

## Some Remarks on Ekedahl–Oort Stratifications

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Abstract. We study independence of symplectic embeddings of the theory of Ekedahl–Oort stratifications on Shimura varieties of Hodge type, by comparing two different embeddings with a third one. The main results are as follows.

1. The Ekedahl–Oort stratification is independent of the choices of symplectic embeddings.
2. Under certain reasonable assumptions, there is certain functoriality for Ekedahl–Oort stratifications with respect to morphisms of Shimura varieties.

### 1. Introduction

Let  $(G, X)$  be a Shimura datum of Hodge type, and  $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$  be the complex Shimura variety attached to a compact open subgroup  $K \subseteq G(\mathbb{A}_f)$ . We fix a prime number  $p > 2$  and assume that  $K = K_p K^p$ , where  $K_p$  is hyperspecial, i.e., there is a reductive group  $G_{\mathbb{Z}_p}$  over  $\mathbb{Z}_p$  such that  $G_{\mathbb{Z}_p} \otimes \mathbb{Q}_p = G_{\mathbb{Q}_p}$  and that  $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ . By works of Shimura and Deligne,  $\mathrm{Sh}_K(G, X)_{\mathbb{C}}$  is defined over a number field  $E$ , denoted by  $\mathrm{Sh}_K(G, X)$ . It is the Shimura variety attached to  $(G, X, K)$ . Let  $v$  be a place of  $E$  over  $p$ , Kisin proved in [1] that  $\mathrm{Sh}_K(G, X)$  has a smooth model  $\mathcal{S}_K(G, X)$  over  $O_{E,(v)}$ . Moreover,  $\mathcal{S}_K(G, X)$  is uniquely determined by the Shimura datum as  $\varprojlim_{K^p} \mathcal{S}_K(G, X)$  satisfies a certain extension property (see [1, (2.3.7)] for the precise statement).

Ekedahl–Oort stratifications for good reductions of Shimura varieties of Hodge type were defined and studied in [8] (see also [7]) using technics developed in [1, 3]. Let  $\kappa = O_{E,(v)}/(v)$  and  $G_0$  (resp.  $\mathcal{S}_{0,K}(G, X)$  or simply  $\mathcal{S}_0$ ) be the special fiber of  $G_{\mathbb{Z}_p}$  (resp.  $\mathcal{S}_K(G, X)$ ). The Shimura datum determines a cocharacter  $\mu: \mathbb{G}_{m,\kappa} \rightarrow G_{0,\kappa}$  which is unique up to  $G_0(\kappa)$ -conjugacy. We constructed in [8] a morphism  $\zeta: \mathcal{S}_0 \rightarrow G_0\text{-zip}_{\kappa}^{\mu}$ , where  $G_0\text{-zip}_{\kappa}^{\mu}$  is the stack of  $G_0$ -zips of type  $\mu$  (see [3] or the paragraph before §2.3 in this paper). Fibers of  $\zeta$  are the Ekedahl–Oort strata of  $\mathcal{S}_0$ . We emphasize that the construction of  $\zeta$  in [8] (as we recalled right before §2.4 here) depends on a symplectic embedding.

There are many questions that we can ask about  $\zeta$ . Here we mention two of them.

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1. Whether the morphism  $\zeta$  is independent of the choices of symplectic embeddings?
2. How to study behavior of stratifications under morphisms of Shimura varieties?

The philosophy behind the first question is as follows. The scheme  $\mathcal{S}_0$  should have a moduli interpretation in terms of motives depending only on  $(G, X, K^p)$ , and one could obtain the morphism  $\zeta$  by taking (geometric) isomorphism types of de Rham realizations of those motives. This would give an intrinsic construction for  $\zeta$ . Unfortunately, no moduli interpretation is known so far for reductions of general Shimura varieties of Hodge type, let alone an intrinsic one. A reasonable candidate for de Rham realizations of those motives is given in [3, Definition 7.1], and we constructed a morphism  $\zeta$  by fixing a symplectic embedding. It would be a great evidence if one could conclude that  $\zeta$  is actually independent of the choices of symplectic embeddings.

The first question is solved by the following theorem.

**Theorem 1.1.** *The morphism  $\zeta$  is independent of choices of symplectic embeddings.*

Section 2 of this note is devoted to a proof of the above statement, by comparing the  $\zeta$ s induced by different symplectic embeddings.

The second question is too general and too inexplicit to study, so we raise the following question. Let  $f: (G, X) \rightarrow (G', X')$  be a morphism of Shimura data of Hodge type. Let  $E$  and  $E'$  be their reflex fields. Then  $E \supseteq E'$ . Let  $K \subseteq G(\mathbb{A}_f)$  and  $K' \subseteq G'(\mathbb{A}_f)$  be such that  $K_p$  and  $K'_p$  are hyperspecial. Assume that  $f(K) \subseteq K'$ , then there is a morphism  $f: \text{Sh}_K(G, X) \rightarrow \text{Sh}_{K'}(G', X')_E$ . Let  $v'$  be a place of  $E'$  over  $p$  with residue field  $\kappa'$  and  $v$  be a place of  $E$  over  $v'$  with residue field  $\kappa$ , then there is a morphism  $\mathcal{S}_K(G, X) \rightarrow \mathcal{S}_{K'}(G', X')_{O_{E,(v)}}$  extending  $f$ . Still write  $f$  for the morphism on special fibers  $\mathcal{S}_{0,K}(G, X) \rightarrow \mathcal{S}_{0,K'}(G', X')_\kappa$ . Let  $G_0$  (resp.  $G'_0$ ) be the reduction of  $G$  (resp.  $G'$ ), and let  $\mu$  (resp.  $\mu'$ ) be the cocharacter determined by the Shimura datum unique up to conjugacy. Then there is a morphism  $\zeta: \mathcal{S}_{0,K}(G, X) \rightarrow G_0\text{-zip}_\kappa^\mu$  (resp.  $\zeta': \mathcal{S}_{0,K'}(G', X') \rightarrow G'_0\text{-zip}_{\kappa'}^{\mu'}$ ) giving the Ekedahl–Oort strata on  $\mathcal{S}_{0,K}(G, X)$  (resp.  $\mathcal{S}_{0,K'}(G', X')$ ).

The question is, whether there is any compatibility between  $f, \zeta$  and  $\zeta'$ . We have the following result.

**Theorem 1.2.** *Assume that  $f_{\mathbb{Q}_p}$  extends to a morphism of reductive group schemes over  $\mathbb{Z}_p$ , then there is a canonical commutative diagram*

$$\begin{array}{ccc}
 \mathcal{S}_{0,K}(G, X) & \xrightarrow{f} & \mathcal{S}_{0,K'}(G', X')_\kappa \\
 \downarrow \zeta & & \downarrow \zeta' \otimes \kappa \\
 G_0\text{-zip}_\kappa^\mu & \xrightarrow{\alpha} & G'_0\text{-zip}_{\kappa'}^{\mu'} \otimes \kappa.
 \end{array}$$

The proof of this result will be given in Section 3.

This note was written and posted on ArXiv in early 2014, during the author’s stay in the Max Planck Institute (Bonn). It was not intended to be published at first, and most of the main results here are recovered 3 years later in our work [4], based on Tom Lovering’s conceptual but very difficult works which uses highly non-trivial  $p$ -adic theories. We finally decided to get it published, as the arguments in this note is more elementary and direct, and hence is completely different from the one in [4]. Moreover, we believe that the ideas and methods here could be helpful for considerations on other questions.

## 2. Independence of symplectic embeddings

Notations as in the introduction, let  $\mathcal{S}_0$  be the special fiber of  $\mathcal{S}_K(G, X)$ . By [8], there is a theory of Ekedahl–Oort stratification on  $\mathcal{S}_0$ . To define the stratification, we need to fix a symplectic embedding, while the variety  $\mathcal{S}_0$  is independent of symplectic embeddings. A natural question is whether different symplectic embeddings give the same stratification.

Let us first recall the construction of the integral models and Ekedahl–Oort stratifications on their special fibers.

