mathematics

Article

A Duality Theorem for Hopf Quasimodule Algebras

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Abstract: In this paper, we introduce and study two smash products $A \bigstar H$ for a left H-quasimodule algebra A over a Hopf quasigroup H over a field $\mathbb{K}$ and $B\#U$ for a coquasi U-module algebra B over a Hopf coquasigroup U, respectively. Then, we prove our duality theorem $(A \bigstar H)\#H^* \cong A \otimes (H\#H^*) \cong A \otimes M_n(\mathbb{K}) \cong M_n(A)$ in the setting of a Hopf quasigroup H of dimension n. As an application of our result, we consider a special case of a finite quasigroup.

Keywords: quasigrups; Hopf (co)quasigroups; duality theorem; quasimodule algebras; coquasi module algebras

MSC: 16W50; 17A60

Citation: Guo, H.; Wan, S. A Duality Theorem for Hopf Quasimodule Algebras. *Mathematics* **2023**, *11*, 1401. https://doi.org/10.3390/math11061401

Academic Editor: Gaetano Fiore

Received: 30 January 2023
Revised: 11 March 2023
Accepted: 12 March 2023
Published: 14 March 2023

1. Introduction

The notion of Hopf quasigroup was introduced by Klim and Majid in [1], which is a particular case of the notion of unital counital coassociative bialgebra introduced in [2]. Dually, the vector space of linear functionals on a finite quasigroup carries the structure of a Hopf coquasigroup (cf. [1]), which is a counital unital associative bialgebra. These Hopf quasigroups and Hopf coquasigroups are generalizations of Hopf algebras (see [3]). These notions are related to cohomology modules [4], Yetter–Drinfeld Modules [5–8], and coalgebras [9] based on digital images.

Given a locally compact abelian group G and a von Neumann algebra N, let G act on N via a homomorphism α of G into $Aut(N)$. Then, we have the smash product algebra $N \times_\alpha G$. Takesaki in 1973 introduced an action $\widehat{\alpha}$ of the dual group $\widehat{G}$ and proved the duality theorem: $(N \times_\alpha G) \times_{\widehat{\alpha}} \widehat{G} \cong N \overline{\otimes} M$ (as algebras), where M is the algebra of all bounded operators on $L_2(G)$ (see [10]). This result was extended to arbitrary locally compact groups G independently by Landstad and Nakagami in the mid 1970s (see [11,12]).

In the above duality theorem, if we replace G by a Hopf algebra and N by an algebra, then the authors in [13] constructed a duality theorem for any Hopf algebra H-module algebra A over a field $\mathbb{K}$ under some condition "locally finite". In particular, if H is finite-dimensional, then the duality theorem is

$$(A\#H)\#H^* \cong A \otimes (H\#H^*) \cong A \otimes M_n(\mathbb{K}) \cong M_n(A).$$

It is now natural to ask whether the duality theorem above in [13] holds in the framework of Hopf quasigroups. This becomes our motivation of writing this paper. We will overcome non-associativity in Hopf quasigroups and non-coassociativity in Hopf coquasigroups by introducing some new notions and developing new ways.

This article is organized as follows: In Section 2, we recall and investigate some basic definitions and properties related to Hopf (co)quasigroups.

In Section 3, we introduce and study two smash products $A \bigstar H$ for a left H-quasimodule algebra A over a Hopf quasigroup H and $B\#U$ for a coquasi U-module algebra B over a Hopf coquasigroup U, respectively. In Section 4, we prove our duality theorem in the setting of finite-dimensional Hopf quasigroups (see Theorem 1). As an application of our result, we consider a special case of a finite quasigroup.

Throughout this paper, $\mathbb{K}$ is a fixed field, and all vector spaces are over $\mathbb{K}$. By linear maps, we mean $\mathbb{K}$-linear maps. Unadorned $\otimes$ means $\otimes_{\mathbb{K}}$. Let C be a coalgebra with a coproduct Δ. We will use the Heyneman–Sweedler's notation (see [3]), $\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$ for all $c \in C$, for coproduct.

2. Preliminaries

In this section, some basic definitions and properties of Hopf (co)quasigroups and smash products are recalled and investigated.

2.1. Algebras and Coalgebras

The following notions can be found in [2]. An *algebra* (A, ∇) is a vector space A equipped with a linear map $\nabla : A \otimes A \longrightarrow A$. The algebra (A, ∇) is called *associative* if $\nabla(id \otimes \nabla) = \nabla(\nabla \otimes id)$. It is customary to write $\nabla(x \otimes y) = xy$, $\forall x, y \in A$. A *unital algebra* (A, ∇, μ) is a vector space A equipped with two linear maps $\nabla : A \otimes A \longrightarrow A$ and $\mu : \mathbb{K} \longrightarrow A$ such that $\nabla(id \otimes \mu) = id = \nabla(\mu \otimes id)$. Generally, we write $1 \in A$ for $\mu(1_{\mathbb{K}})$.

Dually, a *coalgebra* (C, Δ) is a vector space C equipped with a linear map $\Delta : C \longrightarrow C \otimes C$. The coalgebra (C, Δ) is called *coassociative* if $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$. A *counital coalgebra* (C, Δ, ε) is a vector space C equipped with two linear maps $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow \mathbb{K}$ such that $(id \otimes \varepsilon)\Delta = id = (\varepsilon \otimes id)\Delta$.

A *bialgebra* (A, ∇, Δ) is an algebra (A, ∇) and a coalgebra (A, Δ) such that $\Delta(xy) = \Delta(x)\Delta(y)$ for all $x, y \in A$. A *unital bialgebra* (A, ∇, μ, Δ) is a coalgebra (A, Δ) and a unital (A, ∇, μ) such that $\Delta(xy) = \Delta(x)\Delta(y)$ and $\Delta(1) = 1$ for all $x, y \in A$. A *counital bialgebra* $(A, \nabla, \Delta, \varepsilon)$ is a counital coalgebra (A, Δ, ε) and an algebra (A, ∇) such that $\Delta(xy) = \Delta(x)\Delta(y)$ and $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$ for all $x, y \in A$. A *unital counital bialgebra* $(A, \Delta, \varepsilon, \nabla, \mu)$ is both a unital bialgebra (A, Δ, ∇, μ) and a counital bialgebra $(A, \Delta, \varepsilon, \nabla)$ such that $\varepsilon(1) = 1$. A *Hopf algebra* always means a unital counital associative coassociative bialgebra with an antipode (cf. [3]).

2.2. Hopf (Co)quasigroups

Recall from [1] that a *Hopf quasigroup* is a unital counital coassociative bialgebra $(H, \nabla, \mu, \Delta, \varepsilon)$ armed with a linear map $S : H \longrightarrow H$ (called *antipode*) such that

$$\sum S(h_{(1)})(h_{(2)}g) = \varepsilon(h)g = \sum h_{(1)}(S(h_{(2)})g), \tag{1}$$

$$\sum (hg_{(1)})S(g_{(2)}) = h\varepsilon(g) = \sum (hS(g_{(1)}))g_{(2)} \tag{2}$$

for any $h, g \in H$.

Dually, a *Hopf coquasigroup* is a counital unital associative bialgebra $(H, \nabla, \mu, \Delta, \varepsilon)$ equipped with a linear map $S : H \longrightarrow H$ (called *antipode*) such that

$$\sum S(h_{(1)})h_{(2)(1)} \otimes h_{(2)(2)} = 1 \otimes h = \sum h_{(1)}S(h_{(2)(1)}) \otimes h_{(2)(2)}, \tag{3}$$

$$\sum h_{(1)(1)} \otimes S(h_{(1)(2)})h_{(2)} = h \otimes 1 = \sum h_{(1)(1)} \otimes h_{(1)(2)}S(h_{(2)}) \tag{4}$$

for all $h \in H$.

The following remark is helpful to compute something later.

