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On the boundary blow-up problem for real $(n-1)$ Monge–Ampère equation

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ABSTRACT

In this paper, we establish a necessary and sufficient condition for the solvability of the real $(n-1)$ Monge–Ampère equation $\det^{1/n}(\Delta uI - D^2u) = g(x,u)$ in bounded domains with infinite Dirichlet boundary condition. The $(n-1)$ Monge–Ampère operator is derived from geometry and has recently received much attention. Our result embraces the case $g(x,u) = h(x)f(u)$ where $h \in C^\infty(\bar{\Omega})$ is positive and f satisfies the Keller–Osserman type condition. We describe the asymptotic behavior of the solution by constructing suitable sub-solutions and super-solutions, and obtain a uniqueness result in star-shaped domains by using a scaling technique.

1. Introduction

In this paper, we investigate the $(n-1)$ Monge–Ampère equation

$$\det{}^{1/n}(\Delta uI - D^2u) = g(x,u), \quad x \in \Omega, \tag{1.1}$$

with an infinite Dirichlet boundary condition

$$u(x) \to +\infty, \quad \text{as } d(x) := \operatorname{dist}(x, \partial\Omega) \to 0, \tag{1.2}$$

where $\det^{1/n}$ denotes the nth root of $\det$ and g meets some natural regularity and growth conditions. In (1.1) and (1.2), Ω is a bounded domain in $\mathbb{R}^n (n \geq 2)$, u is the unknown function with D^2u and Δu being its Hessian matrix and Laplacian operator, I denotes the $n\times n$ identity matrix and $\operatorname{dist}(x,\partial\Omega)$ denotes the distance function of the point x to the boundary $\partial\Omega$. Since the solution of (1.1)–(1.2) has infinite Dirichlet boundary value at the boundary, problems with infinite Dirichlet boundary conditions (1.2) are commonly denoted as the boundary blow-up problem.

Letting $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\tilde{\lambda} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \dots, \tilde{\lambda}_n)$ be the eigenvalues of D^2u and $\Delta uI - D^2u$ respectively, it is obvious that $\tilde{\lambda}_i = \sum_{k\neq i} \lambda_k$. Then the $(n-1)$ Monge–Ampère operator

$$\det(\Delta uI - D^2u) = \tilde{\lambda}_1\tilde{\lambda}_2\cdots\tilde{\lambda}_n = \prod_{1\leq i_1<\cdots<i_{n-1}\leq n} (\lambda_{i_1} + \cdots + \lambda_{i_{n-1}}),$$

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where $i_1, \ldots, i_{n-1} \in \{1, \ldots, n\}$.

The complex form of Eq. (1.1) comes from the Gauduchon conjecture [1] in complex geometry. The related complex Monge–Ampère type equation has the form

$$\det\left(\left(\sum_{k=1}^{n}\frac{\partial^2 u}{\partial z^k \partial \bar{z}^k}\right)\delta_{x_i x_j} - \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) = f, \tag{1.3}$$

see Tosatti and Weinkove [2]. This kind of Eqs. (1.3) are closely related to the form-type Monge–Ampère equations studied by Fu, Wang and Wu [3,4], whose solutions are $(n-1)$-plurisubharmonic functions in the sense of Harvey and Lawson [5–7]. In this paper, we focus on the real setting to study Eq. (1.1). Following the recent study of the Dirichlet problem in the real case by Jiao and Liu [8], we refer to (1.1) as a *real* $(n-1)$ *Monge–Ampère equation*, which corresponds to $(n-1)$-*form Monge–Ampère equation* in [2,9] in the complex case. Since $\det(\Delta u I - D^2 u) = \det D^2 u$ when $n = 2$, the real $(n-1)$ Monge–Ampère equation reduces to the standard Monge–Ampère equation in the two dimensional case.

The boundary blow-up problem for the elliptic partial differential equations has been an interesting topic for a long time. To our knowledge, it was first studied by Bieberbach [10] in 1916 for the boundary blow-up problem of two-dimensional semilinear elliptic equations

$$\begin{cases} \Delta u = f(u), & \text{in } \Omega, \\ u(x) \to +\infty, & \text{as } d(x) \to 0, \end{cases} \tag{1.4}$$

where $f(u) = e^u$. This problem plays an important role in the theory of Riemannian surfaces with negative constant curvature. Later, Rademacher [11] extended this result to the case $n = 3$ due to its application in physics. It was not until 1957 that such a problem was considered for general nonlinearities in arbitrary dimensions. Keller [12] and Osserman [13] separately studied the problem (1.4) and provided a necessary and sufficient condition on f for the existence of solutions in bounded domains. Since then, many related problems have been proposed and studied, readers can refer to [14–18].

Motivated by geometric problems, Cheng and Yau [19,20] considered the boundary blow-up problem for fully nonlinear elliptic equations

$$\begin{cases} \det D^2 u = g(x,u), & \text{in } \Omega, \\ u(x) \to +\infty, & \text{as } d(x) \to 0, \end{cases} \tag{1.5}$$

where $g(x,u) = b(x)e^{Ku}$ in bounded convex domains and $g(x,u) = e^{2u}$ in unbounded domains were studied, respectively. When $g(x,u) = b(x)u^p$, Lazer and McKenna [21] established an existence and uniqueness theorem for (1.5) in the case $p > n$ and a nonexistence theorem in the case $0 < p < n$. Generalizing the result in [12,13] for the Laplace operator to the Monge–Ampère operator, Matero [22] and Mohammed [23] treated the case $g(x,u) = b(x)f(u)$ in bounded strictly convex domains. Their results were extended to k-Hessian equations [24] and k-curvature equations [25].

Here in this paper, we investigate the boundary blow-up problem for the $(n-1)$ Monge–Ampère equation with general $g(x,u)$ in bounded domains. The existence, uniqueness, and asymptotic behavior of solutions $u(x)$ to the boundary blow-up problem (1.1)–(1.2) will be studied.

We say a function $u \in C^2(\Omega)$ is $(n-1)$-convex if the matrix

$$\Delta u I - D^2 u > 0$$

for every $x \in \Omega$. An $(n-1)$-convex function $\underline{u} \in C^2(\bar{\Omega})$ is said to be a sub-solution of (1.1) if $\underline{u}$ satisfies

$$\det{}^{1/n}(\Delta u I - D^2 u) \geq g(x,u), \quad \text{in } \Omega. \tag{1.6}$$

Note that (1.1) is elliptic with respect to $(n-1)$-convex solutions. We will look for $(n-1)$-convex solutions in $C^\infty(\Omega)$.

Before stating the main theorems, we shall assume that $g(x,z) \in C^\infty(\bar{\Omega} \times [\eta, +\infty))$ $(\eta \in \mathbb{R} \cup \{-\infty\})$ satisfies a subset of the following conditions:

- $(G1)$ there exist $h \in C^\infty(\bar{\Omega})$ and $f \in C^\infty[\eta, +\infty)$ such that

$$\lim_{z\to\infty}\frac{g(x,z)}{f(z)} = h(x) \text{ uniformly in } \Omega$$

and

$$c_1 f(z) \leq g(x,z) \leq c_2 f(z) \quad \text{in } \Omega \times [\eta, +\infty),$$

where c_1, c_2 are two positive constants;
- $(G2)$ $g'_z \geq 0$ in $\bar{\Omega} \times [\eta, +\infty)$;
- $(G3)$ there exists a constant $\gamma > 1$ such that $g(x, \beta z) \leq \beta^\gamma g(x,z)$ for every $\beta \in (0,1)$ and $z \in [\eta, +\infty)$.

The function f in $(G1)$ will be further assumed to satisfy a subset of the following conditions:

- $(F1)$ $f : \mathbb{R} \to (0, +\infty)$ is non-decreasing (or $f : [\eta, +\infty) \to [0, +\infty)$ is non-decreasing, $f(\eta) = 0$, $f(s) > 0$ as $s > \eta$);

$(F2)$ the function

$$\Psi(a)=\int_a^{+\infty}(F(t))^{-\frac{n-1}{2n-1}}\,\mathrm{d}t$$

is well defined for every $a\in[\eta,+\infty)$, where $F(t)=\int_\eta^t f^{\frac{n}{n-1}}(s)\,\mathrm{d}s$ if $\eta\in\mathbb{R}$, and $F(t)=\int_0^t f^{\frac{n}{n-1}}(s)\,\mathrm{d}s$ if $\eta=-\infty$.

