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Article

A New Approach to Braided T-Categories and Generalized Quantum Yang–Baxter Equations

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Abstract: We introduce and study a large class of coalgebras (possibly (non)coassociative) with group-algebraic structures Hopf (non)coassociative group-algebras. Hopf (non)coassociative group-algebras provide a unifying framework for classical Hopf algebras and Hopf group-algebras and Hopf coquasigroups. We introduce and discuss the notion of a *quasitriangular Hopf (non)coassociative π-algebra* and show some of its prominent properties, e.g., antipode S is bijective. As an application of our theory, we construct a new braided T-category and give a new solution to the generalized quantum Yang–Baxter equation.

Keywords: braided T-category; quantum Yang–Baxter equation; Hopf (non)coassociative group-algebra; quasitriangular Hopf (non)coassociative π-algebra

MSC: 16T05; 16W99

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1. Introduction

Topological quantum field theories (TQFT's) realize topological invariants of manifolds using ideas from quantum field theory (QFT), see [1,2]. Turaev introduced in [3] a homotopy quantum field theory (HQFT) as a version of a TQFT for manifolds endowed with maps into a fixed topological space and found an algebraic characterization of 2-dimensional HQFT's whose target space is the Eilenberg–MacLane space $K(\pi, 1)$ determined by a group π. Furthermore, he established a 3-dimensional HQFT with target space $K(\pi, 1)$ by introducing the notion of a modular π-category based on a deep connection between the theory of braided categories and invariants of knots, links and 3-manifolds (see [4]). This connection has been essential in the construction of quantum invariants of knots and 3-manifolds from quantum groups, see [2,5].

Turaev proposed the following open problem in [4]; Can one systematically produce interesting modular π-categories?

Examples of such modular π-categories can be constructed from the so-called *Hopf π-(co)algebras* which can be regarded as a generalization of a Hopf algebra, see [6–8]. At present, many research works have been done for Hopf π-(co)algebras, such as Turaev's Hopf group-coalgebras (cf. [9]), group coalgebra Galois extensions (cf. [10]), Larson-Sweedler theorem (cf. [11]), twisted Drinfel'd doubles (cf. [12]), double construction and Yetter-Drinfel'd modules (cf. [13–15]). We mention that a Hopf π-coalgebra can be regarded as a π-cograded multiplier Hopf algebra, see [16].

In 2010, Klim and Majid in [17] introduced the notion of a Hopf (co)quasigroup which is a particular case of the notion of an H-bialgebra introduced in [18]. The further research of this mathematical object can be found in the references about many topics, such as Hopf modules (cf. [19]), actions (cf. [20]), twisted smash products (cf. [21]), Yetter–Drinfel'd modules (cf. [22]), and Hopf quasicomodules (cf. [23]).

To highlight Turaev's achievements on the modular π-categories, in this article we prefer using the notion of a braided T-category (over π) appeared in [13] to using a modular

π-category [6]. We will provide a new approach to a braided T-category (over π) based on the notion of a quasitriangular Hopf (non)coassociative π-algebra.

An outline of the paper is as follows.

Section 2 provides some preliminary background needed in the paper, such as group-algebras, group-convolution algebras, Hopf group-algebras and Turaev's braided categories.

In Section 3, we give a new characterization of Hopf group-algebras based on the idea from [24,25]. We mainly prove that (H,Δ) is a Hopf π-algebra if and only if Δ is a π-algebra homomorphism and the right and left π-Galois maps are bijective.

In Section 4, we introduce and study the notion of a Hopf non-coassociative π-algebra which is a large class of coalgebras (possibly non-coassociative) with group-algebraic structures unifying the notions of a classical Hopf algebra, a Hopf π-algebra and a Hopf coquasigroup. We study its algebraic properties, such as anti-(co)multiplicativity of the antipode.

In Section 5 we mainly study the notion of a crossed Hopf non-coassociative π-algebra and give some properties of the crossing map. In addition, in Section 6, we discuss the definition and properties of an almost cocommutative Hopf non-coassociative π-algebra and obtain its equivalent characterization.

In the final section, we will introduce and discuss the definition of a quasitriangular Hopf non-coassociative π-algebra H and study some main properties of H. We construct a new braided T-category $Rep_\pi(H)$ over H.

Throughout the paper, we let π be a fixed group and $\Bbbk$ be a field (although much of what we do is valid over any commutative ring). We use the Sweedler's notation to express the coproduct of a coalgebra C as $\Delta(c) = \sum c_1 \otimes c_2$ (cf. [26]).

We set $\Bbbk^* = \Bbbk\backslash\{0\}$. All algebras are supposed to be over $\Bbbk$ and unitary, but not necessarily associative. The tensor product $\otimes = \otimes_\Bbbk$ is always assumed to be over $\Bbbk$. If U and V are $\Bbbk$-spaces, $\sigma_{U,V} : U \otimes V \longrightarrow V \otimes U$ will denote the flip map defined by $\sigma_{U,V}(u \otimes v) = v \otimes u$.

We use id_U for the identity map on U, although sometimes, we also write U for this map. We use id_U^n for the map $\underbrace{id \otimes \cdots \otimes id}_{n} : U \underbrace{\otimes \cdots \otimes}_{n-1} U \longrightarrow U \underbrace{\otimes \cdots \otimes}_{n-1} U$. The identity element in a quasigroup is denoted by e.

2. Preliminaries

In this section, we recall some basic notions used later, such as group-algebras, group-convolution algebras, Hopf group-algebras and braided T-categories.

2.1. Group-Algebras

We recall the definition of a π-algebra, following [4]. A π-algebra (over $\Bbbk$) is a family $A = \{A_\alpha\}_{\alpha\in\pi}$ of $\Bbbk$-spaces endowed with a family $m = \{m_{\alpha,\beta} : A_\alpha \otimes A_\beta \longrightarrow A_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ of $\Bbbk$-linear maps (the multiplication) and a $\Bbbk$-linear map $\eta : \Bbbk \longrightarrow A_1$ (the unit) such that m is associative in the sense that, for any $\alpha, \beta, \gamma \in \pi$,

$$m_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes id_{A_\gamma}) = m_{\alpha,\beta\gamma}(id_{A_\alpha} \otimes m_{\beta,\gamma}); \tag{1}$$

$$m_{\alpha,1}(id_{A_\alpha} \otimes \eta) = id_{A_\alpha} = m_{1,\alpha}(\eta \otimes id_{A_\alpha}). \tag{2}$$

Note that $(A_1, m_{1,1}, \eta)$ is an algebra in the usual sense of the word.

For all $\alpha, \beta \in \pi, h \in A_\alpha, k \in A_\beta$, we write $hk = m_{\alpha,\beta}(h \otimes k)$. The associativity axiom gives that

$$(hk)l = h(kl), \forall \alpha, \beta, \gamma \in \pi, h \in A_\alpha, k \in A_\beta, l \in A_\gamma.$$

Set $\eta(1_\Bbbk) = 1$. The unit axiom gives that $h1 = h = 1h, \forall \alpha \in \pi, h \in A_\alpha$.

For all $\alpha \in \pi$, the $\Bbbk$-space A_α is called the α-th component of A.

A π-algebra morphism between two π-algebras A and A' (with multiplications m and m', respectively) is a family $f = \{f_\alpha : A_\alpha \longrightarrow A'_\alpha\}_{\alpha\in\pi}$ of $\Bbbk$-linear maps such that $f_{\alpha\beta} m_{\alpha,\beta} = m'_{\alpha,\beta}(f_\alpha \otimes f_\beta)$ and $f_1(1) = 1'$, for all $\alpha, \beta \in \pi$. The π-algebra isomorphism $f = \{f_\alpha : A_\alpha \longrightarrow A'_\alpha\}_{\alpha\in\pi}$ is a π-algebra morphism in which each f_α is a linear isomorphism.

Set $\overline{A}_\alpha = A_{\alpha^{-1}}$ and $\overline{m}_{\alpha,\beta} = m^{op}_{\alpha^{-1},\beta^{-1}} = m_{\beta^{-1},\alpha^{-1}} \circ \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}$. Then comes a π-algebra $\overline{A} = \{\overline{A}_\alpha\}_{\alpha\in\pi}$ with the same unit element 1 as in A and the multiplication given by $\overline{m} = \{\overline{m}_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$.

2.2. Group-Convolution Algebra

Let $A = (\{A_\alpha\}, m, \eta)_{\alpha\in\pi}$ be a π-algebra and (C, Δ, ε) be a (not necessarily coassociative) coalgebra with comultiplication Δ and counit ε. For any $f \in \mathrm{Hom}_{\Bbbk}(C, A_\alpha)$ and $g \in \mathrm{Hom}_{\Bbbk}(C, A_\beta)$, we define their convolution product by

$$f * g = m_{\alpha,\beta}(f \otimes g)\Delta \in \mathrm{Hom}_{\Bbbk}(C, A_{\alpha\beta}). \tag{3}$$

Using Equation (3), one verifies that the $\Bbbk$-space

$$\mathrm{Conv}(C, A) = \bigoplus_{\alpha\in\pi} \mathrm{Hom}_{\Bbbk}(C, A_\alpha)$$

endowed with the convolution product $*$ and the unit element $\varepsilon 1$, is called *π-convolution algebra*, which is not necessarily a coassociative π-graded algebra.

In particular, for $C = \Bbbk$, the associative π-graded algebra $\mathrm{Conv}(C, A) = \bigoplus_{\alpha\in\pi} \mathrm{Hom}_{\Bbbk}(\Bbbk, A_\alpha) = \bigoplus_{\alpha\in\pi} A_\alpha$ is denoted by A_*.

2.3. Hopf Group-Algebras

Recall from [3] that a Hopf group-algebra over π is a π-algebra $H = (\{H_\alpha\}, m = \{m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}\}_{\alpha,\beta\in\pi}, \eta)_{\alpha\in\pi}$, endowed with a family $S = \{S_\alpha : H_\alpha \longrightarrow H_{\alpha^{-1}}\}_{\alpha\in\pi}$ of $\Bbbk$-linear maps (the antipode) such that the following conditions hold:

$$\text{each } (H_\alpha, \Delta_\alpha, \varepsilon_\alpha) \text{ is a counital coassociative coalgebra} \\ \text{with comultiplication } \Delta_\alpha \text{ and counit element } \varepsilon_\alpha; \tag{4}$$

$$\text{for all } \alpha, \beta \in \pi, \eta : \Bbbk \longrightarrow H_1 \textit{ and } m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta} \\ \text{are coalgebra homomorphisms,} \tag{5}$$

$$\text{for all } \alpha \in \pi, m_{\alpha^{-1},\alpha}(S_\alpha \otimes id_{H_\alpha})\Delta_\alpha = 1\varepsilon_\alpha = m_{\alpha,\alpha^{-1}}(id_{H_\alpha} \otimes S_\alpha)\Delta_\alpha. \tag{6}$$

Let $H = (\{H_\alpha, \Delta_\alpha, \varepsilon_\alpha\}_{\alpha\in\pi}, m, \eta, S)$ be a Hopf π-algebra. Then

$$S_{\alpha\beta}(ab) = S_\beta(b)S_\alpha(a), \forall \alpha, \beta \in \pi, a \in H_\alpha, b \in H_\beta; \tag{7}$$

$$S_1(1) = 1; \tag{8}$$

$$\Delta_{\alpha^{-1}} S_\alpha = \sigma_{H_{\alpha^{-1}},H_{\alpha^{-1}}}(S_\alpha \otimes S_\alpha)\Delta_\alpha, \forall \alpha \in \pi; \tag{9}$$

$$\varepsilon_{\alpha^{-1}} S_\alpha = \varepsilon_\alpha, \forall \alpha \in \pi. \tag{10}$$

2.4. Braided T-Categories

Let π be a group. A *pre-T-category $\mathcal{T}$ (over π)* is given by the following datum:

- A tensor category $\mathcal{T}$.
- A family of sub categories $\{\mathcal{T}_\alpha\}_{\alpha\in\pi}$ such that $\mathcal{T}$ is a disjoint union of this family and that $U \otimes V \in \mathcal{T}_{\alpha\beta}$, for any $\alpha, \beta \in \pi, U \in \mathcal{T}_\alpha$, and $V \in \mathcal{T}_\beta$.
  Furthermore, $\mathcal{T} = \{\mathcal{T}_\alpha\}$ satisfies the following condition:
- Denote by $aut(\mathcal{T})$ the group of the invertible strict tensor functors from $\mathcal{T}$ to itself, a group homomorphism $\varphi : \pi \longrightarrow aut(\mathcal{T}) : \beta \mapsto \varphi_\beta$, the *conjugation* such that $\varphi_\beta(\mathcal{T}_\alpha) = \mathcal{T}_{\beta\alpha\beta^{-1}}$ for any $\alpha, \beta \in \pi$. Then we call $\mathcal{T}$ a *crossed T-category*.

We will use the left index notation in Turaev: Given $\beta \in \pi$ and an object $V \in \mathcal{T}_\beta$, the functor φ_β will be denoted by ${}^V(\cdot)$ or ${}^\beta(\cdot)$. We use the notation ${}^{\overline{V}}(\cdot)$ for ${}^{\beta^{-1}}(\cdot)$. Then we have ${}^V id_U = id_{{}^V U}$ and ${}^V(g \circ f) = {}^V g \circ {}^V f$. We remark that since the conjugation

$\varphi : \pi \longrightarrow aut(\mathcal{T})$ is a group homomorphism, for any $V, W \in \mathcal{T}$, we have ${}^{V\otimes W}(\cdot) = {}^{V}({}^{W}(\cdot))$ and ${}^{1}(\cdot) = {}^{V}({}^{\overline{V}}(\cdot)) = {}^{\overline{V}}({}^{V}(\cdot)) = id_{\mathcal{T}}$ and that since, for any $V \in \mathcal{T}$, the functor ${}^{V}(\cdot)$ is strict, we have ${}^{V}(f \otimes g) = {}^{V}f \otimes {}^{V}g$, for any morphism f and g in $\mathcal{T}$, and ${}^{V}1 = 1$. In addition, we will use $\mathcal{T}(U, V)$ for a set of morphisms (or arrows) from U to V in $\mathcal{T}$.

Recall from [13] or [6] that a *braided T-category* (over π) is a crossed T-category $\mathcal{T}$ endowed with a braiding, i.e., with a family of isomorphisms

$$c = \{c_{U,V} \in \mathcal{T}(U \otimes V, ({}^{U}V) \otimes V)\}_{U,V \in \mathcal{T}}$$

satisfying the following conditions:

- for any arrow $f \in \mathcal{T}_\alpha(U, U')$ with $\alpha \in \pi$, $g \in \mathcal{T}_\beta(V, V')$, we have

$$(({}^{\alpha}g) \otimes f) \circ c_{U,V} = c_{U',V'} \circ (f \otimes g);$$

- for all $U, V, W \in \mathcal{T}$, we have

$$c_{U\otimes V,W} = a_{{}^{U\otimes V}W,U,V} \circ (c_{U,{}^{V}W} \otimes id_V) \circ a^{-1}_{U,{}^{V}W,V} \circ (id_U \otimes c_{V,W}) \circ a_{U,V,W}, \tag{11}$$

$$c_{U,V\otimes W} = a^{-1}_{{}^{U}V,{}^{U}W,U} \circ (id_{{}^{U}V} \otimes c_{U,W}) \circ a_{{}^{U}V,U,W} \circ (c_{U,V} \otimes id_W) \circ a^{-1}_{U,V,W}; \tag{12}$$

- for any $U, V \in \mathcal{T}$, $\alpha \in \pi$, $\varphi_\alpha(c_{U,V}) = c_{\varphi_\alpha(U),\varphi_\alpha(V)}$.