Remark 1. *Let H be a Hopf quasigroup or a coquasigroup with antipode S.*

(1) A Hopf (co)quasigroup is a Hopf algebra if and only its (co)product is (co)associative;

(2) About S, we have

$$\sum S(h_{(1)})h_{(2)} = \sum h_{(1)}S(h_{(2)}) = \varepsilon(h)1,$$
$$S(hg) = S(g)S(h), \quad \Delta(S(h)) = \sum S(h_{(2)}) \otimes S(h_{(1)})$$

for all $h, g \in H$;

(3) *If S is a bijective antipode S with an inverse* S^{-1}*, then*

$$\sum S^{-1}(h_{(2)})h_{(1)} = \sum h_{(2)}S^{-1}(h_{(1)}) = \varepsilon(h)1,$$
$$S^{-1}(hg) = S^{-1}(g)S^{-1}(h), \quad S^{-1}(1) = 1,$$
$$\Delta(S^{-1}(h)) = \sum S^{-1}(h_{(2)}) \otimes S^{-1}(h_{(1)}), \quad \varepsilon(S^{-1}(h)) = \varepsilon(h)$$

for all $h, g \in H$.

If H is a finite dimensional Hopf quasigroup with antipode S, then its linear dual H^* is not Hopf quasigroup but a Hopf coquasigroup with antipode S^*, and one has the non-degenerate bilinear form

$$\langle , \rangle : H^* \times H \longrightarrow \mathbb{K}$$

given by $\langle h^*, h\rangle = h^*(h)$ for all $h^* \in H^*$ and $h \in H$. Let $h^* \in H^*$ and $h \in H$. Then, we have $\langle S^*(h^*), h\rangle = \langle h^*, S(h)\rangle$, and the left action of h^* on h (denoted by $h^* \rightharpoonup h$) is given by

$$h^* \rightharpoonup h = \sum \langle h^*, h_{(2)}\rangle h_{(1)}. \tag{5}$$

Similarly the right action of h^* on h is denoted by $h \leftharpoonup h^*$ and is given by

$$h \leftharpoonup h^* = \sum \langle h^*, h_{(1)}\rangle h_{(2)}. \tag{6}$$

Proposition 1. *Let H be a finite dimensional Hopf quasigroup. Let* $h^*, l^* \in H$ *and* $h \in H$*. Then*

(a) $h^* \rightharpoonup (l^* \rightharpoonup h) = (h^* l^*) \rightharpoonup h$;

(b) $(h \leftharpoonup h^*) \leftharpoonup l^* = h \leftharpoonup (h^* l^*)$.

Proof. (a) We compute:

$$\begin{aligned} h^* \rightharpoonup (l^* \rightharpoonup h) &= \sum \langle l^*, h_{(2)}\rangle \langle h^*, h_{(1)(2)}\rangle h_{(1)(1)} \\ &= \sum \langle l^*, h_{(3)}\rangle \langle h^*, h_{(2)}\rangle h_{(1)} \\ &= \sum \langle h^* l^*, h_{(2)}\rangle h_{(1)} = (h^* l^*) \rightharpoonup h; \end{aligned}$$

(b) follows similarly. □

Proposition 2. *Let H be a finite dimensional Hopf quasigroup. Let* $h^* \in H$ *and* $h, l \in H$*. Then*

(a) $\Delta(h^* \rightharpoonup h) = \sum h_{(1)} \otimes (h^* \rightharpoonup h_{(2)})$;

(b) $\Delta(h \leftharpoonup h^*) = \sum (h_{(1)} \leftharpoonup h^*) \otimes h_{(2)}$;

(c) $h^* \rightharpoonup (hl) = \sum (h^*_{(1)} \rightharpoonup h)(h^*_{(2)} \rightharpoonup l)$;

(d) $(hl) \leftharpoonup h^* = \sum (h \leftharpoonup h^*_{(1)})(l \leftharpoonup h^*_{(2)})$.

Proof. (a) We compute:

$$\begin{aligned} \Delta(h^* \rightharpoonup h) &= \sum \langle h^*, h_{(2)}\rangle h_{(1)(1)} \otimes h_{(1)(2)} \\ &= \sum \langle h^*, h_{(2)(2)}\rangle h_{(1)} \otimes h_{(2)(1)} = \sum h_{(1)} \otimes (h^* \rightharpoonup h_{(2)}); \end{aligned}$$

(b)–(d) follow similarly. □

2.3. Quasimodules

Let H be a Hopf quasigroup. The following notion is given in [5].

We say that $(M, \triangleright)$ is a left H-*quasimodule* if M is a vector space and $\triangleright : H \otimes M \longrightarrow M$ is a linear map (called the left *quasiaction*) satisfying $1 \triangleright m = m$ and

$$\sum h_{(1)} \triangleright (S(h_{(2)}) \triangleright m) = \sum S(h_{(1)}) \triangleright (h_{(2)} \triangleright m) = \varepsilon(h)(l \triangleright m) \tag{7}$$

for all $h \in H$ and $m \in M$.

Remark 2. *(1) If H has an invertible antipode S with an inverse S^{-1}, then*

$$\sum S^{-1}(h_{(2)}) \triangleright (h_{(1)} \triangleright m) = \sum (h_{(2)}) \triangleright (S^{-1}(h_{(1)}) \triangleright m) = \varepsilon(h)(l \triangleright m) \tag{8}$$

for all $h \in H$ and $m \in M$;

(2) Similarly, we can define a right H-quasimodule. We say that $(M, \triangleleft)$ is a right H-quasimodule if M is a vector space, and $\triangleleft : M \otimes H \longrightarrow M$ is a linear map (called the right quasiaction) satisfying $1 \triangleleft m = m$ and

$$\sum (m \triangleleft h_{(1)}) \triangleleft S(h_{(2)}) = \sum (m \triangleleft S(h_{(1)})) \triangleleft h_{(2)} = m\varepsilon(h)$$

for all $h \in H, m \in M$.

Example 1. *Let H be a finite dimensional Hopf quasigroup.*

(1) Then, $(H^*, \rightharpoonup)$ is a left H-quasimodule. In fact, e.g., we have

$$\begin{aligned} \sum h_{(1)} &\rightharpoonup (S(h_{(2)}) \rightharpoonup h^*) = \sum \langle S(h_{(2)}), h^*_{(2)} \rangle \langle h_{(1)}, h^*_{(1)(2)} \rangle h^*_{(1)(1)} \\ &= \sum \langle h_{(2)}, S^*(h^*_{(2)}) \rangle \langle h_{(1)}, h^*_{(1)(2)} \rangle h^*_{(1)(1)} \\ &= \sum \langle h, h^*_{(1)(2)} S^*(h^*_{(2)}) \rangle h^*_{(1)(1)} \\ &\overset{(4)}{=} \sum \varepsilon(h) h^*. \end{aligned}$$

for any $h, l \in H$ and $h^* \in H^*$;

(2) Similarly, $(H^*, \leftharpoonup)$ is a right H-quasimodule.

3. Two Smash Products

In this section, we will consider two smash products for Hopf quasigroups and Hopf coquasigroup in order to obtain our duality theorem.

3.1. Quasimodule Algebra over Hopf Quasigroup

Definition 1. *Let H be a Hopf quasigroup. Then,*

(1) A unital algebra A is said to be a left H-quasimodule algebra if A is a left H-quasimodule such that, for all $a, b \in A$,

$$h \triangleright 1_A = \varepsilon(h) 1_A; \tag{9}$$

$$h \triangleright (ab) = \sum (h_{(1)} \triangleright a)(h_{(2)} \triangleright b); \tag{10}$$

(2) If A is a left H-quasimodule algebra, one can define a smash product $A \bigstar H = A \otimes H$ with a product given by

$$(a \bigstar x)(b \bigstar y) = \sum a(x_{(1)} \triangleright b) \bigstar x_{(2)} y \tag{11}$$

for any $a, b \in A$ and $x, y \in H$.