### 2.1. Integral canonical models

Notations as before, by [5, Proposition 3.1.2.1(e)],  $G_{\mathbb{Z}_p}$  is the unique reductive group extending  $G_{\mathbb{Q}_p}$  such that  $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$ . Let  $i: (G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^\pm)$  be a symplectic embedding. Then by [1, Lemma 2.3.1], there exists a  $\mathbb{Z}_p$ -lattice  $V_{\mathbb{Z}_p} \subseteq V_{\mathbb{Q}_p}$ , such that  $i_{\mathbb{Q}_p}: G_{\mathbb{Q}_p} \subseteq \mathrm{GL}(V_{\mathbb{Q}_p})$  extends uniquely to a closed embedding  $G_{\mathbb{Z}_p} \hookrightarrow \mathrm{GL}(V_{\mathbb{Z}_p})$ . So there is a  $\mathbb{Z}$ -lattice  $V_{\mathbb{Z}} \subseteq V$  such that  $G_{\mathbb{Z}(p)}$ , the Zariski closure of  $G$  in  $\mathrm{GL}(V_{\mathbb{Z}(p)})$ , is reductive, as the base change to  $\mathbb{Z}_p$  of  $G_{\mathbb{Z}(p)}$  is  $G_{\mathbb{Z}_p}$ . Moreover, we can assume  $V_{\mathbb{Z}}$  is such that  $V_{\mathbb{Z}}^\vee \supseteq V_{\mathbb{Z}}$ . Let  $d = |V_{\mathbb{Z}}^\vee/V_{\mathbb{Z}}|$  and  $g = \frac{1}{2} \dim(V)$ , then the integral canonical model  $\mathcal{S}_K(G, X)$  of  $\mathrm{Sh}_K(G, X)$  is constructed as follows. We can choose  $K' \subseteq \mathrm{GSp}(V, \psi)(\mathbb{A}_f)$  small enough such that  $K' \supseteq K$  and that  $\mathrm{Sh}_{K'}(\mathrm{GSp}(V, \psi), S^\pm)$  affords a moduli interpretation. There is a finite morphism  $f: \mathrm{Sh}_K(G, X) \rightarrow \mathrm{Sh}_{K'}(\mathrm{GSp}(V, \psi), S^\pm)_E$ .

Let  $\mathcal{A}_{g,d,K'}$  be the moduli scheme of abelian schemes over  $\mathbb{Z}_{(p)}$ -schemes with a polarization of degree  $d$  and level  $K'$  structure. Then  $\mathrm{Sh}_{K'}(\mathrm{GSp}(V, \psi), S^\pm)$  is a closed subscheme of  $\mathcal{A}_{g,d,K'} \otimes \mathbb{Q}$ . Let us write  $\mathcal{A}_{g,d,K'}$  for its base extensions to  $E$  for simplicity, the integral canonical model  $\mathcal{S}_K(G, X)$  is the normalization of the Zariski closure of  $\mathrm{Sh}_K(G, X)$  in  $\mathcal{A}_{g,d,K'}$ . Here the word “normalization” makes sense. As  $\mathrm{Sh}_K(G, X)$  is regular, and on each open affine,  $O_{\mathcal{S}_K(G, X)}$  is obtained by taking elements in  $O_{\mathrm{Sh}_K(G, X)}$  that is integral over  $O_{\mathcal{A}_{g,d,K'}}$ .

Note that we didn’t assume that  $K'$  is such that the morphism  $f$  is a closed embedding. Because if we take  $K'' \subseteq K'$  small enough such that the induced morphism

$g: \text{Sh}_K(G, X) \rightarrow \text{Sh}_{K''}(\text{GSp}(V, \psi), S^\pm)_E$  is a closed embedding, then  $f$  factors through  $g$ . The natural morphism  $\mathcal{A}_{g,d,K''} \rightarrow \mathcal{A}_{g,d,K'}$  is finite, so the normalization gives the same  $\mathcal{S}_K(G, X)$ .

### 2.2. $G_0$ -zips

Let  $G_0$  (resp.  $V_0$ ) be the special fiber of  $G_{\mathbb{Z}(p)}$  (resp.  $V_{\mathbb{Z}(p)}$ ). We remark that  $G_0$  is uniquely determined by  $(G, K_p)$ . But  $V_0$  is not uniquely determined by  $(G, K_p)$ , there might be many choices. As in [8, 3.2.2], the Shimura datum  $(G, X)$  determines a cocharacter  $\mu: \mathbb{G}_{m,W(\kappa)} \rightarrow G_{W(\kappa)}$  which is unique up to  $G_{\mathbb{Z}_p}(W(\kappa))$ -conjugacy. This cocharacter is of weights 0 and 1 on  $V_{W(\kappa)}^\vee$ . The special fiber of  $\mu$  will still be denoted by  $\mu$ .

**Setting 2.1.** We start with  $G_0$  and  $\mu: \mathbb{G}_{m,\kappa} \rightarrow G_{0,\kappa}$ . For an  $\mathbb{F}_p$ -scheme  $S$ , let  $\sigma: S \rightarrow S$  be the absolute Frobenius. For an  $S$ -scheme  $T$ , we will write  $T^{(p)}$  for the pull back of  $T$  via  $\sigma$ . In particular, we will write  $\mu^{(p)}$  for the pull back via Frobenius of  $\mu$ . Note that it is a cocharacter of  $G_{0,\kappa}$ .

Let  $P_+$  (resp.  $P_-$ ) be the parabolic subgroup of  $G_{0,\kappa}$  whose Lie algebra is the subspace of non-negative weights (resp. non-positive weights) in  $\text{Lie}(G_{0,\kappa})$  under  $\text{Ad} \circ \mu$ . Let  $U_+$  (resp.  $U_-$ ) be the unipotent radical of  $P_+$  (resp.  $P_-$ ), and  $M$  be the common Levi subgroup of  $P_+$  and  $P_-$ . Note that  $M$  is also the centralizer of  $\mu$ .

**Definition 2.2.** [3, Definition 3.1] Let  $S$  be a scheme over  $\kappa$ . A  $G_0$ -zip of type  $\mu$  over  $S$  is a tuple  $\underline{I} = (I, I_+, I_-, \iota)$  consisting of a right  $G_\kappa$ -torsor  $I$  over  $S$ , a right  $P_+$ -torsor  $I_+ \subseteq I$ , a right  $P_-^{(p)}$ -torsor  $I_- \subseteq I$ , and an isomorphism of  $M^{(p)}$ -torsors  $\iota: I_+^{(p)}/U_+^{(p)} \rightarrow I_-/U_-^{(p)}$ .

A morphism  $(I, I_+, I_-, \iota) \rightarrow (I', I'_+, I'_-, \iota')$  of  $G_0$ -zips of type  $\mu$  over  $S$  consists of equivariant morphisms  $I \rightarrow I'$  and  $I_\pm \rightarrow I'_\pm$  that are compatible with inclusions and the isomorphisms  $\iota$  and  $\iota'$ .

Let  $G_0\text{-zip}_\kappa^\mu(S)$  be the category of  $G_0$ -zips of type  $\mu$  over a  $\kappa$ -scheme  $S$ . They form a fibered category  $G_0\text{-zip}_\kappa^\mu$  over the category of  $\kappa$ -schemes. By [3, Corollary 3.12], it is a smooth algebraic stack of dimension 0.

### 2.3. Ekedahl–Oort strata

Now we explain how to construct Ekedahl–Oort stratification following [8]. Let  $\mathcal{A}$  be the pull back to  $\mathcal{S}_K(G, X)$  of the universal abelian scheme on  $\mathcal{A}_{g,d,K'}$ , and  $\mathcal{V}$  be  $H_{\text{dR}}^1(\mathcal{A}/\mathcal{S}_K(G, X))$ . Let  $V_{\mathbb{Z}} \subseteq V$  and  $G_{\mathbb{Z}(p)}$  be as in §2.1. Then by [1, Proposition 1.3.2], there is a tensor  $s \in V_{\mathbb{Z}(p)}^\otimes$  defining  $G_{\mathbb{Z}(p)} \subseteq \text{GL}(L_{\mathbb{Z}(p)})$ , i.e.,  $G_{\mathbb{Z}(p)}$  is the subgroup of  $\text{GL}(V_{\mathbb{Z}(p)})$  acting trivially on  $s$ . By [1, Corollary 2.3.9], the tensor  $s \in V_{\mathbb{Z}(p)}^\otimes$  induces a section  $s_{\text{dR}} \in \mathcal{V}^\otimes$ . Moreover, the scheme

$$I = \mathbf{Isom}_{\mathcal{S}_K(G, X)}((V_{\mathbb{Z}(p)}^\vee, s) \otimes \mathcal{O}_{\mathcal{S}_K(G, X)}, (\mathcal{V}, s_{\text{dR}}))$$

is a right  $G_{\mathbb{Z}(p)}$ -torsor.

