t) \ge 0, t \in [0, 1]\}$. It is easy to see that $K \subset P$, where K is defined in (2.3). We define T_i $(i = 1, \dots, n)$ and composite operator $T = T_1T_2 \cdots T_n$ as in Section 2. The existence of nontrivial radial convex solutions to system (1.2) is obtained in Theorem 1.1 and therefore investigate T has at most one fixed-point in K is enough. By Lemma 4.1, it suffices to verify that $T : K \to K$ is an increasing and u_0-sublinear for some u_0 positive in $C[0, 1]$. By the definitions of T_i, it is clear that each T_i $(i = 1, \dots, n)$ is a increasing operator, so is the composite operator T, then we just need to prove that T satisfies the Definition 4.1.

Firstly, we show that T satisfies the Definition 4.1 (i).

$$\begin{aligned}
Tv_1(t) = & \int_t^1 \left(\frac{k_1}{\tau^{N-k_1}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}} v_2^{\gamma_1}(s)\, \mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau \\
\le & ||v_2||^{\frac{\gamma_1}{k_1}} \int_t^1 \left(\frac{k_1}{\tau^{N-k_1}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}} \mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau \\
\le & ||T_2(v_3)||^{\frac{\gamma_1}{k_1}} \left(\frac{k_1}{NC_{N-1}^{k_1-1}} \right)^{\frac{1}{k_1}} \int_t^1 \tau\, \mathrm{d}\tau \\
\le & \left[\int_0^1 \left(\frac{k_2}{\tau^{N-k_2}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_2-1}} v_3^{\gamma_2}(s)\, \mathrm{d}s \right)^{\frac{1}{k_2}} \mathrm{d}\tau \right]^{\frac{\gamma_1}{k_1}} (1-t)
\end{aligned}$$

$$
\begin{aligned}
&\leq ||v_3||^{\frac{\gamma_1\gamma_2}{k_1k_2}}(1-t)\\
&\quad\vdots\\
&\leq ||v_1||^{\frac{\Pi_{i=1}^{n}\gamma_i}{\Pi_{i=1}^{n}k_i}}(1-t).
\end{aligned}
$$

Here, let $u_0 = 1-t$, $t \in [0,1)$ and $\theta_2 = ||v_1||^{\frac{\Pi_{i=1}^{n}\gamma_i}{\Pi_{i=1}^{n}k_i}}$, then $Tv_1(t) \leq \theta_2 u_0$. Set

$$
\Gamma := \left(\frac{9}{32}\right)^{\frac{\gamma_1}{k_1}+\cdots+\frac{\Pi_{i=1}^{n-1}\gamma_i}{\Pi_{i=1}^{n-1}k_i}} \left(\frac{k_1}{4^{\gamma_1}NC_{N-1}^{k_1-1}}\right)^{\frac{1}{k_1}} \cdots \left(\frac{k_n}{4^{\gamma_n}NC_{N-1}^{k_n-1}}\right)^{\frac{\Pi_{i=1}^{n-1}\gamma_i}{\Pi_{i=1}^{n}k_i}} ||v_1||^{\frac{\Pi_{i=1}^{n}\gamma_i}{\Pi_{i=1}^{n}k_i}}
$$

be a positive constant which depends only on $||v_1||$. Next, let $c \in (\frac{1}{4}, \frac{3}{4})$ be a fixed number. Notice that $Tv_1(t)$ is decreasing with t, we have for $t \in [0, c)$,