Remark 3. *(1) It follows immediately from Equation (3) above that $(a \bigstar x)(1 \bigstar y) = \sum a \bigstar xy$ and $(a \bigstar 1)(b \bigstar y) = \sum ab \bigstar y$, for any $a, b \in A$ and $x, y \in H$;*

(2) The study of smash product is also referred to the papers [14,15].

Example 2. *(1) Let H be a Hopf quasigroup. Then $\mathbb{K}$ is a left H-quasimodule algebra with the trivial action given by $h \triangleright a = \varepsilon(h)a$, for $h \in H, a \in \mathbb{K}$. Thus, we have $\mathbb{K} \bigstar H \cong H$;*

(2) Let H be a finite-dimensional Hopf coquasigroup. Then, H^* is a finite-dimensional Hopf quasigroup. A unital associative algebra H can be regarded as a left H^*-quasimodule algebra with $\rightharpoonup$. Hence, we can form smash product $H\bigstar H^*$ with the following product:

$$(h\bigstar h^*)(l\bigstar l^*) = \sum\langle h^*_{(1)}, l_{(2)}\rangle hl_{(1)}\bigstar h^*_{(2)}l^* \tag{12}$$

for any $h, l \in A$ and $h^*, l^* \in H^*$.

In fact, by Example 1, we just check (1) and (2). In fact, for any $h, l \in H$ and $h^* \in H^*$, it is obvious that $h^* \rightharpoonup 1_H = \langle h^*, 1_H\rangle 1_H$. For (2), we have

$$\begin{aligned} h^* \rightharpoonup (hl) &= \sum\langle h^*, h_{(2)}l_{(2)}\rangle h_{(1)}l_{(1)} \\ &= \sum\langle h^*_{(1)}, h_{(2)}\rangle\langle h^*_{(2)}, l_{(2)}\rangle h_{(1)}l_{(1)} = (h^*_{(1)} \rightharpoonup h)(h^*_{(2)} \rightharpoonup l). \end{aligned}$$

Furthermore, it is easy to check that H is not a left H^*-module.

Proposition 3. *With notations as above, then, $A\bigstar H$ is a unital algebra with unit $1_A\bigstar 1_H$. Furthermore, $A\bigstar H$ is an associative algebra if and only if H is a Hopf algebra, and A is the usual left H-module algebra.*

Proof. Obviously, $A\bigstar H$ is a unital algebra with unit $1_A\bigstar 1_H$. Furthermore, for any $a, b, \in A$ and $x, y, z \in H$, we have

$$\begin{aligned} [(a\bigstar x)(b\bigstar y)](c\bigstar z) &= \sum[a(x_{(1)} \triangleright b)\bigstar x_{(2)}y](c\bigstar z) \\ &= \sum[a(x_{(1)} \triangleright b)][(x_{(2)}y)_{(1)} \triangleright c]\bigstar (x_{(2)}y)_{(2)}z \\ &= \sum[a(x_{(1)} \triangleright b)][x_{(2)(1)}y_{(1)} \triangleright c]\bigstar [x_{(2)(2)}y_{(2)}]z \\ &= \sum[a(x_{(1)} \triangleright b)][x_{(2)}y_{(1)} \triangleright c]\bigstar [x_{(3)}y_{(2)}]z \end{aligned}$$

and

$$\begin{aligned} (a\bigstar x)[(b\bigstar y)(c\bigstar z)] &= \sum(a\bigstar x)[b(y_{(1)} \triangleright c)\bigstar y_{(2)}z] \\ &= \sum a[x_{(1)} \triangleright (b(y_{(1)} \triangleright c))]\bigstar x_{(2)}[y_{(2)}z] \\ &= \sum a[(x_{(1)(1)} \triangleright b)(x_{(1)(2)} \triangleright (y_{(1)} \triangleright c))]\bigstar x_{(2)}[y_{(2)}z] \\ &= \sum a[(x_{(1)} \triangleright b)(x_{(2)} \triangleright (y_{(1)} \triangleright c))]\bigstar x_{(3)}[y_{(2)}z]. \end{aligned}$$

If $A\bigstar H$ is associative, then we have

$$(x_{(1)}y_{(1)}) \triangleright c \otimes [x_{(2)}y_{(2)}]z = (x_{(1)} \triangleright (y_{(1)} \triangleright c)) \otimes x_{(2)}[y_{(2)}z]$$

by taking $a = b = 1$, and

$$(ab)c = a(bc)$$

by taking $x = y = z = 1$. It is easy to obtain that H is a Hopf algebra, and A is the usual left H-module algebra.

Conversely, it is obvious. □

3.2. Coquasi Module Algebra over Hopf Coquasigroup

The following notion is different from the one in [15,16].

Definition 2. *Let H be a Hopf coquasigroup. Then,*

(1) A unital algebra A is called a left coquasi H-module algebra *if A is a left H-module such that, for all $a, b \in A$,*

$$h \cdot 1_A = \varepsilon(h)1_A, \tag{13}$$

and

$$h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b); \tag{14}$$

(2) *If A is a left coquasi H-module algebra, we can define the smash product $A\#H = A \otimes H$ with a multiplication given by*

$$(a\#h)(b\#l) = \sum a(h_{(1)} \cdot b)\#h_{(2)}l \tag{15}$$

where $a, b \in A$ and $h, l \in H$.

Example 3. *(1) Let H be a Hopf coquasigroup. Then $\mathbb{K}$ is a left coquasi H-module algebra with the trivial action given by $h \cdot a = \varepsilon(h)a$, for $h \in H, a \in \mathbb{K}$. Thus, we have $\mathbb{K}\#H \cong H$;*

(2) Let H be a finite-dimensional Hopf quasigroup. Then, H^* be a finite-dimensional Hopf coquasigroup. A unital algebra H can be regarded as a left coquasi H^*-module algebra with $\rightharpoonup$. Hence, we can form smash product $H\#H^*$ with the following product:

$$(h\#h^*)(l\#l^*) = \sum \langle h^*_{(1)}, l_{(2)} \rangle h l_{(1)} \# h^*_{(2)} l^* \tag{16}$$

for any $h, l \in H$ and $h^*, l^* \in H^*$. In fact, for any $h, l \in H$ and $h^*, l^* \in H^*$

$$\begin{aligned}
(h^*l^*) \rightharpoonup h &= \sum h_{(1)} \langle h^*l^*, h_{(2)} \rangle \\
&= \sum h_{(1)} \langle h^*, h_{(2)(1)} \rangle \langle l^*, h_{(2)(2)} \rangle \\
&= \sum h_{(1)(1)} \langle h^*, h_{(1)(2)} \rangle \langle l^*, h_{(2)} \rangle \\
&= \sum (h^* \rightharpoonup h_{(1)}) \langle l^*, h_{(2)} \rangle \\
&= h^* \rightharpoonup (l^* \rightharpoonup h)
\end{aligned}$$

and so H is a left H^*-module. It is easy to see that $h^* \rightharpoonup 1_H = \langle h^*, 1_H \rangle 1_H$. By Proposition 2(c), we obtain Equation (6).