We remark that, one can also work with a collection of tensors  $s_\alpha \in V_{\mathbb{Z}(p)}^\otimes$  such that  $G_{\mathbb{Z}(p)}$  is the subgroup of  $\mathrm{GL}(V_{\mathbb{Z}(p)})$  acting trivially on each  $s_\alpha$ . All the above constructions work in this situation.

**Setting 2.3.** Still write  $\mathcal{V}$ ,  $s$ ,  $s_{\mathrm{dR}}$  and  $I$  for its reduction mod  $p$ . Let  $\varphi: \mathcal{V}^{(p)} \rightarrow \mathcal{V}$  and  $v: \mathcal{V} \rightarrow \mathcal{V}^{(p)}$  be the Frobenius and Verschiebung on  $\mathcal{V}$  respectively. Let  $\delta: \mathcal{V} \rightarrow \mathcal{V}^{(p)}$  be the semi-linear map sending  $v$  to  $v \otimes 1$ . Then we have a semi-linear map  $\varphi \circ \delta: \mathcal{V} \rightarrow \mathcal{V}$ . There is a descending filtration  $\mathcal{V} \supseteq \ker(\varphi \circ \delta) \supseteq 0$  and an ascending filtration  $0 \subseteq \mathrm{im}(\varphi) \subseteq \mathcal{V}$ . The morphism  $v$  induces an isomorphism  $\mathcal{V}/\mathrm{im}(\varphi) \rightarrow \ker(\varphi)$  whose inverse will be denoted by  $v^{-1}$ . Then  $\varphi$  and  $v^{-1}$  induce isomorphisms

$$\varphi_0: (\mathcal{V}/\ker(\varphi \circ \delta))^{(p)} \rightarrow \mathrm{im}(\varphi) \quad \text{and} \quad \varphi_1: (\ker(\varphi \circ \delta))^{(p)} \rightarrow \mathcal{V}/(\mathrm{im}(\varphi)).$$

**Setting 2.4.** Let  $V_0$  be the special fiber of  $V_{\mathbb{Z}(p)}$ . The cocharacter

$$\mu: \mathbb{G}_{m,\kappa} \rightarrow G_{0,\kappa} \subseteq \mathrm{GL}(V_{0,\kappa}) \cong \mathrm{GL}(V_{0,\kappa}^\vee)$$

induces an  $F$ -zip<sup>1</sup> structure on  $V_{0,\kappa}^\vee$  as follows. Let  $(V_{0,\kappa}^\vee)^0$  (resp.  $(V_{0,\kappa}^\vee)^1$ ) be the subspace of  $V_{0,\kappa}^\vee$  of weight 0 (resp. 1) with respect to  $\mu$ , and  $(V_{0,\kappa}^\vee)_0$  (resp.  $(V_{0,\kappa}^\vee)_1$ ) be the subspace of  $V_{0,\kappa}^\vee$  of weight 0 (resp. 1) with respect to  $\mu^{(p)}$ . Then we have a descending filtration  $V_{0,\kappa}^\vee \supseteq (V_{0,\kappa}^\vee)^1 \supseteq 0$  and an ascending filtration  $0 \subseteq (V_{0,\kappa}^\vee)_0 \subseteq V_{0,\kappa}^\vee$ . Let  $\xi: V_{0,\kappa}^\vee \rightarrow (V_{0,\kappa}^\vee)^{(p)}$  be the isomorphism given by  $v \otimes k \mapsto v \otimes 1 \otimes k$  for all  $v \in V_0^\vee$  and  $k \in \kappa$ . Then  $\xi$  induces isomorphisms

$$\phi_0: (V_{0,\kappa}^\vee)^{(p)}/((V_{0,\kappa}^\vee)^1)^{(p)} \xrightarrow{\mathrm{pr}_2} ((V_{0,\kappa}^\vee)^0)^{(p)} \xrightarrow{\xi^{-1}} (V_{0,\kappa}^\vee)_0$$

and

$$\phi_1: ((V_{0,\kappa}^\vee)^1)^{(p)} \xrightarrow{\xi^{-1}} ((V_{0,\kappa}^\vee)_1 \simeq V_{0,\kappa}^\vee/(V_{0,\kappa}^\vee)_0).$$

The first main result of [8] is as follows.

**Theorem 2.5.** [8, Theorem 3.4.1]

(1) Let  $I_+ \subseteq I$  be the closed subscheme

$$I_+ := \mathbf{Isom}_{\mathcal{S}_0}((V_{0,\kappa}^\vee \supseteq (V_{0,\kappa}^\vee)^1, s) \otimes O_{\mathcal{S}_0}, (\mathcal{V} \supseteq \ker(\varphi \circ \delta), s_{\mathrm{dR}})).$$

Then  $I_+$  is a  $P_+$ -torsor over  $\mathcal{S}_0$ .

(2) Let  $I_- \subseteq I$  be the closed subscheme

$$I_- := \mathbf{Isom}_{\mathcal{S}_0}(((V_{0,\kappa}^\vee)_0 \subseteq V_{0,\kappa}^\vee, s) \otimes O_{\mathcal{S}_0}, (\mathrm{im}(\varphi) \subseteq \mathcal{V}, s_{\mathrm{dR}})).$$

Then  $I_-$  is a  $P_-^{(p)}$ -torsor over  $\mathcal{S}_0$ .

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<sup>1</sup>We are not going to recall what are  $F$ -zips but just referring to [2]. It should be fine, as we are using them in explicit ways.

(3) Let  $\iota: I_+^{(p)}/U_+^{(p)} \rightarrow I_-/U_-^{(p)}$  be the morphism induced by

$$I_+^{(p)} \rightarrow I_-/U_-^{(p)}$$

$$f \mapsto (\varphi_0 \oplus \varphi_1) \circ \text{gr}(f) \circ (\phi_0^{-1} \oplus \phi_1^{-1}) \quad \text{for all } S/\mathcal{S}_0 \text{ and } f \in I_+^{(p)}(S).$$

Then  $\iota$  is an isomorphism of  $M^{(p)}$ -torsors.

Hence the tuple  $(I, I_+, I_-, \iota)$  is a  $G_0$ -zip of type  $\mu$  over  $\mathcal{S}_0$ .

The  $G_0$ -zip  $(I, I_+, I_-, \iota)$  as above induces a morphism  $\zeta: \mathcal{S}_0 \rightarrow G_0\text{-zip}_k^\mu$ . As we have seen, to construct  $\zeta$ , we need to choose a  $\mathbb{Z}_{(p)}$ -model  $G_{\mathbb{Z}_{(p)}}$  of  $G$ , a symplectic embedding  $i: (G, X) \hookrightarrow (\text{GSp}(V, \psi), S^\pm)$ , a  $\mathbb{Z}_{(p)}$ -lattice  $V_{\mathbb{Z}_{(p)}} \subseteq V$ , and a tensor  $s \in V_{\mathbb{Z}_{(p)}}^\otimes$  defining  $G_{\mathbb{Z}_{(p)}}$ . So, by independence of symplectic embeddings, we actually mean that  $\zeta$  is independent of the choices of  $G_{\mathbb{Z}_{(p)}}$ ,  $i$ ,  $V_{\mathbb{Z}_{(p)}}$  and  $s$ .

### 2.4. Uniqueness of $G_{\mathbb{Z}_{(p)}}$

We should point out, before getting started, that to analyse Ekedahl–Oort strata, the uniqueness of  $G_{\mathbb{Z}_p}$ , which follows from [5, Proposition 3.1.2.1(e)], would be enough. But we prefer to work with  $G_{\mathbb{Z}_{(p)}}$  here. The  $\mathbb{Z}_{(p)}$ -model  $G_{\mathbb{Z}_{(p)}}$  of  $G$  is independent of the choices made above. More precisely, we have the followings.