3. A New Characterization of Hopf Group-Algebras

Based on the idea from [24,25], in this section we mainly show that H is a Hopf π-algebra if and only if Δ is a π-algebra homomorphism and the right and left π-Galois maps both have inverses.

Proposition 1. *If H is a Hopf π-algebra, then the families of linear maps $T_1 = \{T_1^{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_\alpha \otimes H_{\alpha\beta}\}$ (called the left π-*Galois *map) and $T_2 = \{T_2^{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta} \otimes H_\beta\}$ (called the right π-*Galois *map), defined, respectively, by*

$$T_1^{\alpha,\beta}(a \otimes b) = \Delta_\alpha(a)(1 \otimes b) \text{ and } T_2^{\alpha,\beta}(a \otimes b) = (a \otimes 1)\Delta_\beta(b)$$

are bijective.

Proof. Define two families of linear maps

$$R_1 = \{R_1^{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_\alpha \otimes H_{\alpha^{-1}\beta}\}, \quad \text{and} \quad R_2 = \{R_2^{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta^{-1}} \otimes H_\beta\},$$

respectively, by

$$R_1^{\alpha,\beta}(a \otimes b) = ((id_{H_\alpha} \otimes S_\alpha)\Delta_\alpha(a))(1 \otimes b) \quad \text{and} \quad R_2^{\alpha,\beta}(a \otimes b) = (a \otimes 1)((S_\beta \otimes id_{H_\beta})\Delta_\beta(b)).$$

By a straightforward application of the properties of S one can show that $R_1^{\alpha,\alpha\beta}$ is the inverse of $T_1^{\alpha,\beta}$ and that $R_2^{\alpha\beta,\beta}$ is the inverse of $T_2^{\alpha,\beta}$. □

If the antipode S has an inverse, then also the other families of linear maps, defined by

$$T_3^{\alpha,\beta}(a \otimes b) = \Delta_\alpha(a)(b \otimes 1) \text{ and } T_4^{\alpha,\beta}(a \otimes b) = (1 \otimes a)\Delta_\beta(b)$$

are bijections. This follows, e.g., from the fact that $S^{cop} = \{S_\alpha^{cop} = S^{-1}_{\alpha^{-1}}\}_{\alpha\in\pi}$ will be the antipode if we set $H_\alpha^{cop} = H_\alpha$ as an algebra and replace Δ_α by the opposite comultiplication $\Delta_\alpha^{cop} = \sigma_{H_\alpha,H_\alpha}\Delta_\alpha$.

We now discuss some results if $H = \{H_\alpha\}_{\alpha\in\pi}$ is a unital, but not necessarily associative, group algebra over $\Bbbk$ with a family $\Delta = \{\Delta_\alpha : H_\alpha \longrightarrow H_\alpha \otimes H_\alpha\}_{\alpha\in\pi}$ of coassociative comultiplications, a family of linear maps, such that the families of linear maps T_1 and T_2 are bijections.

Define a family of maps $E = \{E_\alpha : H_\alpha \longrightarrow H_1\}_{\alpha\in\pi}$ by

$$E_\alpha(a)b = m_{\alpha,\alpha^{-1}\beta}(T_1^{\alpha,\alpha^{-1}\beta})^{-1}(a \otimes b)$$

where $m_{\alpha,\alpha^{-1}\beta}$ denotes multiplication, considered as a linear map from $H_\alpha \otimes H_\beta$ to H_β and where $T_1^{\alpha,\alpha^{-1}\beta}$ is defined as before by $T_1^{\alpha,\alpha^{-1}\beta}(a \otimes b) = \Delta_\alpha(a)(1 \otimes b) \in H_\alpha \otimes H_\beta$.

Lemma 1. *For all $a, b \in H_\alpha$, we have $(H_{\alpha\beta} \otimes E_\beta)((a \otimes 1)\Delta_\beta(b)) = ab \otimes 1$.*

Proof. By the coassociativity of Δ_α, one can easily obtain

$$\begin{aligned}&(m_{\alpha,\beta} \otimes H_\beta \otimes H_{\beta\gamma})(H_\alpha \otimes \Delta_\beta \otimes H_{\beta\gamma})(H_\alpha \otimes H_\beta \otimes m_{\beta,\gamma})(H_\alpha \otimes \Delta_\beta \otimes H_\gamma)\\ =&(H_{\alpha\beta} \otimes H_\beta \otimes m_{\beta,\gamma})(H_{\alpha\beta} \otimes \Delta_\beta \otimes H_\gamma)(m_{\alpha,\beta} \otimes H_\beta \otimes H_\gamma)(H_\alpha \otimes \Delta_\beta \otimes H_\gamma).\end{aligned}$$

Assume $a \in H_\alpha, b \in H_\beta$ and, since T_1 is surjective, let

$$a \otimes b = \sum_{i=1}^{n} \Delta_\alpha(a_i)(1 \otimes b_i).$$

If we apply $\Delta_\alpha \otimes H_\beta$ and then multiply with $c \otimes 1 \otimes 1$ to the both sides of the above equation, where $c \in H_\gamma$, on the left, by the direct conclusion of coassociativity given above, we can obtain

$$(c \otimes 1)\Delta_\alpha(a) \otimes b = T_1^{\alpha,\alpha^{-1}\beta}\big(\sum(\varphi \otimes H_\alpha)((c \otimes 1)\Delta_\alpha(a_i)) \otimes b_i\big).$$

By the definition of E_α we get

$$E_\alpha((\varphi \otimes H_\alpha)((c \otimes 1)\Delta_\alpha(a)))b = \sum(\varphi \otimes H_\alpha)((c \otimes 1)\Delta_\alpha(a_i))b_i.$$

So

$$\begin{aligned}(\varphi \otimes H_\beta)((H_{\gamma\alpha} \otimes E_\alpha)((c \otimes 1)\Delta_\alpha(a))(1 \otimes b)) &= E_\alpha((\varphi \otimes H_\alpha)((c \otimes 1)\Delta_\alpha(a)))b\\ &= \sum(\varphi \otimes H_\alpha)((c \otimes 1)\Delta_\alpha(a_i))b_i\\ &= (\varphi \otimes H_\beta)\big((c \otimes 1)\sum \Delta_\alpha(a_i)(1 \otimes b_i)\big)\\ &= (\varphi \otimes H_\beta)((c \otimes 1)(a \otimes b)).\end{aligned}$$

Because this holds for all φ, we get

$$(H_{\gamma\alpha} \otimes E_\alpha)((c \otimes 1)\Delta_\alpha(a))(1 \otimes b) = (ca \otimes 1)(1 \otimes b).$$

This gives the required formula. □

Lemma 2. $E_\alpha(H_\alpha) \subseteq \Bbbk 1$.

Proof. By the surjectivity of T_2, defined by $T_2^{\alpha\beta^{-1},\beta}(x \otimes y) = (x \otimes 1)\Delta_\beta(y) \in H_\alpha \otimes H_\beta$, we see that $a \otimes E_\beta(b) \in H_\alpha \otimes 1$ for all a in H_α and b in H_β. This gives the result. □

Define a family of linear maps

$$\varepsilon = \{\varepsilon_\alpha : H_\alpha \longrightarrow \Bbbk\}_{\alpha\in\pi}$$

by $\varepsilon_\alpha(a)1 = E_\alpha(a)$.

Remark 1. *The formula in Lemma 1 can be rewritten as*

$$(H_{\alpha\beta} \otimes \varepsilon_\beta)((a \otimes 1)\Delta_\beta(b)) = ab.$$

It can be concluded that the associativity of $m = \{m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ holds if and only if $(H_{\alpha,\beta\gamma} \otimes \varepsilon_{\beta\gamma})((a \otimes 1)\Delta_{\beta\gamma}(bc)) = (H_{\alpha\beta} \otimes \varepsilon_\beta)((a \otimes 1)\Delta_\beta(b))c$.

By the definition of ε, we also get

$$(\varepsilon_\alpha \otimes H_\beta)(a \otimes b) = E_\alpha(a)b = m_{\alpha,\alpha^{-1}\beta}(T_1^{\alpha,\alpha^{-1}\beta})^{-1}(a \otimes b)$$

and, by the surjectivity of T_1; hence,

$$(\varepsilon_\alpha \otimes H_{\alpha\beta})(\Delta_\alpha(x)(1 \otimes y)) = xy.$$

These formulas just mean

$$(H_\alpha \otimes \varepsilon_\alpha)\Delta_\alpha = H_\alpha = (\varepsilon_\alpha \otimes H_\alpha)\Delta_\alpha.$$

It shows that, for any $\alpha \in \pi$, $(H_\alpha, \Delta_\alpha, \varepsilon_\alpha)$ is a coalgebra.

Define a family of maps $F = \{F_\alpha : H_\alpha \longrightarrow H_1\}_{\alpha\in\pi}$ by $aF_\beta(b) = m_{\alpha\beta^{-1},\beta}(T_2^{\alpha\beta^{-1},\beta})^{-1}(a \otimes b)$ where $m_{\alpha\beta^{-1},\beta}$ denotes multiplication, considered as a linear map from $H_{\alpha\beta^{-1}} \otimes H_\beta$ to H_α and where $T_2^{\alpha\beta^{-1},\beta}$ is defined as before by $T_2^{\alpha\beta^{-1},\beta}(a \otimes b) = (a \otimes 1)\Delta_\beta(b)$.

Similar to Lemmas 1 and 2, we have

Lemma 3. *For all $a \in H_\alpha$ and $b \in H_\beta$, we have $(F_\alpha \otimes H_{\alpha\beta})(\Delta_\alpha(a)(1 \otimes b)) = 1 \otimes ab$.*

Lemma 4. $F_\alpha(H_\alpha) \subseteq \Bbbk 1$.

Define a family of linear maps

$$\epsilon = \{\epsilon_\alpha : H_\alpha \longrightarrow \Bbbk\}_{\alpha\in\pi}$$

by $\epsilon_\alpha(a)1 = F_\alpha(a)$.

Remark 2. *The formula in Lemma 3 can be rewritten as*

$$(\epsilon_\alpha \otimes H_{\alpha\beta})(\Delta_\alpha(a)(1 \otimes b)) = ab.$$

It can be concluded that the associativity of $m = \{m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}\}_{\alpha,\beta\in\pi}$ holds if and only if $(\epsilon_{\alpha\beta} \otimes H_{\alpha\beta\gamma})(\Delta_{\alpha\beta}(ab)(1 \otimes c)) = a(\epsilon_\beta \otimes H_{\beta\gamma})(\Delta_\beta(b)(1 \otimes c))$.

By the definition of ϵ, we also get

$$(H_\alpha \otimes \epsilon_\beta)(a \otimes b) = aF_\beta(b) = m_{\alpha\beta^{-1},\beta}(T_2^{\alpha\beta^{-1},\beta})^{-1}(a \otimes b)$$

and, by the surjectivity of T_2; hence,

$$(H_{\alpha\beta} \otimes \epsilon_\beta)((x \otimes 1)\Delta_\beta(y)) = xy.$$

These formulas just mean

$$(H_\alpha \otimes \epsilon_\alpha)\Delta_\alpha = H_\alpha = (\epsilon_\beta \otimes H_\alpha)\Delta_\alpha.$$

It shows that, for any $\alpha \in \pi$, $(H_\alpha, \Delta_\alpha, \epsilon_\alpha)$ is a coalgebra.

Due to the loss of the associativity of $m = \{m_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$, the counit family $\varepsilon = \{\varepsilon_\alpha\}_{\alpha\in\pi}$ is not necessarily a π-algebra homomorphism even when $\Delta = \{\Delta_\alpha\}_{\alpha\in\pi}$ is a π-algebra homomorphism. From now on, every group algebra will tacitly be assumed to carry the associativity of its multiplication and we also suppose that $\Delta = \{\Delta_\alpha\}_{\alpha\in\pi}$ is a π-algebra homomorphism.

We will show that $\varepsilon = \{\varepsilon_\alpha\}_{\alpha\in\pi}$ satisfies the usual properties of the counit family in Hopf group-algebra theory.

Lemma 5. *$\varepsilon = \{\varepsilon_\alpha\}_{\alpha\in\pi}$ is a π-algebra homomorphism.*

Proof. By Lemma 1, we have

$$(H_{\alpha\beta\gamma} \otimes \varepsilon_{\beta\gamma})((a \otimes 1)\Delta_{\beta\gamma}(bc)) = a(bc)$$

for all $a \in H_\alpha, b \in H_\beta, c \in H_\gamma$. Then

$$(H_{\alpha\beta\gamma} \otimes \varepsilon_{\beta\gamma})((a \otimes 1)\Delta_\beta(b)\Delta_\gamma(c)) = a(bc) = (ab)c = (H_{\alpha\beta} \otimes \varepsilon_\beta)((a \otimes 1)\Delta_\beta(b))c.$$

By the surjectivity of T_2 we get

$$\begin{aligned}(H_{\alpha\gamma} \otimes \varepsilon_{\beta\gamma})((a \otimes b)\Delta_\gamma(c)) &= (H_\alpha \otimes \varepsilon_\beta)(a \otimes b)c = a\varepsilon_\beta(b)c = \varepsilon_\beta(b)ac \\ &= \varepsilon_\beta(b)(H_{\alpha\gamma} \otimes \varepsilon_\gamma)((a \otimes 1)\Delta_\gamma(c))\end{aligned}$$

for all $a \in H_\alpha, b \in H_\beta, c \in H_\gamma$. Again by the surjectivity of T_2 we get

$$(H_\alpha \otimes \varepsilon_{\beta\gamma})(a \otimes bc) = \varepsilon_\beta(b)(H_\alpha \otimes \varepsilon_\gamma)(a \otimes c).$$

This means

$$a\varepsilon_{\beta\gamma}(bc) = a\varepsilon_\beta(b)\varepsilon_\gamma(c).$$

Set $\alpha = 1$ and $a = 1$, we have

$$\varepsilon_{\beta\gamma}(bc) = \varepsilon_\beta(b)\varepsilon_\gamma(c).$$

Since $(H_\alpha \otimes \varepsilon_\alpha)\Delta_\alpha = H_\alpha = (\varepsilon_\alpha \otimes H_\alpha)\Delta_\alpha$, we obtain

$$1 = H_1(1) = (\varepsilon_1 \otimes H_1)\Delta_1(1) = (\varepsilon_1 \otimes H_1)(1 \otimes 1) = \varepsilon_1(1)1$$

whereby $\varepsilon_1(1) = 1_{\mathbb{k}}$. □

Remark that, by a similar reasoning, we can also claim that $\epsilon = \{\epsilon_\alpha\}_{\alpha\in\pi}$ is a π-algebra homomorphism.

In fact, for all $\alpha \in \pi$, $\varepsilon_\alpha = \epsilon_\alpha$. In order to check this result, we need the following lemma.