Proposition 4. *Let A be a left coquasi H-module algebra. If A is a unital associative algebra, then $A\#H$ is a unital associative algebra with unit $1_A\#1_H$ if and only if*

$$\sum h_{(1)} \cdot (ab) \otimes h_{(2)} = \sum (h_{(1)} \cdot a)(h_{(2)(1)} \cdot b) \otimes h_{(2)(2)}. \tag{17}$$

Proof. Obviously, $A\#H$ is a unital algebra with unit $1_A\#1_H$. Furthermore, for any $a, b, \in A$ and $x, y, z \in H$, we have

$$\begin{aligned}
[(a\#x)(b\#y)](c\#z) &= \sum [a(x_{(1)} \cdot b)\#x_{(2)}y](c\#z) \\
&= \sum [a(x_{(1)} \cdot b)][(x_{(2)}y)_{(1)} \cdot c]\#(x_{(2)}y)_{(2)}z \\
&= \sum [a(x_{(1)} \cdot b)][x_{(2)(1)}y_{(1)} \cdot c]\#[x_{(2)(2)}y_{(2)}]z \\
&= \sum a[(x_{(1)} \cdot b)(x_{(2)(1)} \cdot (y_{(1)} \cdot c))]\#x_{(2)(2)}[y_{(2)}z] \\
&\overset{(16)}{=} \sum a[x_{(1)} \cdot (b(y_{(1)} \cdot c))]\#x_{(2)}[y_{(2)}z] \\
&= \sum (a\#x)[b(y_{(1)} \cdot c)\#y_{(2)}z] \\
&= (a\#x)[(b\#y)(c\#z)].
\end{aligned}$$

Obviously, $A\#H = (A\#1_H)(1_A\#H)$. □

Remark 4. *(1) In [16], we replace Equation (13) with Equation (16) to define a left quasi H-module algebra;*

(2) The unital algebra $H\#H^*$ in Example 3 is not associative.

Proposition 5. *Let H be a finite-dimensional Hopf quasigroup. If A is a left H-quasimodule algebra, then* $A\bigstar H$ *becomes a left coquasi* H^**-module algebra via*

$$h^* \cdot (a\bigstar h) = a\bigstar(h^* \rightharpoonup h)$$

for any $h^* \in H^*, a \in A$ *and* $h \in H$*. Then, we have a smash product* $(A\bigstar H)\#H^*$*.*

Proof. By the proof of Example 3, it is easy to obtain that $A\bigstar H$ is a left H^*-module.

We now prove Equations (13) and (14). For Equation (13), we have, for $h^* \in H^*$,

$$h^* \cdot (1_A\bigstar 1_H) = 1_A\bigstar(h^* \rightharpoonup 1_H) = \langle h^*, 1_H\rangle 1_A\bigstar 1_H.$$

As for Equation (14), we compute for $h^* \in H^*, a, b \in A$ and $h, l \in H$,

$$\begin{aligned}
& h^* \cdot [(a\bigstar h)(b\bigstar l)] \\
= & \sum h^* \cdot [a(h_{(1)} \cdot b)\bigstar h_{(2)}l] \\
= & \sum a(h_{(1)} \cdot b)\bigstar h^* \rightharpoonup (h_{(2)}l) \\
= & \sum a(h_{(1)} \cdot b)\bigstar(h^*_{(1)} \rightharpoonup h_{(2)})(h^*_{(2)} \rightharpoonup l) \\
= & \sum a(h_{(1)} \cdot b)\bigstar h_{(2)}l_{(1)}\langle h^*_{(1)}, h_{(3)}\rangle\langle h^*_{(2)}, l_{(2)}\rangle \\
= & \sum a(h_{(1)(1)} \cdot b)\bigstar h_{(1)(2)}l_{(1)}\langle h^*_{(1)}, h_{(2)}\rangle\langle h^*_{(2)}, l_{(2)}\rangle \\
= & \sum (a\bigstar h_{(1)})(b\bigstar l_{(1)})\langle h^*_{(1)}, h_{(2)}\rangle\langle h^*_{(2)}, l_{(2)}\rangle \\
= & \sum [a\bigstar h^* \rightharpoonup h][b\bigstar h^*_{(2)(1)} \rightharpoonup l] \\
= & \sum [h^*_{(1)} \cdot (a\bigstar h)][h^*_{(2)} \cdot (b\bigstar l)].
\end{aligned}$$

Then, we have a smash product $(A\bigstar H)\#H^*$. □

4. Duality Theorem

In this section, let H be a finite-dimensional Hopf quasigroup and A a left H-quasimodule algebra. We will prove our duality theorem.

Lemma 1. *Let Q be a Hopf coquasigroup and B a left coquasi H-module algebra. Then, there is a unital homomorphism*

$$\Lambda_{B,Q} : B\#Q \longrightarrow End(B) \tag{18}$$

given by $\Lambda_{B,Q}(a\#h)(b) = a(h \cdot b)$ *for any* $h \in Q$ *and* $a, b \in B$*. Furthermore, if B is associative, then* $\Lambda_{B,Q}$ *is an algebra homomorphism.*

Proof. It is easy to see that $\Lambda_{B,Q}(1_B\#1_Q)(1_B) = 1_B$. If B is associative, then, for any $h, l \in Q$ and $a, b, c \in B$, we have

$$\begin{aligned}
\Lambda_{B,Q}[(a\#h)(b\#l)](c) &= \sum \Lambda_{B,Q}(a(h_{(1)} \cdot b)\#h_{(2)}l)(c) \\
&= \sum [(a(h_{(1)}) \cdot b)][(h_{(2)}l) \cdot c] \\
&= \sum a[(h_{(1)} \cdot b)(h_{(2)} \cdot (l \cdot c))] \\
&= a[h \cdot [b(l \cdot c)]] \\
&= [\Lambda_{B,Q}(a\#h) \circ \Lambda_{B,Q}(b\#l)](c).
\end{aligned}$$

□

Remark 5. *For Equation (9), in the special case where* $B = H$*, a Hopf quasigroup, and* $Q = H^*$*, the Hopf coquasigroup. By Example 3(2), we have* $H\#H^*$*. Then, there is a unital algebra anti-homomorphism*

$$\Gamma_{H,H^*} : H^* \longrightarrow End(H) \tag{19}$$

given by $\Gamma_{H,H^*}(h^*)(h) = h \leftharpoonup h^*$ *for any* $h \in H$ *and* $h^* \in H^*$.

Let H be a finite-dimensional Hopf quasigroup and A a left H-quasimodule algebra. Then, we have the smash product $A\bigstar H$. By using the map Λ from Equation (9) and Proposition 2, we define:

$$\Phi = \Lambda_{A\bigstar H,} : (A\bigstar H)\#H^* \longrightarrow End(A\bigstar H)$$

and by the left regular representation $\lambda_l : A\ End(A), a \mapsto (x \mapsto ax)$, one defines

$$\Psi = \lambda_l \otimes \Lambda_{H,H^*} : A \otimes (H\#H^*) \longrightarrow End(A\bigstar H).$$

That is,

$$\Phi((a\bigstar h)\#h^*)(b\bigstar l) = (a\bigstar h)(b\bigstar(h^* \rightharpoonup l))$$

and

$$\Psi(a \otimes (h\#h^*))(b\bigstar l) = ab\bigstar h(h^* \rightharpoonup l)$$

for any $a, b \in A, h, l \in H$ and $h^* \in H^*$.

Remark 6. *We notice that* $id \otimes \Gamma_{H,H^*}(h^*) = \Psi(1 \otimes v)$ *for some* $v \in H\#H^*$ *for* $h^* \in H^*$.

Lemma 2. *With notations above,* Φ *and* Ψ *do not preserve multiplication. However, we have*

(i) $\Phi((a\bigstar h)\#h^*) = \Phi((a\bigstar 1)\#\varepsilon) \circ \Phi((1\bigstar h)\#h^*)$;
(ii) $\Psi(a \otimes (h\#h^*)) = \Psi(a \otimes (1\#\varepsilon)) \circ \Psi(1 \otimes (h\#h^*))$;
(iii) $\Phi((1\bigstar h)\#h^*) = \Phi((1\bigstar h)\#\varepsilon) \circ \Phi((1\bigstar 1)\#h^*)$;
(iv) $\Psi(1 \otimes (h\#h^*)) = \Psi(1 \otimes (h\#\varepsilon)) \circ \Psi(1 \otimes (1\#h^*))$.