The condition $(F2)$ can be regarded as the Keller–Osserman type condition for $(n-1)$ Monge–Ampère equation, see [18] for the detailed discussion. Conditions $(F1)$-$(F2)$ can include two special kinds of nonlinearities

$$f(z)=a+e^{bz}\ (a\geq 0,b>0),\quad\text{and}\quad f(z)=k(z-\eta)^p\ (k>0,p>1).$$

The former one allows f to be increasing and strictly positive in $\mathbb{R}$. The latter f is increasing and positive for $z>\eta$ and $f(\eta)=0$. These two particular nonlinearities correspond to the two alternative cases in condition $(F1)$.

Now, we establish the main results of this paper.

Theorem 1.1. *Let Ω be a bounded, strictly convex domain with smooth boundary $\partial\Omega$. Assume that $(G1)$, $(G2)$ and $(F1)$ hold, then there exists an $(n-1)$-convex solution $u\in C^\infty(\Omega)$ of the problem (1.1)–(1.2) if and only if $(F2)$ holds.*

Since Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, there exists $R_1>0$ such that for any $y\in\partial\Omega$, $B_{R_1}(x_1)\subset\Omega$ with $y\in\partial B_{R_1}(x_1)$ and $x_1\in\Omega$. Furthermore, since Ω is strictly convex, there exists $R_2>0$ such that for any $y\in\partial\Omega$, $B_{R_2}(x_2)\supset\Omega$ is tangent to $\partial\Omega$ at y and $x_2\in\mathbb{R}^n$. Then we can characterize the boundary blow-up rate of $u(x)$ in terms of the distance of x to $\partial\Omega$.

Theorem 1.2. *Let Ω be a bounded, strictly convex domain with smooth boundary $\partial\Omega$. Assume that $(G1)$, $(G2)$, $(F1)$ and $(F2)$ hold. Then for any $(n-1)$-convex solution u of the problem (1.1)–(1.2), there exist positive constants R_1, R_2 such that*

$$\left[c_1^n a(n)R_1\right]^{\frac{1}{2n-1}}\leq\liminf_{x\to\partial\Omega}\frac{\Psi(u(x))}{d(x)}\leq\limsup_{x\to\partial\Omega}\frac{\Psi(u(x))}{d(x)}\leq\left[c_2^n a(n)R_2\right]^{\frac{1}{2n-1}},\tag{1.7}$$

where $a(n)=\frac{(2n-1)^{n-1}}{(n-1)^n}$, $d(x)=\mathrm{dist}(x,\partial\Omega)$, c_1,c_2 are the constants in condition $(G1)$, and Ψ is the function in condition $(F2)$.

In Theorem 1.2, the limit $\lim_{x\to\partial\Omega}\frac{\Psi(u(x))}{d(x)}$ exists provided that Ω is a ball.

When Ω is a bounded star-shaped domain, we state the uniqueness of the problem (1.1)–(1.2) under the $(G3)$ condition.

Theorem 1.3. *Suppose Ω is star-shaped (with respect to a point $x_0\in\Omega$) and g satisfies $(G3)$, then the problem (1.1)–(1.2) has at most one $(n-1)$-convex solution.*

A direct consequence of Theorems 1.1 and 1.3 is that there exists a unique $(n-1)$-convex solution $u\in C^\infty(\Omega)$ of the problem (1.1)–(1.2) in a bounded, strictly convex domain Ω with smooth boundary $\partial\Omega$ when $(G1)$, $(G2)$, $(G3)$, $(F1)$ and $(F2)$ hold, see Remark 5.1.

Building upon the work [8] for the finite Dirichlet boundary value problem, this paper delves deeper into the infinite Dirichlet boundary value problem for real $(n-1)$ Monge–Ampère equation in the bounded strictly convex domain, using a Keller–Osserman type condition in [18]. The key point is to construct sub-solutions based on the strict convexity of the domain. A highlight of this paper is that the (F2) condition is not only a sufficient condition for the existence of $(n-1)$-convex solution but also a necessary condition for the existence of $(n-1)$-convex solution.

The paper is organized as follows. In Section 2, we recall the existence result of the Dirichlet boundary value problem for the $(n-1)$ Monge–Ampère equation in [8] and comparison principle in [26]. A computation on the determinant is also presented as a lemma, which will be used to construct sub-solutions in the subsequent sections. In Section 3, we first obtain a uniform upper bound of solutions to (1.1) in any bounded domain. Then we establish a necessary and sufficient condition for the existence of radial solutions to the problem (1.1)–(1.2) in a ball and their explicit asymptotic estimates. In Section 4, we prove that the condition $(F2)$ is the necessary and sufficient condition for the solvability of the problem (1.1)–(1.2) in strictly convex domains, based on the construction of suitable sub- and super-solutions. In Section 5, we establish the asymptotic behavior of the problem (1.1)–(1.2) near the boundary in terms of the distance of x to $\partial\Omega$, and further prove the uniqueness result under the condition $(G3)$ in star-shaped domains.

2. Preliminaries

In this section, we present some preliminary lemmas and propositions that will be useful in the subsequent sections.

We consider the Dirichlet problem

$$\begin{cases}\det^{1/n}(\Delta uI-D^2u)=g(x,u), & \text{in }\Omega,\\ u(x)=k, & \text{on }\partial\Omega,\end{cases}\tag{2.1}$$

where k is a positive integer. The existence of classical solutions for the problem (2.1) can be found in [8], where the right hand side term g also depends on ∇u. In particular, we state the existence result for problem (2.1).

Lemma 2.1 (*Existence*). *Let Ω be a bounded domain with smooth boundary $\partial\Omega$, and g be a positive smooth function satisfying $(G2)$. Assume that there exists an $(n-1)$-convex sub-solution $\underline{u} \in C^2(\bar{\Omega})$ satisfying (1.6) with $\underline{u} = k$ on $\partial\Omega$. Then there exists a unique $(n-1)$-convex solution $u \in C^\infty(\bar{\Omega})$ of the problem (2.1).*

In [8], the existence of solutions to (2.1) was proved by the continuity method and degree theory under the assumptions that $\underline{u}$ is a sub-solution and g satisfies $\sup_{(x,z)\in\Omega\times\mathbb{R}} \frac{-g_z'(x,z)}{g(x,z)} < \infty$. In addition, the uniqueness of the solution can be obtained by $(G2)$, and in Lemma 2.1 the condition $(G2)$ naturally implies that $\sup_{(x,z)\in\Omega\times\mathbb{R}} \frac{-g_z'(x,z)}{g(x,z)} < \infty$ holds.

Under condition $(G2)$, the comparison principle holds for the $(n-1)$ Monge–Ampère Eq. (1.1). For convenience, we present a comparison principle for general second-order elliptic partial differential equations, namely Theorem 17.1 in [26].

Lemma 2.2 (*Comparison principle*). *Suppose $F[u] := F(x, u, Du, D^2u)$ is a real function defined on the set $\Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$, we denote (x, z, p, r) as a point in Γ. Let $u, v \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfy $F[u] \geq F[v]$ in Ω, $u \leq v$ on $\partial\Omega$, if*

(i) the function F is continuously differentiable with respect to the z, p, r variables in Γ;
(ii) the operator F is elliptic on all functions of the form $\theta u + (1-\theta)v$, $0 \leq \theta \leq 1$;
(iii) the function F is non-increasing in z for each $(x, p, r) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$,

then we have

$$u \leq v, \quad \text{in } \bar{\Omega}.$$

If for any positive integer k, there exists an $(n-1)$-convex sub-solution $\underline{u}_k \in C^2(\bar{\Omega})$ satisfying (1.6) with $\underline{u}_k = k$ on $\partial\Omega$, by Lemma 2.1, there exists a unique $(n-1)$-convex solution u_k of the problem (2.1). Then we have a family of solutions $u_k \in C^\infty(\bar{\Omega})$, $k = 1, 2, \ldots$. Based on the existence result in Lemma 2.1 and the comparison principle in Lemma 2.2, we get the following observation on the solution of (2.1).