**Lemma 2.6.** *The group scheme  $G_{\mathbb{Z}_{(p)}}$  is the unique reductive model of  $G$  over  $\mathbb{Z}_{(p)}$  such that  $G_{\mathbb{Z}_{(p)}}(\mathbb{Z}_p) = K_p$ .*

*Proof.* Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two reductive models such that  $\mathcal{G}_1(\mathbb{Z}_p) = \mathcal{G}_2(\mathbb{Z}_p) = K_p$ . We consider the morphism  $G \rightarrow \mathcal{G}_1 \times \mathcal{G}_2$  which is the diagonal embedding over  $\mathbb{Q}$ . Let  $\mathcal{G}_3$  be the Zariski closure of the image. By [5, Proposition 3.1.2.1(e)],  $\mathcal{G}_{1, \mathbb{Z}_p} = \mathcal{G}_{2, \mathbb{Z}_p}$ , and hence flat base-change implies that  $\mathcal{G}_{3, \mathbb{Z}_p}$  is the diagonal subgroup of  $\mathcal{G}_{1, \mathbb{Z}_p} \times \mathcal{G}_{1, \mathbb{Z}_p}$ . In particular, the morphism  $\mathcal{G}_3 \rightarrow \mathcal{G}_1 \times \mathcal{G}_2 \xrightarrow{p_1} \mathcal{G}_1$  is an isomorphism and hence  $\mathcal{G}_3 \cong \mathcal{G}_1$ . Similarly,  $\mathcal{G}_3 \cong \mathcal{G}_2$ . □

### 2.5. Comparing $G_0$ -zips (I)

We will first show that the morphism  $\zeta$  does not depend on the choices of  $s$ , once  $i$  and  $V_{\mathbb{Z}_{(p)}}$  are fixed. As in §2.3, for a different choice of  $s' \in V_{\mathbb{Z}_{(p)}}^\otimes$ , we have another section  $s'_{\text{dR}} \in \mathcal{V}^\otimes$ , and hence two  $G_{\mathbb{Z}_{(p)}}$ -torsors

$$I = \mathbf{Isom}_{\mathcal{S}_K(G, X)}((V_{\mathbb{Z}_{(p)}}^\vee, s) \otimes O_{\mathcal{S}_K(G, X)}, (\mathcal{V}, s_{\text{dR}}))$$

and

$$I' = \mathbf{Isom}_{\mathcal{S}_K(G, X)}((V_{\mathbb{Z}_{(p)}}^\vee, s') \otimes O_{\mathcal{S}_K(G, X)}, (\mathcal{V}, s'_{\text{dR}})).$$

**Lemma 2.7.** *The two  $G_{\mathbb{Z}(p)}$ -torsors  $I$  and  $I'$  are canonically isomorphic.*

*Proof.*  $I$  and  $I'$  are both closed subscheme of  $\mathbf{Isom}_{\mathcal{S}_K(G,X)}(V_{\mathbb{Z}(p)}^\vee \otimes O_{\mathcal{S}_K(G,X)}, \mathcal{V})$ . Let  $I'' := \mathbf{Isom}_{\mathcal{S}_K(G,X)}((V_{\mathbb{Z}(p)}^\vee, s, s') \otimes O_{\mathcal{S}_K(G,X)}, (\mathcal{V}, s_{\text{dR}}, s'_{\text{dR}}))$ , then it is a closed subscheme of both  $I$  and  $I'$ . But  $I''$  is also a  $G_{\mathbb{Z}(p)}$ -torsor over  $\mathcal{S}_K(G, X)$ , so  $I = I'' = I'$ .  $\square$

Let us still write  $I$  (resp.  $I'$ ) for its special fiber. The construction in §2.3, especially Theorem 2.5, gives a  $G_0$ -zip (of type  $\mu$ )  $(I, I_+, I_-, \iota)$  on  $\mathcal{S}_0$ , using  $V_{\mathbb{Z}(p)}^\vee, s, \mathcal{V}, s_{\text{dR}}$  and the  $F$ -zip structure on  $\mathcal{V}$ . Similarly, there is a  $G_0$ -zip  $(I', I'_+, I'_-, \iota')$  attached to  $V_{\mathbb{Z}(p)}^\vee, s', \mathcal{V}, s'_{\text{dR}}$ .

**Corollary 2.8.** *The  $G_0$ -zips  $(I, I_+, I_-, \iota)$  and  $(I', I'_+, I'_-, \iota')$  on  $\mathcal{S}_{0,K}(G, X)$  are canonically isomorphic.*

*Proof.* By Lemma 2.7, the torsors  $I$  and  $I'$  are canonically isomorphic. Noting that  $(I_+, I_-, \iota)$  and  $(I'_+, I'_-, \iota')$  are constructed using Frobenius and Verschiebung on  $\mathcal{V}$ , the two  $G_0$ -zips are canonically isomorphic.  $\square$

### 2.6. Sum of symplectic embeddings

Let  $i_1: (G, X) \hookrightarrow (\text{GSp}(V_1, \psi_1), S_1^\pm)$  and  $i_2: (G, X) \hookrightarrow (\text{GSp}(V_2, \psi_2), S_2^\pm)$  be two symplectic embeddings. We can construct another symplectic embedding as follows. By the definition of symplectic similitude groups, there is a character  $\chi_1: \text{GSp}(V_1, \psi_1) \rightarrow \mathbb{G}_m$ , such that  $\text{GSp}(V_1, \psi_1)$  acts on  $\psi_1$  via  $\chi_1$ . Note that changing  $\chi_1$  to a power of it will not change the symplectic similitude group. Similarly, we have  $\chi_2: \text{GSp}(V_2, \psi_2) \rightarrow \mathbb{G}_m$ . Let  $w: \mathbb{G}_m \rightarrow G$  be the weight cocharacter attached to  $(G, X)$ . Then  $\chi_1 \circ w$  and  $\chi_2 \circ w$  are two characters  $\mathbb{G}_m \rightarrow \mathbb{G}_m$  of weights, say  $m_1$  and  $m_2$ , respectively. After changing  $\chi_1$  to  $\chi_1^{m_2}$  and  $\chi_2$  to  $\chi_2^{m_1}$ , we see that  $G$  acts on  $\psi_1$  and  $\psi_2$  via the same character. Let  $V = V_1 \oplus V_2$  and  $\psi: V \times V \rightarrow \mathbb{Q}$  be such that

$$\psi((v_1, v_2), (v'_1, v'_2)) = \psi_1(v_1, v'_1) + \psi_2(v_2, v'_2) \quad \text{for all } v_1, v'_1 \in V_1 \text{ and } v_2, v'_2 \in V_2.$$

Then  $G \subseteq \text{GSp}(V, \psi)$ , and this embedding induces an embedding of Shimura data

$$(G, X) \subseteq (\text{GSp}(V, \psi), S^\pm).$$

### 2.7. Comparing $G_0$ -zips (II)

Let  $\zeta: \mathcal{S}_{0,K}(G, X) \rightarrow G_0\text{-zip}_K^\mu$  be as before, we will show that it is independent of choices of symplectic embeddings and lattices. Note that  $\zeta$  is independent of  $K^p$ . More precisely,

for  $K \subseteq K'$ , there is a commutative diagram

$$\begin{array}{ccc} \mathcal{S}_{0,K}(G, X) & \longrightarrow & \mathcal{S}_{0,K'}(G, X) \\ & \searrow \zeta & \downarrow \zeta' \\ & & G_0\text{-zip}_{\kappa}^{\mu} \end{array}$$

inducing a  $G(\mathbb{A}_f^p)$ -equivariant morphism

$$\mathcal{S}_{0,K_p}(G, X) = \varprojlim_{K^p} \mathcal{S}_{K_p K^p}(G, X) \rightarrow G_0\text{-zip}_{\kappa}^{\mu}.$$

Here  $G(\mathbb{A}_f^p)$  acts on  $G_0\text{-zip}_{\kappa}^{\mu}$  trivially. So, we can shrink  $K^p$  if necessary.