Proof. For any $a, b \in A, h, l \in H$ and $h^* \in H^*$, we have

$$\begin{aligned} & \Phi((a\bigstar h)\#h^*)(b\bigstar l) \\ = \ & (a\bigstar h)(b\bigstar(h^* \rightharpoonup l)) \\ = \ & a(h_{(1)} \triangleright b)\bigstar h_{(2)}(h^* \rightharpoonup l)) \\ = \ & (a\bigstar 1)[(h_{(1)} \triangleright b)\bigstar h_{(2)}(h^* \rightharpoonup l)] \\ = \ & \Phi((a\bigstar 1)\#\varepsilon)[(h_{(1)} \triangleright b)\bigstar h_{(2)}(h^* \rightharpoonup l)] \\ = \ & \Phi((a\bigstar 1)\#\varepsilon)((1\bigstar h)(b\bigstar h^* \rightharpoonup l) \\ = \ & [\Phi((a\bigstar 1)\#\varepsilon) \circ \Phi((1\bigstar h)\#h^*)](b\bigstar l). \end{aligned}$$

Similarly,

$$\begin{aligned} & \Psi(a \otimes (h\#h^*))(b\bigstar l) \\ = \ & ab\bigstar h(h^* \rightharpoonup l) \\ = \ & \Psi(a \otimes (1\#\varepsilon))(b \otimes h(h^* \rightharpoonup l)) \\ = \ & [\Psi(a \otimes (1\#\varepsilon)) \circ \Psi(1 \otimes (h\#h^*))](b\bigstar l). \end{aligned}$$

(iii) and (iv). Straightforward. □

Lemma 3. *With notations above,* Φ *and* Ψ *are injective linear maps.*

Proof. In order to prove that Φ and Ψ are injective maps, we consider the following injective linear maps Φ', Ψ' and Θ:

$$\Phi' : (A\bigstar H)\#H^* \longrightarrow End(A\bigstar H), \quad \Phi'((a\bigstar h)\#h^*)(b\bigstar l) = \langle h^*, l\rangle(a\bigstar h)(b\bigstar 1),$$
$$\Psi' : A \otimes (H\#H^*) \longrightarrow End(A\bigstar H), \quad \Psi'(a \otimes (h\#h^*))(b\bigstar l) = \langle h^*, l\rangle ab\bigstar h,$$
$$\Theta : End(A\bigstar H) \longrightarrow End(A\bigstar H), \quad \Theta(f)(b\bigstar l) = \sum[f(b\bigstar l_{(2)})](1\bigstar l_{(1)})$$

for any $a, b \in A, h, l \in H, h^* \in H^*$ and $f \in End(A\bigstar H)$.

Let $x \in Ker(\Phi')$ and write $x = \sum_{i=1}^{n} y_i\bigstar h^*i$, where $y_i \in A\bigstar H$ and $\{h_1^*, h_2^*, \cdots, h_n^*\}$ is a linearly independent subset of H^*. Choose $h_1, h_2, \cdot h_n$ such that $h_i^*(h_j) = \delta_{ij}$, with $1 \leq i, j, \leq n$. Then, $0 = \Phi'(x)(1\bigstar k_i) = y_i$ for all i, so that $x = 0$. Thus, Φ' is injective.

Similarly, we can prove that Ψ' is injective.

To see that Θ is injective (actually it is bijective), we construct a left inverse for Θ. We define $\Upsilon : End(A\bigstar H) \longrightarrow End(A\bigstar H)$ by

$$\Upsilon(f)(b\bigstar l) = \sum[f(b\bigstar l_{(2)})](1\bigstar S^{-1}(l_{(1)})).$$

Then, we compute

$$\begin{aligned}
(\Upsilon \circ \Theta)(f)(b\bigstar l) &= \sum[\Theta(f)(b\bigstar l_{(2)})](1\bigstar S^{-1}(l_{(1)})) \\
&= \sum[[f(b\bigstar l_{(2)(2)})](1\bigstar l_{(2)(1)})](1\bigstar S^{-1}(l_{(1)})) \\
&= \sum[[f(b\bigstar l_{(2)})](1\bigstar l_{(1)(2)})](1\bigstar S^{-1}(l_{(1)(1)})) \\
&= \sum u_i\bigstar(v_i l_{(1)(2)})S^{-1}(l_{(1)(1)})) \\
&\overset{(3)}{=} \sum u_i\bigstar v_i\varepsilon(l_{(1)}) \\
&= \sum f(b\bigstar l_{(2)})\varepsilon(l_{(1)}) \\
&= f(b\bigstar l)
\end{aligned}$$

where we write $f(b\bigstar l_{(2)}) = \sum_i u_i \otimes v_i$, use the Remark after Definition 1, and use the coassociativity in the Hopf quasigroup.

Similarly, we also have:

$$\begin{aligned}
(\Theta \circ \Upsilon)(f)(b\bigstar l) &= \sum[\Upsilon(f)(b\bigstar l_{(2)})](1\bigstar l_{(1)}) \\
&= \sum[[f(b\bigstar l_{(2)(2)})](1\bigstar S^{-1}(l_{(2)(1)})](1\bigstar l_{(1)}) \\
&= \sum[[f(b\bigstar l_{(2)})](1\bigstar S^{-1}(l_{(1)(2)})](1\bigstar l_{(1)(1)}) \\
&= \sum u_i\bigstar(v_i S^{-1}(l_{(1)(2)}))l_{(1)(1)} \\
&\overset{(3)}{=} \sum u_i\bigstar v_i\varepsilon(l_{(1)}) \\
&= \sum f(b\bigstar l_{(2)})\varepsilon(l_{(1)}) \\
&= f(b\bigstar l).
\end{aligned}$$

Therefore, Υ is a two-sided inverse for Θ.

Next, we will show that $\Phi = \Theta \circ \Phi'$ and $\Psi = \Theta \circ \Psi'$. For the first one, we have

$$
\begin{aligned}
(\Theta\circ\Phi')((a\bigstar h)\#h^*)(b\bigstar l) &= \Theta[\Phi'((a\bigstar h)\#h^*)](b\bigstar l)\\
&= \sum \Phi'((a\bigstar h)\#h^*)(b\bigstar l_{(2)})(1\bigstar l_{(1)})\\
&= \sum [(a\bigstar h)(b\bigstar 1)](1\bigstar l_{(1)})\langle h^*, l_{(2)}\rangle\\
&= \sum [a(h_{(1)}\triangleright b)\bigstar h_{(2)}l_{(1)}]\langle h^*, l_{(2)}\rangle\\
&= \sum (a\bigstar h)(b\bigstar l_{(1)})\langle h^*, l_{(2)}\rangle\\
&= (a\bigstar h)(b\bigstar (h^*\rightharpoonup l))\\
&= \Phi((a\bigstar h)\#h^*)(b\bigstar l),
\end{aligned}
$$

and, for the second one, we compute as follows:

$$
\begin{aligned}
(\Theta\circ\Psi')(a\otimes(h\#h^*))(b\bigstar l) &= \Theta[\Psi'(a\otimes(h\#h^*))](b\bigstar l)\\
&= \sum [\Psi'(a\otimes(h\#h^*))(b\bigstar l_{(2)})](1\bigstar l_{(1)})\\
&= \sum (ab\bigstar h)(1\bigstar l_{(1)})\langle h^*, l_{(2)}\rangle\\
&= \sum (ab\bigstar hl_{(1)})\langle h^*, l_{(2)}\rangle\\
&= ab\bigstar h(h^*\rightharpoonup l)\\
&= \Psi(a\otimes(h\#h^*))(b\bigstar l).
\end{aligned}
$$

This shows that Φ and Ψ are injective linear maps. □

Corollary 1. *Let H be a finite-dimensional Hopf quasigroup of dimension $n<\infty$. Then, Λ_{H,H^*} is a bijective linear map, so that $H\#H^*\cong End(H)\cong M_n(\mathbb{K})$, the algebra of $n\times n$ matrices over $\mathbb{K}$.*

Proof. By Example 2(1), we have $\mathbb{K}\bigstar H\cong H$ and so $\Psi=\Lambda_{H,H^*}$, so that Λ_{H,H^*} is injective. Observe that $dim(H\#H^*)=n^2=dim(End(H))$ and so Λ_{H,H^*} is a bijective linear map. □

We next define a map $\Xi\in End(A\bigstar H)$ by

$$\Xi(b\bigstar l)=\sum (S^{-1}(l_{(1)})\triangleright b)\bigstar l_{(2)}$$

for any $b\in A$ and $l\in H$.