Proposition 2.1. *Let Ω be a bounded domain with smooth boundary $\partial\Omega$, and g be a positive smooth function satisfying $(G2)$. Assume that there exist $(n-1)$-convex sub-solutions $\underline{u}_k, \underline{u}_{k+1} \in C^2(\bar{\Omega})$ satisfying (1.6) with $\underline{u}_k = k$ on $\partial\Omega$ and $\underline{u}_{k+1} = k+1$ on $\partial\Omega$, respectively. Then there exist $(n-1)$-convex solutions $u_k, u_{k+1} \in C^\infty(\bar{\Omega})$ of problem (2.1) satisfying $u_k = k$ on $\partial\Omega$ and $u_{k+1} = k+1$ on $\partial\Omega$, respectively. Moreover, we have*

$$u_k \leq u_{k+1}, \quad \text{in } \bar{\Omega}.$$

Proof. Under the assumptions of Ω, g, and the sub-solutions $\underline{u}_k, \underline{u}_{k+1} \in C^2(\bar{\Omega})$, the solutions $u_k, u_{k+1} \in C^\infty(\bar{\Omega})$ can be obtained immediately from Lemma 2.1. Let $F[u] := F(x, u, Du, D^2u) = \det^{1/n}(\Delta u I - D^2u) - g(x, u)$. It is clear that F is elliptic with respect to $(n-1)$-convex functions. Due to $g_z'(x,z) \geq 0$ in $\Omega \times \mathbb{R}$, the function F is non-increasing in z at each point $x \in \Omega$. Since $F[u_k] = F[u_{k+1}] = 0$ in Ω and $k = u_k < u_{k+1} = k+1$ on $\partial\Omega$, it follows from Lemma 2.2 that $u_k \leq u_{k+1}$ in $\bar{\Omega}$. □

Proposition 2.1 shows that u_k is non-decreasing in k. In fact, the non-decreasing monotonicity of u_k with respect to k still holds even if k is a general positive constant, (not just a positive integer).

Finally, we provide a computation of the determinant that will be helpful in constructing sub-solutions in Section 4.

Lemma 2.3. *Let u be a C^2 strictly convex function in an open set Ω in $\mathbb{R}^n$, let $\phi \in C^2$ be a function defined on an interval containing the range of u. If $w = \phi(u)$, then*

$$\det D^2 w = \left[(\phi'(u))^n + \phi''(u)(\phi'(u))^{n-1}(Du)^T(D^2u)^{-1}Du\right] \det D^2u, \quad \text{in } \Omega, \tag{2.2}$$

where A^T denotes the transpose of matrix A.

Proof. For any point $x \in \Omega$, by rotating the coordinates, we let

$$Du(x) = (u_{x_1}(x), 0, \ldots, 0), \quad u_{x_ix_j}(x) = u_{x_ix_i}(x)\delta_{ij} \quad \text{for } i, j = 2, \ldots, n,$$

where δ_{ij} denotes the usual Kronecker delta. Then at the point x, we have

$$w_{x_1x_1}(x) = \phi''(u)u_{x_1}^2(x) + \phi'(u)u_{x_1x_1}(x),$$

$$w_{x_1x_i}(x) = w_{x_ix_1}(x) = \phi'(u)u_{x_1x_i}(x) = \phi'(u)u_{x_ix_1}(x) \quad \text{for } i = 2, \ldots, n,$$

$$w_{x_ix_j}(x) = \phi'(u)u_{x_ix_i}(x)\delta_{ij} \quad \text{for } i, j = 2, \ldots, n,$$

where $u_{x_i} = \frac{\partial u}{\partial x_i}$ and $u_{x_ix_j} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, $i,j = 1,\dots,n$. The determinant of $\{w_{x_ix_j}\}$ can be expressed as

$$\begin{aligned}
\det D^2 w &= \begin{vmatrix} \phi'' u_{x_1}^2 + \phi' u_{x_1x_1} & \phi' u_{x_1x_2} & \phi' u_{x_1x_3} & \cdots & \phi' u_{x_1x_2} \\ \phi' u_{x_2x_1} & \phi' u_{x_2x_2} & 0 & \cdots & 0 \\ \phi' u_{x_3x_1} & 0 & \phi' u_{x_3x_3} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi' u_{x_nx_1} & 0 & 0 & \cdots & \phi' u_{x_nx_n} \end{vmatrix} \\
&= \begin{vmatrix} \phi'' u_{x_1}^2 & \phi' u_{x_1x_2} & \cdots & \phi' u_{x_1x_2} \\ 0 & \phi' u_{x_2x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi' u_{x_nx_n} \end{vmatrix} + \begin{vmatrix} \phi' u_{x_1x_1} & \phi' u_{x_1x_2} & \cdots & \phi' u_{x_1x_2} \\ \phi' u_{x_2x_1} & \phi' u_{x_2x_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \phi' u_{x_nx_1} & 0 & \cdots & \phi' u_{x_nx_n} \end{vmatrix} \\
&= \phi''(u)(\phi'(u))^{n-1} u_{x_1}^2 \prod_{i=2}^n u_{x_ix_i} + (\phi'(u))^n \det D^2 u \\
&= \phi''(u)(\phi'(u))^{n-1} u^*_{x_1x_1} u_{x_1}^2 + (\phi'(u))^n \det D^2 u,
\end{aligned} \tag{2.3}$$

where $u^*_{x_ix_j}$ denotes the cofactor of the (i,j)-th entry of the matrix $\{u_{x_ix_j}\}$ for $i,j = 1,\dots,n$. Since u is strictly convex and the matrix $\{u_{x_ix_j}\}$ is symmetric, we use the formula for the inverse of a matrix to deduce

$$u^*_{x_1x_1} u_{x_1}^2 = (\det D^2 u) u^{x_1x_1} u_{x_1}^2 = \det D^2 u \sum_{i,j=1}^n u^{x_ix_j} u_{x_i} u_{x_j} = (\det D^2 u)(Du)^T (D^2 u)^{-1} Du, \tag{2.4}$$

at x, where $\{u^{x_ix_j}\}$ is the inverse of the matrix $\{u_{x_ix_j}\}$ for $i,j = 1,\dots,n$. By substituting (2.4) into (2.3), we thus get (2.2). □

Note that the formula (2.2) in Lemma 2.3 is already used in [23]. An alternative proof of Lemma 2.3 can be found in [21].

3. Blow-up estimates in a ball

In this section, we present three lemmas on radially symmetric solutions of (1.1) when $g(x,z)$ is independent of x and Ω is a ball of radius R. In Lemma 3.1, we give a uniform upper bound for this solution. As a consequence, in Lemma 3.2, a radial solution of (1.1)–(1.2) exists if and only if condition $(F2)$ holds. Finally, we establish an explicit version of the blow-up rate of a radial solution to (3.1) in Lemma 3.3.

We study the classical radial solution of the problem (1.1)–(1.2) with g depending only on u in a finite ball $B_R(0)$, $R < \infty$. Setting $r = |x|$, the corresponding problem of (1.1)–(1.2) can be written as

$$\begin{cases} \det^{1/n}(\Delta v(r) I - D^2 v(r)) = f(v(r)), & r \in [0, R), \\ v(r) \to +\infty, & \text{as } r \to R. \end{cases} \tag{3.1}$$

Lemma 3.1. *Assume $(F1)$ and $(F2)$ hold, then for every $(n-1)$-convex solution of $\det^{1/n}(\Delta v I - D^2 v) = f(v)$ in a ball $B_R(0)$ with constant value $\alpha > 0$ on the boundary, there exists a decreasing function $\zeta(\delta)$ with $\lim_{\delta\to 0}\zeta(\delta) = +\infty$ such that*

$$v(r) \le \zeta(R - r), \quad x \in \Omega. \tag{3.2}$$

Proof. We consider the Dirichlet boundary problem:

$$\begin{cases} \det^{1/n}(\Delta v I - D^2 v) = f(v), & \text{in } B_R(0), \\ v(x) = \alpha, & \text{on } \partial B_R(0), \end{cases} \tag{3.3}$$

where f is defined as in $(G1)$. In fact, the existence and uniqueness of $(n-1)$-convex solution $v \in C^\infty(\overline{B_R(0)})$ of (3.3) is guaranteed by Lemma 2.1 under the existence of a strict sub-solution $\underline{v}$ to (3.3). More specifically, this strict sub-solution has the form

$$\underline{v}(r) = \alpha - \int_r^R \frac{t^{2-n}}{n-1} \left[\int_0^t n s^{n-1} (f(\alpha)+1)^{\frac{n}{n-1}} \, ds \right]^{\frac{n-1}{n}} dt, \quad r \in (0, R), \tag{3.4}$$

which satisfies (3.3) with $f(v)$ replaced by $f(\alpha)+1$. We notice that v must be a radial function. Otherwise, if v is not spherically symmetric, a different solution could be obtained by rotating v, which contradicts the uniqueness of the solution. Therefore $v(x) = v(r)$, $r = |x|$, $r \in [0, R]$. Since v is a function depending on α, we can define a function $\zeta(\delta)$ by

$$\zeta(\delta) := \zeta(R - r) = \lim_{\alpha\to+\infty} v(r),$$

where $\delta = R - r$. Then $\lim_{\delta\to 0}\zeta(\delta) = \lim_{\alpha\to+\infty} v(R) = +\infty$. Since f satisfies $(F1)$, it follows from Proposition 2.1 that v is a non-decreasing function of α. For every α, we obtain the desired inequality (3.2) at each point in $B_R(0)$.