Let  $G_{\mathbb{Z}(p)}$  be the reductive model of  $G$  over  $\mathbb{Z}(p)$  such that  $G_{\mathbb{Z}(p)}(\mathbb{Z}_p) = K_p$ . Notations as in §2.6, there are lattices  $V_{t,\mathbb{Z}} \subseteq V_t$ ,  $t = 1, 2$ , such that

1.  $\psi_t$  takes integral value on  $V_{t,\mathbb{Z}}$ ;
2. the Zariski closure of  $G$  in  $\text{GL}(V_{t,\mathbb{Z}(p)})$  is  $G_{\mathbb{Z}(p)}$ .

Let  $d_t = |V_{t,\mathbb{Z}}^{\vee}/V_{t,\mathbb{Z}}|$ ,  $g_t = \frac{1}{2} \dim(V_t)$ , and  $n \geq 3$  be an integer such that  $(n, p) = 1$ . Let  $\mathcal{A}_{g_t, d_t, n}$  be the moduli scheme of abelian schemes over  $\mathbb{Z}(p)$ -schemes of relative dimension  $g_t$  with a polarization  $\lambda_t$  of degree  $d_t$  and a level  $n$  structure  $\tau_t$ . We write  $(\mathcal{A}_t, \lambda_t, \tau_t)$  for the universal family on  $\mathcal{A}_{g_t, d_t, n}$ . Let  $K^p \subseteq G(\mathbb{A}_f^p)$  be small enough such that there are morphisms  $\text{Sh}_K(G, X) \rightarrow \mathcal{A}_{g_1, d_1, n}$  and  $\text{Sh}_K(G, X) \rightarrow \mathcal{A}_{g_2, d_2, n}$ . Then by the construction of the integral canonical model, there are finite morphisms  $i_1: \mathcal{S}_K(G, X) \rightarrow \mathcal{A}_{g_1, d_1, n}$  and  $i_2: \mathcal{S}_K(G, X) \rightarrow \mathcal{A}_{g_2, d_2, n}$ .

Let  $(V, \psi)$  be as in §2.6, and  $V_{\mathbb{Z}}$  be  $V_{1,\mathbb{Z}} \oplus V_{2,\mathbb{Z}}$ . There is an embedding of Shimura data  $i: (G, X) \rightarrow (\text{GSp}(V, \psi), S^{\pm})$ . Consider the diagram

$$\mathcal{A}_{g_1, d_1, n} \xleftarrow{p_1} \mathcal{A}_{g_1, d_1, n} \times \mathcal{A}_{g_2, d_2, n} \xrightarrow{p_2} \mathcal{A}_{g_2, d_2, n}.$$

The universal family  $p_1^*(\mathcal{A}_1, \lambda_1, \tau_1) \times p_2^*(\mathcal{A}_2, \lambda_2, \tau_2)$  on  $\mathcal{A}_{g_1, d_1, n} \times \mathcal{A}_{g_2, d_2, n}$  is an abelian scheme of dimension  $g_3 := g_1 + g_2$  with a polarization of degree  $d_3 := d_1 d_2$  and level  $n$  structure. There is a unique morphism

$$i': \mathcal{A}_{g_1, d_1, n} \times \mathcal{A}_{g_2, d_2, n} \rightarrow \mathcal{A}_{g_3, d_3, n}$$

such that

$$i'^*(\mathcal{A}, \lambda, \tau) = p_1^*(\mathcal{A}_1, \lambda_1, \tau_1) \times p_2^*(\mathcal{A}_2, \lambda_2, \tau_2),$$

where  $(\mathcal{A}, \lambda, \tau)$  is the universal family on  $\mathcal{A}_{g_3, d_3, n}$ .



By the construction of  $\mathcal{S}_K(G, X)$ , we have a commutative diagram

$$\begin{array}{ccccc}
 & & \mathcal{A}_{g_2, d_2, n} & & \\
 & \nearrow^{i_2} & & \nwarrow_{p_2} & \\
 \mathcal{S}_K(G, X) & \xrightarrow{(i_1, i_2)} & \mathcal{A}_{g_1, d_1, n} \times \mathcal{A}_{g_2, d_2, n} & \xrightarrow{i'} & \mathcal{A}_{g_3, d_3, n} \\
 & \searrow_{i_1} & & \nearrow_{p_1} & \\
 & & \mathcal{A}_{g_1, d_1, n} & & 
 \end{array}$$

such that the generic fiber of  $i' \circ (i_1, i_2)$  is induced by  $i$ . We will write  $i$  for  $i' \circ (i_1, i_2)$ . The pull back via  $i$  of the universal family on  $\mathcal{A}_{g_3, d_3, n}$  is precisely  $i_1^*(\mathcal{A}_1, \lambda_1, \tau_1) \times i_2^*(\mathcal{A}_2, \lambda_2, \tau_2)$ .

For simplicity, let's write  $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}$  for the pull back to  $\mathcal{S}_K(G, X)$  of the universal abelian schemes on  $\mathcal{A}_{g_1, d_1, n}, \mathcal{A}_{g_2, d_2, n}$  and  $\mathcal{A}_{g_3, d_3, n}$  respectively. Then  $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2$ . Let  $\mathcal{V}_t = H_{\text{dR}}^1(\mathcal{A}_t/\mathcal{S}_K(G, X))$ ,  $t = 1, 2$ , and  $\mathcal{V} = H_{\text{dR}}^1(\mathcal{A}/\mathcal{S}_K(G, X))$ . Then  $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ . A tensor  $s_1 \in V_{1, \mathbb{Z}(p)}^\otimes$  (resp.  $s \in V_{\mathbb{Z}(p)}^\otimes$ ) defining  $G_{\mathbb{Z}(p)} \subseteq \text{GL}(V_{1, \mathbb{Z}(p)})$  (resp.  $G_{\mathbb{Z}(p)} \subseteq \text{GL}(V_{\mathbb{Z}(p)})$ ) induces a section  $s_{1, \text{dR}} \in \mathcal{V}_1^\otimes$  (resp.  $s_{\text{dR}} \in \mathcal{V}^\otimes$ ).

Let  $I$  be  $\mathbf{Isom}_{\mathcal{S}_K(G, X)}((V_{\mathbb{Z}(p)}^\vee \supseteq V_{1, \mathbb{Z}(p)}^\vee, s_1, s) \otimes \mathcal{O}_{\mathcal{S}_K(G, X)}, (\mathcal{V} \supseteq \mathcal{V}_1, s_{1, \text{dR}}, s_{\text{dR}}))$ .

**Lemma 2.9.** *The scheme  $I$  is a right  $G_{\mathbb{Z}(p)}$ -torsor over  $\mathcal{S}_K(G, X)$ .*

*Proof.* Let  $I_3 = \mathbf{Isom}_{\mathcal{S}_K(G, X)}((V_{\mathbb{Z}(p)}^\vee, s) \otimes \mathcal{O}_{\mathcal{S}_K(G, X)}, (\mathcal{V}, s_{\text{dR}}))$ , then  $I \subseteq I_3$ . To prove the statement, we only need to check that  $\mathcal{V}_1 = V_{1, \mathbb{Z}(p)}^\vee \times^{G_{\mathbb{Z}(p)}} I_3$ , and that  $s_{1, \text{dR}}: \mathcal{O}_{\mathcal{S}} \rightarrow \mathcal{V}_1^\otimes$  is the same as  $(\mathbb{Z}(p) \xrightarrow{s_1} V_{1, \mathbb{Z}(p)}^\otimes) \times^{G_{\mathbb{Z}(p)}} I_3$ . Noting that  $\mathcal{V}_1$  and  $V_{1, \mathbb{Z}(p)}^\vee \times^{G_{\mathbb{Z}(p)}} I_3$  are both locally direct summands of  $\mathcal{V}$ , and that  $\mathcal{S}$  is integral, it suffices to work over the generic fiber, and everything follows from the first three paragraphs of [1, §2.2].  $\square$

We remark that the proof above also implies that  $I = I_3$ . We write  $i_1, i_2, p_1, p_2, i$  for the morphisms of the special fibers. To prove that the Ekedahl–Oort stratifications are independent of choices of symplectic embedding, it suffices to prove that the stratifications induced by  $i_1$  and  $i$  coincide. By Corollary 2.8 and the proof of Lemma 2.9, the special fiber of  $I$  is precisely the  $G_0$ -torsor in the  $G_0$ -zip over  $\mathcal{S}_{0, K}(G, X)$  constructed using  $i$ . Let us write  $I$  for this special fiber and  $(I, I_+, I_-, \iota)$  for the  $G_0$ -zip constructed using  $i$ . Let  $(I_1, I_{1,+}, I_{1,-}, \iota_1)$  be the  $G_0$ -zip over  $\mathcal{S}_{0, K}(G, X)$  constructed using  $i_1$ .

