Recent developments in double-scaled SYK

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based on
KO [arXiv: 2212.09213, 2304.01522, 2305.12674, 2306.15981, 2312.00880, 2401.07403, 2404.02833, 2408.03726, 2501.15501]
KO and K. Suzuki [JHEP 05 (2023) 117]
KO and T. Suyama [JHEP 04 (2024) 094]
M. Miyaji, S. Mori, and KO [to appear]

Plan of my talk

1. SYK and JT gravity
2. Double-scaled SYK (DSSYK)
3. End of the world brane in DSSYK
4. ETH matrix model of DSSYK
5. Summary

Holographic principle

- Quantum many body system without gravity is expected to be holographically dual to quantum gravity

Holographic duality

d-dim quantum system $\Leftrightarrow$ $(d+1)$-dim quantum gravity

- In particular, duality between a critical point of quantum many body system (CFT) and quantum gravity on AdS (AdS/CFT correspondence) has been widely studied
[Maldacena 1997]

SYK model

- Sachdev-Ye-Kitaev (SYK) model is an interesting toy model of holographic duality [Sachdev-Ye 1993, Kitaev 2015]

- SYK model is a quantum system of N Majorana fermions

$$H = i^{p/2} \sum_{1 \le i_1 < \cdots < i_p \le N} J_{i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p}$$

- Coupling constant J of p-body interaction is randomly distributed with Gaussian weight

Properties of SYK model

- 2-point function of fermions can be computed in the large N limit
- Approximate conformal symmetry emerges at low energy [Sachdev-Ye 1993]
- This conformal symmetry is spontaneously broken
- Nambu-Goldstone mode for this SSB is called Schwarzian mode, which governs the low energy behavior

Relation to JT gravity

- Jackiw-Teitelboim (JT) gravity is a 2d dilaton gravity
- Dynamical DOF of JT gravity is Schwarzian mode describing the fluctuation of AdS_2 boundary

Holographic duality

SYK model at low energy $\Leftrightarrow$ JT gravity on AdS_2

- More generally, JT gravity is dual to a random matrix model in the double scaling limit [Saad-Shenker-Stanford 2019]

Double-scaled SYK (DSSYK)

Double scaled SYK (DSSYK)

- Certain scaling limit of SYK model is exactly solvable without low energy approximation

  [Berkooz-Isachenkov-Narovlansky-Torrents 2018]

- Take a large N limit with p-body interaction $p \sim \sqrt{N}$

$$N, p \to \infty, \quad \lambda = \frac{2p^2}{N} = \text{fixed}$$

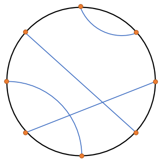
- This is called double scaled SYK model (DSSYK)

Chord diagram

- Average $\langle \cdots \rangle_J$ over random coupling $J_{i_1 \cdots i_p}$ boils down to the computation of chord diagrams

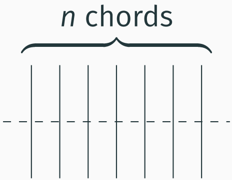
Wick contraction of J $\quad \langle HH \rangle_J = \overset{\sqcap}{HH} = \text{chord}$

$$\left\langle \operatorname{tr} H^{2k} \right\rangle_J = \sum_{\text{chord diagrams}} q^{\#(\text{intersections})}, \quad q = e^{-\lambda}$$

$\left\langle \operatorname{tr} H^8 \right\rangle_J \supset$  $= q^2$

Transfer matrix

- Combinatorics of chord diagrams is solved by introducing the transfer matrix T
- Transfer matrix T acts on a chord number state $|n\rangle$

$|n\rangle =$ 

q-oscillator

- T is given by the q-deformed oscillator $A_\pm$

$$T = \frac{A_+ + A_-}{\sqrt{1-q}}$$

- $A_\pm$ creates/annihilates the chords

$$A_-|n\rangle = \sqrt{1-q^n}|n-1\rangle$$
$$A_+|n\rangle = \sqrt{1-q^{n+1}}|n+1\rangle$$

Partition function of DSSYK

- Partition function of DSSYK is written in terms of the transfer matrix T

$$\left\langle \operatorname{tr} e^{-\beta H} \right\rangle_J = \langle 0 | e^{-\beta T} | 0 \rangle$$

- 0-chord state $|0\rangle$ is interpreted as Hartle-Hawking vacuum of bulk quantum gravity [Lin 2022]

Matter operator

- We can introduce matter operators in DSSYK [Berkooz-Isachenkov-Narovlansky-Torrents 2018]

$$\mathcal{O}_\Delta = i^{s/2} \sum_{1\le i_1<\cdots<i_s\le N} K_{i_1\cdots i_s}\psi_{i_1}\cdots\psi_{i_s}$$

- K is assumed to be Gaussian random and independent of J

- We also take a scaling limit $s \sim \sqrt{N}$

$$\Delta = \frac{2ps}{N} = \text{fixed}$$

Matter chord

- Two types of chord arise from Wick contraction of J and K

$$\contraction{}{H}{}{H} HH = H\text{-chord}$$

$$\contraction{}{\mathcal{O}_\Delta}{}{\mathcal{O}_\Delta} \mathcal{O}_\Delta \mathcal{O}_\Delta = \text{matter chord}$$