It remains to show that $\zeta(R-r)$ is finite so that (3.2) is not trivial, and that $\zeta(\delta)$ is a decreasing function of δ. To this end, we must examine v, the solution of (3.3). With some proper calculations, we can see that v satisfies

$$(n-1)\frac{v'(r)}{r}\left(v''(r)+(n-2)\frac{v'(r)}{r}\right)^{n-1} = f^n(v(r)), \quad r\in(0,R), \tag{3.5}$$

$$v'(0)=0, \quad \text{and } v(R)=\alpha. \tag{3.6}$$

We see that each α uniquely determines the initial value $v(0)$. In this sense, we can regard $v(0)$ as a function of α, which is increasing in α. Therefore α itself is uniquely determined by $v(0)$. Thus we can replace $v(R)=\alpha$ in (3.6) by $v(0) =: \tilde{v}_0$. Setting

$$v_0 := \lim_{\alpha\to+\infty}\tilde{v}_0 = \lim_{\alpha\to+\infty} v(0),$$

we shall show that $v(R)=+\infty$ for some finite v_0.

Multiplying by $r^{\frac{(2n-1)(n-2)}{n-1}}v'(r)$, we can rewrite (3.5) in the form

$$\begin{aligned}\left[\left(r^{n-2}v'(r)\right)^{\frac{2n-1}{n-1}}\right]' &= \frac{2n-1}{(n-1)^{\frac{n}{n-1}}} r^{\frac{(2n-1)(n-2)}{n-1}}\left[rf^n(v(r))\right]^{\frac{1}{n-1}} v'(r)\\ &= \frac{2n-1}{(n-1)^{\frac{n}{n-1}}} r^{2n-3}F'(v(r)).\end{aligned} \tag{3.7}$$

Integrating (3.7) from 0 to r and taking the first equality in (3.6) into account, we have

$$\left(r^{n-2}v'(r)\right)^{\frac{2n-1}{n-1}} = \frac{2n-1}{(n-1)^{\frac{n}{n-1}}}\left[r^{2n-3}F(v(r)) - (2n-3)\int_0^r t^{2n-4}F(v(t))\,\mathrm{d}t\right]. \tag{3.8}$$

Since the integral term in (3.8) is positive, we can obtain

$$v'(r) \le \left[a(n)r(F(v(r)))^{n-1}\right]^{\frac{1}{2n-1}},$$

which provides a lower bound for $r^{\frac{1}{2n-1}}$,

$$[a(n)r]^{\frac{1}{2n-1}} \ge [F(v(r))]^{-\frac{n-1}{2n-1}} v'(r), \tag{3.9}$$

where $a(n)=\frac{(2n-1)^{n-1}}{(n-1)^n}$. In order to get an upper bound for $r^{\frac{1}{2n-1}}$, by rewriting (3.5) we get

$$\left[(r^{n-2}v'(r))^{\frac{n}{n-1}}\right]' = \frac{n}{(n-1)^{\frac{n}{n-1}}} r^{n-1} f^{\frac{n}{n-1}}(v(r)), \quad r\in(0,R). \tag{3.10}$$

From (3.10), we get

$$v'(r) = \frac{r^{2-n}}{n-1}\left(\int_0^r n s^{n-1}(f(v(s)))^{\frac{n}{n-1}}\,\mathrm{d}s\right)^{\frac{n-1}{n}}. \tag{3.11}$$

According to $(F1)$, we see that $v'(r)\ge 0$ for $r\ge 0$. Thus, v is a non-decreasing function. From (3.11) and $(F1)$ it follows immediately

$$\frac{v'(r)}{r} \le \frac{f(v(r))}{n-1}, \quad r\in(0,R). \tag{3.12}$$

Substituting (3.12) into (3.5), we get

$$\begin{aligned} v''(r)\left(\frac{v'(r)}{r}\right)^{\frac{1}{n-1}} &= \left(\frac{1}{n-1}(f(v(r))^n)\right)^{\frac{1}{n-1}} - (n-2)\left(\frac{v'(r)}{r}\right)^{\frac{n}{n-1}}\\ &\ge \left(\frac{1}{n-1}\right)^{\frac{n}{n-1}}[f(v(r))]^{\frac{n}{n-1}}.\end{aligned}$$

Multiplying both sides of the above inequality by $v'(r)$, we have

$$\left[(v'(r))^{\frac{2n-1}{n-1}}\right]' \ge \frac{2n-1}{n-1}\left(\frac{1}{n-1}\right)^{\frac{n}{n-1}} r^{\frac{1}{n-1}} F'(v(r)).$$

Integrating the above inequality from 0 to r and using the first equality in (3.6), we get

$$\begin{aligned} v'(r) &\ge \left(\frac{2n-1}{n-1}\right)^{\frac{n-1}{2n-1}}\left(\frac{1}{n-1}\right)^{\frac{n}{2n-1}}\left[\int_0^r s^{\frac{1}{n-1}}F'(v(s))\,\mathrm{d}s\right]^{\frac{n-1}{2n-1}}\\ &= \left(\frac{2n-1}{n-1}\right)^{\frac{n-1}{2n-1}}\left(\frac{1}{n-1}\right)^{\frac{n}{2n-1}}\left[r^{\frac{1}{n-1}}F(v(r)) - \frac{1}{n-1}\int_0^r s^{\frac{2-n}{n-1}}F(v(s))\,\mathrm{d}s\right]^{\frac{n-1}{2n-1}}.\end{aligned} \tag{3.13}$$

Noticing that $v(r)$ is convex in $B_R(0)$, then the term $s^{-1}F(v(s))$ is non-decreasing for $0 < s < R$, so is $s^{\frac{2-n}{n-1}}F(v(s))$. By (3.13) we have

$$v'(r) \geq \left(\frac{2n-1}{n-1}\right)^{\frac{n-1}{2n-1}} \left(\frac{1}{n-1}\right)^{\frac{n}{2n-1}} \left[r^{\frac{1}{n-1}}F(v(r)) - \frac{1}{n-1}r^{\frac{1}{n-1}}F(v(r))\right]^{\frac{n-1}{2n-1}}$$
$$= \left[b(n)r(F(v(r)))^{n-1}\right]^{\frac{1}{2n-1}},$$

which shows that

$$[b(n)r]^{\frac{1}{2n-1}} \leq [F(v(r))]^{-\frac{n-1}{2n-1}} v'(r), \tag{3.14}$$

where $b(n) = \frac{[(2n-1)(n-2)]^{n-1}}{(n-1)^{3n-2}}$.