There is a natural morphism  $\epsilon: I \rightarrow I_1$  given by

$$f \in I(S) \mapsto f|_{V_{1, \kappa}^\vee \otimes \mathcal{O}_S} \in I_1(S) \quad \text{for all } \mathcal{S}_{0, K}(G, X)\text{-scheme } S.$$

**Theorem 2.10.**  *$\epsilon$  induces an isomorphism  $(I, I_+, I_-, \iota) \rightarrow (I_1, I_{1,+}, I_{1,-}, \iota_1)$  of  $G_0$ -zips. In particular,  $i_1$  and  $i$  give the same Ekedahl–Oort stratification.*

*Proof.* The morphism  $\epsilon: I \rightarrow I_1$  is clearly  $G_0$ -equivariant, and hence an isomorphism of  $G_0$ -torsors. For any  $S/\mathcal{S}_{0,K}(G, X)$ , and any  $f \in I_+(S) \subseteq I(S)$ ,  $f$  maps the weight 1 subspace of  $V_{\kappa}^{\vee} \otimes O_S$  to  $\ker(\varphi)$ , where  $\varphi$  is the Frobenius on  $\mathcal{V}$ . Let  $\varphi'$  be the Frobenius on  $\mathcal{V}_1$ , then  $\ker(\varphi') = \ker(\varphi) \cap \mathcal{V}_1$ , as  $\mathcal{V}_1 \subseteq \mathcal{V}$  is induced by a morphism of abelian schemes and hence compatible with Frobenius. So  $\epsilon(f) = f|_{V_{1,\kappa}^{\vee} \otimes O_S}$  maps the weight 1 subspace of  $V_{1,\kappa}^{\vee} \otimes O_S$  to  $\ker(\varphi) \cap \mathcal{V}_1 = \ker(\varphi')$ , and hence lies in  $I_{1,+}(S)$ . But then  $\epsilon|_{I_+}$  will automatically be an isomorphism of  $P_+$ -torsors. Similarly,  $\epsilon|_{I_-}$  is an isomorphism of  $P_-^{(p)}$ -torsors.

Now we check the compatibility between  $\iota$  and  $\iota_1$ . We apply the constructions in Settings 2.3 and 2.4 to  $\mathcal{V}_1$  and  $V_{1,\kappa}^{\vee}$  respectively, and denote the obtained morphisms by  $\varphi'_0, \varphi'_1, \phi'_0$  and  $\phi'_1$ . Let  $(V_{1,\kappa}^{\vee})^0$  (resp.  $(V_{1,\kappa}^{\vee})^1$ ) be the subspace of  $V_{1,\kappa}^{\vee}$  of weight 0 (resp. 1) with respect to  $\mu$ , and  $(V_{1,\kappa}^{\vee})_0$  (resp.  $(V_{1,\kappa}^{\vee})_1$ ) be the subspace of  $V_{1,\kappa}^{\vee}$  of weight 0 (resp. 1) with respect to  $\mu^{(p)}$ . Then  $\phi'_0$  and  $\phi'_1$  are compatible with  $\phi_0$  and  $\phi_1$ , in the sense that

$$\phi_0|_{(V_{1,\kappa}^{\vee})^{(p)}/((V_{1,\kappa}^{\vee})^1)^{(p)}} = \phi'_0: (V_{1,\kappa}^{\vee})^{(p)}/((V_{1,\kappa}^{\vee})^1)^{(p)} \rightarrow (V_{1,\kappa}^{\vee})_0$$

and

$$\phi_1|_{((V_{1,\kappa}^{\vee})^1)^{(p)}} = \phi'_1: ((V_{1,\kappa}^{\vee})^1)^{(p)} \rightarrow V_{0,\kappa}^{\vee}/(V_{1,\kappa}^{\vee})_0.$$

Let  $\varphi': \mathcal{V}_1^{(p)} \rightarrow \mathcal{V}_1$  and  $v': \mathcal{V}_1 \rightarrow \mathcal{V}_1^{(p)}$  be the Frobenius and Verschiebung on  $\mathcal{V}_1$  respectively. Then they are the restrictions of  $\varphi$  and  $v$  to  $\mathcal{V}_1$ . Notations as in Setting 2.3, we then have

$$\varphi_0|_{(\mathcal{V}_1/\ker(\varphi' \circ \delta))^{(p)}} = \varphi'_0: (\mathcal{V}_1/\ker(\varphi' \circ \delta))^{(p)} \rightarrow \ker(\varphi')$$

and

$$\varphi_1|_{(\ker(\varphi' \circ \delta))^{(p)}} = \varphi'_1: (\ker(\varphi' \circ \delta))^{(p)} \rightarrow \mathcal{V}_1/(\text{im}(\varphi')).$$

Now we first how  $\iota$  and  $\iota'$  are defined in Theorem 2.5. Notations as above,  $\iota: I_+^{(p)}/U_+^{(p)} \rightarrow I_-/U_-^{(p)}$  is the morphism induced by

$$I_+^{(p)} \rightarrow I_-/U_-^{(p)} \\ f \mapsto (\varphi_0 \oplus \varphi_1) \circ \text{gr}(f) \circ (\phi_0^{-1} \oplus \phi_1^{-1}) \quad \text{for all } S/\mathcal{S}_0 \text{ and } f \in I_+^{(p)}(S),$$

and similarly for  $\iota'$ . Then for all  $S/\mathcal{S}_0$  and  $f \in I_+^{(p)}(S)$ , we have

$$\iota' \circ \epsilon(f) = \iota'(f|_{V_{1,\kappa}^{\vee} \otimes O_S}) = (\varphi'_0 \oplus \varphi'_1) \circ \text{gr}(f|_{V_{1,\kappa}^{\vee} \otimes O_S}) \circ (\phi_0'^{-1} \oplus \phi_1'^{-1}) = \epsilon \circ \iota(f).$$

This shows that  $\epsilon$  is an isomorphism of  $G_0$ -zips. □

*Remark 2.11.* Bruhat stratifications are defined and studied by Wedhorn in [6] (actually, we need the morphism  $\zeta$  to define the Bruhat stratification on  $\mathcal{S}_{0,K}(G, X)$ ). In the case of Siegel modular varieties, the Bruhat stratification is precisely the  $a$ -number stratification. Theorem 2.10 also implies that the Bruhat stratification is independent of choices of symplectic embeddings.

### 3. Functoriality

Let  $p \geq 3$  be a prime, and  $(G, X)$  and  $(G', X')$  be two Shimura data of Hodge type such that they both have good reduction at  $p$ . Let  $E$  (resp.  $E'$ ) be the reflex field of  $(G, X)$  (resp.  $(G', X')$ ). Let  $K$  (resp.  $K'$ ) be a compact open subgroup of  $G(\mathbb{A}_f)$  (resp.  $G'(\mathbb{A}_f)$ ) such that  $K_p$  (resp.  $K'_p$ ) is hyperspecial. Let  $f: (G, X) \rightarrow (G', X')$  be a morphism of Shimura data, then  $E \supseteq E'$ . If  $K$  and  $K'$  are such that  $f(K) \subseteq K'$ , then  $f$  induces a morphism of Shimura varieties  $f: \text{Sh}_K(G, X) \rightarrow \text{Sh}_{K'}(G', X')_E$ .

Let  $v'$  be a place of  $E'$  over  $p$  with residue field  $\kappa'$  and  $v$  be a place of  $E$  over  $v'$  with residue field  $\kappa$ . Let  $\mathcal{S}_K(G, X)$  (resp.  $\mathcal{S}_{K'}(G', X')$ ) be the integral canonical model of  $\text{Sh}_K(G, X)$  (resp.  $\text{Sh}_{K'}(G', X')$ ). Then  $f$  extends uniquely to a morphism  $\mathcal{S}_K(G, X) \rightarrow \mathcal{S}_{K'}(G', X')_{\mathcal{O}_{E, (v)}}$  whose special fiber  $\mathcal{S}_{0, K}(G, X) \rightarrow \mathcal{S}_{0, K'}(G', X')_{\kappa}$  will still be denoted by  $f$ .