$$
\begin{aligned}
Tv_1(t) \geq &Tv_1(c) = \int_c^1 \left(\frac{k_1}{\tau^{N-k_1}}\int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}} v_2^{\gamma_1}(s)\,\mathrm{d}s\right)^{\frac{1}{k_1}} \mathrm{d}\tau\\
&\geq \int_c^{\frac{3}{4}} \left(\frac{k_1}{\tau^{N-k_1}}\int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}} (\frac{1}{4}||v_2||)^{\gamma_1}\,\mathrm{d}s\right)^{\frac{1}{k_1}} \mathrm{d}\tau\\
&\geq \left(\frac{k_1}{4^{\gamma_1}NC_{N-1}^{k_1-1}}\right)^{\frac{1}{k_1}} \left(\frac{9}{32}-\frac{1}{2}c^2\right) ||T_2(v_3)||^{\frac{\gamma_1}{k_1}}\\
&\quad\vdots\\
&\geq \Gamma\left(\frac{9}{32}-\frac{1}{2}c^2\right)\\
&\geq \Gamma\left(\frac{9}{32}-\frac{1}{2}c^2\right)(1-t),
\end{aligned}
$$

where we use the fact $\min_{0\leq t\leq \frac{3}{4}} v(t) \geq \frac{1}{4}||v||$ in the above inequality which follows from (2.3) combining with the concavity of v_i $(i = 1, \ldots, n)$ and $v_i'(0) = 0$. For $t \in [c, 1)$, we let

$$
\zeta(\tau) := \left(\frac{1}{\tau^{N-k_1}}\int_0^\tau s^{N-1}(1-s)^{\gamma_1}\,\mathrm{d}s\right)^{\frac{1}{k_1}},\ \tau \in [c, 1].
$$

Notice that $\zeta(\tau) \in C[c, 1]$ and $\zeta(\tau) > 0,\ \tau \in [c, 1]$ is well-defined, then $\zeta([c, 1])$ is the image of a compact set and so is compact which shows that it is both closed and bounded. So $\zeta([c, 1])$ has a positive absolute minimum. Besides, by the concavity of $v_i(t)$ $(i = 1, \ldots, n)$ and $v_i'(0) = v_i(1) = 0$, we have

$$
v_i(t) \geq v_i(0)(1-t), \quad \forall t \in [0, 1]. \tag{4.1}
$$

Then we have for $t \in [c, 1)$,

$$
\begin{aligned}
Tv_1(t) &= \int_t^1 \left(\frac{k_1}{\tau^{N-k_1}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}} v_2^{\gamma_1}(s)\, \mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau \\
&\geq \int_t^1 \left(\frac{k_1}{\tau^{N-k_1}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}} [v_2(0)(1-s)]^{\gamma_1}\, \mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau \\
&= ||v_2||^{\frac{\gamma_1}{k_1}} \left(\frac{k_1}{C_{N-1}^{k_1-1}} \right)^{\frac{1}{k_1}} \int_t^1 \left(\frac{1}{\tau^{N-k_1}} \int_0^\tau s^{N-1}(1-s)^{\gamma_1}\, \mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau \\
&\geq ||T_2(v_3)||^{\frac{\gamma_1}{k_1}} \left(\frac{k_1}{C_{N-1}^{k_1-1}} \right)^{\frac{1}{k_1}} \min_{\tau \in [c,1]} \zeta(\tau) \int_t^1 \mathrm{d}\tau \\
&= \left[\int_0^1 \left(\frac{k_2}{\tau^{N-k_2}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_2-1}} v_3^{\gamma_2}(s)\, \mathrm{d}s \right)^{\frac{1}{k_2}} \mathrm{d}\tau \right]^{\frac{\gamma_1}{k_1}} \left(\frac{k_1}{C_{N-1}^{k_1-1}} \right)^{\frac{1}{k_1}} \min_{\tau \in [c,1]} \zeta(\tau)(1-t) \\
&\geq \left[\int_0^{\frac{3}{4}} \left(\frac{k_2}{\tau^{N-k_2}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_2-1}} (\frac{1}{4}||v_3||)^{\gamma_2}\, \mathrm{d}s \right)^{\frac{1}{k_2}} \mathrm{d}\tau \right]^{\frac{\gamma_1}{k_1}} \left(\frac{k_1}{C_{N-1}^{k_1-1}} \right)^{\frac{1}{k_1}} \min_{\tau \in [c,1]} \zeta(\tau)(1-t) \\
&= ||T_3(v_4)||^{\frac{\gamma_1\gamma_2}{k_1k_2}} \left(\frac{k_1}{C_{N-1}^{k_1-1}} \right)^{\frac{1}{k_1}} \left(\frac{k_2}{4^{\gamma_2} C_{N-1}^{k_2-1}} \right)^{\frac{\gamma_1}{k_1k_2}} \left(\frac{9}{32} \right)^{\frac{\gamma_1}{k_1}} \min_{\tau \in [c,1]} \zeta(\tau)(1-t) \\
&\quad \vdots \\
&\geq \Gamma 4^{\frac{\gamma_1}{k_1}} \min_{\tau \in [c,1]} \zeta(\tau)(1-t).
\end{aligned}
$$