Integrating (3.9) and (3.14), we have

$$[b(n)]^{\frac{1}{2n-1}} \int_0^r t^{\frac{1}{2n-1}}\,dt \leq \int_{\tilde{v}_0}^{v(r)} (F(t))^{-\frac{n-1}{2n-1}}\,dt \leq [a(n)]^{\frac{1}{2n-1}} \int_0^r t^{\frac{1}{2n-1}}\,dt. \tag{3.15}$$

From (3.15) we have

$$c(n)\left[\int_{\tilde{v}_0}^{v(r)} (F(t))^{-\frac{n-1}{2n-1}}\,dt\right]^{\frac{2n-1}{2n}} \leq r \leq C(n)\left[\int_{\tilde{v}_0}^{v(r)} (F(t))^{-\frac{n-1}{2n-1}}\,dt\right]^{\frac{2n-1}{2n}}, \tag{3.16}$$

where $c(n) := [a(n)]^{-\frac{1}{2n}}(\frac{2n}{2n-1})^{\frac{2n-1}{2n}}$, and $C(n) := [b(n)]^{-\frac{1}{2n}}(\frac{2n}{2n-1})^{\frac{2n-1}{2n}}$. Letting $r \to R$ in (3.16) and $\alpha \to +\infty$, we get

$$c(n)\left[\int_{v_0}^{+\infty} (F(t))^{-\frac{n-1}{2n-1}}\,dt\right]^{\frac{2n-1}{2n}} \leq R \leq C(n)\left[\int_{v_0}^{+\infty} (F(t))^{-\frac{n-1}{2n-1}}\,dt\right]^{\frac{2n-1}{2n}}. \tag{3.17}$$

If $v_0 = +\infty$, by (F2), the integral $\int_{v_0}^{+\infty}(F(t))^{-\frac{n-1}{2n-1}}\,dt = 0$, which contracts with the second inequality in (3.17). Then we know that $v_0 < +\infty$. We have proved that $\alpha = v(R)$ is infinite for some finite v_0.

We can see that for each value v_0, v becomes infinite at some value of r in the range shown by (3.16). Let us denote by $\delta(v_0)$ the corresponding value of r when v becomes infinite.

The function $\delta(v_0)$ is continuous and non-increasing. If it increases, then two solutions corresponding to different values of v_0 must equal some value of r. However, this is impossible because the solution of the ordinary differential Eq. (3.5) with prescribed values on the sphere is unique. For $r = \delta(v_0)$, letting $\alpha \to +\infty$ in (3.16), we have

$$c(n)\left[\int_{v_0}^{+\infty} (F(t))^{-\frac{n-1}{2n-1}}\,dt\right]^{\frac{2n-1}{2n}} \leq r \leq C(n)\left[\int_{v_0}^{+\infty} (F(t))^{-\frac{n-1}{2n-1}}\,dt\right]^{\frac{2n-1}{2n}}, \tag{3.18}$$

By (F2), the integral $\int_{v_0}^{+\infty}(F(t))^{-\frac{n-1}{2n-1}}\,dt \to 0$ as $v_0 \to +\infty$. Thus, according to (3.18), $\delta(v_0)$ has the same behavior as $\int_{v_0}^{+\infty}(F(t))^{-\frac{n-1}{2n-1}}\,dt$, that is

$$\lim_{v_0 \to +\infty} \delta(v_0) = 0.$$

We now define $\zeta(\delta)$ as the "inverse" of $\delta(v_0)$, namely, $\zeta(\delta) = \min\{v_0 | \delta(v_0) = R\}$. This function is the desired $\zeta(\delta)$ of this lemma, which is decreasing and satisfies $\lim_{\delta\to 0}\zeta(\delta) = +\infty$. Thus, we complete the proof. □

In Lemma 3.1, we get a uniform upper bound estimate for v that satisfies $\det^{1/n}(\Delta v I - D^2 v) = f(v)$ in a ball with prescribed constant boundary, which is an extension of the result of Keller [12] for the Laplace operator to the $(n-1)$ Monge–Ampère operator.

The next lemma shows that the condition $(F2)$ is a necessary and sufficient condition for the solvability of the problem (3.1).

Lemma 3.2. *Assume f satisfies $(F1)$, the problem (3.1) admits an $(n-1)$-convex solution if and only if $(F2)$ holds.*

Proof. *Sufficiency.* Indeed, by the argument in Lemma 3.1, if $(F1)$ holds and $\underline{v}$ exists as in (3.4), then (3.3) admits a unique $(n-1)$-convex solution $v_\alpha \in C^\infty(\overline{B_R(0)})$ for any constant α. This solution v_α is radially symmetric. Otherwise, we could get another solution by rotating v_α, but by comparison principle, v_α is unique. Moreover, by Proposition 2.1 the sequence $\{v_\alpha\}$ is increasing in α at every point of $B_R(0)$. If f satisfies $(F1)$ and $(F2)$, then Lemma 3.1 shows that all of the $\{v_\alpha\}$ are uniformly bounded above (in α) at each point x in $B_R(0)$. By Theorem 1.2 in [8], we get an estimate of the C^2-norm of v_α,

$$\|v_\alpha\|_{C^2(\bar{\Omega}')} \leq C,$$

where the domain Ω' is any closed subdomain of $B_R(0)$ that does not contain a point of $\partial B_R(0)$, the constant C depends on $|v_\alpha|_{C^1(\bar{\Omega}')}$, $|\underline{v}|_{C^2(\bar{\Omega}')}$, $|f|_{C^2}$ and $\inf f$. Thus the Arzelà–Ascoli theorem asserts that there exists a subsequence $\{v_{\alpha_i}\}_{i=1}^{\infty}$ of $\{v_\alpha\}$ which converges uniformly to a limit v. Since $\{v_\alpha\}$ is $(n-1)$-convex radial solution of (3.3) and the convergence $v_{\alpha_i}(x) \to v(x)$ holds in $C^2(B_R(0))$, then the limit $v \in C^2(B_R(0))$ is an $(n-1)$-convex radial solution of (3.3). As x approaches $\partial B_R(0)$, $v(x)$ increases infinitely, since $v_\alpha = \alpha$ becomes infinite on $\partial B_R(0)$. Thus v is the desired solution of the problem (3.1).

Necessity. Assume on the contrary that there exists $a_0 \geq \eta$ such that

$$\Phi(t) := \int_{a_0}^{t} (F(\tau))^{-\frac{n-1}{2n-1}} \, d\tau \to +\infty, \quad \text{as } t \to +\infty, \tag{3.19}$$

and (3.1) admits an $(n-1)$-convex solution v. We assume $a_0 > v(0)$. From *Step* 3 of the proof of Theorem 1.1 in [18], we know that $(F1)$ and (3.19) imply the Cauchy problem

$$\begin{cases} u'(r) = \dfrac{r^{2-n}}{n-1} \left(\displaystyle\int_0^r n s^{n-1} f^{\frac{n}{n-1}}(u(s)) \, ds \right)^{\frac{n-1}{n}}, & r \in [0, \infty), \\ u(0) = a_0, \end{cases} \tag{3.20}$$

admits an entire solution $u(r) \in C^2[0, \infty)$. In other words, $u(r)$ is a radial solution to the equation in (3.1) and can be extended to the whole space, where $R = \infty$. According to $(F1)$ and (3.20), u is positive and bounded in $\overline{B_R(0)}$. By comparison principle, we get $u \leq v$ in $B_R(0)$, which contradicts the fact that $u(0) > v(0)$. □

We can characterize the boundary blow-up rate of $v(r)$ to the problem (3.1) in terms of $R - r$ as follows.

Lemma 3.3. *Let $v(r)$ be a solution of problem (3.1), if $(F1)$ and $(F2)$ hold, then*

$$\lim_{r \to R} \frac{\Psi(v(r))}{R-r} = [a(n)R]^{\frac{1}{2n-1}},$$

where $a(n) = \frac{(2n-1)^{n-1}}{(n-1)^n}$.

Proof. By Lemma 3.1, we have that (3.5) and the first equality in (3.6) hold, and that $v(R) = +\infty$. Thus, from (3.7), we can also obtain (3.9). Integrating (3.9) from r to R, we get

$$\begin{aligned} \int_{v(r)}^{v(R)} (F(t))^{-\frac{n-1}{2n-1}} \, dt &\leq [a(n)]^{\frac{1}{2n-1}} \int_r^R t^{\frac{1}{2n-1}} \, dt \\ &= [a(n)]^{\frac{1}{2n-1}} \frac{2n-1}{2n} (R^{\frac{2n}{2n-1}} - r^{\frac{2n}{2n-1}}). \end{aligned} \tag{3.21}$$

Since $\lim_{r \to R} \frac{R^{\frac{2n}{2n-1}} - r^{\frac{2n}{2n-1}}}{R-r} = \frac{2n}{2n-1} R^{\frac{1}{2n-1}}$ and $v(R) = +\infty$, from (3.21) we obtain that

$$\limsup_{r \to R} \frac{\Psi(v(r))}{R-r} \leq [a(n)R]^{\frac{1}{2n-1}}. \tag{3.22}$$