- Correlator of $\mathcal{O}_\Delta$ also reduces to the computation of chord diagrams

$$\langle \mathcal{O}_\Delta \mathcal{O}_\Delta \rangle = \sum_{\text{chord}} q^{\#(H\text{-}H \text{ intersections})} e^{-\Delta \#(H\text{-}\mathcal{O} \text{ intersections})}$$

Matter 2-point function

- Combinatorics of matter correlator is also solved by the technique of transfer matrix

- 2-point function of $\mathcal{O}_\Delta$

$$\left\langle \mathrm{tr}(e^{-\beta_1 H}\mathcal{O}_\Delta e^{-\beta_2 H}\mathcal{O}_\Delta)\right\rangle_{J,K} = \langle 0|e^{-\beta_1 T}e^{-\Delta \widehat{N}}e^{-\beta_2 T}|0\rangle$$

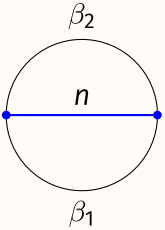
- $\widehat{N}$ is the number operator of chords

$$\widehat{N}|n\rangle = n|n\rangle$$

2-point function

- 2-point function is expanded as

$$\langle 0|e^{-\beta_1 T}e^{-\Delta\widehat{N}}e^{-\beta_2 T}|0\rangle = \sum_{n=0}^{\infty} e^{-\Delta n}$$



- n is interpreted as a discretized bulk geodesic length

Diagonalization of T

- Transfer matrix T is diagonalized in the $|\theta\rangle$-basis

$$T|\theta\rangle = E_0 \cos\theta |\theta\rangle, \quad E_0 = \frac{2}{\sqrt{1-q}}$$

- θ-representation of chord number state $|n\rangle$ is q-Hermite polynomial $H_n(x|q)$

$$\langle\theta|n\rangle = \frac{H_n(\cos\theta|q)}{\sqrt{(q;q)_n}}$$

End of the world brane in DSSYK

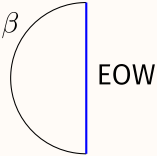
Boundary state

- We can consider end of the world (EOW) brane in DSSYK
- Boundary state of EOW brane $|B_a\rangle$ is a coherent state of q-oscillator $A_\pm$ [KO 2023]

$$A_-|B_a\rangle = a|B_a\rangle$$

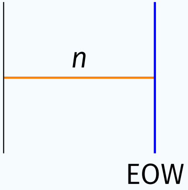
- Half-disk amplitude is written in terms of $|B_a\rangle$

$$\langle 0|e^{-\beta T}|B_a\rangle =$$



Big q-Hermite and EOW brane

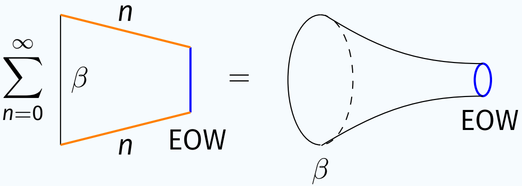
- Wavefunction of bulk quantum gravity in the presence of EOW brane is big q-Hermite polynomial $H_n(x, a|q)$ [KO 2023]

$H_n(x, a|q) =$ 

- The parameter a is related to the tension μ of brane

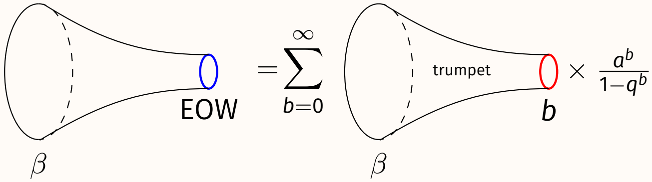
$$a = q^{\mu + \frac{1}{2}}$$

- Amplitude of half-wormhole ending on the EOW brane



- Sum over n represents a trace
  $\Rightarrow$ top and bottom of the LHS are identified

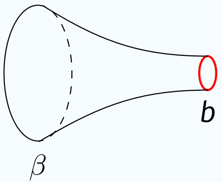
- Half-wormhole can be decomposed into trumpet and the factor coming from EOW brane



- Half-wormhole in JT gravity has a similar decomposition
  [Gao-Jafferis-Kolchmeyer 2021]

Trumpet of DSSYK

- Trumpet of DSSYK is given by the modified Bessel function [Jafferis-Kolchmeyer-Mukhametzhanov-Sonner 2022, KO 2023]



$$= \ I_b(\beta E_0)$$

- Trumpet also arises in JT gravity, but there is a difference
  - In DSSYK, length of geodesic loop b is discrete
  - In JT gravity, length b is continuous [Saad-Shenker-Stanford 2019]

ETH matrix model of DSSYK

ETH matrix model

- We can find an $L \times L$ hermitian matrix model which reproduces the disk amplitude of DSSYK at large $L = 2^{N/2}$
- Such a matrix model is dubbed ETH matrix model
  [Jafferis-Kolchmeyer-Mukhametzhanov-Sonner 2022]

$$\mathcal{Z} = \int_{L\times L} dH\, e^{-L\,\mathrm{Tr}\,V(H)},$$

$$V(E) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} q^{\frac{1}{2}n^2} \left(q^{\frac{1}{2}n} + q^{-\frac{1}{2}n}\right) \underbrace{T_{2n}(E/E_0)}_{\text{Chebyshev poly}}$$