Let $\theta_1 = \min\left\{\Gamma\left(\frac{9}{32} - \frac{1}{2}c^2\right), \Gamma 4^{\frac{\gamma_1}{k_1}} \min_{\tau\in[c,1]} \zeta(\tau)\right\}$, then we have $Tv_1(t) \geq \theta_1 u_0$.

To verify the Definition 4.1 (ii), we have for any $\theta_1 u_0 \leq v_1 \leq \theta_2 u_0$ and $\xi \in (0,1)$, $T_1(\xi v_2) = \xi^{\frac{\gamma_1}{k_1}} T_1(v_2)$, $T_2(\xi v_3) = \xi^{\frac{\gamma_2}{k_2}} T_2(v_3)$, $\dots$, $T_n(\xi v_1) = \xi^{\frac{\gamma_n}{k_n}} T_n(v_1)$. Notice that $\prod_{i=1}^n \gamma_i < \prod_{i=1}^n k_i$, then there exists $\eta > 0$ such that

$$
T(\xi v_1) = T_1 T_2 \cdots T_n(\xi v_1) = T_1 T_2 \cdots T_{n-1}(\xi^{\frac{\gamma_n}{k_n}} T_n(v_1)) = \cdots = \xi^{\frac{\prod_{i=1}^n \gamma_i}{\prod_{i=1}^n k_i}} Tv_1 \geq (1+\eta)\xi T v_1.
$$

Thus T is a u_0-sublinear operator and T has at most one fixed-point in K by Lemma 4.1 which shows that the system (1.2) has a unique nontrivial radial convex solution. □

Remark 4.1. Note that we also use the convexity of $u_i = -v_i$ $(i = 1, \dots, n)$ in this subsection, namely, the inequality (4.1).

4.2. Nonexistence

In the case of $\prod_{i=1}^n \gamma_i = \prod_{i=1}^n k_i$, we can get nonexistence result by contradiction.

Proof of Theorem 1.4. Suppose, to the contrary, that v_0 is a fixed-point of T in K, then $Tv_0 = v_0$. It follows immediately from the definition of T that v_0 is a concave function satisfying $v_0(1) = 0$ and $v_0(t) > 0, t \in [0,1)$.

On the other hand, for any $v_1 \in K$, we have

$$\begin{aligned}||T(v_1)|| &= \int_0^1 \left(\frac{k_1}{\tau^{N-k_1}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}} v_2^{\gamma_1}(s)\,\mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau \\ &\leq ||v_2||^{\frac{\gamma_1}{k_1}} \int_0^1 \left(\frac{k_1}{\tau^{N-k_1}} \int_0^\tau \frac{s^{N-1}}{C_{N-1}^{k_1-1}}\,\mathrm{d}s \right)^{\frac{1}{k_1}} \mathrm{d}\tau \\ &= \frac{1}{2} \left(\frac{k_1}{NC_{N-1}^{k_1-1}} \right)^{\frac{1}{k_1}} ||v_2||^{\frac{\gamma_1}{k_1}} \\ &< ||v_2||^{\frac{\gamma_1}{k_1}} \\ &\vdots \\ &< ||v_1||^{\frac{\prod_{i=1}^n \gamma_i}{\prod_{i=1}^n k_i}} = ||v_1||.\end{aligned}$$