Lemma 6. *For all $a \in H_\alpha, b \in H_\beta$, $m_{\alpha\beta^{-1},\beta}\left(\left(T_2^{\alpha\beta^{-1},\beta}\right)^{-1}(a \otimes b)\right) = a\varepsilon_\beta(b)$.*

Proof. Assume $a \in H_\alpha, b \in H_\beta$ and, since T_2 is surjective, let

$$a \otimes b = T_2^{\alpha\beta^{-1},\beta}\left(\sum_{i=1}^{n} a_i \otimes b_i\right) = \sum_{i=1}^{n}(a_i \otimes 1)\Delta_\beta(b_i).$$

Hence,

$$m_{\alpha\beta^{-1},\beta}\left(\left(T_2^{\alpha\beta^{-1},\beta}\right)^{-1}(a\otimes b)\right)=m_{\alpha\beta^{-1},\beta}\left(\left(T_2^{\alpha\beta^{-1},\beta}\right)^{-1}T_2^{\alpha\beta^{-1},\beta}\left(\sum_{i=1}^{n}a_i\otimes b_i\right)\right)$$
$$=m_{\alpha\beta^{-1},\beta}\left(\sum_{i=1}^{n}a_i\otimes b_i\right)=\sum_{i=1}^{n}a_ib_i=\sum_{i=1}^{n}a_ib_{i(1)}\varepsilon_\beta(b_{i(2)})=(H_\alpha\otimes\varepsilon_\beta)\sum_{i=1}^{n}(a_ib_{i(1)}\otimes b_{i(2)})$$
$$=(H_\alpha\otimes\varepsilon_\beta)\sum_{i=1}^{n}(m_{\alpha\beta^{-1},\beta}\otimes H_\beta)(H_{\alpha\beta^{-1}}\otimes\Delta_\beta)(a_i\otimes b_i)=(H_\alpha\otimes\varepsilon_\beta)(a\otimes b)=a\varepsilon_\beta(b).\quad\square$$

Lemma 7. *For all $\alpha\in\pi$, $\varepsilon_\alpha=\epsilon_\alpha$.*

Proof. For all $a\in H_\alpha, b\in H_\beta$, by the definition of F, we have

$$a\epsilon_\beta(b)=aF_\beta(b)=m_{\alpha\beta^{-1},\beta}\left(\left(T_2^{\alpha\beta^{-1},\beta}\right)^{-1}(a\otimes b)\right).$$

By Lemma 6, we also get

$$m_{\alpha\beta^{-1},\beta}\left(\left(T_2^{\alpha\beta^{-1},\beta}\right)^{-1}(a\otimes b)\right)=a\varepsilon_\beta(b).$$

It follows that $a\epsilon_\beta(b)=a\varepsilon_\beta(b)$. $\square$

We have constructed a counit family $\varepsilon=\{\varepsilon_\alpha\}_{\alpha\in\pi}$ satisfying the usual properties of the counit family in Hopf group-algebra theory.

We will construct an antihomomorphism $S=\{S_\alpha\}_{\alpha\in\pi}$ that has the properties of the antipode in the Hopf group-algebra theory.

Definition 1. *Define a family of linear maps $S=\{S_\alpha:H_\alpha\to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ by*

$$S_\alpha(a)b=(\varepsilon_\alpha\otimes H_{\alpha^{-1}\beta})(T_1^{\alpha,\alpha^{-1}\beta})^{-1}(a\otimes b)$$

for all $a\in H_\alpha, b\in H_\beta$.

Lemma 8. $(H_{\gamma\alpha}\otimes S_\alpha)((c\otimes 1)\Delta_\alpha(a))(1\otimes b)=(c\otimes 1)(T_1^{\alpha,\alpha^{-1}\beta})^{-1}(a\otimes b)$.

Proof. As in the proof of Lemma 1, for $\varphi\in H_{\gamma\alpha}^*$ and $a\in H_\alpha, b\in H_\beta, c\in H_\gamma$, we get

$$(\varphi\otimes H_\alpha)((c\otimes 1)\Delta_\alpha(a))\otimes b=T_1^{\alpha,\alpha^{-1}\beta}(\sum(\varphi\otimes H_\alpha)((c\otimes 1)\Delta_\alpha(a_i))\otimes b_i$$

if $a\otimes b=\sum_{i=1}^{n}\Delta_\alpha(a_i)(1\otimes b_i)$. Then, by the definition of S, we get

$$\begin{aligned}S_\alpha((\varphi\otimes H_\alpha)((c\otimes 1)\Delta_\alpha(a)))b&=(\varepsilon_\alpha\otimes H_{\alpha^{-1}\beta})(\sum(\varphi\otimes H_\alpha)((c\otimes 1)\Delta_\alpha(a_i))\otimes b_i)\\&=(\varphi\otimes H_{\alpha^{-1}\beta})(\sum(H_{\gamma\alpha}\otimes\varepsilon_\alpha)((c\otimes 1)\Delta_\alpha(a_i))\otimes b_i)\\&=(\varphi\otimes H_{\alpha^{-1}\beta})(\sum ca_i\otimes b_i)\\&=(\varphi\otimes H_{\alpha^{-1}\beta})\left((c\otimes 1)(T_1^{\alpha,\alpha^{-1}\beta})^{-1}(a\otimes b)\right).\end{aligned}$$

Hence,

$$(\varphi\otimes H_{\alpha^{-1}\beta})((H_{\gamma\alpha}\otimes S_\alpha)((c\otimes 1)\Delta_\alpha(a))(1\otimes b))=(\varphi\otimes H_{\alpha^{-1}\beta})\left((c\otimes 1)(T_1^{\alpha,\alpha^{-1}\beta})^{-1}(a\otimes b)\right).$$

This is true for all $\varphi \in H^*_{\gamma\alpha}$ and hence proves the result. □

Lemma 9. *For all $a \in H_\alpha, b \in H_\beta$ and $c \in H_\gamma$, we have*

$$m_{\gamma\alpha,\alpha^{-1}\beta}((H_{\gamma\alpha} \otimes S_\alpha)((c \otimes 1)\Delta_\alpha(a))(1 \otimes b)) = c\varepsilon_\alpha(a)b.$$

Proof. We get this formula if we apply $m_{\gamma\alpha,\alpha^{-1}\beta}$ on the equation in Lemma 8 because

$$\begin{aligned} m_{\gamma\alpha,\alpha^{-1}\beta}((H_{\gamma\alpha} \otimes S_\alpha)((c \otimes 1)\Delta_\alpha(a))(1 \otimes b)) &= m_{\gamma\alpha,\alpha^{-1}\beta}((c \otimes 1)(T_1^{\alpha,\alpha^{-1}\beta})^{-1}(a \otimes b)) \\ &= cm_{\alpha,\alpha^{-1}\beta}((T_1^{\alpha,\alpha^{-1}\beta})^{-1}(a \otimes b)) = c(E_\alpha(a)b) = c\varepsilon_\alpha(a)b. \quad \square \end{aligned}$$

Lemma 10. $S_{\alpha\beta}(ab) = S_\beta(b)S_\alpha(a)$ *for all $a \in H_\alpha$ and $b \in H_\beta$.*

Proof. We have

$$\begin{aligned} &m_{\gamma\alpha\beta,\beta^{-1}\alpha^{-1}\delta}((H_{\gamma\alpha\beta} \otimes S_{\alpha\beta})((c \otimes 1)\Delta_\alpha(a)\Delta_\beta(b))(1 \otimes d)) \\ &= m_{\gamma\alpha\beta,\beta^{-1}\alpha^{-1}\delta}((H_{\gamma\alpha\beta} \otimes S_{\alpha\beta})((c \otimes 1)\Delta_{\alpha\beta}(ab))(1 \otimes d)) \\ &= c\varepsilon_{\alpha\beta}(ab)d = c\varepsilon_\alpha(a)d\varepsilon_\beta(b) = m_{\gamma\alpha,\alpha^{-1}\delta}((H_{\gamma\alpha} \otimes S_\alpha)((c \otimes 1)\Delta_\alpha(a))(1 \otimes d))\varepsilon_\beta(b) \end{aligned}$$

for all $a \in H_\alpha, b \in H_\beta, c \in H_\gamma$ and $d \in H_\delta$. By the surjectivity of T_2, we get

$$\begin{aligned} &m_{\gamma\beta,\beta^{-1}\alpha^{-1}\delta}((H_{\gamma\beta} \otimes S_{\alpha\beta})((c \otimes a)\Delta_\beta(b))(1 \otimes d)) \\ &= m_{\gamma,\alpha^{-1}\delta}((H_\gamma \otimes S_\alpha)(c \otimes a)(1 \otimes d))\varepsilon_\beta(b) = cS_\alpha(a)d\varepsilon_\beta(b) = c\varepsilon_\beta(b)S_\alpha(a)d \\ &= m_{\gamma\beta,\beta^{-1}\alpha^{-1}\delta}((H_{\gamma\beta} \otimes S_\beta)((c \otimes 1)\Delta_\beta(b))(1 \otimes S_\alpha(a)d)) \end{aligned}$$

for all $a \in H_\alpha, b \in H_\beta, c \in H_\gamma$ and $d \in H_\delta$. Again by the surjectivity of T_2, we get

$$m_{\gamma,\beta^{-1}\alpha^{-1}\delta}((H_\gamma \otimes S_{\alpha\beta})(c \otimes ab)(1 \otimes d)) = m_{\gamma,\beta^{-1}\alpha^{-1}\delta}((H_\gamma \otimes S_\beta)(c \otimes b)(1 \otimes S_\alpha(a)d))$$

whence $cS_{\alpha\beta}(ab)d = cS_\beta(b)S_\alpha(a)d$. □

Define another one family of linear maps $\bar{S} = \{\bar{S}_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha\in\pi}$ by

$$a\bar{S}_\beta(b) = (H_{\alpha\beta^{-1}} \otimes \varepsilon_\beta)(T_2^{\alpha\beta^{-1},\beta})^{-1}(a \otimes b)$$

for all $a \in H_\alpha, b \in H_\beta$.

Completely similar as in Lemma 8, we get here that

Lemma 11. $(c \otimes 1)(\bar{S}_\alpha \otimes H_{\alpha\beta})(\Delta_\alpha(a)(1 \otimes b)) = \Big((T_2^{\gamma\alpha^{-1},\alpha})^{-1}(c \otimes a)\Big)(1 \otimes b).$

Lemma 12. $m_{\gamma\alpha^{-1},\alpha\beta}((c \otimes 1)(S_\alpha \otimes H_{\alpha\beta})(\Delta_\alpha(a)(1 \otimes b))) = c\varepsilon_\alpha(a)b.$

Proof. By Lemma 11, we get

$$(c \otimes 1)(\bar{S}_\alpha \otimes H_{\alpha\beta})(\Delta_\alpha(a)(1 \otimes b)) = \Big((T_2^{\gamma\alpha^{-1},\alpha})^{-1}(c \otimes a)\Big)(1 \otimes b).$$

And if we apply $m_{\gamma\alpha^{-1},\alpha\beta}$ to the both sides of the above equation, we get the formula in the statement of the lemma with $\bar{S}$ instead of S because

$$m_{\gamma\alpha^{-1},\alpha}\Big((T_2^{\gamma\alpha^{-1},\alpha})^{-1}(c \otimes a)\Big) = c\varepsilon_\alpha(a).$$

We now show that $S = S'$. Indeed, we have, by definition,

$$a\bar{S}_\beta(b) = \sum a_i \varepsilon_\beta(b_i)$$

if $a \otimes b = \sum (a_i \otimes 1)\Delta_\beta(b_i)$. If we apply $H_\alpha \otimes S_\beta$ and multiply with $1 \otimes c$ to the both sides of the equation: $a \otimes b = \sum (a_i \otimes 1)\Delta_\beta(b_i)$, we get

$$a \otimes S_\beta(b)c = \sum (H_\alpha \otimes S_\beta)\big((a_i \otimes 1)\Delta_\beta(b_i)\big)(1 \otimes c).$$

And if we apply $m_{\alpha,\beta^{-1}\gamma}$ to the both sides of the above equation, we obtain, using Lemma 9, that

$$aS_\beta(b)c = \sum a_i \varepsilon_\beta(b_i)c = a\bar{S}_\beta(b)c.$$

This shows that $S_\beta(b) = \bar{S}_\beta(b)$. This proves the lemma; the formula was already proven for $\bar{S}$. □

Apropos of Lemmas 9 and 12, by setting $\beta = \gamma = 1$ and $b = c = 1$, we have the usual formulas

$$m_{\alpha,\alpha^{-1}}(H_\alpha \otimes S_\alpha)\Delta_\alpha(a) = \varepsilon_\alpha(a)1, \quad m_{\alpha^{-1},\alpha}(S_\alpha \otimes H_\alpha)\Delta_\alpha(a) = \varepsilon_\alpha(a)1.$$

We have constructed an antihomomorphism $S = \{S_\alpha\}_{\alpha\in\pi}$ that has the properties of the antipode in the Hopf group-algebra theory.

From the above discussion, we get the following the main result.

Theorem 1. *If $H = \{H_\alpha\}_{\alpha\in\pi}$ is a unital associative group algebra over $\Bbbk$ with a family $\Delta = \{\Delta_\alpha : H_\alpha \longrightarrow H_\alpha \otimes H_\alpha\}_{\alpha\in\pi}$ of coassociative comultiplications, then H is a* Hopf *π-algebra if and only if Δ is a π-algebra homomorphism and the right and left π-*Galois *maps both have inverses.*

4. Hopf (Non)coassociative Group-Algebras

We begin by the main definition of this paper which is slightly dual to the notion of a quasigroup Hopf group-coalgebra studied in [27].

Definition 2. *A Hopf non-coassociative group-algebra over π is a π-algebra $H = (\{H_\alpha\}, m = \{m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}\}_{\alpha,\beta\in\pi}, \eta)_{\alpha\in\pi}$, endowed with a family $S = \{S_\alpha : H_\alpha \longrightarrow H_{\alpha^{-1}}\}_{\alpha\in\pi}$ of $\Bbbk$-linear maps (the antipode) such that the following conditions hold:*

- *Each $(H_\alpha, \Delta_\alpha, \varepsilon_\alpha)$ with comultiplication Δ_α and counit ε_α is a not necessarily coassociative coalgebra;* (13)

- *for all $\alpha, \beta \in \pi$, $\eta : \Bbbk \longrightarrow H_1$ and $m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}$ are coalgebra homomorphisms;* (14)

- *for $\alpha \in \pi$,* $(m_{\alpha^{-1},\alpha} \otimes id_{H_\alpha})(S_\alpha \otimes id_{H_\alpha} \otimes id_{H_\alpha})(id_{H_\alpha} \otimes \Delta_\alpha)\Delta_\alpha = \eta \otimes id_{H_\alpha} = (m_{\alpha,\alpha^{-1}} \otimes id_{H_\alpha})(id_{H_\alpha} \otimes S_\alpha \otimes id_{H_\alpha})(id_{H_\alpha} \otimes \Delta_\alpha)\Delta_\alpha;$ (15)

- *for $\alpha \in \pi$,* $(id_{H_\alpha} \otimes m_{\alpha^{-1},\alpha})(id_{H_\alpha} \otimes S_\alpha \otimes id_{H_\alpha})(\Delta_\alpha \otimes id_{H_\alpha})\Delta_\alpha = id_{H_\alpha} \otimes \eta = (id_{H_\alpha} \otimes m_{\alpha,\alpha^{-1}})(id_{H_\alpha} \otimes id_{H_\alpha} \otimes S_\alpha)(\Delta_\alpha \otimes id_{H_\alpha})\Delta_\alpha.$ (16)

We remark that the notion of a Hopf non-coassociative group-algebra is not self-dual and that $(H_1, m_{1,1}, \eta, \Delta_1, \varepsilon_1, S_1)$ is a (classical) Hopf coquasigroup. Let $\pi = \{1\}$, $H = H_1$ is a (classical) Hopf coquasigroup. One can easily verify that a Hopf non-coassociative group-algebra is a Hopf π-algebra if and only if its coproduct is coassociative.