Lemma 4. *With notations above, Ξ is invertible with inverse Ω given by*

$$\Omega(b\bigstar l)=\sum (l_{(1)}\triangleright b)\bigstar l_{(2)}$$

for any $b\in A$ and $l\in H$.

Proof. For any $b\in A$ and $l\in H$, we have

$$
\begin{aligned}
\Xi(\Omega(b\bigstar l)) &= \sum \Xi[(l_{(1)}\triangleright b)\bigstar l_{(2)}]\\
&= \sum (S^{-1}(l_{(2)(1)})\triangleright(l_{(1)}\triangleright b))\bigstar l_{(2)(2)}\\
&= \sum (S^{-1}(l_{(1)(2)})\triangleright(l_{(1)(1)}\triangleright b))\bigstar l_{(2)}\\
&= \sum \varepsilon(l_{(1)})b\bigstar l_{(2)}\\
&= \sum b\bigstar l,
\end{aligned}
$$

and so $\Xi\circ\Omega=id$. Meanwhile, we have

$$
\begin{aligned}
\Omega(\Xi(b\bigstar l)) &= \sum \Omega[(S^{-1}(l_{(1)}) \triangleright b)\bigstar l_{(2)}] \\
&= \sum (l_{(2)(1)} \triangleright (S^{-1}(l_{(1)}) \triangleright b))\bigstar l_{(2)(2)} \\
&= \sum (l_{(1)(2)} \triangleright (S^{-1}(l_{(1)(1)}) \triangleright b))\bigstar l_{(2)} \\
&= \sum \varepsilon(l_{(1)}) b\bigstar l_{(2)} \\
&= \sum b\bigstar l,
\end{aligned}
$$

and thus $\Omega \circ \Xi = id$. □

Let H be a finite-dimensional Hopf quasigroup of dimension $n < \infty$. Let $\{h_1^*, h_2^*, \cdots, h_n^*\}$ be a basis of H^* and $\{h_1, h_2, \cdots, h_n\}$ be its dual basis for H, i.e., so that $h_i^*(h_j) = \delta_{ij}$, with $1 \leq i, j, \leq n$.

Remark 7. *With notations above, let $a \in A$ and $a_i = h_i \triangleright a$ with $i \in \{1, 2, \cdots, n\}$. For any $h \in H$, we let $h = \sum_i k_i h_i$ with $k_i \in \mathbb{K}$. Then, $h \triangleright a = \sum_i k_i (h_i \triangleright a) = \sum_i k_i a_i$. However, $\langle h_j^*, h\rangle = \langle h_j^*, \sum_i k_i h_i\rangle = k_j$, and so*

$$
h \triangleright a = \sum_i \langle h_i^*, h\rangle a_i. \tag{20}
$$

Lemma 5. *With the notations above, fix $a \in A$ so that $a_i = h_i \triangleright a$ with $i \in \{1, 2, \cdots, n\}$ and, for any $b \in A, h, l \in H$ and $h^* \in H^*$, we have the following identities:*

(i) $\Omega \circ \Psi(1 \otimes (1\#h^*)) \circ \Xi = \Phi((1\bigstar 1)\#h^*)$;

(ii) $\Omega \circ \Psi(a \otimes (1\#\varepsilon)) \circ \Xi = \sum_{i=1}^n \Phi((a_i \bigstar 1)\#\varepsilon) \circ \Omega \circ (id \otimes \Gamma_{H,H^*}(h_i^*)) \circ \Xi$;

(iii) $\Xi \circ \Phi((a\bigstar 1)\#\varepsilon) \circ \Omega = \sum_{i=1}^n \Psi((a_i \otimes (1\#\varepsilon)) \circ (id \otimes \Gamma_{H,H^*}(S^{*-1}(h_i^*))$;

(iv) $[\Omega \circ \Psi(1 \otimes (h\#\varepsilon)) \circ \Xi](b\bigstar l) = \Phi((1\bigstar hl_{(2)})\#\varepsilon)(S^{-1}(l_{(1)}) \triangleright b)\bigstar 1)$.

Proof. For any $b \in A$ and $l \in H$.

(i) We check as follows:

$$
\begin{aligned}
& [\Omega \circ \Psi(1 \otimes (1\#h^*)) \circ \Xi](b\bigstar l) \\
=& \sum [\Omega \circ \Psi(1 \otimes (1\#h^*))](S^{-1}(l_{(1)}) \triangleright b)\bigstar l_{(2)} \\
=& \sum \Omega[(S^{-1}(l_{(1)}) \triangleright b)\bigstar (h^* \rightharpoonup l_{(2)})] \\
=& \sum [[(h^* \rightharpoonup l_{(2)})]_{(1)} \triangleright (S^{-1}(l_{(1)}) \triangleright b)]\bigstar [(h^* \rightharpoonup l_{(2)})]_{(2)} \\
=& \sum [(h^* \rightharpoonup l_{(2)})_{(1)} \triangleright (S^{-1}(l_{(1)}) \triangleright b)]\bigstar [(h^* \rightharpoonup l_{(2)})_{(2)}] \\
=& \sum [l_{(2)(1)} \triangleright (S^{-1}(l_{(1)}) \triangleright b)]\bigstar (h^* \rightharpoonup l_{(2)(2)}) \quad \text{by Proposition 1.3(a)} \\
=& \sum [l_{(1)(2)} \triangleright (S^{-1}(l_{(1)(1)}) \triangleright b)]\bigstar (h^* \rightharpoonup l_{(2)}) \\
=& \sum b\bigstar (h^* \rightharpoonup l) \\
=& \Phi((1\bigstar 1)\#h^*)(b\bigstar l).
\end{aligned}
$$

For (ii), we have

$$
\begin{aligned}
& [\Omega \circ \Psi(a \otimes (1\#\varepsilon)) \circ \Xi](b \bigstar l) \\
= \ & \sum [\Omega \circ \Psi(a \otimes (1\#\varepsilon))](S^{-1}(l_{(1)}) \triangleright b) \bigstar l_{(2)} \\
= \ & \sum \Omega[a(S^{-1}(l_{(1)}) \triangleright b) \bigstar (\varepsilon \rightharpoonup l_{(2)})] \\
= \ & \sum \Omega[a(S^{-1}(l_{(1)}) \triangleright b) \bigstar l_{(2)}] \\
= \ & \sum [l_{(2)(1)}[a(S^{-1}(l_{(1)}) \triangleright b)]] \bigstar l_{(2)(2)} \\
= \ & \sum [l_{(2)(1)(1)} \triangleright a][l_{(2)(1)(2)} \triangleright (S^{-1}(l_{(1)}) \triangleright b)] \bigstar l_{(2)(2)} \\
= \ & \sum [l_{(2)} \triangleright a][l_{(3)} \triangleright (S^{-1}(l_{(1)}) \triangleright b)] \bigstar l_{(4)} \\
\overset{(11)}{=} \ & \sum \sum_{i=1}^{n} \langle h_i^*, l_{(2)} \rangle a_i [l_{(3)} \triangleright (S^{-1}(l_{(1)}) \triangleright b)] \bigstar l_{(4)} \\
= \ & \sum \sum_{i=1}^{n} (a_i \bigstar 1)[l_{(3)} \triangleright (S^{-1}(l_{(1)}) \triangleright b) \bigstar l_{(4)}] \langle h_i^*, l_{(2)} \rangle \\
= \ & \sum \sum_{i=1}^{n} [\Phi((a_i \bigstar 1)\#\varepsilon)][l_{(3)} \triangleright (S^{-1}(l_{(1)}) \triangleright b) \bigstar l_{(4)}] \langle h_i^*, l_{(2)} \rangle \\
= \ & \sum \sum_{i=1}^{n} [\Phi((a_i \bigstar 1)\#\varepsilon)][l_{(2)} \leftharpoonup h_i^* \triangleright (S^{-1}(l_{(1)}) \triangleright b) \bigstar l_{(3)}] \\
= \ & \sum \sum_{i=1}^{n} [\Phi((a_i \bigstar 1)\#\varepsilon)][((l_{(2)} \leftharpoonup h_i^*)_{(1)} \triangleright (S^{-1}(l_{(1)}) \triangleright b) \bigstar ((l_{(2)} \leftharpoonup h_i^*)_{(2)}] \\
& \qquad \text{by Proposition 2(b)} \\
= \ & \sum \sum_{i=1}^{n} [\Phi((a_i \bigstar 1)\#\varepsilon)]\Omega[(S^{-1}(l_{(1)}) \triangleright b) \bigstar (l_{(2)} \leftharpoonup h_i^*] \\
= \ & \sum \sum_{i=1}^{n} [\Phi((a_i \bigstar 1)\#\varepsilon) \circ \Omega][(S^{-1}(l_{(1)}) \triangleright b) \bigstar (l_{(2)} \leftharpoonup h_i^*] \\
= \ & \sum \sum_{i=1}^{n} [\Phi((a_i \bigstar 1)\#\varepsilon) \circ \Omega \circ (id \otimes \Gamma_{H,H^*}(h_i^*))](S^{-1}(l_{(1)}) \triangleright b) \bigstar l_{(2)} \\
= \ & \sum_{i=1}^{n} [\Phi((a_i \bigstar 1)\#\varepsilon) \circ \Omega \circ (id \otimes \Gamma_{H,H^*}(h_i^*)) \circ \Xi](b \bigstar l).
\end{aligned}
$$