Here, we also use the fact $\frac{1}{2}\left(\frac{k_1}{NC_{N-1}^{k_1-1}}\right)^{\frac{1}{k_1}} < 1$ for the same reason in (3.1). Thus if we take $v_1 = v_0$ in the above estimate, we have $||v_0||$ is strictly larger than $||Tv_0||$. This contradicts $Tv_0 = v_0$ and concludes the proof. $\square$

Remark 4.2. Due to the fact $\frac{1}{2}\left(\frac{k_1}{NC_{N-1}^{k_1-1}}\right)^{\frac{1}{k_1}} < 1$, we have a direct proof of the nonexistence theorem by reduction to absurdity without using the fixed-point theorem in Theorem 2.1. Therefore, we can obtain the nonexistence for $\boldsymbol{k}$-admissible solutions of system (1.2) in the assumption of $\prod_{i=1}^n \gamma_i = \prod_{i=1}^n k_i$, (not just for the convex solutions of system (1.2)).

5. Eigenvalue problem

In the previous section, we proved the nonexistence of nontrivial radial convex solution to the power-type system (1.2) in a unit ball. Then by imposing a suitable condition on positive parameters of eigenvalue problem (1.3), we can also get the existence of $\boldsymbol{k}$-admissible solution in a general strictly $(k-1)$-convex domain. In this section, our main tool is the generalized Krein-Rutman theorem in [13].

We first recall some basic concepts:

Let E be a Banach space, $M \subset E$ be a cone.

Definition 5.1. The cone M introduces a partial order in E by the relation

$$u \prec v \text{ if and only if } u - v \in M.$$

Definition 5.2. Define an operator $A : E \to E$. Then

(i) A is called positive if $A(M) \subset M$;
(ii) A is said to be homogeneous if it is positively homogeneous with degree 1;
(iii) A is monotone if it satisfies $x \prec y \Rightarrow A(x) \prec A(y)$;
(iv) A is called strong (relative to M), if for all $u, v \in Im(A) \cap M \setminus \{\theta\}$, there exist positive constants δ and γ that depend on u and v such that $u - \delta v \in M$, $v - \gamma u \in M$.

The following is the generalized Krein-Rutman theorem developed by Jacobsen in [13].

Lemma 5.1. *Let E contain a cone M, $A : E \to E$ be a completely continuous operator with $A|_M : M \to M$ homogeneous, monotone, and strong. Furthermore, assume that there exists a nonzero element ω, $A(\omega) \in Im(A) \cap M$. Then there exists a constant $\lambda_0 > 0$ with the following properties:*

(i) There exists $u \in M \setminus \{\theta\}$, with $u = \lambda_0 A(u)$;
(ii) If $v \in M \setminus \{\theta\}$ and $\lambda > 0$ such that $v = \lambda A(v)$, then $\lambda = \lambda_0$.

For the convenience of the reader, we also present the existence theorems in [20,25].

Lemma 5.2. *(see [20]) Let Ω be a uniformly $(k-1)$-convex domain in $\mathbb{R}^N$, $k = 2, \dots, N$, $\varphi \in C^0(\overline{\Omega})$ and $\psi \geq 0, \in L^p(\Omega)$, for $p > \frac{N}{2k}$. Then there exists a unique admissible weak solution $u \in C^0(\overline{\Omega})$ to the problem*

$$\begin{cases} S_k\left(D^2 u\right) = \psi, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega. \end{cases}$$

Lemma 5.3. *(see [25]) Assume that Ω is $(k-1)$-convex, φ, $\Omega \in C^{3,1}$, $f \in C^{1,1}(\overline{\Omega})$, and $f \geq f_0 > 0$. Then there is a unique k-admissible solution $u \in C^{3,\alpha}(\overline{\Omega})$ to the Dirichlet problem*

$$\begin{cases} S_k\left(D^2 u\right) = f(x), & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega. \end{cases}$$

Proof of Theorem 1.5. Let X be a Banach space $C(\overline{\Omega})$ equipped with the supremum norm. Define a cone $P := \{u \in X : u(x) \leq 0, \forall x \in \Omega\}$. Then by the Definition 5.1, we notice that the partial order induced by P implies that $u \prec v \Longleftrightarrow u(x) \leq v(x), \forall x \in \Omega$.