In this paper, a Hopf non-coassociative group-algebra over π is called *Hopf non-coassociative π-algebra*.

Remark 3.

(1) *The axiom (14) amounts to that, for any $\alpha, \beta \in \pi, a \in H_\alpha$ and $b \in H_\beta$,*

$$\Delta_1(1) = 1 \otimes 1, \qquad \varepsilon_1(1) = 1_{\Bbbk},$$
$$\Delta_{\alpha\beta}(ab) = \Delta_\alpha(a)\Delta_\beta(b), \qquad \varepsilon_{\alpha\beta}(ab) = \varepsilon_\alpha(a)\varepsilon_\beta(b).$$

(2) *In terms of Sweedler's notation, the axiom (15) gives that, for any $\alpha \in \pi, h \in H_\alpha$,*

$$S_\alpha(h_{(1)})h_{(2)(1)} \otimes h_{(2)(2)} = 1 \otimes h = h_{(1)}S_\alpha(h_{(2)(1)}) \otimes h_{(2)(2)}. \tag{17}$$

(3) *In terms of Sweedler's notation, the axiom (16) gives that, for any $\alpha \in \pi, h \in H_\alpha$,*

$$h_{(1)(1)} \otimes S_\alpha(h_{(1)(2)})h_{(2)} = h \otimes 1 = h_{(1)(1)} \otimes h_{(1)(2)}S_\alpha(h_{(2)}). \tag{18}$$

Definition 3. *Let H be a Hopf non-coassociative group-algebra. Then, for all $\alpha \in \pi$ and $a \in H_\alpha$,*

(1) *H is* commutative *if $m_{\alpha,\alpha^{-1}} = m_{\alpha^{-1},\alpha}$.*
(2) *H is* cocommutative *if each Δ_α is cocommutative.*
(3) *H is* flexible *if*

$$a_{(1)}a_{(2)(2)} \otimes a_{(2)(1)} = a_{(1)(1)}a_{(2)} \otimes a_{(1)(2)}.$$

(4) *H is* alternative *if*

$$a_{(1)}a_{(2)(1)} \otimes a_{(2)(2)} = a_{(1)(1)}a_{(1)(2)} \otimes a_{(2)}, a_{(1)} \otimes a_{(2)(1)}a_{(2)(2)} = a_{(1)(1)} \otimes a_{(1)(2)}a_{(2)}.$$

(5) *H is called* Moufang *if*

$$a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = a_{(1)(1)(1)}a_{(1)(2)} \otimes a_{(1)(1)(2)} \otimes a_{(2)}.$$

A Hopf non-coassociative group-algebra H is said to be of finite type if, for all $\alpha \in \pi, H_\alpha$ is finite dimensional (over $\Bbbk$). Note that it does not mean that $\bigoplus_{\alpha\in\pi} H_\alpha$ is finite-dimensional (unless $H_\alpha \neq 0$, for all but a finite number of $\alpha \in \pi$).

The antipode $S = \{S_\alpha\}_{\alpha\in\pi}$ of H is said to be bijective if each S_α is bijective. We will later show that it is bijective whenever H is quasitriangular (see Theorem 12).

Example 1. *Let $(H, m, \Delta, \varepsilon, S)$ be a Hopf coquasigroup and the group π act on H by Hopf coquasigroup endomorphisms.*

(1) *Set $H^\pi = \{H_\alpha\}_{\alpha\in\pi}$ where the coalgebra H_α is a copy of H for each $\alpha \in \pi$. Fix an identification isomorphism of coalgebras $i_\alpha : H \longrightarrow H_\alpha$. For $\alpha, \beta \in \pi$, one defines a multiplication $m_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}$ by*

$$m_{\alpha,\beta}(i_\alpha(h) \otimes i_\beta(a)) = (i_{\alpha\beta}(ha))$$

for any $h, a \in H$. The counit $\varepsilon_1 : H_1 \longrightarrow \Bbbk$ is defined by $\varepsilon_1(i_1(h)) = \varepsilon(h)$ for $h \in H$. For any $\alpha \in \pi$, the antipode $S_\alpha : H_\alpha \longrightarrow H_{\alpha^{-1}}$ is given by $S_\alpha(i_\alpha(h)) = i_{\alpha^{-1}}(S(h))$. All the axioms of a Hopf non-coassociative π-algebra for H^π follow directly from definitions.

(2) *Let $\overline{H}^\pi$ be the same family of coalgebras $\{H_\alpha = H\}$ with the same counit, the multiplication $\overline{m}_{\alpha,\beta} : H_\alpha \otimes H_\beta \longrightarrow H_{\alpha\beta}$ and the antipode $S_\alpha : H_\alpha \longrightarrow H_{\alpha^{-1}}$ defined by*

$$\overline{m}_{\alpha,\beta}i_\alpha(\beta(h)) \otimes i_\beta(a) = i_{\alpha\beta}(h)$$
$$\overline{S}_\alpha(i_\alpha(h)) = i_{\alpha^{-1}}(\alpha(S(h))) = i_{\alpha^{-1}}(S(\alpha(h))$$

where $h, a \in H$. The axioms of a Hopf non-coassociative π-algebra for $\overline{H}^\pi$ follow from definitions. Both H^π and $\overline{H}^\pi$ are extensions of H since $H_1^\pi = \overline{H}_1^\pi = H_1$ as Hopf coquasigroups.

Example 2.

(1) *Let* $A = (\{A_\alpha\}, m, \eta)_{\alpha\in\pi}$ *be a* π*-algebra. Set*

$$A^{op}_\alpha = A_{\alpha^{-1}} \text{ and } m^{op}_{\alpha,\beta} = m_{\beta^{-1},\alpha^{-1}} \circ \sigma_{A_{\alpha^{-1}},A_{\beta^{-1}}}.$$

Then $A^{op} = (\{A^{op}_\alpha\}, m^{op}, \eta)_{\alpha\in\pi}$ *is a* π*-algebra, called opposite to A.*
If $H = \{H_\alpha\}_{\alpha\in\pi}$ *is a Hopf non-coassociative group-algebra whose antipode* $S = \{S_\alpha\}_{\alpha\in\pi}$ *is bijective, then the opposite* π*-algebra* H^{op}*, where* $H^{op}_\alpha = H_{\alpha^{-1}}$ *as a coalgebra, is a Hopf non-coassociative* π*-algebra with antipode* $S^{op} = \{S^{op}_\alpha = S^{-1}_\alpha\}_{\alpha\in\pi}$*.*

(2) *Let* $H = (\{H_\alpha, \Delta_\alpha, \varepsilon_\alpha\}_{\alpha\in\pi}, m, \eta, S)$ *be a Hopf non-coassociative* π*-algebra. Suppose that the antipode* $S = \{S_\alpha\}_{\alpha\in\pi}$ *of H is bijective. For any* $\alpha \in \pi$*, let* H^{cop}_α *be the coopposite coalgebra to* H_α*. Then* $H^{cop} = \{H^{cop}_\alpha\}_{\alpha\in\pi}$*, endowed with the multiplication and unit of H and with the antipode* $S^{cop} = \{S^{cop}_\alpha = S^{-1}_{\alpha^{-1}}\}_{\alpha\in\pi}$*, is a Hopf non-coassociative* π*-algebra called coopposite to H.*

(3) *Let* $H = (\{H_\alpha, \Delta_\alpha, \varepsilon_\alpha\}_{\alpha\in\pi}, m, \eta, S)$ *be a Hopf non-coassociative* π*-algebra. Even if the antipode of H is not bijective, one can always define a Hopf non-coassociative* π*-algebra opposite and coopposite to H by setting*

$$H^{op,cop}_\alpha = H^{cop}_{\alpha^{-1}}, m^{op,cop}_{\alpha,\beta} = m^{op}_{\alpha,\beta}, 1^{op,cop} = 1, \text{ and } S^{op,cop}_\alpha = S_{\alpha^{-1}}.$$

Definition 4. *Let* $H = \{H_\alpha\}_{\alpha\in\pi}$ *and* $H' = \{H'_\alpha\}_{\alpha\in\pi}$ *be Hopf non-coassociative* π*-algebras. A Hopf non-coassociative* π*-algebra morphism between H and H′ is a* π*-algebra morphism* $f = \{f_\alpha : H_\alpha \longrightarrow H'_\alpha\}_{\alpha\in\pi}$ *between H and H′ such that, for any* $\alpha \in \pi$*,* f_α *is a coalgebra morphism and* $f_{\alpha^{-1}} \circ S_\alpha = S'_\alpha \circ f_\alpha$*. The Hopf non-coassociative* π*-algebra isomorphism* $f = \{f_\alpha : H_\alpha \longrightarrow H'_\alpha\}_{\alpha\in\pi}$ *is a Hopf non-coassociative* π*-algebra morphism in which each* f_α *is a linear isomorphism.*

Let us first remark that, when π is a finite group, there is a one-to-one correspondence between (isomorphic classes of) π-algebras and (isomorphic classes of) π-graded algebras. Recall that an algebra (A, m, η) is π-graded if A admits a decomposition as a direct sum of $\Bbbk$-spaces $A = \bigoplus_{\alpha\in\pi} A_\alpha$ such that

$$A_\alpha A_\beta \subset A_{\alpha\beta}, \forall \alpha, \beta \in \pi.$$
$$1 \in A_1.$$

Let us denote by $\pi_\alpha : A_\alpha \longrightarrow A$ the canonical injection. Then $\{A_\alpha\}_{\alpha\in\pi}$ is a π-algebra with multiplication $\{m(\pi_\alpha \otimes \pi_\beta)|A_\alpha \otimes A_\beta\}$ and unit η. Conversely, if $A = (\{A_\alpha\}, m, \eta)_{\alpha\in\pi}$ is a π-algebra, then $\widetilde{A} = \bigoplus_{\alpha\in\pi} A_\alpha$ is a π-graded algebra with multiplication $\widetilde{m}$ and unit $\widetilde{\eta}$ given on the summands by

$$\widetilde{m}|A_\alpha \otimes A_\beta = m_{\alpha,\beta} \text{ and } \widetilde{\eta} = \eta.$$

Let now $H = (\{H_\alpha, \Delta_\alpha, \varepsilon_\alpha\}, m, 1, S)_{\alpha\in\pi}$ be a Hopf non-coassociative group-algebra, where π is a finite group. Then the algebra $(\widetilde{H}, \widetilde{m}, \widetilde{\eta})$, defined as above, is a Hopf coquasigroup with comultiplication $\widetilde{\Delta}$, counit element $\widetilde{\varepsilon}$, and antipode $\widetilde{S}$ given by

$$\widetilde{\Delta}|H_\alpha = \Delta_\alpha \qquad \widetilde{\varepsilon} = \sum_{\alpha\in\pi} \varepsilon_\alpha, \qquad \widetilde{S} = \sum_{\alpha\in\pi} S_\alpha.$$

In what follows, we study structure properties for a Hopf non-coassociative π-algebra.

Theorem 2. *Let $H = (\{H_\alpha, \Delta_\alpha, \varepsilon_\alpha\}_{\alpha\in\pi}, m, \eta, S)$ be a Hopf non-coassociative π-algebra. Then*

$$m_{\alpha^{-1},\alpha}(S_\alpha \otimes id_{H_\alpha})\Delta_\alpha = 1\varepsilon_\alpha = m_{\alpha,\alpha^{-1}}(id_{H_\alpha} \otimes S_\alpha)\Delta_\alpha, \forall\alpha \in \pi. \tag{19}$$

$$S_{\alpha\beta}(ab) = S_\beta(b)S_\alpha(a), \forall\alpha, \beta \in \pi, a \in H_\alpha, b \in H_\beta; \tag{20}$$

$$S_1(1) = 1; \tag{21}$$

$$\Delta_{\alpha^{-1}}S_\alpha = \sigma_{H_{\alpha^{-1}},H_{\alpha^{-1}}}(S_\alpha \otimes S_\alpha)\Delta_\alpha, \forall\alpha \in \pi; \tag{22}$$

$$\varepsilon_{\alpha^{-1}}S_\alpha = \varepsilon_\alpha, \forall\alpha \in \pi. \tag{23}$$

Proof. Equation (19) is directly obtained by applying $id_{H_1} \otimes \varepsilon_\alpha$ to Equation (17) in the definition of a Hopf nonassociative π-coalgebra. We now show Equation (20) as follows:

$$\begin{aligned}
&S_{\alpha\beta}(ab) = S_{\alpha\beta}m_{\alpha,\beta}(a \otimes b) = S_{\alpha\beta}m_{\alpha,\beta}\sigma_{H_\beta,H_\alpha}(b \otimes a) = 1S_{\alpha\beta}m_{\alpha,\beta}\sigma_{H_\beta,H_\alpha}(b \otimes a)\\
=&m_{1,\beta^{-1}\alpha^{-1}}(1 \otimes S_{\alpha\beta}m_{\alpha,\beta}\sigma_{H_\beta,H_\alpha}(b \otimes a))\\
=&m_{1,\beta^{-1}\alpha^{-1}}(id_{H_1} \otimes S_{\alpha\beta}m_{\alpha,\beta}\sigma_{H_\beta,H_\alpha})(1 \otimes b \otimes a)\\
=&m_{1,\beta^{-1}\alpha^{-1}}(id_{H_1} \otimes S_{\alpha\beta}m_{\alpha,\beta}\sigma_{H_\beta,H_\alpha})(S_\beta(b_{(1)})b_{(2)(1)} \otimes b_{(2)(2)} \otimes a)\\
=&m_{1,\beta^{-1}\alpha^{-1}}(S_\beta(b_{(1)})b_{(2)(1)} \otimes S_{\alpha\beta}m_{\alpha,\beta}\sigma_{H_\beta,H_\alpha}(b_{(2)(2)} \otimes a))\\
=&m_{1,\beta^{-1}\alpha^{-1}}(S_\beta(b_{(1)})b_{(2)(1)} \otimes S_{\alpha\beta}m_{\alpha,\beta}(a \otimes b_{(2)(2)}))\\
=&m_{1,\beta^{-1}\alpha^{-1}}(S_\beta(b_{(1)})b_{(2)(1)} \otimes S_{\alpha\beta}(ab_{(2)(2)}))\\
=&(S_\beta(b_{(1)})b_{(2)(1)})S_{\alpha\beta}(ab_{(2)(2)}) = (S_\beta(b_{(1)})1b_{(2)(1)})S_{\alpha\beta}(ab_{(2)(2)})\\
=&m_{1,\beta^{-1}\alpha^{-1}}(m_{\beta^{-1},1,\beta}(S_\beta(b_{(1)}) \otimes 1 \otimes b_{(2)(1)}) \otimes S_{\alpha\beta}m_{\alpha,\beta}(a \otimes b_{(2)(2)}))\\
=&m_{1,\beta^{-1}\alpha^{-1}}(m_{\beta^{-1},1,\beta} \otimes S_{\alpha\beta}m_{\alpha,\beta})(S_\beta(b_{(1)}) \otimes 1 \otimes b_{(2)(1)} \otimes a \otimes b_{(2)(2)}))\\
=&m_{1,\beta^{-1}\alpha^{-1}}(m_{\beta^{-1},1,\beta} \otimes S_{\alpha\beta}m_{\alpha,\beta})(id_{H_{\beta^{-1}}} \otimes id_{H_1} \otimes \sigma_{H_\alpha,H_\beta} \otimes id_{H_\beta})\\
&(S_\beta(b_{(1)}) \otimes 1 \otimes a \otimes b_{(2)(1)} \otimes b_{(2)(2)}))\\
=&m_{1,\beta^{-1}\alpha^{-1}}(m_{\beta^{-1},1,\beta} \otimes S_{\alpha\beta}m_{\alpha,\beta})(id_{H_{\beta^{-1}}} \otimes id_{H_1} \otimes \sigma_{H_\alpha,H_\beta} \otimes id_{H_\beta})\\
&(S_\beta(b_{(1)}) \otimes S_\alpha(a_{(1)})a_{(2)(1)} \otimes a_{(2)(2)} \otimes b_{(2)(1)} \otimes b_{(2)(2)}))\\
=&m_{1,\beta^{-1}\alpha^{-1}}(m_{\beta^{-1},1,\beta} \otimes S_{\alpha\beta}m_{\alpha,\beta})(S_\beta(b_{(1)}) \otimes S_\alpha(a_{(1)})a_{(2)(1)} \otimes b_{(2)(1)} \otimes a_{(2)(2)} \otimes b_{(2)(2)}))\\
=&m_{1,\beta^{-1}\alpha^{-1}}(m_{\beta^{-1},1,\beta}(S_\beta(b_{(1)}) \otimes S_\alpha(a_{(1)})a_{(2)(1)} \otimes b_{(2)(1)}) \otimes S_{\alpha\beta}m_{\alpha,\beta}(a_{(2)(2)} \otimes b_{(2)(2)})))\\
=&m_{1,\beta^{-1}\alpha^{-1}}(S_\beta(b_{(1)})(S_\alpha(a_{(1)})a_{(2)(1)})b_{(2)(1)} \otimes S_{\alpha\beta}(a_{(2)(2)}b_{(2)(2)})))\\
=&(S_\beta(b_{(1)})(S_\alpha(a_{(1)})a_{(2)(1)})b_{(2)(1)})S_{\alpha\beta}(a_{(2)(2)}b_{(2)(2)})\\
=&S_\beta(b_{(1)})S_\alpha(a_{(1)})(a_{(2)(1)}b_{(2)(1)}S_{\alpha\beta}(a_{(2)(2)}b_{(2)(2)}))\\
=&S_\beta(b_{(1)})S_\alpha(a_{(1)})((a_{(2)}b_{(2)})_{(1)}S_{\alpha\beta}((a_{(2)}b_{(2)})_{(2)}))\\
=&S_\beta(b_{(1)})S_\alpha(a_{(1)})\varepsilon(a_{(2)}b_{(2)}) = S_\beta(b_{(1)})S_\alpha(a_{(1)})\varepsilon(a_{(2)})\varepsilon(b_{(2)})\\
=&S_\beta(b_{(1)}\varepsilon(b_{(2)}))S_\alpha(a_{(1)}\varepsilon(a_{(2)})) = S_\beta(b)S_\alpha(a).
\end{aligned}$$

Thus, $S_{\alpha\beta}(ab) = S_\beta(b)S_\alpha(a), \forall\alpha, \beta \in \pi, a \in H_\alpha, b \in H_\beta$.

To show Equation (22), for all, $\alpha \in \pi, h \in H_\alpha$, we have that

$$
\begin{aligned}
&(S_\alpha \otimes S_\alpha)\Delta_\alpha(h) = S_\alpha(h_{(1)}) \otimes S_\alpha(h_{(2)}) \\
=&(m_{\alpha^{-1},1} \otimes m_{\alpha^{-1},1})(id_{H_{\alpha^{-1}}} \otimes \sigma_{H_{\alpha^{-1}},H_1} \otimes id_{H_1})(id_{H_{\alpha^{-1}}} \otimes id_{H_{\alpha^{-1}}} \otimes \sigma_{H_1,H_1}) \\
&(S_\alpha(h_{(1)}) \otimes S_\alpha(h_{(2)}) \otimes 1 \otimes 1) \\
=&(m_{\alpha^{-1},1} \otimes m_{\alpha^{-1},1})(id_{H_{\alpha^{-1}}} \otimes \sigma_{H_{\alpha^{-1}},H_1} \otimes id_{H_1})(id_{H_{\alpha^{-1}}} \otimes id_{H_{\alpha^{-1}}} \otimes \sigma_{H_1,H_1}) \\
&(S_\alpha(h_{(1)}) \otimes S_\alpha(h_{(2)}) \otimes \Delta(1)) \\
=&(m_{\alpha^{-1},1} \otimes m_{\alpha^{-1},1})(id_{H_{\alpha^{-1}}} \otimes \sigma_{H_{\alpha^{-1}},H_1} \otimes id_{H_1})(id_{H_{\alpha^{-1}}} \otimes id_{H_{\alpha^{-1}}} \otimes \sigma_{H_1,H_1}) \\
&(S_\alpha(h_{(1)}) \otimes S_\alpha(h_{(2)(1)(1)}) \otimes \Delta(h_{(2)(1)(2)} S_\alpha(h_{(2)(2)}))) \\
=&(m_{\alpha^{-1},1} \otimes m_{\alpha^{-1},1})(id_{H_{\alpha^{-1}}} \otimes \sigma_{H_{\alpha^{-1}},H_1} \otimes id_{H_1})(id_{H_{\alpha^{-1}}} \otimes id_{H_{\alpha^{-1}}} \otimes \sigma_{H_1,H_1}) \\
&(S_\alpha(h_{(1)}) \otimes S_\alpha(h_{(2)(1)(1)}) \otimes \Delta(h_{(2)(1)(2)})\Delta(S_\alpha(h_{(2)(2)}))) \\
=&(m_{\alpha^{-1},1} \otimes m_{\alpha^{-1},1})(id_{H_{\alpha^{-1}}} \otimes \sigma_{H_{\alpha^{-1}},H_1} \otimes id_{H_1})(id_{H_{\alpha^{-1}}} \otimes id_{H_{\alpha^{-1}}} \otimes \sigma_{H_1,H_1}) \\
&(S_\alpha(h_{(1)}) \otimes S_\alpha(h_{(2)(1)(1)}) \otimes \Delta(h_{(2)(1)(2)})\Delta S_\alpha(h_{(2)(2)})) \\
=&(m_{\alpha^{-1},1} \otimes m_{\alpha^{-1},1})(id_{H_{\alpha^{-1}}} \otimes \sigma_{H_{\alpha^{-1}},H_1} \otimes id_{H_1})(id_{H_{\alpha^{-1}}} \otimes id_{H_{\alpha^{-1}}} \otimes \sigma_{H_1,H_1}) \\
&(S_\alpha(h_{(1)}) \otimes S_\alpha(h_{(2)(1)(1)}) \otimes (h_{(2)(1)(2)(1)} \otimes h_{(2)(1)(2)(2)})(S_\alpha(h_{(2)(2)})_{(1)} \otimes S_\alpha(h_{(2)(2)})_{(2)})) \\
=&(m_{\alpha^{-1},1} \otimes m_{\alpha^{-1},1})(id_{H_{\alpha^{-1}}} \otimes \sigma_{H_{\alpha^{-1}},H_1} \otimes id_{H_1})(id_{H_{\alpha^{-1}}} \otimes id_{H_{\alpha^{-1}}} \otimes \sigma_{H_1,H_1}) \\
&(S_\alpha(h_{(1)}) \otimes S_\alpha(h_{(2)(1)(1)}) \otimes h_{(2)(1)(2)(1)} S_\alpha(h_{(2)(2)})_{(1)} \otimes h_{(2)(1)(2)(2)} S_\alpha(h_{(2)(2)})_{(2)}) \\
=&S_\alpha(h_{(1)})(h_{(2)(1)(2)(2)} S_\alpha(h_{(2)(2)})_{(2)}) \otimes S_\alpha(h_{(2)(1)(1)})(h_{(2)(1)(2)(1)} S_\alpha(h_{(2)(2)})_{(1)}) \\
=&S_\alpha(h_{(1)})(h_{(2)(1)(2)(2)} S_\alpha(h_{(2)(2)})_{(2)}) \otimes (S_\alpha(h_{(2)(1)(1)})h_{(2)(1)(2)(1)}) S_\alpha(h_{(2)(2)})_{(1)} \\
=&S_\alpha(h_{(1)})(h_{(2)(1)} S_\alpha(h_{(2)(2)})_{(2)}) \otimes S_\alpha(h_{(2)(2)})_{(1)} = (S_\alpha(h_{(1)})h_{(2)(1)}) S_\alpha(h_{(2)(2)})_{(2)} \otimes S_\alpha(h_{(2)(2)})_{(1)} \\
=&(S_\alpha(h_{(1)})h_{(2)(1)} \otimes 1)(S_\alpha(h_{(2)(2)})_{(2)} \otimes S_\alpha(h_{(2)(2)})_{(1)}) \\
=&(S_\alpha(h_{(1)})h_{(2)(1)} \otimes 1)\sigma_{H_{\alpha^{-1}},H_{\alpha^{-1}}} \Delta_{\alpha^{-1}} S_\alpha(h_{(2)(2)}) \\
=&(1 \otimes 1)\sigma_{H_{\alpha^{-1}},H_{\alpha^{-1}}} \Delta_{\alpha^{-1}} S_\alpha(h) = \sigma_{H_{\alpha^{-1}},H_{\alpha^{-1}}} \Delta_{\alpha^{-1}} S_\alpha(h).
\end{aligned}
$$

Thus, $\Delta_{\alpha^{-1}} S_\alpha = \sigma_{H_{\alpha^{-1}},H_{\alpha^{-1}}}(S_\alpha \otimes S_\alpha)\Delta_\alpha, \forall \alpha \in \pi$.

Using Equation (19), we obtain Equation (21):

$$m_{1,1}(S_1 \otimes id_{H_1})\Delta_1(1) = 1\varepsilon_1(1) \Longrightarrow S_1(1)1 = 1 \Longrightarrow S_1(1) = 1.$$

We can obtain Equation (23) also by Equation (19): $\forall \alpha \in \pi, h \in H_\alpha$,

$$
\begin{aligned}
&m_{\alpha^{-1},\alpha}(S_\alpha \otimes id_{H_\alpha})\Delta_\alpha(h) = 1\varepsilon_\alpha(h) \Longrightarrow S_\alpha(h_{(1)})h_{(2)} = 1\varepsilon_\alpha(h) \\
\Longrightarrow&\varepsilon_1(S_\alpha(h_{(1)})h_{(2)}) = \varepsilon_1(1\varepsilon_\alpha(h)) \Longrightarrow \varepsilon_{\alpha^{-1}}(S_\alpha(h_{(1)}))\varepsilon_\alpha(h_{(2)}) = \varepsilon_1(1)\varepsilon_\alpha(h) \\
\Longrightarrow&\varepsilon_{\alpha^{-1}}(S_\alpha(h_{(1)}\varepsilon_\alpha(h_{(2)}))) = 1_{\Bbbk}\varepsilon_\alpha(h) \Longrightarrow \varepsilon_{\alpha^{-1}}(S_\alpha(h)) = \varepsilon_\alpha(h),
\end{aligned}
$$

i.e., $\varepsilon_{\alpha^{-1}} S_\alpha = \varepsilon_\alpha, \forall \alpha \in \pi$. □

Corollary 1. *The antipode of a Hopf non-coassociative π-algebra is unique.*

Proof. If $S, \widehat{S}$ are two antipodes on a Hopf non-coassociative π-algebra H, then they are equal in that, for any $\alpha \in \pi$ and $h \in H_\alpha$,

$$
\begin{aligned}
\widehat{S}_\alpha(h) &= \widehat{S}_\alpha(h_{(1)}\varepsilon_\alpha(h_{(2)})) = \widehat{S}_\alpha(h_{(1)})\varepsilon_\alpha(h_{(2)})1 = \widehat{S}_\alpha(h_{(1)})\Big(h_{(2)(1)} S_\alpha(h_{(2)(2)})\Big) \\
&= \Big(\widehat{S}_\alpha(h_{(1)})h_{(2)(1)}\Big) S_\alpha(h_{(2)(2)}) = 1S_\alpha(h) = S_\alpha(h). \quad \square
\end{aligned}
$$

Corollary 2. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a Hopf non-coassociative π-algebra with the antipode $S = \{S_\alpha\}_{\alpha\in\pi}$. Then S_α is the unique convolution inverse of id_{H_α} in the convolution algebra* $\mathrm{Conv}(H_\alpha, H)$*, for all $\alpha \in \pi$.*

Proof. Equation (19) says that S_α is a convolution inverse of id_{H_α} in the convolution algebra $\mathrm{Conv}(H_\alpha, H)$, for all $\alpha \in \pi$. Fix $\alpha \in \pi$. Let T_α be a right convolution inverse of id_{H_α} in the convolution algebra $\mathrm{Conv}(H_\alpha, H)$. For all $h \in H_\alpha$, we compute

$$\begin{aligned} S_\alpha(h) =& S_\alpha * (id_{H_\alpha} * T_\alpha)(h) = S_\alpha(h_{(1)})(id_{H_\alpha} * T_\alpha)(h_{(2)}) = S_\alpha(h_{(1)})(h_{(2)(1)}T_\alpha(h_{(2)(2)})) \\ =& (S_\alpha(h_{(1)})h_{(2)(1)})T_\alpha(h_{(2)(2)}) \xlongequal{\text{by Equation (17)}} 1T_\alpha(h) = T_\alpha(h) \\ \Longrightarrow T_\alpha =& S_\alpha. \end{aligned}$$

Fix $\alpha \in \pi$. Let T_α now be a left convolution inverse of id_{H_α} in the convolution algebra $\mathrm{Conv}(H_\alpha, H)$. Similarly, we have $T_\alpha = S_\alpha$. Therefore, S_α is the unique convolution inverse of id_{H_α} in the convolution algebra $\mathrm{Conv}(H_\alpha, H)$, for all $\alpha \in \pi$. □

Similarly, one can get

Corollary 3. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ be a Hopf non-coassociative π-algebra with the antipode $S = \{S_\alpha\}_{\alpha\in\pi}$. Then id_{H_α} is the unique convolution inverse of S_α in the convolution algebra* $\mathrm{Conv}(H_\alpha, H)$*, for all $\alpha \in \pi$.*

Corollary 4. *Let $H = (\{H_\alpha, \Delta_\alpha, \varepsilon_\alpha\}_{\alpha\in\pi}, m, \eta, S)$ be a Hopf non-coassociative π-algebra. Then $\{\alpha \in \pi, |H_\alpha \neq 0\}$ is a subgroup of π.*

Proof. Set $G = \{\alpha \in \pi, |H_\alpha \neq 0\}$. Since $\varepsilon_1(1) = 1_{\Bbbk} \neq 0$, we first have $0 \neq 1 \in H_1$, i.e., $H_1 \neq 0$, and so $1 \in G$.