To prove the reverse inequality, we use (3.7) again, but this time we integrate the equality from r_1 to r. For $0 < r_1 < r < R$, we have

$$\left(r^{n-2} v'(r)\right)^{\frac{2n-1}{n-1}} = \left(r_1^{n-2} v'(r_1)\right)^{\frac{2n-1}{n-1}} + [a(n)r]^{\frac{1}{n-1}} \int_{r_1}^{r} t^{2n-3} F'(v(t)) \, dt,$$

namely

$$\begin{aligned} \left(v'(r)\right)^{\frac{2n-1}{n-1}} &= \left[\left(\frac{r_1}{r}\right)^{n-2} v'(r_1)\right]^{\frac{2n-1}{n-1}} + [a(n)r]^{\frac{1}{n-1}} \int_{r_1}^{r} \left(\frac{t}{r}\right)^{2n-3} F'(v(t)) \, dt \\ &= \left[\left(\frac{r_1}{r}\right)^{n-2} v'(r_1)\right]^{\frac{2n-1}{n-1}} + [a(n)r]^{\frac{1}{n-1}} \left[F(v(r)) - F(v(r_1))\right] \\ &\quad + [a(n)r]^{\frac{1}{n-1}} \int_{r_1}^{r} \left[\left(\frac{t}{r}\right)^{2n-3} - 1\right] F'(v(t)) \, dt. \end{aligned} \tag{3.23}$$

It is readily checked that the last integral in (3.23) is negative. Moreover, we have

$$\left| \int_{r_1}^{r} \left[\left(\frac{t}{r}\right)^{2n-3} - 1\right] F'(v(t)) \, dt \right| \leq \left[1 - \left(\frac{r_1}{R}\right)^{2n-3}\right] \left[F(v(r)) - F(v(r_1))\right]. \tag{3.24}$$

Then for any $\varepsilon > 0$, we choose r_1 such that $1 - \left(\frac{r_1}{R}\right)^{2n-3} < \varepsilon$. Substituting (3.24) into (3.23), we have

$$\begin{aligned} \left(v'(r)\right)^{\frac{2n-1}{n-1}} &\geq \left[\left(\frac{r_1}{r}\right)^{n-2} v'(r_1)\right]^{\frac{2n-1}{n-1}} + [a(n)r]^{\frac{1}{n-1}} (1-\varepsilon) \left[F(v(r)) - F(v(r_1))\right] \\ &\geq [a(n)r]^{\frac{1}{n-1}} (1-\varepsilon) \left[F(v(r)) - F(v(r_1))\right], \quad r_1 < r < R. \end{aligned} \tag{3.25}$$

Since $F(v)$ is increasing function with $F(v) \to +\infty$ as $v \to +\infty$, for $\varepsilon > 0$, there exists $r_2 > 0$ such that

$$F(v(r)) > \frac{1-\varepsilon}{\varepsilon} F(v(r_1)), \quad r > r_2.$$

Then for $\max\{r_1, r_2\} < r < R$, (3.25) yields

$$\left(v'(r)\right)^{\frac{2n-1}{n-1}} \geq [a(n)r]^{\frac{1}{n-1}}(1-2\varepsilon)F(v(r)),$$

which shows that

$$[F(v(r))]^{-\frac{n-1}{2n-1}}v'(r) \geq [a(n)]^{\frac{1}{2n-1}}(1-2\varepsilon)^{\frac{n-1}{2n-1}}r^{\frac{1}{2n-1}}. \tag{3.26}$$

Integrating (3.26) from r to R, we get

$$\begin{aligned}\int_{v(r)}^{v(R)}[F(t)]^{-\frac{n-1}{2n-1}}\,\mathrm{d}t &\geq [a(n)]^{\frac{1}{2n-1}}(1-2\varepsilon)^{\frac{n-1}{2n-1}}\int_r^R t^{\frac{1}{2n-1}}\,\mathrm{d}t \\ &= [a(n)]^{\frac{1}{2n-1}}(1-2\varepsilon)^{\frac{n-1}{2n-1}}\frac{2n-1}{2n}\left(R^{\frac{2n}{2n-1}} - r^{\frac{2n}{2n-1}}\right).\end{aligned} \tag{3.27}$$

When r_1 is close enough to R, there exists r_2 sufficiently close to R such that $\varepsilon \to 0$. Since ε is arbitrary and $v(R) = +\infty$, by taking the limit on both sides of (3.27), we obtain

$$\liminf_{r\to R}\frac{\Psi(v(r))}{R-r} \geq [a(n)R]^{\frac{1}{2n-1}}. \tag{3.28}$$

Combining (3.22) and (3.28), the proof is done. □

4. Existence

In this section, we give a proof of Theorem 1.1, which states that the condition $(F2)$ is a necessary and sufficient condition for the solvability of the boundary blow-up problem (1.1)–(1.2).

Proof of Theorem 1.1. First, we prove the sufficiency. From Lemma 2.1, we can see that (2.1) admits a solution $u_k \in C^\infty(\bar{\Omega})$, provided that there exists a corresponding sub-solution $\underline{u} \in C^2(\bar{\Omega})$ satisfying

$$\det{}^{1/n}(\Delta\underline{u}I - D^2\underline{u}) \geq g(x,\underline{u}) \quad \text{in } \Omega, \quad \underline{u}|_{\partial\Omega} = k, \tag{4.1}$$

for $k \in \mathbb{Z}^+$. Therefore, there exists a sequence $\{u_k\}_{k=1}^\infty$, which is increasing with respect to k in $\bar{\Omega}$ in view of Proposition 2.1. Indeed, we can prove that such sub-solution $\underline{u}$ in (4.1) exists for each $k \in \mathbb{Z}^+$ under the strict convexity of the domain Ω.

Since Ω is a bounded, strictly convex domain in $\mathbb{R}^n$ with smooth boundary, from Theorem 1.1 in [27], there exists a unique strictly convex solution $\psi \in C^\infty(\bar{\Omega})$ satisfying

$$\det D^2\psi = 1 \text{ in } \Omega, \quad \psi|_{\partial\Omega} = 0. \tag{4.2}$$

Thus ψ can be referred as the defining function for Ω such that $\psi = 0, D\psi \neq 0$ on $\partial\Omega$ and the matrix $\{\psi_{x_ix_j}\}$ is positive definite in $\bar{\Omega}$. Then no eigenvalue of $\{\psi_{x_ix_j}\}$ can be zero at any point of $\bar{\Omega}$ and the trace of $\{\psi_{x_ix_j}\}$ is positive in $\bar{\Omega}$. By maximum principle, $\psi < 0$ in Ω and the eigenvalues of $\{\psi_{x_ix_j}\}$ are strictly positive in $\bar{\Omega}$. Therefore, there exists a positive constant α such that

$$\{\psi_{x_ix_j}\} \geq \alpha I, \quad \text{in } \Omega.$$

We take

$$\underline{u} = k + \mu(e^{\rho\psi} - 1), \tag{4.3}$$

where μ and ρ are positive constants to be determined. Then

$$\underline{u} = k, \quad \text{on } \partial\Omega \tag{4.4}$$

automatically, and the (i,j)th entry of the second derivative of $\underline{u}$ is

$$\underline{u}_{x_ix_j} = \mu\rho e^{\rho\psi}(\psi_{x_ix_j} + \rho\psi_{x_i}\psi_{x_j}), \quad \text{in } \Omega. \tag{4.5}$$

According to $\psi_{x_ix_j} > 0$, we see from (4.5) that $\underline{u}_{x_ix_j} > 0$ in Ω, that is $\underline{u}$ is strictly convex. By the definition of $(n-1)$ Monge–Ampère operator, we deduce that

$$\det(\Delta\underline{u}I - D^2\underline{u}) > \det D^2\underline{u}, \quad \text{in } \Omega. \tag{4.6}$$

By Lemma 2.3, the determinant of $D^2\underline{u}$ can be written as

$$\det D^2\underline{u} = (\mu\rho e^{\rho\psi})^n \det D^2\psi\left[1 + \rho(D\psi)^T(D^2\psi)^{-1}D\psi\right], \quad \text{in } \Omega.$$

Note that the matrix $\{\psi^{-1}_{x_ix_j}\}$ is strictly positive in $\bar{\Omega}$, it can be inferred that there exists a constant $\bar{\alpha} > 0$ such that

$$(D\psi)^T(D^2\psi)^{-1}D\psi \geq \bar{\alpha}|D\psi|^2, \quad \text{in } \Omega. \tag{4.7}$$

Then we have

$$\det D^2\underline{u} \geq (\mu\rho e^{\rho\psi})^n\left[1 + \rho\bar{\alpha}|D\psi|^2\right], \quad \text{in } \Omega, \tag{4.8}$$

where (4.2) and (4.7) are used. Furthermore, according to conditions $(G1)$ and $(F1)$, we see that

$$g(x,\underline{u}) \leq c_2 f(\underline{u}) \leq c_2 f(k), \quad \text{in } \bar{\Omega}, \tag{4.9}$$

where $\underline{u}|_{\partial\Omega} = k$ and the strict convexity $\underline{u}$ are used. Since $\psi < 0$ in Ω, we can first fix ρ to be some positive constant, and then choose μ sufficiently large, such that

$$(\mu\rho e^{\rho\psi})^n \left[1 + \rho\bar{\alpha}|D\psi|^2\right] \geq c_2^n f^n(k), \quad \text{in } \Omega. \tag{4.10}$$

Combining (4.4), (4.6), (4.8), (4.9) and (4.10), we have proved that $\underline{u}$ in (4.3) is the required sub-solution satisfying (4.1).