For $i = 1, \dots, n$, we define $\overline{T}_i : X \to X$, $\overline{T}_i(u_{i+1}) = u_i$, where u_i is the unique admissible weak solution of the problem

$$\begin{cases} S_{k_i}\left(D^2 u_i\right) = |u_{i+1}|^{\gamma_i}, & \text{in } \Omega, \\ u_i = 0, & \text{on } \partial\Omega. \end{cases} \tag{5.1}$$

Notice that we denote $u_1 := u_{n+1}$ here. It follows from Lemma 5.2 that the admissible weak solution $\overline{T}_i(u_{i+1}) \in C^0(\overline{\Omega})(i = 1, \dots, n)$. Define a composite operator $\overline{T} := \overline{T}_1\overline{T}_2 \cdots \overline{T}_n$, which is a completely continuous operator. Next, we verify that $\overline{T}$ satisfies the assumptions of Lemma 5.1.

Due to the k-convexity property of the admissible weak solution and the boundary data, we have $\overline{T}(X) \subseteq P$, which implies that $\overline{T}$ is positive and the operator $\overline{T}$ maps P into itself. For $t > 0$, we have

$$\overline{T}_1(tu_2) = t^{\frac{\gamma_1}{k_1}}\overline{T}_1(u_2), \quad \overline{T}_2(tu_3) = t^{\frac{\gamma_2}{k_2}}\overline{T}_2(u_3), \quad \cdots, \quad \overline{T}_n(tu_1) = t^{\frac{\gamma_n}{k_n}}\overline{T}_n(u_1).$$

Since the assumption $\prod_{i=1}^n \gamma_i = \prod_{i=1}^n k_i$, we deduce that

$$\overline{T}(tu_1) = t^{\frac{\prod_{i=1}^n \gamma_i}{\prod_{i=1}^n k_i}}\overline{T}(u_1) = t\overline{T}(u_1),$$

which implies that $\overline{T}$ is homogeneous. By comparison principle in Lemma 2.1 in [20] and the definition of $\overline{T}_i$, we get that $\overline{T}_i$ $(i = 1, \dots, n)$ are all monotone, so is $\overline{T}$. Finally, we just have to verify that $\overline{T}$ is strong, that is, for all $u, v \in Im(\overline{T}) \cap P \setminus \{\theta\}$, there exist $\delta > 0$ and $\gamma > 0$ such that $u - \delta v \leq 0$ in Ω and $v - \gamma u \leq 0$

in Ω. If $u \in Im(\overline{T}) \cap P \setminus \{\theta\}$, then there exists a function $v \in X \setminus \{\theta\}$ such that $u = \overline{T}v = \overline{T}_1\overline{T}_2 \cdots \overline{T}_n(v)$, where u is nonzero k-admissible and strictly negative in Ω satisfying

$$\begin{cases} S_{k_1}\left(D^2u\right) = |v|^{\gamma_1}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

It follows from Lemma 5.2 that $u = \overline{T}v \in C^0(\overline{\Omega})$. Notice that v is also the solution of (5.1) satisfying $v \in C^0(\overline{\Omega})$, we let v attains its minimum at $x_0 \in \overline{\Omega}$ and $G := \max |v|^{\gamma_1} = (-v(x_0))^{\gamma_1}$, then we have $0 < |v|^{\gamma_1} \leq G$ in Ω. Consider a function ω which satisfies

$$\begin{cases} S_{k_1}\left(D^2\omega\right) = G, & \text{in } \Omega, \\ \omega = 0, & \text{on } \partial\Omega. \end{cases}$$