Now let $\alpha \in G$ whereby $H_\alpha \neq 0$, then there exists $0 \neq a \in H_\alpha$. Using Equation (13), one can see that $a_{(1)}\varepsilon_\alpha(a_{(2)}) = \varepsilon_\alpha(a_{(1)})a_{(2)} = a \neq 0$. It follows that $\exists h \in H_\alpha, s.t.\ \varepsilon_\alpha(h) \neq 0$. Then let $\beta \in G$. In a similar manner, one can also obtain that $\exists g \in H_\beta, s.t.\ \varepsilon_\beta(g) \neq 0$. Thus, $\varepsilon_{\alpha\beta}(hg) = \varepsilon_\alpha(h)\varepsilon_\beta(g) \neq 0$, i.e., $0 \neq hg \in H_{\alpha\beta}$ and so $\alpha\beta \in G$.

Finally, let $\alpha \in G$. By Equation (23), $\varepsilon_{\alpha^{-1}}S_\alpha(h) = \varepsilon_\alpha(h) \neq 0$. Therefore $0 \neq S_\alpha(h) \in H_{\alpha^{-1}}$ and hence $\alpha^{-1} \in G$. □

The following theorem sheds considerable light on the concept of a Hopf non-coassociative π-algebra morphism.

Theorem 3. *Let $H = \{H_\alpha\}_{\alpha\in\pi}$ and $H' = \{H'_\alpha\}_{\alpha\in\pi}$ be Hopf non-coassociative π-algebras. A π-algebra morphism $f = \{f_\alpha : H_\alpha \longrightarrow H'_\alpha\}_{\alpha\in\pi}$ between H and H' such that, for any $\alpha \in \pi$, f_α is a coalgebra morphism satisfies $f_{\alpha^{-1}} \circ S_\alpha = S'_\alpha \circ f_\alpha$, for all $\alpha \in \pi$.*

Proof. Consider the convolution inverse of f_α in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$,

$$S'_\alpha \circ f_\alpha * f_\alpha(h) = S'_\alpha \circ f_\alpha(h_{(1)})f_\alpha(h_{(2)}) = S'_\alpha\Big((f_\alpha(h))_{(1)}\Big)(f_\alpha(h))_{(2)} = \varepsilon'_\alpha(f_\alpha(h))1' = \varepsilon_\alpha(h)1',$$

whence $S'_\alpha \circ f_\alpha$ is a left convolution inverse of f_α in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$,

$$f_\alpha * f_{\alpha^{-1}} \circ S_\alpha(h) = f_\alpha(h_{(1)})f_{\alpha^{-1}} \circ S_\alpha(h_{(2)}) = f_1\Big(h_{(1)}S_\alpha(h_{(2)})\Big) = f_1(\varepsilon_\alpha(h)1) = \varepsilon_\alpha(h)1',$$

whence $f_{\alpha^{-1}} \circ S_\alpha$ is a right convolution inverse of f_α in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$,

$$
\begin{aligned}
&f_{\alpha^{-1}} \circ S_\alpha(h) = (S'_\alpha \circ f_\alpha * f_\alpha) * f_{\alpha^{-1}} \circ S_\alpha(h) = S'_\alpha \circ f_\alpha * f_\alpha(h_{(1)}) f_{\alpha^{-1}} \circ S_\alpha(h_{(2)}) \\
&= \Big(S'_\alpha \circ f_\alpha(h_{(1)(1)}) f_\alpha(h_{(1)(2)})\Big) f_{\alpha^{-1}} \circ S_\alpha(h_{(2)}) = S'_\alpha \circ f_\alpha(h_{(1)(1)}) \Big(f_\alpha(h_{(1)(2)}) f_{\alpha^{-1}} \circ S_\alpha(h_{(2)})\Big) \\
&= S'_\alpha \circ f_\alpha(h_{(1)(1)}) f_1\Big(h_{(1)(2)} S_\alpha(h_{(2)})\Big) = S'_\alpha \circ f_\alpha(h) f_1(1) = S'_\alpha \circ f_\alpha(h) 1' = S'_\alpha \circ f_\alpha(h),
\end{aligned}
$$

from which we obtain $f_{\alpha^{-1}} \circ S_\alpha = S'_\alpha \circ f_\alpha$. This completes the proof. □

By looking into the proof of Theorem 3, we note that $f_{\alpha^{-1}} \circ S_\alpha = S'_\alpha \circ f_\alpha$ and f_α are convolution inverses in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$. More precisely, we claim:

Corollary 5. *If $f = \{f_\alpha : H_\alpha \longrightarrow H'_\alpha\}_{\alpha \in \pi}$ is a Hopf non-coassociative π-algebra morphism between H and H'. Then:*

(1) $f_{\alpha^{-1}} \circ S_\alpha = S'_\alpha \circ f_\alpha$ is the unique convolution inverse of f_α in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$;

(2) f_α is the unique convolution inverse of $f_{\alpha^{-1}} \circ S_\alpha = S'_\alpha \circ f_\alpha$ in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$.

Proof. We first establish part (1). Fix $\alpha \in \pi$. Let T_α be a right convolution inverse of f_α in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$.

$$
\begin{aligned}
&f_{\alpha^{-1}} \circ S_\alpha(h) = f_{\alpha^{-1}} \circ S_\alpha * (f_\alpha * T_\alpha)(h) = f_{\alpha^{-1}} \circ S_\alpha(h_{(1)}) f_\alpha * T_\alpha(h_{(2)}) \\
&\quad = f_{\alpha^{-1}} \circ S_\alpha(h_{(1)}) \Big(f_\alpha(h_{(2)(1)}) T_\alpha(h_{(2)(2)})\Big) = \Big(f_{\alpha^{-1}} \circ S_\alpha(h_{(1)}) f_\alpha(h_{(2)(1)})\Big) T_\alpha(h_{(2)(2)}) \\
&\quad = f_1\Big(S_\alpha(h_{(1)}) h_{(2)(1)}\Big) T_\alpha(h_{(2)(2)}) = f_1(1) T_\alpha(h) = 1' T_\alpha(h) = T_\alpha(h).
\end{aligned}
$$

Fix $\alpha \in \pi$. Let T_α now be a left convolution inverse of f_α in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$. Similarly, we have $f_{\alpha^{-1}} \circ S_\alpha(h) = T_\alpha(h)$.

$f_{\alpha^{-1}} \circ S_\alpha = S'_\alpha \circ f_\alpha$ is therefore the unique convolution inverse of f_α in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$, for all $\alpha \in \pi$.

We now turn to part (2). Fix $\alpha \in \pi$. Let T_α be a right convolution inverse of $f_{\alpha^{-1}} \circ S_\alpha = S'_\alpha \circ f_\alpha$ in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$.

$$
\begin{aligned}
&f_\alpha(h) = f_\alpha * (f_{\alpha^{-1}} \circ S_\alpha * T_\alpha)(h) = f_\alpha(h_{(1)}) f_{\alpha^{-1}} \circ S_\alpha * T_\alpha(h_{(2)}) \\
&\quad = f_\alpha(h_{(1)}) \Big(f_{\alpha^{-1}} \circ S_\alpha(h_{(2)(1)}) T_\alpha(h_{(2)(2)})\Big) = \Big(f_\alpha(h_{(1)}) f_{\alpha^{-1}} \circ S_\alpha(h_{(2)(1)})\Big) T_\alpha(h_{(2)(2)}) \\
&\quad = f_1\Big(h_{(1)} S_\alpha(h_{(2)(1)})\Big) T_\alpha(h_{(2)(2)}) = f_1(1) T_\alpha(h) = 1' T_\alpha(h) = T_\alpha(h).
\end{aligned}
$$

Fix $\alpha \in \pi$. Let T_α now be a left convolution inverse of $f_{\alpha^{-1}} \circ S_\alpha = S'_\alpha \circ f_\alpha$ in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$. Similarly, we have $f_\alpha(h) = T_\alpha(h)$. Therefore, f_α is the unique convolution inverse of $f_{\alpha^{-1}} \circ S_\alpha = S'_\alpha \circ f_\alpha$ in the convolution algebra $\mathrm{Conv}(H_\alpha, H')$, for all $\alpha \in \pi$. □

The following two corollaries can be directly deduced from Theorems 2 and 3.

Corollary 6. *If H is a Hopf non-coassociative π-algebra, then the map $S : H \longrightarrow H^{op,cop}$ (where both are opposite and $S^{op,cop} = \{S^{op,cop}_\alpha = S_{\alpha^{-1}}\}_{\alpha \in \pi}$) is a Hopf non-coassociative π-algebra isomorphism.*

Corollary 7. *If H is a Hopf non-coassociative π-algebra with an invertible antipode S, then the map $S : H \longrightarrow H^{op,cop}$ (where both are opposite and $S^{op,cop} = \{S^{op,cop}_\alpha = S_{\alpha^{-1}}\}_{\alpha \in \pi}$) is a Hopf non-coassociative π-algebra isomorphism.*

Theorem 4. *Let H be a Hopf non-coassociative π-algebra. Then for any $\alpha \in \pi$, $S_{\alpha^{-1}} S_\alpha = id_{H_\alpha}$ if H is commutative or cocommutative.*

Proof. For any $\alpha \in \pi$. Let $h \in H_\alpha$. If H is commutative, we have

$$\begin{aligned} S_{\alpha^{-1}}S_\alpha(h) &= S_{\alpha^{-1}}S_\alpha(h_{(1)}\varepsilon(h_{(2)})) = S_{\alpha^{-1}}S_\alpha(h_{(1)})\varepsilon(h_{(2)}) \\ &= S_{\alpha^{-1}}S_\alpha(h_{(1)})(S_\alpha(h_{(2)(1)})h_{(2)(2)}) = (S_{\alpha^{-1}}S_\alpha(h_{(1)})S_\alpha(h_{(2)(1)}))h_{(2)(2)} \\ &= S_1(h_{(2)(1)}S_\alpha(h_{(1)}))h_{(2)(2)} = S_1(S_\alpha(h_{(1)})h_{(2)(1)})h_{(2)(2)} = S_1(1)h = 1h = h. \end{aligned}$$

It follows that $S_{\alpha^{-1}}S_\alpha = id_{H_\alpha}$.

Similar to the case of H being cocommutative. □

Theorem 5. *Let H be a Hopf non-coassociative π-algebra such that each S_α^{-1} exists, for all $\alpha \in \pi$. Then the following identities are equivalent*:

(1) $a_{(2)(1)}S_\alpha(a_{(1)}) \otimes a_{(2)(2)} = a_{(2)}S_\alpha(a_{(1)(2)}) \otimes a_{(1)(1)} = 1 \otimes a$, *for all* $\alpha \in \pi, a \in H_\alpha$.

(2) $a_{(2)(2)} \otimes S_\alpha(a_{(2)(1)})a_{(1)} = a_{(1)(1)} \otimes S_\alpha(a_{(2)})a_{(1)(2)} = a \otimes 1$, *for all* $\alpha \in \pi, a \in H_\alpha$.

(3) $S_{\alpha^{-1}}S_\alpha = id_{H_\alpha}$, *for all* $\alpha \in \pi$.

Proof. Let $\alpha \in \pi$ and $a \in H_\alpha$. We have

$$\begin{aligned} &S_{\alpha^{-1}}S_\alpha(a) = S_{\alpha^{-1}}S_\alpha(a_{(1)}\varepsilon_\alpha(a_{(2)})) = S_{\alpha^{-1}}S_\alpha(a_{(1)})\varepsilon_\alpha(a_{(2)}) = S_{\alpha^{-1}}S_\alpha(a_{(1)})(\varepsilon_\alpha(a_{(2)})1) \\ =&S_{\alpha^{-1}}S_\alpha(a_{(1)})(S_\alpha(a_{(2)(1)})a_{(2)(2)}) = S_1(a_{(2)(1)}S_\alpha(a_{(1)}))a_{(2)(2)}. \end{aligned}$$

If (1) holds, we then find that $S_{\alpha^{-1}}S_\alpha(a) = S_1(1)a = a$, which implies that (3) holds. If (3) is satisfied, then one has

$$\begin{aligned} 1 \otimes a =& \ a_{(1)}S_\alpha(a_{(2)(1)}) \otimes a_{(2)(2)} = \ a_{(1)}S_\alpha(a_{(2)(1)}) \otimes S_{\alpha^{-1}}S_\alpha(a_{(2)(2)}) \\ =& \ a_{(1)}S_\alpha(a_{(2)})_{(2)} \otimes S_{\alpha^{-1}}(S_\alpha(a_{(2)})_{(1)}) \\ =& \ S_{\alpha^{-1}}S_\alpha(a_{(1)})S_\alpha(a_{(2)})_{(2)} \otimes S_{\alpha^{-1}}(S_\alpha(a_{(2)})_{(1)}) \\ =& \ S_{\alpha^{-1}}(S_\alpha(a)_{(2)})S_\alpha(a)_{(1)(2)} \otimes S_{\alpha^{-1}}(S_\alpha(a)_{(1)(1)}). \end{aligned}$$

Applying S_α to the second tensor factor we obtain

$$\begin{aligned} 1 \otimes S_\alpha(a) =& \ S_{\alpha^{-1}}(S_\alpha(a)_{(2)})S_\alpha(a)_{(1)(2)} \otimes S_\alpha S_{\alpha^{-1}}(S_\alpha(a)_{(1)(1)}) \\ =& \ S_{\alpha^{-1}}(S_\alpha(a)_{(2)})S_\alpha(a)_{(1)(2)} \otimes S_\alpha(a)_{(1)(1)}. \end{aligned}$$

So (2) holds since S is bijective.

We have shown (1) $\Longrightarrow$ (3) $\Longrightarrow$ (2).

Similarly one proves (2) $\Longrightarrow$ (3) $\Longrightarrow$ (1). □

Theorem 6. *Let H be a Hopf non-coassociative π-algebra with a bijective antipode S and S^{-1} the composite inverse to S. Then*

$$\begin{aligned} &S^{-1}_{\alpha^{-1}}(h_{(2)})h_{(1)} = \ h_{(2)}S^{-1}_{\alpha^{-1}}(h_{(1)}) = \varepsilon_\alpha(h)1, \quad S^{-1}_{(\alpha\beta)^{-1}}(hg) = S^{-1}_{\beta^{-1}}(g)S^{-1}_{\alpha^{-1}}(h), \quad S^{-1}_1(1) = \ 1 \\ &\Delta_{\alpha^{-1}}(S^{-1}_{\alpha^{-1}}(h)) = S^{-1}_{\alpha^{-1}}(h_{(2)}) \otimes S^{-1}_{\alpha^{-1}}(h_{(1)}), \quad \varepsilon_{\alpha^{-1}}(S^{-1}_{\alpha^{-1}}(h)) = \varepsilon_\alpha(h) \end{aligned}$$

for all $\alpha, \beta \in \pi$, $h \in H_\alpha$ and $g \in H_\beta$.