By “functoriality”, we mean a certain kind of compatibility of Ekedahl–Oort stratifications with respect to  $f$ . But it seems that we need some extra assumptions. The reason is as follows. For a morphism  $f: G_{\mathbb{Q}_p} \rightarrow G'_{\mathbb{Q}_p}$  such that  $f(K_p) \subseteq K'_p$ , it is NOT always possible to extend  $f$  to a morphism  $G_{\mathbb{Z}_p} \rightarrow G'_{\mathbb{Z}_p}$  (see [5, Proposition 3.1.2.1(b)]). So there is NO natural morphism  $G_0 \rightarrow G'_0$ , and hence there is NO direct way to compare  $G_0$ -zips and  $G'_0$ -zips.

#### 3.1. Basic settings

Let  $\mathcal{G}/\mathbb{Z}_{(p)}$  (resp.  $\mathcal{G}'/\mathbb{Z}_{(p)}$ ) be the reductive model of  $G$  (resp.  $G'$ ) with special fiber  $G_0$  (resp.  $G'_0$ ). Let  $E, E', \kappa$  and  $\kappa'$  be as at the beginning of this section. Then by [8, 3.2.2], the Shimura datum  $(G, X)$  (resp.  $(G', X')$ ) determines a cocharacter  $\mu$  (resp.  $\mu'$ ) of  $\mathcal{G}_{W(\kappa)}$  (resp.  $\mathcal{G}'_{W(\kappa')}$ ) which is unique up to conjugacy. The reduction of  $\mu$  (resp.  $\mu'$ ) will still be denoted by  $\mu$  (resp.  $\mu'$ ).

Besides the conditions stated at the beginning of this section, we make the following assumption on  $f: (G, X) \rightarrow (G', X')$ .

**Assumption 3.1.** There exists a homomorphism  $\mathcal{G}_{\mathbb{Z}_p} \rightarrow \mathcal{G}'_{\mathbb{Z}_p}$  extending  $f_{\mathbb{Q}_p}$ . This homomorphism will be denoted by  $\underline{f}$ .

#### 3.2. The morphism $\alpha$

The morphism  $\underline{f}$  induces a natural morphism

$$\alpha: G_0\text{-zip}_{\kappa}^{\mu} \rightarrow G'_0\text{-zip}_{\kappa'}^{\mu'} \otimes \kappa$$

which we will now explain. Still write  $\mu$  for the cocharacter  $\mathbb{G}_{m, \kappa} \rightarrow G_{0, \kappa} \rightarrow G'_{0, \kappa}$ , then  $\mu$  and  $\mu'$  are  $G'_0(\kappa)$ -conjugate.

There is a natural morphism  $\alpha_1: G_0\text{-zip}_\kappa^\mu \rightarrow G'_0\text{-zip}_\kappa^\mu$  as follows. The cocharacter  $\mu$  induces homomorphisms  $P_+ \rightarrow P'_+$ ,  $P_- \rightarrow P'_-$  and  $M \rightarrow M'$ . For any  $\kappa$ -scheme  $S$  and any  $S$ -point  $(I, I_+, I_-, \iota)$  of  $\in G_0\text{-zip}_\kappa^\mu$ ,

$$\alpha_1(I, I_+, I_-, \iota) := (I \times^{G_{0,S}} G'_{0,S}, I_+ \times^{P_{+,S}} P'_{+,S}, I_- \times^{P_{-,S}^{(p)}} P'_{-,S}{}^{(p)}, \iota'),$$

where  $?_1 \times^{?_2} ?_3$  is the quotient of  $?_1 \times ?_3$  equalizing the  $?_2$ -action on  $?_1$  given by the torsor structure and that on  $?_3$  induced by  $\underline{f}$ , and  $\iota'$  is the composition of  $M'^{(p)}$ -equivariant isomorphisms

$$\begin{aligned} (I_+ \times^{P_{+,S}} P'_{+,S})^{(p)} / U_+^{(p)} &\xrightarrow{\simeq} (I_+^{(p)} / U_+^{(p)}) \times^{L_S^{(p)}} L_S'^{(p)}, \\ (I_+^{(p)} / U_+^{(p)}) \times^{L_S^{(p)}} L_S'^{(p)} &\xrightarrow{\iota \times \text{id}} (I_- / U_-^{(p)}) \times^{L_S^{(p)}} L_S'^{(p)} \end{aligned}$$

and

$$(I_- / U_-^{(p)}) \times^{L_S^{(p)}} L_S'^{(p)} \xrightarrow{\simeq} (I_- \times^{P_{-,S}^{(p)}} P'_{-,S}{}^{(p)}) / U_-'^{(p)}.$$

Let  $\mu' \otimes 1$  be the base change to  $\kappa$  of the cocharacter  $\mu'$ , then by [7, Remark 5.16(1)], there is an obvious isomorphism  $\alpha_2: G'_0\text{-zip}'_{\kappa'} \otimes \kappa \rightarrow G'_0\text{-zip}'_{\kappa'} \otimes^1$  given by base change. Let  $g \in G'_0(\kappa)$  be such that  $\text{int}(g) \circ (\mu' \otimes 1) = \mu$ , then  $g$  induces an isomorphism of algebraic stacks

$$\begin{aligned} \alpha_3^g: G'_0\text{-zip}'_{\kappa'} \otimes^1 &\rightarrow G'_0\text{-zip}'_{\kappa} \\ (I, I_+, I_-, \iota) &\mapsto (I', I'_+, I'_-, \iota') := (I, (I_+) \cdot g^{-1}, (I_-) \cdot \sigma(g)^{-1}, r_{\sigma(g)^{-1}} \circ \iota \circ r_{\sigma(g)}), \end{aligned}$$

where  $r_{\sigma(g)}$  and  $r_{\sigma(g)^{-1}}$  are the obvious morphisms  $I_+^{(p)} / U_+^{(p)} \simeq I_+^{(p)} / U_+^{(p)}$  and  $I_- / U_-^{(p)} \simeq I'_- / U_-'^{(p)}$  given by multiplication with  $\sigma(g)$  and  $\sigma(g)^{-1}$  on the right respectively.

*Remark 3.2.* The morphism  $\alpha_3^g$  is canonical, in the sense that it is uniquely determined by  $\mu$  and  $\mu' \otimes 1$  and does not depend on the choices of  $g$ . For an  $h \in G'_0(\kappa)$  such that  $\text{int}(h) \circ (\mu' \otimes 1) = \mu$ , there exists an  $l \in M'(\kappa)$ , such that  $h = gl$ . Here  $M'$  is, as before, the centralizer in  $G'_{0,\kappa}$  of  $\mu'$ . Then

$$\begin{aligned} &\alpha_3^h(I, I_+, I_-, \iota) \\ &= (I, (I_+) \cdot h^{-1}, (I_-) \cdot \sigma(h)^{-1}, r_{\sigma(h)^{-1}} \circ \iota \circ r_{\sigma(h)}) \\ &= (I, (I_+) \cdot l^{-1}g^{-1}, (I_-) \cdot \sigma(l)^{-1}\sigma(g)^{-1}, r_{\sigma(g)^{-1}} \circ r_{\sigma(l)^{-1}} \circ \iota \circ r_{\sigma(l)} \circ r_{\sigma(g)}) \\ &= (I, (I_+) \cdot g^{-1}, (I_-) \cdot \sigma(g)^{-1}, r_{\sigma(g)^{-1}} \circ \iota \circ r_{\sigma(g)}). \end{aligned}$$

The last equality is because of that  $I_+$  (resp.  $I_-$ ) is  $M'$  (resp.  $M'^{(p)}$ ) stable and that  $\iota$  is  $M'^{(p)}$ -equivariant. We will simply write  $\alpha_3$  for  $\alpha_3^g$ , as it is independent of  $g$ .

The morphism  $\alpha$ , at the beginning of this subsection, is then defined to be  $\alpha_2^{-1} \circ \alpha_3^{-1} \circ \alpha_1$ .