Then, it follows from Lemma 5.3 that $\omega \in C^{2,\alpha}(\overline{\Omega})$. By comparison principle in Lemma 2.1 in [20], we have $\omega \leq u \leq 0$, in $\overline{\Omega}$ and $\omega = u = 0$, on $\partial\Omega$. Thus, for some small $t > 0$, we have

$$0 \leq \frac{u(x - t\nu) - u(x)}{-t} \leq \frac{\omega(x - t\nu) - \omega(x)}{-t}, \quad \text{for } x \in \partial\Omega,$$

where ν is the unit outer normal vector field on $\partial\Omega$. Take a limit in the last inequality, we have

$$\begin{aligned} 0 \leq \limsup_{t\to 0^+} \frac{u(x - t\nu) - u(x)}{-t} &\leq \limsup_{t\to 0^+} \frac{\omega(x - t\nu) - \omega(x)}{-t} \\ &= \frac{\partial\omega(x)}{\partial\nu}, \quad \text{for } x \in \partial\Omega. \end{aligned}$$

With the same argument, there also exists $\tilde{\omega} \in C^{2,\alpha}(\overline{\Omega})$ such that

$$\begin{aligned} 0 \leq \limsup_{t\to 0^+} \frac{v(x - t\nu) - v(x)}{-t} &\leq \limsup_{t\to 0^+} \frac{\tilde{\omega}(x - t\nu) - \tilde{\omega}(x)}{-t} \\ &= \frac{\partial\tilde{\omega}(x)}{\partial\nu}, \quad \text{for } x \in \partial\Omega, \end{aligned} \tag{5.2}$$

then by Hopf Lemma in [11], we have

$$\liminf_{t\to 0^+} \frac{u(x - t\nu) - u(x)}{-t} > 0, \quad \text{on } \partial\Omega. \tag{5.3}$$

By choosing a sufficiently small constant $\delta_1 > 0$, we have

$$\begin{aligned} &\liminf_{t\to 0^+} \frac{(u - \delta_1 v)(x - t\nu) - (u - \delta_1 v)(x)}{-t} \\ =&\liminf_{t\to 0^+} \frac{u(x - t\nu) - u(x)}{-t} - \delta_1 \limsup_{t\to 0^+} \frac{v(x - t\nu) - v(x)}{-t} \\ >&0, \quad \text{on } \partial\Omega, \end{aligned}$$

where (5.2) and (5.3) are used in the last inequality. Since u, v are the solutions of (5.1) which satisfy $u = v = 0$ on $\partial\Omega$. Then for $x \in \partial\Omega$, there exists a constant $t_0 > 0$ such that

$$(u - \delta_1 v)(x - t\nu) < 0, \quad \text{for } t < t_0. \tag{5.4}$$

Now, (5.4) implies that

$$u - \delta_1 v < 0, \quad \text{in } \Omega_{t_0} := \{x \in \Omega | \text{dist}(x, \partial\Omega) < t_0\},$$

where $dist(x, \partial\Omega)$ denotes the distance from x to $\partial\Omega$. For $x \in \Omega \setminus \Omega_{t_0}$, we set

$$\delta_2 := \inf_{x \in \Omega \setminus \Omega_{t_0}} \frac{u(x)}{v(x)} > 0.$$

Fixing a constant $\delta \leq \min\{\delta_1, \delta_2\}$, then we have

$$u - \delta v \leq u - \delta_1 v < 0, \quad \text{in } \Omega_{t_0} \quad \text{and} \quad u - \delta v \leq u - \delta_2 v \leq 0, \quad \text{in } \Omega \setminus \Omega_{t_0},$$

which implies

$$u - \delta v \leq 0, \quad \text{in } \Omega.$$

The same argument shows that there exists a constant γ such that $v - \gamma u \leq 0$ in Ω. Now we have shown that $\overline{T}$ is strong. Moreover, $\mathcal{N}(\overline{T}) := \{u \in P | \overline{T}(u) = 0\} = \{0\}$.