Proof. The proof is straightforward. □

Theorem 7. *Let H be a Hopf non-coassociative π-algebra such that each S_α^{-1} exists, for all $\alpha \in \pi, a \in H_\alpha$. Then the following identities are equivalent*:

(1) $a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = a_{(1)(1)(1)}a_{(1)(2)} \otimes a_{(1)(1)(2)} \otimes a_{(2)}$.

(2) $a_{(1)(1)(1)} \otimes a_{(1)(1)(2)}a_{(2)} \otimes a_{(1)(2)} = a_{(1)} \otimes a_{(2)(1)}a_{(2)(2)(2)} \otimes a_{(2)(2)(1)}$.

(3) $a_{(1)(1)}a_{(2)(2)} \otimes a_{(1)(2)} \otimes a_{(2)(1)} = a_{(1)(1)}a_{(2)} \otimes a_{(1)(2)(1)} \otimes a_{(1)(2)(2)}$.

Proof. (1) $\Longrightarrow$ (2) Let $T = (id_{H_\alpha} \otimes \sigma_{H_\alpha,H_\alpha} \otimes id_{H_\alpha})$. Then

$$
\begin{aligned}
& a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = a_{(1)(1)(1)}a_{(1)(2)} \otimes a_{(1)(1)(2)} \otimes a_{(2)} \\
\Longrightarrow & (m_{\alpha,\alpha} \otimes id^2_{H_\alpha})T(id^2_{H_\alpha} \otimes \Delta_\alpha)(id_{H_\alpha} \otimes \Delta_\alpha)\Delta_\alpha(a) \\
= & (m_{\alpha,\alpha} \otimes id_{H_\alpha} \otimes id_{H_\alpha})T(\Delta_\alpha \otimes id^2_{H_\alpha})(\Delta_\alpha \otimes id_{H_\alpha})\Delta_\alpha(a) \\
\Longrightarrow & (m_{\alpha,\alpha} \otimes id_{H_\alpha} \otimes id_{H_\alpha})T(id^2_{H_\alpha} \otimes \Delta_\alpha)(id_{H_\alpha} \otimes \Delta_\alpha)\Delta_\alpha(S_{\alpha^{-1}}(b)) \\
= & (m_{\alpha,\alpha} \otimes id^2_{H_\alpha})T(\Delta_\alpha \otimes id^2_{H_\alpha})(\Delta_\alpha \otimes id_{H_\alpha})\Delta_\alpha(S_{\alpha^{-1}}(b)) \\
\Longrightarrow & (m_{\alpha,\alpha} \otimes id^2_{H_\alpha})T(id^2_{H_\alpha} \otimes \Delta_\alpha)(id_{H_\alpha} \otimes \Delta_\alpha)(S_{\alpha^{-1}}(b_{(2)}) \otimes S_{\alpha^{-1}}(b_{(1)})) \\
= & (m_{\alpha,\alpha} \otimes id_{H_\alpha} \otimes id_{H_\alpha})T(\Delta_\alpha \otimes id^2_{H_\alpha})(\Delta_\alpha \otimes id_{H_\alpha})(S_{\alpha^{-1}}(b_{(2)}) \otimes S_{\alpha^{-1}}(b_{(1)})) \\
\Longrightarrow & (m_{\alpha,\alpha} \otimes id^2_{H_\alpha})T(id^2_{H_\alpha} \otimes \Delta_\alpha)(S_{\alpha^{-1}}(b_{(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(1)})) \\
= & (m_{\alpha,\alpha} \otimes id^2_{H_\alpha})T(\Delta_\alpha \otimes id^2_{H_\alpha})(S_{\alpha^{-1}}(b_{(2)(2)}) \otimes S_{\alpha^{-1}}(b_{(2)(1)}) \otimes S_{\alpha^{-1}}(b_{(1)})) \\
\Longrightarrow & (m_{\alpha,\alpha} \otimes id^2_{H_\alpha})T(S_{\alpha^{-1}}(b_{(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(1)(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(1)(1)})) \\
= & (m_{\alpha,\alpha} \otimes id^2_{H_\alpha})T(S_{\alpha^{-1}}(b_{(2)(2)(2)}) \otimes S_{\alpha^{-1}}(b_{(2)(2)(1)}) \otimes S_{\alpha^{-1}}(b_{(2)(1)}) \otimes S_{\alpha^{-1}}(b_{(1)})) \\
\Longrightarrow & (m_{\alpha,\alpha} \otimes id^2_{H_\alpha})(S_{\alpha^{-1}}(b_{(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(1)(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(1)(1)})) \\
= & (m_{\alpha,\alpha} \otimes id^2_{H_\alpha})(S_{\alpha^{-1}}(b_{(2)(2)(2)}) \otimes S_{\alpha^{-1}}(b_{(2)(1)}) \otimes S_{\alpha^{-1}}(b_{(2)(2)(1)}) \otimes S_{\alpha^{-1}}(b_{(1)})) \\
\Longrightarrow & S_{\alpha^{-1}}(b_{(2)})S_{\alpha^{-1}}(b_{(1)(1)(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(1)(1)}) \\
= & S_{\alpha^{-1}}(b_{(2)(2)(2)})S_{\alpha^{-1}}(b_{(2)(1)}) \otimes S_{\alpha^{-1}}(b_{(2)(2)(1)}) \otimes S_{\alpha^{-1}}(b_{(1)}) \\
\Longrightarrow & S_{\alpha^{-1}\alpha^{-1}}(b_{(1)(1)(2)}b_{(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(2)}) \otimes S_{\alpha^{-1}}(b_{(1)(1)(1)}) \\
= & S_{\alpha^{-1}\alpha^{-1}}(b_{(2)(1)}b_{(2)(2)(2)}) \otimes S_{\alpha^{-1}}(b_{(2)(2)(1)}) \otimes S_{\alpha^{-1}}(b_{(1)}) \\
\Longrightarrow & b_{(1)(1)(2)}b_{(2)} \otimes b_{(1)(2)} \otimes b_{(1)(1)(1)} = b_{(2)(1)}b_{(2)(2)(2)} \otimes b_{(2)(2)(1)} \otimes b_{(1)} \\
\Longrightarrow & b_{(1)(1)(1)} \otimes b_{(1)(1)(2)}b_{(2)} \otimes b_{(1)(2)} = b_{(1)} \otimes b_{(2)(1)}b_{(2)(2)(2)} \otimes b_{(2)(2)(1)} \\
\Longrightarrow & a_{(1)(1)(1)} \otimes a_{(1)(1)(2)}b_{(2)} \otimes a_{(1)(2)} = a_{(1)} \otimes a_{(2)(1)}b_{(2)(2)(2)} \otimes a_{(2)(2)(1)}.
\end{aligned}
$$

Similarly, (1) implies (2).

$$
\begin{aligned}
&(3) \Longrightarrow (1)\\
&a_{(1)(1)}a_{(2)(2)} \otimes a_{(1)(2)} \otimes a_{(2)(1)} = a_{(1)(1)}a_{(2)} \otimes a_{(1)(2)(1)} \otimes a_{(1)(2)(2)}\\
\Longrightarrow & a_{(1)(1)(1)}a_{(1)(2)(2)}S_\alpha(a_{(2)})_{(2)} \otimes a_{(1)(1)(2)} \otimes a_{(1)(2)(1)}S_\alpha(a_{(2)})_{(1)}\\
=& a_{(1)(1)(1)}a_{(1)(2)}S_\alpha(a_{(2)})_{(2)} \otimes a_{(1)(1)(2)(1)} \otimes a_{(1)(1)(2)(2)}S_\alpha(a_{(2)})_{(1)}\\
\Longrightarrow & a_{(1)(1)(1)}(a_{(1)(2)}S_\alpha(a_{(2)}))_{(2)} \otimes a_{(1)(1)(2)} \otimes (a_{(1)(2)}S_\alpha(a_{(2)}))_{(1)}\\
=& a_{(1)(1)(1)}a_{(1)(2)}S_\alpha(a_{(2)})_{(2)} \otimes a_{(1)(1)(2)(1)} \otimes a_{(1)(1)(2)(2)}S_\alpha(a_{(2)})_{(1)}\\
\Longrightarrow & a_{(1)} \otimes a_{(2)} \otimes 1 = a_{(1)(1)(1)}a_{(1)(2)}S_\alpha(a_{(2)})_{(2)} \otimes a_{(1)(1)(2)(1)} \otimes a_{(1)(1)(2)(2)}S_\alpha(a_{(2)})_{(1)}\\
\Longrightarrow & a_{(1)} \otimes a_{(2)} \otimes 1 = a_{(1)(1)(1)}a_{(1)(2)}S_\alpha(a_{(2)(1)}) \otimes a_{(1)(1)(2)(1)} \otimes a_{(1)(1)(2)(2)}S_\alpha(a_{(2)(2)})\\
\Longrightarrow & a_{(1)} \otimes a_{(2)(1)} \otimes S_\alpha(a_{(2)(2)})1 = a_{(1)(1)(1)}a_{(1)(2)}S_\alpha(a_{(2)(1)}) \otimes a_{(1)(1)(2)} \otimes 1S_\alpha(a_{(2)(2)})\\
\Longrightarrow & a_{(1)} \otimes a_{(2)(1)} \otimes S_\alpha(a_{(2)(2)}) = a_{(1)(1)(1)}a_{(1)(2)}S_\alpha(a_{(2)(1)}) \otimes a_{(1)(1)(2)} \otimes S_\alpha(a_{(2)(2)})\\
\Longrightarrow & a_{(1)} \otimes a_{(2)(1)} \otimes S_\alpha^{-1}(S_\alpha(a_{(2)(2)})) = a_{(1)(1)(1)}a_{(1)(2)}S_\alpha(a_{(2)(1)}) \otimes a_{(1)(1)(2)} \otimes S_\alpha^{-1}(S_\alpha(a_{(2)(2)}))\\
\Longrightarrow & a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} = a_{(1)(1)(1)}a_{(1)(2)}S_\alpha(a_{(2)(1)}) \otimes a_{(1)(1)(2)} \otimes a_{(2)(2)}\\
\Longrightarrow & a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = a_{(1)(1)(1)}a_{(1)(2)}S_\alpha(a_{(2)(1)})a_{(2)(2)(1)} \otimes a_{(1)(1)(2)} \otimes a_{(2)(2)(2)}\\
\Longrightarrow & a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = a_{(1)(1)(1)}a_{(1)(2)}1 \otimes a_{(1)(1)(2)} \otimes a_{(2)}\\
\Longrightarrow & a_{(1)}a_{(2)(2)(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)(2)} = a_{(1)(1)(1)}a_{(1)(2)} \otimes a_{(1)(1)(2)} \otimes a_{(2)}.
\end{aligned}
$$

Similarly, (1) implies (3). □

We have observed that if H is a Hopf non-coassociative π-algebra with antipode S then $H^{op,cop}$ is a Hopf non-coassociative π-algebra with antipode $S^{op,cop} = \{S_\alpha^{op,cop} = S_{\alpha^{-1}}\}_{\alpha\in\pi}$. Furthermore, the following theorem says, if H^{op} or H^{cop} is a Hopf non-coassociative π-algebra, then S is bijective, and vice versa.

Proposition 2. *Suppose that H is a Hopf non-coassociative π-algebra with antipode S over the field $\Bbbk$. Then the following are equivalent:*

(a) $H^{op} = \{H_\alpha^{op} = H_{\alpha^{-1}}\}_{\alpha\in\pi}$ *is a Hopf non-coassociative π-algebra.*
(b) $H^{cop} = \{H_\alpha^{cop} = H_\alpha\}_{\alpha\in\pi}$ *is a Hopf non-coassociative π-algebra.*
(c) *S is bijective.*

If S is bijective, then H^{op} and H^{cop} have antipodes $S^{op} = \{S_\alpha^{op} = S_\alpha^{-1}\}_{\alpha\in\pi}$ and $S^{cop} = \{S_\alpha^{cop} = S_{\alpha^{-1}}^{-1}\}_{\alpha\in\pi}$, respectively.

Proof. Since $H^{cop} = (H^{op})^{op,cop}$ and $H^{op} = (H^{cop})^{op,cop}$, the parts (a) and (b) are equivalent.

If the part (c) holds, then it is easy to check that Part (a) holds. Conversely, suppose that H^{op} is a Hopf non-coassociative π-algebra with antipode $T = \{T_\alpha\}_{\alpha\in\pi}$. Then $T_\alpha(h_{(1)})h_{(2)} = \varepsilon_\alpha^{op}(h)1 = h_{(1)}T_\alpha(h_{(2)})$, or equivalently, $h_{(2)}T_\alpha(h_{(1)}) = \varepsilon_{\alpha^{-1}}(h)1 = T_\alpha(h_{(2)})h_{(1)}$, for $h \in H_\alpha^{op} = H_{\alpha^{-1}}$. Applying S_1 to the left-hand side of the above equation, we have

$$(S_\alpha \circ T_\alpha)(h_{(1)})S_{\alpha^{-1}}(h_{(2)}) = \varepsilon_{\alpha^{-1}}(h)1.$$

Replacing h with $S_\alpha(h)$ in this equation, one has

$$\varepsilon_{\alpha^{-1}}(S_\alpha(h))1 = T_\alpha(S_\alpha(h)_{(2)})S_\alpha(h)_{(1)},$$

or equivalently,

$$\varepsilon_\alpha(h)1 = T_\alpha \circ S_\alpha(h_{(1)})S_\alpha(h_{(2)}).$$

Therefore $T_\alpha \circ S_\alpha$ and $S_{\alpha^{-1}} \circ T_{\alpha^{-1}}$ are both left inverses of S_α in the convolution algebra $\mathrm{Conv}(H_\alpha, H)$. It follows from Corollary 3 that $S_{\alpha^{-1}} \circ T_{\alpha^{-1}} = \mathrm{id}_{H_\alpha} = T_\alpha \circ S_\alpha$ which establishes that the part (a) implies that the part (c). □

Theorem 8. *Let H be a commutative flexible Hopf non-coassociative π-algebra. Then*

$$a_{(1)}S_\alpha(a_{(2)(2)}) \otimes a_{(2)(1)} = a_{(1)(1)}S_\alpha(a_{(2)}) \otimes a_{(1)(2)}, \forall \alpha \in \pi, a \in H_\alpha.$$

Proof. $\forall \alpha \in \pi, a \in H_\alpha$. Since H is flexible, we have that

$$a_{(1)(1)}a_{(2)} \otimes a_{(1)(2)} = a_{(1)}a_{(2)(2)} \otimes a_{(2)(1)}$$

$$\Longrightarrow (a_{(1)(1)}a_{(2)})S_\alpha(a_{(1)(2)(2)}) \otimes a_{(1)(2)(1)} = (a_{(1)}a_{(2)(2)})S_\alpha(a_{(2)(1)(2)}) \otimes a_{(2)(1)(1)}$$
$$\Longrightarrow a_{(1)(1)}(a_{(2)}S_\alpha(a_{(1)(2)(2)})) \otimes a_{(1)(2)(1)} = a_{(1)}(a_{(2)(2)}S_\alpha(a_{(2)(1)(2)})) \otimes a_{(2)(1)(1)}$$
$$\Longrightarrow a_{(1)(1)}(S_\alpha(a_{(1)(2)(2)})a_{(2)}) \otimes a_{(1)(2)(1)} = a_{(1)(1)(1)}(S_\alpha(a_{(1)(2)})a_{(2)}) \otimes a_{(1)(1)(2)}$$
$$\Longrightarrow a_{(1)(1)}S_\alpha(a_{(1)(2)(2)})a_{(2)} \otimes a_{(1)(2)(1)} = a_{(1)}S_\alpha(a_{(2)(1)(2)})a_{(2)(2)} \otimes a_{(2)(1)(1)}$$
$$\Longrightarrow a_{(1)(1)(1)}S_\alpha(a_{(1)(1)(2)(2)})a_{(1)(2)}S_\alpha(a_{(2)}) \otimes a_{(1)(1)(2)(1)} = a_{(1)(1)}S_\alpha(a_{(2)}) \otimes a_{(1)(2)}u \quad \square$$

In the end of this section, we study how to construct an coassociator for any Hopf non-coassociative π-algebra.