Large L expansion of ETH matrix model

- Large L expansion of the correlator of ETH matrix model
  - trumpets
  - discrete volume of the moduli space of Riemann surfaces

    [Norbury-Scott 2010, KO 2023]

$$
\left\langle \prod_{i=1}^{n} \mathrm{Tr}\, e^{-\beta_i H} \right\rangle_{\mathrm{conn}} = \sum_{g=0}^{\infty} L^{2-2g-n} \sum_{b_1,\cdots,b_n=0}^{\infty} \underbrace{N_{g,n}(b_1,\cdots,b_n)}_{\text{discrete volume}} \prod_{i=1}^{n} \underbrace{b_i I_{b_i}(\beta_i E_0)}_{\text{trumpet}}
$$

- This is a discrete version of JT gravity amplitude

  [Saad-Shenker-Stanford 2019]

Inner product at finite L

- Inner product of chord number state $|n\rangle$ can be generalized to finite L

$$\begin{aligned} &\langle n|m\rangle = \delta_{nm} \\ \Rightarrow \quad & G_{nm} = \frac{1}{L}\,\mathrm{Tr}\big[\psi_n(H)\psi_m(H)\big], \quad \psi_n(E) = \frac{H_n(E/E_0|q)}{\sqrt{(q;q)_n}} \end{aligned}$$

- Expectation value of G_{nm} has a genus expansion

$$\langle G_{nm}\rangle = \frac{1}{\mathcal{Z}}\int_{L\times L} dH\, e^{-L\,\mathrm{Tr}\,V(H)} G_{nm} = \delta_{nm} + \mathcal{O}(L^{-2})$$

Null states at finite L

- Determinant of Gram matrix G_{nm} $(0 \leq n, m \leq K-1)$ vanishes when $K > L$

$$\det(G_{nm})_{K\times K} = 0 \quad \text{if} \quad K > L$$

  - This follows from the Cayley-Hamilton theorem

H^L is written as a linear combination of $H^0, \cdots, H^{L-1}$

- This implies the existence of null states at finite L, and Hilbert space is truncated to $\{\psi_n\}_{n<L}$
[Miyaji-Mori-KO, to appear]

Recap of null states in DSSYK

Mechanism of null states in DSSYK

1. Wavefunction $\psi_n(E)$ is a degree n polynomial in E
2. Cayley–Hamilton at finite L : $\quad H^L = \sum_{k=0}^{L-1} c_k H^k$
3. Hilbert space is truncated to $\{\psi_n\}_{n<L}$

- This mechanism of null states is different from JT gravity [Iliesiu-Levine-Lin-Maxfield-Mezei 2024]
  - In JT gravity, wavefunction is not a polynomial in E
  - JT matrix model is defined by a double-scaling limit

Length of two-sided black hole

- Classically, the geodesic length between the boundaries of two-sided black hole grows linearly as a function of t
- In DSSYK, the above length is given by

$$\bar{n} = \langle 0 | e^{itT} \widehat{N} e^{-itT} | 0 \rangle$$

- This length stops growing due to baby universe emission [Stanford-Yang 2022]
  - This is a higher genus effect
  - What happens at finite L?

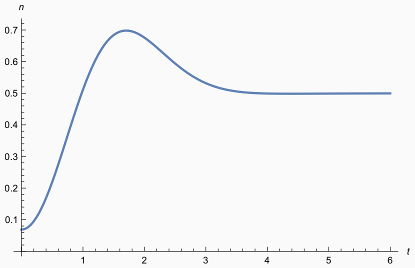
Non-perturbative length

- Non-perturbative length at finite L [Miyaji-Mori-KO, to appear]

$$\bar{n} = \sum_{k,n,m=0}^{L-1} \left\langle \mathrm{Tr}\left[e^{itH}\psi_k(H)\right] G_{kn}^{-\frac{1}{2}} n G_{nm}^{-\frac{1}{2}} \mathrm{Tr}\left[\psi_m(H)e^{-itH}\right] \right\rangle$$

- $\bar{n}$ exhibits a linear growth, slope, and plateau

This is consistent with the spread complexity [Balasubramanian-Caputa-Magan-Wu 2022]



(We set $L = 2$, $q = 0$ in this figure)

Summary

- DSSYK is a solvable example of holography
- Combinatorics of chord diagrams is solved by the technique of transfer matrix
- One can define EOW brane, trumpet, and volume of moduli space in a similar manner as JT gravity
- In the ETH matrix model, there appear null states due to the finite L matrix relation
  - reminiscent of the fortuitous BPS states in $\mathcal{N} = 4$ SYM

$$\text{fortuitous: } Q|\text{state}\rangle = 0 \text{ up to matrix relation}$$
$$\text{null: } \langle\text{state}|\text{state}\rangle = 0 \text{ up to matrix relation}$$