Next, we show that the sequence $\{u_k\}_{k=1}^{\infty}$ is uniformly bounded in Ω. For each $y \in \partial\Omega$, since Ω is a bounded domain in $\mathbb{R}^n$ with C^2 boundary, there exists $R_1 > 0$ such that $B_{R_1}(x_1) \subset \Omega$ with $y \in \partial B_{R_1}(x_1)$ and $x_1 \in \Omega$. It follows from Lemma 3.2 that there exists a solution $U(r)$ of the problem

$$\begin{cases} \det^{1/n}(\Delta U(r) I - D^2 U(r)) = c_1 f(U(r)), & r \in [0, R_1), \\ U(r) \to +\infty, & \text{as } r \to R_1, \end{cases}$$

where $r = |x - x_1|$. Note that $U(r)$ is an increasing function. Define a function $\tilde{U}$ as

$$\tilde{U}(x) = \begin{cases} U(0), & \text{if } \operatorname{dist}(x, \partial\Omega) > R_1, \\ U(R_1 - \operatorname{dist}(x, \partial\Omega)), & \text{if } \operatorname{dist}(x, \partial\Omega) \leq R_1. \end{cases}$$

Since the same radius R_1 may be used for every boundary point $y \in \partial\Omega$, we can use the function $\tilde{U}$ defined above as a uniform upper bound for $\{u_k\}_{k=1}^{\infty}$ where u_k solves (2.1), that is

$$u_k(x) \leq \tilde{U}(x) \tag{4.11}$$

at each point in Ω and for every $k \in \mathbb{Z}^+$. By Theorem 1.2 of [8], we get an estimate of the $C^2(\bar{\Omega})$-norm of u_k,

$$\|u_k\|_{C^2(\bar{\Omega})} \leq C,$$

for each $k \in \mathbb{Z}^+$, where the constant C depends on $|u_k|_{C^1(\bar{\Omega})}$, $|\underline{u}|_{C^2(\bar{\Omega})}$, $|f|_{C^2}$ and $\inf f$. We remark that $\min u_1 \leq \min u_k$ in $\bar{\Omega}$ for every $k \in \mathbb{Z}^+$, and by (4.11), we obtain a uniform bound with respect to k. Thus the Arzelà–Ascoli theorem implies the existence of a C^2 function u and a subsequence $\{u_{k_i}\}$ that converges uniformly to u in every closed subdomain of Ω. By Lemma 2.1, $u_k \in C^\infty(\bar{\Omega})$ is $(n-1)$-convex solution of (2.1), then the limit $u \in C^2(\Omega)$ satisfies

$$\det^{1/n}(\Delta u I - D^2 u) = \lim_{k_i \to \infty} \det^{1/n}(\Delta u_{k_i} I - D^2 u_{k_i}) = \lim_{k_i \to \infty} g(x, u_{k_i}) = g(x, u), \quad \text{in } \Omega.$$

From the standard elliptic theory it follows that $u \in C^\infty(\Omega)$. When x is close to the boundary, the value of u increases infinitely since $u_k = k$ becomes infinite on $\partial\Omega$ as $k \to +\infty$. Thus $u \in C^\infty(\Omega)$ is the desired solution of the problem (1.1)–(1.2).

Finally, we prove the necessity, which is similar to the necessity proof in Lemma 3.2. We assume on the contrary that there exists a constant $a_0 > \eta$ such that (3.19) holds and the problem (1.1)–(1.2) admits an $(n-1)$ convex solution $u \in C^\infty(\Omega)$. For $x_0 \in \Omega$, let $r = |x - x_0|$ and $a_0 > u(x_0)$, under (F1) and (3.19), the Cauchy problem

$$\begin{cases} w'(r) = \dfrac{c_2 r^{2-n}}{n-1} \left(\displaystyle\int_0^r n s^{n-1} f^{\frac{n}{n-1}}(w(s))\, ds \right)^{\frac{n-1}{n}}, & r \in [0, \infty), \\ w(0) = a_0, \end{cases}$$

has an entire solution $w(r) \in C^2[0, \infty)$, where c_2 is the constant in (G1), see *Step* 3 of the proof of Theorem 1.1 in [18]. Since $(F1)$ and (3.19) hold, it follows from Theorem 1.1 in [18] that there exists an entire sub-solution $w \in C^2(\mathbb{R}^n)$ with $w(x_0) = a_0$ satisfying

$$\det^{1/n}(\Delta w I - D^2 w) \geq c_2 f(w), \text{ in } \mathbb{R}^n. \tag{4.12}$$

Using (G1) in (4.12), then w satisfies

$$\det^{1/n}(\Delta w I - D^2 w) \geq g(x, w), \text{ in } \Omega. \tag{4.13}$$

Since $w \in C^2(\bar{\Omega})$, $u \in C^\infty(\Omega)$ satisfying $u \to +\infty$ as $d(x) \to 0$, and $w(x_0) = a_0 > u(x_0)$, there exists an open subset D containing x_0 such that $\bar{D} \subset \Omega$ and

$$u(x) < w(x) \text{ in } D \quad \text{and} \quad u(x) = w(x) \quad \text{on } \partial D.$$

On the other hand, u satisfies

$$\det^{1/n}(\Delta u I - D^2 u) = g(x, u) \text{ in } D \quad \text{and} \quad u(x) = w(x) \quad \text{on } \partial D.$$

By Lemma 2.2 and recall (4.13), we obtain $w \leq u$ in D, which contradicts $w(x_0) > u(x_0)$. Thus we prove the necessity.

In conclusion, we complete the proof of Theorem 1.1. □

Remark 4.1. By constructing the family of solutions $u_k \in C^\infty(\bar{\Omega})$ and the comparison principle, the limit u is the smallest large solution of the problem (1.1)–(1.2), that is $u(x) \leq u^*(x)$ in Ω for every solution u^* of the problem (1.1)–(1.2).

Remark 4.2. Note that in the proof of the sufficiency of Theorem 1.1, we can remove the strict convexity condition of the domains, provided that we alternatively assume that there exist strict sub-solutions $\underline{u}_k \in C^2(\Omega)$ of Eqs. (1.1) with $\underline{u}_k = k$ on $\partial\Omega$ for each integer k.

5. Asymptotic behavior and uniqueness

In this section, we establish the asymptotic behavior of solutions in smooth strictly convex domains. In addition, in star-shaped domains, we prove the uniqueness of solutions under the condition $(G3)$.

By giving upper and lower bounds and applying Lemma 3.3, we prove the boundary asymptotic behavior of boundary blow-up solutions of (1.1).