For (iii), one has

$$
\begin{aligned}
& [\Xi \circ \Phi((a\bigstar 1)\#\varepsilon) \circ \Omega](b\bigstar l) \\
= \;& \sum [\Xi \circ \Phi((a\bigstar 1)\#\varepsilon)]((l_{(1)} \triangleright b)\bigstar l_{(2)}) \\
= \;& \sum \Xi[(a\bigstar 1)((l_{(1)} \triangleright b)\bigstar l_{(2)})] \\
= \;& \sum \Xi[a(l_{(1)} \triangleright b)\bigstar l_{(2)}] \\
= \;& \sum S^{-1}(l_{(2)(1)}) \triangleright [a(l_{(1)} \triangleright b)]\bigstar l_{(2)(2)}] \\
= \;& \sum S^{-1}(l_{(2)}) \triangleright [a(l_{(1)} \triangleright b)]\bigstar l_{(3)}] \\
= \;& \sum (S^{-1}(l_{(3)}) \triangleright a)[S^{-1}(l_{(2)}) \triangleright (l_{(1)} \triangleright b)]\bigstar l_{(4)}] \\
= \;& \sum (S^{-1}(l_{(1)}) \triangleright a) b\bigstar l_{(2)} \\
\overset{(11)}{=} \;& \sum_{i=1}^{n} \langle h_i^*, S^{-1}(l_{(1)})\rangle a_i b\bigstar l_{(2)} \\
= \;& \sum_{i=1}^{n} a_i b\bigstar l_{(2)} \langle S^{*-1}(h_i^*), l_{(1)}\rangle \\
= \;& \sum_{i=1}^{n} a_i b\bigstar (l \leftharpoonup S^{*-1}(h_i^*)) \\
= \;& \sum_{i=1}^{n} [\Psi((a_i \otimes (1\#\varepsilon))](b\bigstar (l \leftharpoonup S^{*-1}(h_i^*))) \\
= \;& \sum_{i=1}^{n} [\Psi((a_i \otimes (1\#\varepsilon)) \circ (id \otimes \Gamma_{H,H^*}(S^{*-1}(h_i^*))](b\bigstar l).
\end{aligned}
$$

In addition, finally, for (iv), we have

$$
\begin{aligned}
& [\Omega \circ \Psi(1 \otimes (h\#\varepsilon)) \circ \Xi](b\bigstar l) \\
= \;& \sum [\Omega \circ \Psi(1 \otimes (h\#\varepsilon))](S^{-1}(l_{(1)}) \triangleright b)\bigstar l_{(2)} \\
= \;& \sum \Omega[(S^{-1}(l_{(1)}) \triangleright b)\bigstar hl_{(2)}] \\
= \;& \sum [(hl_{(2)})_{(1)} \triangleright (S^{-1}(l_{(1)}) \triangleright b)]\bigstar (hl_{(2)})_{(2)} \\
= \;& \sum [h_{(1)} l_{(2)} \triangleright (S^{-1}(l_{(1)}) \triangleright b)]\bigstar (h_{(2)} l_{(3)}) \\
= \;& \sum [(h_{(2)} l)_{(2)} \triangleright (S^{-1}((h_{(2)} l)_{(1)}) h_{(1)} \triangleright b)]\bigstar (h_{(2)} l)_{(3)} \\
= \;& \Phi((1\bigstar (h_{(2)} l)_{(2)})\#\varepsilon)((S^{-1}((h_{(2)} l)_{(1)}) h_{(1)} \triangleright b)\bigstar 1) \\
= \;& \Phi((1\bigstar hl_{(2)})\#\varepsilon)(S^{-1}(l_{(1)} \triangleright b)\bigstar 1).
\end{aligned}
$$

This completes the proof. □

Remark 8. *In general, for any $h \in H$ and $h^* \in H^*$, we have*

$$\Omega \circ \Psi(1 \otimes (h\#h^*)) \circ \Xi \neq \Phi((1\bigstar h)\#h^*).$$

Lemma 6. *Let B be a semigroup with a multiplication. Let $H, J \subseteq B$ be non-empty subsets of B. If there is an invertible element $\chi \in B$ so that $H = \chi^{-1} J \chi$, then there exists a bijective map $\xi : H \longrightarrow J$ that preserves multiplication.*

Proof. Define a map

$$\xi : H \longrightarrow J, \quad h \mapsto \chi h \chi^{-1}.$$

Obviously, we can show that ξ is bijective and $\xi(hl) = \chi(hl)\chi^{-1} = \chi h \chi^{-1} \chi l \chi^{-1} = \xi(h)\xi(l)$ for $h, l \in H$. □

We are now in a position to prove the main theorem of this paper.

Theorem 1. *Let H be a finite-dimensional Hopf quasigroup with bijective antipode and A a left H-quasimodule algebra. Then,*

$$(A\bigstar H)\#H^* \cong A \otimes (H\#H^*).$$

Proof. Let $a \in A, h \in H$ and $h^* \in H^*$. Firstly, we show that $\Omega \circ \Psi(a \otimes (h\#h^*)) \circ \Xi$ belong to $\Phi((A\bigstar H)\#H^*)$.

By the fact that $a \otimes (h\#h^*) = (a \otimes (1\#\varepsilon))(1 \otimes (h\#h^*))$ and Lemma 2, we have

$$\begin{aligned} & \Omega \circ \Psi(a \otimes (h\#h^*)) \circ \Xi \\ = & \Omega \circ \Psi(a \otimes (1\#\varepsilon)) \circ \Psi(1 \otimes (h\#h^*)) \circ \Xi \\ = & [\Omega \circ \Psi(a \otimes (1\#\varepsilon)) \circ \Xi] \circ [\Omega \circ \Psi(1 \otimes (h\#h^*)) \circ \Xi] \quad \text{by Lemma 4}. \end{aligned}$$

Thus, it suffices to show that $\Omega \circ \Psi(a \otimes (1\#\varepsilon)) \circ \Xi$ and $\Omega \circ \Psi(1 \otimes (h\#h^*)) \circ \Xi$ each belong to $\Phi((A\bigstar H)\#H^*)$. The first does by Lemma 5(ii). We also have

$$\begin{aligned} \Omega \circ \Psi(1 \otimes (h\#h^*)) \circ \Xi &= \Omega \circ \Psi(1 \otimes (h\#\varepsilon)) \circ \Psi(1 \otimes (1\#h^*)) \circ \Xi \\ &= [\Omega \circ \Psi(1 \otimes (h\#\varepsilon)) \circ \Xi] \circ [\Omega \circ \Psi(1 \otimes (1\#h^*)) \circ \Xi] \\ &\in \Phi((A\bigstar H)\#H^*) \quad \text{by Lemma 5 (i)(iv)} \end{aligned}$$

which implies that the second does also.