### 3.3. Functoriality

We use the same notations as at the beginning of this section. We have  $\zeta: \mathcal{S}_{0,K}(G, X) \rightarrow G_0\text{-zip}_\kappa^\mu$  and  $\zeta': \mathcal{S}_{0,K'}(G', X') \rightarrow G'_0\text{-zip}_{\kappa'}^{\mu'}$ . Moreover, we assume that the morphism of Shimura data  $f: (G, X) \rightarrow (G', X')$  satisfies Assumption 3.1.

By functoriality, we mean the following.

**Theorem 3.3.** *The diagram*

$$\begin{CD} \mathcal{S}_{0,K}(G, X) @>f>> \mathcal{S}_{0,K'}(G', X')_\kappa \\ @V\zeta VV @VV\zeta' \otimes \kappa V \\ G_0\text{-zip}_\kappa^\mu @>\alpha>> G'_0\text{-zip}_{\kappa'}^{\mu'} \otimes \kappa \end{CD}$$

is commutative.

*Proof.* The proof, which is a variation of that of Theorem 2.10, will be divided into several steps.

*Step 1.* Let  $i: (G, X) \rightarrow (\text{GSp}(V, \psi), S^\pm)$  and  $i': (G', X') \rightarrow (\text{GSp}(V', \psi'), S'^\pm)$  be symplectic embeddings. Note that we do NOT assume that there is any compatibility between  $f$  and the symplectic embeddings. The weight cocharacter  $w: \mathbb{G}_{m,\mathbb{Q}} \rightarrow G$  induces a  $\mathbb{G}_{m,\mathbb{Q}}$ -action on  $V$  of weight 1, and the composition  $f \circ w$  induces a  $\mathbb{G}_{m,\mathbb{Q}}$ -action on  $V'$  of weight 1. Let  $V_1 = V \oplus V'$  and  $\psi_1: V_1 \times V_1 \rightarrow \mathbb{Q}$  be such that

$$\psi_1((v, v'), (w, w')) = \psi(v, w) + \psi'(v', w') \quad \text{for all } v, w \in V \text{ and } v', w' \in V'.$$

Then  $i$  and  $i' \circ f$  induce a faithful representation of  $G$  on  $V_1$ . Moreover, the argument in §2.6 shows that  $G \subseteq \text{GSp}(V_1, \psi_1)$  and this embedding induces an embedding of Shimura data  $i_1: (G, X) \subseteq (\text{GSp}(V_1, \psi_1), S_1^\pm)$ . In sum, we have a commutative diagram

$$\begin{CD} \text{Sh}_K(G, X) @>i \times (i' \circ f)>> \mathcal{A}_{g,d,n} \times \mathcal{A}_{g',d',n} @>>> \mathcal{A}_{g_1,d_1,n} \\ @VfVV @Vp_2VV @VVV \\ \text{Sh}_{K'}(G', X') @>i'>> \mathcal{A}_{g',d',n} \end{CD}$$

*Step 2.* There is a  $\mathbb{Z}$ -lattice  $V_\mathbb{Z} \subseteq V$  (resp.  $V'_\mathbb{Z} \subseteq V'$ ) such that the Zariski closure of  $G$  (resp.  $G'$ ) in  $\text{GL}(V_{\mathbb{Z}(p)})$  (resp.  $\text{GL}(V'_{\mathbb{Z}(p)})$ )  $\mathcal{G}$  (resp.  $\mathcal{G}'$ ) is reductive and such that  $K_p = \mathcal{G}(\mathbb{Z}_p)$  (resp.  $K'_p = \mathcal{G}'(\mathbb{Z}_p)$ ). Let  $V_1$  be  $V_\mathbb{Z} \oplus V'_\mathbb{Z}$ . Consider the sequence of closed embeddings

$$\mathcal{G} \times \mathcal{G}' \subseteq \text{GL}(V_{\mathbb{Z}(p)}) \times \text{GL}(V'_{\mathbb{Z}(p)}) \subseteq \text{GL}(V_{1,\mathbb{Z}(p)}).$$

Let  $\mathcal{G}_1$  be the Zariski closure of  $G$  in  $\text{GL}(V_{1,\mathbb{Z}(p)})$ . Then  $\mathcal{G}_1 \subseteq \mathcal{G} \times \mathcal{G}'$ . Flat base-change implies that  $\mathcal{G}_{1,\mathbb{Z}_p}$  is the graph of  $\underline{f}: \mathcal{G}_{\mathbb{Z}_p} \rightarrow \mathcal{G}'_{\mathbb{Z}_p}$ . So  $\mathcal{G}_1 \cong \mathcal{G}$  and  $\underline{f}$  is defined over  $\mathbb{Z}(p)$ .

*Step 3.* Let  $s \in V_{\mathbb{Z}(p)}^{\otimes}$  (resp.  $s' \in V_{\mathbb{Z}(p)}^{\prime\otimes}$ ,  $s_1 \in V_{1,\mathbb{Z}(p)}^{\otimes}$ ) be a tensor defining  $\mathcal{G} \subseteq \mathrm{GL}(V_{\mathbb{Z}(p)})$  (resp.  $\mathcal{G}' \subseteq \mathrm{GL}(V_{\mathbb{Z}(p)}')$ ,  $\mathcal{G} \subseteq \mathrm{GL}(V_{1,\mathbb{Z}(p)})$ ). Then the symplectic embeddings  $i$ ,  $i'$  and  $i_1$  induce vector bundles  $\mathcal{V}$ ,  $\mathcal{V}_1$  on  $\mathcal{S}_{0,K}(G, X)$  and  $\mathcal{V}'$  on  $\mathcal{S}_{0,K'}(G', X')$ . The tensors  $s$ ,  $s'$  and  $s_1$  induce tensors  $s_{\mathrm{dR}} \in \mathcal{V}^{\otimes}$ ,  $s'_{\mathrm{dR}} \in \mathcal{V}'^{\otimes}$  and  $s_{1,\mathrm{dR}} \in \mathcal{V}_1^{\otimes}$  respectively. Note that we have  $\mathcal{V}_1 = \mathcal{V} \oplus f^*\mathcal{V}'$ . Let  $(I, I_+, I_-, \iota)$  be the  $G_0$ -zip on  $\mathcal{S}_{0,K}(G, X)$  constructed using  $i$ , and  $(I_1, I_{1,+}, I_{1,-}, \iota_1)$  be the one constructed using  $i_1$ . By Theorem 2.10,  $(I, I_+, I_-, \iota) \cong (I_1, I_{1,+}, I_{1,-}, \iota_1)$ . Let  $(I'_1, I'_{1,+}, I'_{1,-}, \iota'_1)$  be the image of  $(I_1, I_{1,+}, I_{1,-}, \iota_1)$  under  $\alpha_1$  as at the beginning of §3.2. It is a  $G'_0$ -zip of type  $\mu$  over  $\mathcal{S}_{0,K}(G, X)$ .

*Step 4.* Let  $(I', I'_+, I'_-, \iota')$  be the  $G'_0$ -zip of type  $\mu'$  over  $\mathcal{S}_{0,K'}(G', X')$  constructed using  $i'$ , and  $(I', I'_+, I'_-, \iota')_{\kappa}$  be its pull back to  $\mathcal{S}_{0,K}(G, X)$ . One gets a  $G'_0$ -zip of type  $\mu$  over  $\mathcal{S}_{0,K}(G, X)$  from it, still denoted by  $(I', I'_+, I'_-, \iota')_{\kappa}$ , by first applying  $\alpha_2$  and then  $\alpha_3$  as in the constructions before Remark 3.2. By the same arguments as in the proof of Theorem 2.10,  $\epsilon$  induces an isomorphism of  $G'_0$ -zips of type  $\mu$ . But this means that the diagram

$$\begin{CD} \mathcal{S}_{0,K}(G, X) @>f>> \mathcal{S}_{0,K'}(G', X')_{\kappa} \\ @V\zeta VV @VV\zeta' \otimes \kappa V \\ G_0\text{-zip}_{\kappa}^{\mu} @>\alpha>> G'_0\text{-zip}_{\kappa'}^{\mu'} \otimes \kappa \end{CD}$$

is commutative. □

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