Combining this with Lemma 5.1 (i), we obtain that there exists $u_1^* \in P \setminus \{\theta\}$ and constant $\lambda_0 > 0$ such that $u_1^* = \lambda_0 \overline{T}(u_1^*) = \lambda_0 \overline{T}_1 \overline{T}_2 \cdots \overline{T}_n(u_1^*)$. Let $u_n^* = \overline{T}_n(u_1^*), \ldots, u_2^* = \overline{T}_2(u_3^*)$. Then $(u_1^*, u_2^*, \ldots, u_n^*)$ is a solution of the following system

$$\begin{cases} S_{k_1}\left(D^2(\frac{u_1}{\lambda_0})\right) = (-u_2)^{\gamma_1}, & \text{in } \Omega, \\ S_{k_2}\left(D^2 u_2\right) = (-u_3)^{\gamma_2}, & \text{in } \Omega, \\ \qquad \vdots \\ S_{k_{n-1}}\left(D^2 u_{n-1}\right) = (-u_n)^{\gamma_{n-1}}, & \text{in } \Omega, \\ S_{k_n}\left(D^2 u_n\right) = (-u_1)^{\gamma_n}, & \text{in } \Omega, \\ u_i = 0, i = 1, \ldots, n, & \text{on } \partial\Omega. \end{cases}$$

By Lemma 5.1 (ii), if there exist $u_0 \in P \setminus \{\theta\}$ and $\lambda_1 > 0$ such that $u_0 = \lambda_1 \overline{T}(u_0)$, then $\lambda_1 = \lambda_0$.

For this reason, the eigenvalue problem

$$\begin{cases} S_{k_1}\left(D^2 u_1\right) = \tilde{\lambda}(-u_2)^{\gamma_1}, & \text{in } \Omega, \\ S_{k_2}\left(D^2 u_2\right) = (-u_3)^{\gamma_2}, & \text{in } \Omega, \\ \qquad \vdots \\ S_{k_{n-1}}\left(D^2 u_{n-1}\right) = (-u_n)^{\gamma_{n-1}}, & \text{in } \Omega, \\ S_{k_n}\left(D^2 u_n\right) = (-u_1)^{\gamma_n}, & \text{in } \Omega, \\ u_i = 0, i = 1, \ldots, n, & \text{on } \partial\Omega, \end{cases} \tag{5.5}$$

admits a solution (**k**-admissible solution) if and only if $\tilde{\lambda} = \lambda_0^{k_1}$.

Next, we prove that the system (1.3) has a nonzero **k**-admissible solution if and only if

$$\lambda_1 \lambda_2^{\frac{\gamma_1}{k_2}} \cdots \lambda_n^{\frac{\prod_{i=1}^{n-1}\gamma_i}{\prod_{i=2}^{n} k_i}} = \lambda_0^{k_1}.$$

In fact, if $(u_1, \ldots, u_n)$ is a $\boldsymbol{k}$-admissible solution of the system (1.3), then

$$S_{k_n}(D^2 u_n) = \lambda_n(-u_1)^{\gamma_n},$$

which implies that

$$S_{k_n}\left(D^2(\lambda_n^{-\frac{1}{k_n}} u_n)\right) = (-u_1)^{\gamma_n}.$$

Let $\tilde{u}_n = \lambda_n^{-\frac{1}{k_n}} u_n$, we have $S_{k_n}(D^2\tilde{u}_n) = (-u_1)^{\gamma_n}$ and

$$S_{k_{n-1}}(D^2 u_{n-1}) = \lambda_{n-1}(-u_n)^{\gamma_{n-1}} = \lambda_{n-1}\lambda_n^{\frac{\gamma_{n-1}}{k_n}}(-\tilde{u}_n)^{\gamma_{n-1}}.$$