Proof of Theorem 1.2. Let x be a point in Ω near the boundary. For $y \in \partial\Omega$ such that $d(x) = |x-y|$, since Ω is bounded strict convex with smooth boundary, there exist positive constants R_1 and R_2 such that $B_{R_1}(x_1) \subset \Omega \subset B_{R_2}(x_2)$ with $y \in \partial B_{R_1} \cap \partial B_{R_2}$, where $x_1 \in \Omega$ and $x_2 \in \mathbb{R}^n$. For $\varepsilon > 0$, we consider the following boundary blow-up problems in balls $B_{R_1-\varepsilon}$ and $B_{R_2+\varepsilon}$:

$$\begin{cases} \det^{1/n}(\Delta v_1^\varepsilon(x) I - D^2 v_1^\varepsilon(x)) = c_1 f(v_1^\varepsilon(x)), & x \in B_{R_1-\varepsilon}(x_1), \\ v_1^\varepsilon(x) \to +\infty, & x \to \partial B_{R_1-\varepsilon}(x_1), \end{cases} \tag{5.1}$$

and

$$\begin{cases} \det^{1/n}(\Delta v_2^\varepsilon(x) I - D^2 v_2^\varepsilon(x)) = c_2 f(v_2^\varepsilon(x)), & x \in B_{R_2+\varepsilon}(x_2), \\ v_2^\varepsilon(x) \to +\infty, & x \to \partial B_{R_2+\varepsilon}(x_2). \end{cases} \tag{5.2}$$

Let L^ε denote a ray going from x to y, and let x_1^ε and x_2^ε denote the points in $L^\varepsilon \cap \partial B_{R_1-\varepsilon}$ and $L^\varepsilon \cap \partial B_{R_2+\varepsilon}$, respectively. The boundary blow-up estimates of the solution u of the problem (1.1)–(1.2) are obtained by comparing with the radial solutions of problems (5.1) and (5.2) in the balls $B_{R_1-\varepsilon}$ and $B_{R_2+\varepsilon}$, respectively.

We first prove the upper bound of u. From Lemma 3.2, (5.1) admits $(n-1)$-convex solution $v_1^\varepsilon(r) = v_1^\varepsilon(|x-x_1|)$ and $v_1^\varepsilon(r) \to +\infty$ as $r \to R_1 - \varepsilon$, i.e.,

$$v_1^\varepsilon(|x-x_1|) \to +\infty, \quad \text{as } |x-y| \to \varepsilon.$$

Since $u \in C^2(\overline{B_{R_1-\varepsilon}})$, and thus u is finite on $\partial B_{R_1-\varepsilon}$. Furthermore, $v_1^\varepsilon(r) = +\infty$ on $\partial B_{R_1-\varepsilon}$, we see that $u \le v_1^\varepsilon$ on $\partial B_{R_1-\varepsilon}$. Since the condition $(G1)$ holds, it can be obtained from Lemma 2.2 that

$$u \le v_1^\varepsilon, \quad \text{in } B_{R_1-\varepsilon}. \tag{5.3}$$

Since ε is independent of x, then for $x \in \Omega$, letting $\varepsilon \to 0$, we get that $\lim_{\varepsilon\to 0} \frac{|x-y|}{|x-x_1^\varepsilon|} = 1$. Combining this with (5.3), we have

$$\lim_{x\in\Omega, x\to y} u(x) \le \lim_{\varepsilon\to 0} \lim_{x\in\Omega, x\to x_1^\varepsilon} v_1^\varepsilon(x) = \lim_{x\in\Omega, x\to y} v_1(x).$$

where $v_1(x)$ is the limit solution of the boundary blow-up problem (5.1) for $\varepsilon \to 0$. Note that the existence of the limit solution $v_1(x)$ is guaranteed by Lemma 3.2. By Lemma 3.3, we have

$$\lim_{x\to y} \frac{\Psi(v_1(x))}{|x-y|} = c_1^{\frac{n}{2n-1}} [a(n)]^{\frac{1}{2n-1}} R_1^{\frac{1}{2n-1}}. \tag{5.4}$$

Thus by using the monotonicity of Ψ and (5.4), we have

$$\begin{aligned} \liminf_{x\to y} \frac{\Psi(u(x))}{|x-y|} &\ge \lim_{\varepsilon\to 0} \liminf_{x\to x_1^\varepsilon} \frac{\Psi(v_1^\varepsilon(x))}{|x-y|} \frac{|x-y|}{|x-x_1^\varepsilon|} \\ &= \lim_{\varepsilon\to 0} \liminf_{|x-y|\to\varepsilon} \frac{\Psi(v_1^\varepsilon(x))}{|x-y|-\varepsilon} \\ &= \liminf_{x\to y} \frac{\Psi(v_1(x))}{|x-y|} \\ &= c_1^{\frac{n}{2n-1}} [a(n)]^{\frac{1}{2n-1}} R_1^{\frac{1}{2n-1}}, \end{aligned}$$

for every $x \in \Omega$. This completes the proof of the left-hand side of the inequality in (1.7).

Similarly, $u \ge v_2^\varepsilon$ in Ω can be proved by (5.2) and Lemma 2.2. Repeating the above steps and using Lemma 3.3 again, the right-hand side of the inequality (1.7) is also obtained. Thus, we have completed the proof. □

Next, we use a scaling technique to investigate the uniqueness of this solution in a star-shaped domain Ω with respect to a point $x_0 \in \Omega$.

Proof of Theorem 1.3. Assume on the contrary that $u_1(x)$ and $u_2(x)$ are both solutions to the problem (1.1)–(1.2), namely

$$\det^{1/n}(\Delta u_1 I - D^2 u_1) = g(x, u_1), \quad \text{in } \Omega,$$

$$\det{}^{1/n}(\Delta u_2 I - D^2 u_2) = g(x, u_2), \quad \text{in } \Omega, \tag{5.5}$$

$$u_1(x), u_2(x) \to +\infty, \quad \text{as } x \to \partial\Omega.$$

Without loss of generality, we let $x_0 = 0$. Given two constants $h > 1$ and $p > 1$, let

$$\Omega_h = \{\frac{1}{h}x | x \in \Omega\} \subset \Omega,$$

and

$$w_h(x) = pu_1(hx)$$

for $x \in \Omega_h$. We have

$$\begin{aligned} \det{}^{1/n}(\Delta w_h(x)I - D^2 w_h(x)) &= ph^2 \det{}^{1/n}(\Delta u_1(hx)I - D^2 u_1(hx)) \\ &= ph^2 g(hx, u_1(hx)) \\ &= ph^2 g(hx, \frac{1}{p} w_h(x)) \\ &\le p^{1-\gamma} h^2 g(hx, w_h(x)), \end{aligned} \tag{5.6}$$

for $x \in \Omega_h$, where the condition $(G3)$ is used to obtain the inequality in (5.6). If we choose $p = p(h)$ such that

$$p^{1-\gamma} = \inf_{(x,z)\in\bar{\Omega}_h\times\mathbb{R}} \frac{g(x,z)}{h^2 g(hx,z)},$$

then

$$\det{}^{1/n}(\Delta w_h(x)I - D^2 w_h(x)) \le g(x, w_h(x)), \quad \text{in } \Omega_h. \tag{5.7}$$

Since $\gamma > 1$ and $p > 1$, we see that

$$p(h) \to 1 \quad \text{as } h \to 1. \tag{5.8}$$

Note that $w_h(x) \to +\infty$ as $d(x, \partial\Omega_h) \to 0$ and $u_2 \to +\infty$ as $d(x) \to 0$ and u_2 is a continuous function on $\bar{\Omega}_h$, thus $u_2(x)$ has a finite value on $\partial\Omega_h$. We claim that $u_2(x) \le w_h(x)$ for all $x \in \Omega_h$. Assuming on the contrary that $u_2(x_0) > w_h(x_0)$ for some $x_0 \in \Omega_h$. There exists a subdomain D of Ω_h such that $x_0 \subset D$, $\bar{D} \subset \Omega_h$, $u_2(x) > w_h(x)$ in D, and $u_2(x) = w_h(x)$ on ∂D. However, since (5.5) and (5.7) hold in D, it is obvious from Lemma 2.2 that for any $h > 1$, we get

$$u_2(x) \le p(h)u_1(hx)$$

for all $x \in \Omega_h$, which is a contradiction to $u_2(x_0) > w_h(x_0)$. Since every $x \in \Omega$ is contained in Ω_h, letting $h \to 1$ and recalling (5.8), we get $u_2(x) \le u_1(x)$ in Ω.

By the same argument, we can also get $u_1(x) \le u_2(x)$ in Ω.

Thus, the solution to the problem (1.1)–(1.2) is unique. □

Remark 5.1. Since any convex domain is a star-shaped domain, it follows from Theorem 1.3 that the solution in Theorem 1.1 is unique when the additional condition $(G3)$ holds. Furthermore, the special case $g(x, u) = b(x)u^p$ $(p > 1)$ naturally satisfies the condition $(G3)$.

Data availability

No data was used for the research described in the article.

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