Then, we prove similarly that $\Xi \circ \Phi((a\bigstar h)\#h^*) \circ \Omega$ belongs to $\Psi(A \otimes (H\#H^*))$. Actually, it follows from Lemma 5(i)(iii)(iv).

We now obtain

$$\Phi((A\bigstar H)\#H^*) = \Xi^{-1} \circ \Psi(A \otimes (H\#H^*)) \circ \Xi.$$

By Lemma 6, our theorem is proved. □

Corollary 2. *Let H be a finite-dimensional Hopf quasigroup with bijective antipode and A a left H-quasimodule algebra. Then,*

$$(A\bigstar H)\#H^* \cong A \otimes (H\#H^*) \cong A \otimes M_n(\mathbb{K}) \cong M_n(A).$$

Proof. It follows Theorem 1 and Corollary 1. □

Example 4. *Let Q be a quasigroup (see [17]). Then, it follows from [1] (Proposition 4.7) that $H = \mathbb{K}Q$ is a Hopf quasigroup with a linear extension of the product and $\Delta(h) = h \otimes h, \varepsilon(h) = 1$ and $S(h) = h^{-1}$ on the basis elements $h \in Q$.*

If Q is a finite quasigroup, then $(\mathbb{K}Q)^*$ is a Hopf coquasigroup (see [1]). Explicitly, a basis of $(\mathbb{K}Q)^*$ is the set of projections $\{p_g \mid g \in Q\}$; that is, for any $g \in Q$ and $x = \sum_{h\in Q} \alpha_h h \in \mathbb{K}Q$, $p_g(x) = \alpha_g \in \mathbb{K}$. The set $\{p_g\}$ consists of orthogonal idempotents whose sum is 1. The comultiplication on $(\mathbb{K}Q)^*$ is given by $\Delta(p_g) = \sum_{h\in Q} p_{gh^{-1}} \otimes p_h$, and the counit is given by $\varepsilon(p_g) = \delta_{1,g}$ (where δ denotes the Kronecker delta).

Let A be a left $\mathbb{K}Q$-quasimodule algebra. Then, we have $1 \triangleright a = a$, $h \triangleright (ab) = (h \triangleright a)(h \triangleright b)$, and

$$h \triangleright (h^{-1} \triangleright a) = h^{-1} \triangleright (h \triangleright a) = a$$

for all $h \in Q$ and $a, b \in A$.

We remark here that Q does not act as automorphism of A like a group acting as automorphism of A. In case of group G, we know that A is a Hopf algebra $\mathbb{K}G$-module algebra if and only if G acts as automorphism of A, and the smash product $A\#\mathbb{K}G = A * G$ is just the skew group ring of G over A (see [18]).

In our case of quasigroup Q, we have a *skew quasigroup ring* $A \star Q$ of Q over A with a product:

$$(a\bigstar x)(b\bigstar y) = a(x \triangleright b)\bigstar xy \tag{21}$$

for any $a, b \in A$ and $x, y \in Q$. We note that a skew quasigroup ring generally is not associative unless $(xy) \triangleright a = x \triangleright (y \triangleright a)$ and Q is a group.

We know that $(\mathbb{K}Q)^*$ is a Hopf coquasigroup. Then, a unital algebra A is a left *coquasi $(\mathbb{K}Q)^*$-module algebra* if A is a left $\mathbb{K}Q$-module, i.e., Q-action, such that $p_g \cdot 1 = \delta_{1,g}1$ and

$$p_g \cdot (ab) = \sum_{h\in Q} (p_{gh^{-1}} \cdot a)(p_h \cdot b)$$

for all $a, b \in A$. Then, we have a smash product $A * Q$ with a multiplication given by

$$(a * p_g)(b * p_l) = a(p_{gl^{-1}} \cdot b) * p_l$$

where $a, b \in A$ and $g, l \in Q$.

In particular, by Example 3(2), when Q is finite, we have the smash product $Q * Q$ with the following product:

$$(h * p_g)(q * p_l) = hq * p_{q^{-1}g}$$

for any $h, g, q, l \in Q$.

By Corollary 2, we have

$$(A \star Q) * Q \cong A \otimes (Q * Q) \cong A \otimes M_n(\mathbb{K}) \cong M_n(A).$$

In the end of this paper, we remark here that, when we consider a finite field $\mathbb{Z}_p$ (Galois field) with a prime p as a finite-dimensional Hopf quasigroup over $\mathbb{Z}_p$, we have a Hopf algebra $\mathbb{Z}_p$ with the coproduct $\Delta([a]) = [a]([1] \otimes [1])$ and the counit $\varepsilon([a]) = [a]$ for any $[a] \in \mathbb{Z}_p$. Then, we have

$$(\mathbb{Z}_p\bigstar\mathbb{Z}_p)\#(\mathbb{Z}_p)^* \cong (\mathbb{Z}_p\#(\mathbb{Z}_p)^*) \cong M_p(\mathbb{Z}_p)$$

where we use the adjoint action of $\mathbb{Z}_p$ in the smash product $\mathbb{Z}_p\bigstar\mathbb{Z}_p$.

5. Conclusions and Further Research

As we mentioned already in the Introduction, Blattner and Montgomery obtained in [13] the duality theorem in the setting of Hopf algebras. In particular, if H is a finite-dimensional Hopf algebra and A is a left H-module algebra, then the duality theorem takes the form: $(A\#H)\#H^* \cong A \otimes (H\#H^*) \cong M_n(A)$.

The dual space of a finite-dimensional Hopf algebra is a Hopf algebra. This duality breaks down for Hopf quasigroups, since the dual coalgebra of the algebra of a Hopf quasigroup is no longer a co-associative. This means that the dual space of a finite-dimensional Hopf quasigroup is not a Hopf quasigroup, but a Hopf coquasigroup, which are generalizations of Hopf algebras.

In this paper, we have studied two kinds of smash products "$\bigstar$" and "#" on the tensor product space $A \otimes H$ associated with a finite-dimensional Hopf quasigroup H and a left H-quasimodule algebra A. We have obtained an analogue of Blattner and Montgomery's duality theorem in the general finite-dimensional Hopf quasigroup case in Section 4: $(A\bigstar H)\#H^* \cong A \otimes (H\#H^*) \cong A \otimes M_n(\mathbb{K}) \cong M_n(A)$ (see Theorem 1 and Corollary 2). In addition, we have paid special attention to the finite quasigroup case (see Example 4). It is still not clear if the nicer results, obtained in the finite case, can be pushed forward to the infinite case so that better results can also be shown there. We expect, however, that this will not be easy, neither to prove these results if they are true nor to find counter examples if they are not.

Finally, constructing an analogue of Blattner and Montgomery's duality theorem in the general finite-dimensional Hopf coquasigroup is not so easy to do.

Author Contributions: Methodology, H.G. and S.W.; Formal analysis, H.G.; Investigation, H.G.; Resources, S.W.; Writing—original draft, S.W.; Writing—review & editing, H.G. and S.W. All authors have read and agreed to the published version of the manuscript.

Funding: The National Natural Science Foundation of China (Grant Nos. 12271089 and 11871144).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors thank the anonymous referee for his/her thorough review of this work and his/her comments. The second author is thankful for the financial support of the National Natural Science Foundation of China (Grant Nos. 12271089 and 11871144).

Conflicts of Interest: The authors declare no conflict of interest.

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