By the same argument, we let

$$\begin{aligned}
\tilde{u}_{n-1} &= \lambda_{n-1}^{-\frac{1}{k_{n-1}}}\lambda_n^{-\frac{\gamma_{n-1}}{k_{n-1}k_n}} u_{n-1},\\
&\vdots\\
\tilde{u}_2 &= \lambda_2^{-\frac{1}{k_2}}\lambda_3^{-\frac{\gamma_2}{k_2k_3}}\cdots\lambda_n^{-\frac{\prod_{i=2}^{n-1}\gamma_i}{\prod_{i=2}^{n}k_i}} u_2,
\end{aligned}$$

therefore we have $S_{k_{n-1}}(D^2\tilde{u}_{n-1}) = (-\tilde{u}_n)^{\gamma_{n-1}}$, $\cdots$, $S_{k_1}(D^2 u_1) = \lambda_1\lambda_2^{\frac{\gamma_1}{k_2}}\cdots\lambda_n^{\frac{\prod_{i=1}^{n-1}\gamma_i}{\prod_{i=2}^{n}k_i}}(-\tilde{u}_2)^{\gamma_1}$. From the previous discussion, we know that (5.5) admits a $\boldsymbol{k}$-admissible solution if and only if $\tilde{\lambda} = \lambda_0^{k_1}$. So, $\lambda_1\lambda_2^{\frac{\gamma_1}{k_2}}\cdots\lambda_n^{\frac{\prod_{i=1}^{n-1}\gamma_i}{\prod_{i=2}^{n}k_i}} = \lambda_0^{k_1}$.

On the other hand, if $\lambda_1\lambda_2^{\frac{\gamma_1}{k_2}}\cdots\lambda_n^{\frac{\prod_{i=1}^{n-1}\gamma_i}{\prod_{i=2}^{n}k_i}} = \lambda_0^{k_1}$, we let $\tilde{\lambda} = \lambda_1\lambda_2^{\frac{\gamma_1}{k_2}}\cdots\lambda_n^{\frac{\prod_{i=1}^{n-1}\gamma_i}{\prod_{i=2}^{n}k_i}}$. Then $\tilde{\lambda} = \lambda_0^{k_1}$, which implies that (5.5) has a $\boldsymbol{k}$-admissible solution $(u_1, \ldots, u_n)$. Define

$$\begin{aligned}
u_2^* &= \lambda_2^{\frac{1}{k_2}}\lambda_3^{\frac{\gamma_2}{k_2k_3}}\cdots\lambda_n^{\frac{\prod_{i=2}^{n-1}\gamma_i}{\prod_{i=2}^{n}k_i}} u_2,\\
&\vdots\\
u_{n-1}^* &= \lambda_{n-1}^{\frac{1}{k_{n-1}}}\lambda_n^{\frac{\gamma_{n-1}}{k_{n-1}k_n}} u_{n-1},\\
u_n^* &= \lambda_n^{\frac{1}{k_n}} u_n.
\end{aligned}$$

Then $(u_1, u_2^*, \ldots, u_n^*)$ is a $\boldsymbol{k}$-admissible solution of system (1.3). □

Remark 5.1. It is necessary to emphasis that if we define different composite operator, for example $\overline{T} := \overline{T}_2\cdots\overline{T}_n\overline{T}_1$, etc, then we can let $\tilde{\lambda}$ be related to each k_i $(i = 1, \ldots, n)$. Here, we only take $\tilde{\lambda} = \lambda_0^{k_1}$ for a detailed explanation.

Remark 5.2. Here, we point out that our proof of the strong property of $\overline{T}$ is different from that in [28]. We overcome the difficult caused by the non-differentiability of solutions to degenerate k-Hessian equations and find the corresponding sub-solutions equipped with the higher regularity. Thanks to the Hopf lemma in [11] which applied to the non-differentiable function, we derive the proof.

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