Definition 5. *In any Hopf non-coassociative π-algebra, we define the coassociator*

$$\Phi = \{\Phi_\alpha : H_\alpha \longrightarrow H_\alpha \otimes H_\alpha \otimes H_\alpha\}_{\alpha \in \pi} \quad by$$

$$(\Delta_\alpha \otimes id_{H_\alpha})\Delta_\alpha(a) = (id_{H_\alpha} \otimes \Delta_\alpha)\Delta_\alpha(a_{(1)(1)})\Phi_\alpha(a_{(1)(2)})(id_{H_{\alpha^{-1}}} \otimes \Delta_{\alpha^{-1}})\Delta_{\alpha^{-1}}(S_\alpha(a_{(2)}))$$

for all $\alpha \in \pi$ and $a \in H_\alpha$.

Remark 4. *For the next theorem, we will use some convenient notation. Let H be a Hopf non-coassociative π-algebra. $\forall \alpha \in \pi, a \in H_\alpha$, we write $\Phi_\alpha(a) = \Phi_a^{(1)} \otimes \Phi_a^{(2)} \otimes \Phi_a^{(3)}$.*

Theorem 9. *Let H be a Hopf non-coassociative π-algebra. Then*

(1) *The associator $\Phi = \{\Phi_\alpha\}_{\alpha \in \pi}$ exists and is uniquely determined as* $\Phi_\alpha(a) = S_\alpha(a_{(1)})_{(1)}a_{(2)(1)(1)(1)}a_{(2)(2)(1)} \otimes S_\alpha(a_{(1)})_{(2)(1)}a_{(2)(1)(1)(2)}a_{(2)(2)(2)(1)}$ $\otimes S_\alpha(a_{(1)})_{(2)(2)}a_{(2)(1)(2)}a_{(2)(2)(2)(2)}, \forall \alpha \in \pi, a \in H_\alpha.$

(2) $(\varepsilon_\alpha \otimes \varepsilon_\alpha \otimes id_{H_\alpha})\Phi_\alpha(a) = (\varepsilon_\alpha \otimes id_{H_\alpha} \otimes \varepsilon_\alpha)\Phi_\alpha(a)$ $= (id_{H_\alpha} \otimes \varepsilon_\alpha \otimes \varepsilon_\alpha)\Phi_\alpha(a) = a, \forall \alpha \in \pi, a \in H_\alpha.$

(3) $\Phi_a^{(1)}S_\alpha(\Phi_a^{(2)}) \otimes \Phi_a^{(3)} = S_\alpha(\Phi_a^{(1)})\Phi_a^{(2)} \otimes \Phi_a^{(3)}, \forall \alpha \in \pi, a \in H_\alpha.$

(4) $\Phi_a^{(1)} \otimes S_\alpha(\Phi_a^{(2)})\Phi_a^{(3)} = \Phi_a^{(1)} \otimes \Phi_a^{(2)}S_\alpha(\Phi_a^{(3)}), \forall \alpha \in \pi, a \in H_\alpha.$

(5) $(\Phi_a^{(1)})_{(1)}S_\alpha(\Phi_a^{(3)}) \otimes (\Phi_a^{(1)})_{(2)}S_\alpha(\Phi_a^{(2)}) = S_\alpha((\Phi_a^{(1)})_{(1)})\Phi_a^{(3)} \otimes S_\alpha((\Phi_a^{(1)})_{(2)})\Phi_a^{(2)}$ $= \Phi_a^{(1)}S_\alpha((\Phi_a^{(3)})_{(2)}) \otimes \Phi_a^{(2)}S_\alpha((\Phi_a^{(3)})_{(1)}) = S_\alpha(\Phi_a^{(1)})(\Phi_a^{(3)})_{(2)} \otimes S_\alpha(\Phi_a^{(2)})(\Phi_a^{(3)})_{(1)}$ $= S_\alpha(\Phi_a^{(1)})(\Phi_a^{(2)})_{(1)} \otimes S_\alpha((\Phi_a^{(2)})_{(2)})\Phi_a^{(3)} = S_\alpha(\Phi_a^{(1)})(\Phi_a^{(3)})_{(2)} \otimes S_\alpha(\Phi_a^{(2)})(\Phi_a^{(3)})_{(1)},$ $\forall \alpha \in \pi, a \in H_\alpha.$

Proof. The proof of this theorem consists of a long tedious computation. We just show readers as follows for the part (1). The other are similar.

(1) $\forall \alpha \in \pi, a \in H_\alpha$, we have that

$$
\begin{aligned}
& S_\alpha(a_{(1)})_{(1)}a_{(2)(1)(1)(1)}a_{(2)(2)(1)} \otimes S_\alpha(a_{(1)})_{(2)(1)}a_{(2)(1)(1)(2)}a_{(2)(2)(2)(1)} \otimes S_\alpha(a_{(1)})_{(2)(2)}a_{(2)(1)(2)}a_{(2)(2)(2)(2)} \\
=& (S_\alpha(a_{(1)})_{(1)} \otimes S_\alpha(a_{(1)})_{(2)(1)} \otimes S_\alpha(a_{(1)})_{(2)(2)}) \\
& (a_{(2)(1)(1)(1)} \otimes a_{(2)(1)(1)(2)} \otimes a_{(2)(1)(2)})(a_{(2)(2)(1)} \otimes a_{(2)(2)(2)(1)} \otimes a_{(2)(2)(2)(2)}) \\
=& (S_\alpha(a_{(1)})_{(1)} \otimes S_\alpha(a_{(1)})_{(2)(1)} \otimes S_\alpha(a_{(1)})_{(2)(2)})(a_{(2)(1)(1)(1)(1)} \otimes a_{(2)(1)(1)(1)(2)(1)} \otimes a_{(2)(1)(1)(1)(2)(2)}) \\
& \Phi_\alpha(a_{(2)(1)(1)(2)}) \\
& (S_\alpha(a_{(2)(1)(2)})_{(1)} \otimes S_\alpha(a_{(2)(1)(2)})_{(2)(1)} \otimes S_\alpha(a_{(2)(1)(2)})_{(2)(2)})(a_{(2)(2)(1)} \otimes a_{(2)(2)(2)(1)} \otimes a_{(2)(2)(2)(2)}) \\
=& (S_\alpha(a_{(1)})_{(1)}a_{(2)(1)(1)(1)(1)} \otimes S_\alpha(a_{(1)})_{(2)(1)}a_{(2)(1)(1)(1)(2)(1)} \otimes S_\alpha(a_{(1)})_{(2)(2)}a_{(2)(1)(1)(1)(2)(2)}) \\
& \Phi_\alpha(a_{(2)(1)(1)(2)}) \\
& (S_\alpha(a_{(2)(1)(2)})_{(1)}a_{(2)(2)(1)} \otimes S_\alpha(a_{(2)(1)(2)})_{(2)(1)}a_{(2)(2)(2)(1)} \otimes S_\alpha(a_{(2)(1)(2)})_{(2)(2)}a_{(2)(2)(2)(2)}) \\
=& ((S_\alpha(a_{(1)})a_{(2)(1)(1)(1)})_{(1)} \otimes (S_\alpha(a_{(1)})a_{(2)(1)(1)(1)})_{(2)(1)} \otimes (S_\alpha(a_{(1)})a_{(2)(1)(1)(1)})_{(2)(2)}) \\
& \Phi_\alpha(a_{(2)(1)(1)(2)}) \\
& ((S_\alpha(a_{(2)(1)(2)})a_{(2)(2)})_{(1)} \otimes (S_\alpha(a_{(2)(1)(2)})a_{(2)(2)})_{(2)(1)} \otimes (S_\alpha(a_{(2)(1)(2)})a_{(2)(2)})_{(2)(2)}) \\
=& ((S_\alpha(a_{(1)})a_{(2)(1)})_{(1)} \otimes (S_\alpha(a_{(1)})a_{(2)(1)})_{(2)(1)} \otimes (S_\alpha(a_{(1)})a_{(2)(1)})_{(2)(2)}) \\
& \Phi_\alpha(a_{(2)(2)})(1_{(1)} \otimes 1_{(2)(1)} \otimes 1_{(2)(2)}) \\
=& ((S_\alpha(a_{(1)})a_{(2)(1)})_{(1)} \otimes (S_\alpha(a_{(1)})a_{(2)(1)})_{(2)(1)} \otimes (S_\alpha(a_{(1)})a_{(2)(1)})_{(2)(2)})\Phi_\alpha(a_{(2)(2)}) \\
=& (1_{(1)} \otimes 1_{(2)(1)} \otimes 1_{(2)(2)})\Phi_\alpha(a) \\
=& \Phi_\alpha(a),
\end{aligned}
$$

and

$$
\begin{aligned}
& (a_{(1)(1)(1)} \otimes a_{(1)(1)(2)(1)} \otimes a_{(1)(1)(2)(2)})\Phi_\alpha(a_{(1)(2)})(S_\alpha(a_{(2)})_{(1)} \otimes S_\alpha(a_{(2)})_{(2)(1)} \otimes S_\alpha(a_{(2)})_{(2)(2)}) \\
=& (a_{(1)(1)(1)} \otimes a_{(1)(1)(2)(1)} \otimes a_{(1)(1)(2)(2)})(S_\alpha(a_{(1)(2)(1)})_{(1)}a_{(1)(2)(2)(1)(1)(1)}a_{(1)(2)(2)(2)(1)} \\
& \otimes S_\alpha(a_{(1)(2)(1)})_{(2)(1)}a_{(1)(2)(2)(1)(1)(2)}a_{(1)(2)(2)(2)(2)(1)} \otimes S_\alpha(a_{(1)(2)(1)})_{(2)(2)}a_{(1)(2)(2)(1)(2)}a_{(1)(2)(2)(2)(2)(2)}) \\
& (S_\alpha(a_{(2)})_{(1)} \otimes S_\alpha(a_{(2)})_{(2)(1)} \otimes S_\alpha(a_{(2)})_{(2)(2)}) \\
=& a_{(1)(1)(1)}S_\alpha(a_{(1)(2)(1)})_{(1)}a_{(1)(2)(2)(1)(1)(1)}a_{(1)(2)(2)(2)(1)}S_\alpha(a_{(2)})_{(1)} \\
& \otimes a_{(1)(1)(2)(1)}S_\alpha(a_{(1)(2)(1)})_{(2)(1)}a_{(1)(2)(2)(1)(1)(2)}a_{(1)(2)(2)(2)(2)(1)}S_\alpha(a_{(2)})_{(2)(1)} \\
& \otimes a_{(1)(1)(2)(2)}S_\alpha(a_{(1)(2)(1)})_{(2)(2)}a_{(1)(2)(2)(1)(2)}a_{(1)(2)(2)(2)(2)(2)}S_\alpha(a_{(2)})_{(2)(2)} \\
=& (a_{(1)(1)}S_\alpha(a_{(1)(2)(1)}))_{(1)}a_{(1)(2)(2)(1)(1)(1)}(a_{(1)(2)(2)(2)}S_\alpha(a_{(2)}))_{(1)} \\
& \otimes (a_{(1)(1)}S_\alpha(a_{(1)(2)(1)}))_{(2)(1)}a_{(1)(2)(2)(1)(1)(2)}(a_{(1)(2)(2)(2)}S_\alpha(a_{(2)}))_{(2)(1)} \\
& \otimes (a_{(1)(1)}S_\alpha(a_{(1)(2)(1)}))_{(2)(2)}a_{(1)(2)(2)(1)(2)}(a_{(1)(2)(2)(2)}S_\alpha(a_{(2)}))_{(2)(2)} \\
=& 1_{(1)}a_{(1)(1)(1)(1)}(a_{(1)(2)}S_\alpha(a_{(2)}))_{(1)} \otimes 1_{(2)(1)}a_{(1)(1)(1)(2)}(a_{(1)(2)}S_\alpha(a_{(2)}))_{(2)(1)} \\
& \otimes 1_{(2)(2)}a_{(1)(1)(2)}(a_{(1)(2)}S_\alpha(a_{(2)}))_{(2)(2)} \\
=& a_{(1)(1)(1)(1)}(a_{(1)(2)}S_\alpha(a_{(2)}))_{(1)} \otimes a_{(1)(1)(1)(2)}(a_{(1)(2)}S_\alpha(a_{(2)}))_{(2)(1)} \otimes a_{(1)(1)(2)}(a_{(1)(2)}S_\alpha(a_{(2)}))_{(2)(2)} \\
=& a_{(1)(1)}1_{(1)} \otimes a_{(1)(2)}1_{(2)(1)} \otimes a_{(2)}1_{(2)(2)} = a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)}.
\end{aligned}
$$

□

5. Crossed Hopf Non-Coassociative π-Algebras

In this section we mainly study the notion of a crossed Hopf non-coassociative π-algebra and give some properties of the crossing map.

Definition 6. *A Hopf non-coassociative π-algebra $H = (\{H_\alpha, \Delta_\alpha, \varepsilon_\alpha\}_{\alpha\in\pi}, m, \eta, S)$ is said to be crossed provided it is endowed with a family $\varphi = \{\varphi_\beta : H_\alpha \longrightarrow H_{\beta\alpha\beta^{-1}}\}_{\alpha,\beta\in\pi}$ of $\Bbbk$-linear maps (the cocrossing) such that*

$$\bigstar \text{each } \varphi_\beta : H_\alpha \longrightarrow H_{\beta\alpha\beta^{-1}} \text{ is a coalgebra isomorphism,} \tag{24}$$

$$\bigstar \text{each } \varphi_\beta \text{ preserves the multiplication, i.e., for all } \alpha, \beta, \gamma \in \pi,$$

$$\bigstar \qquad \varphi_\beta m_{\alpha,\gamma} = m_{\beta\alpha\beta^{-1},\beta\gamma\beta^{-1}}(\varphi_\beta \otimes \varphi_\beta), \tag{25}$$

$$\bigstar \text{each } \varphi_\beta \text{ preserves the unit, i.e., } \varphi_\beta(1) = 1, \tag{26}$$

$$\bigstar \varphi \text{ is multiplicative in the sense that } \varphi_{\beta\beta'} = \varphi_\beta\varphi_{\beta'} \text{ for all } \beta, \beta' \in \pi. \tag{27}$$

The following result is straightforward.

Lemma 13. *Let H be a crossed Hopf non-coassociative π-algebra with cocrossing φ. Then*

(a) $\varphi_{1|H_\alpha} = id_{H_\alpha}$ *for all* $\alpha \in \pi$;
(b) $\varphi_\beta^{-1} = \varphi_{\beta^{-1}}$ *for all* $\beta \in \pi$;
(c) *φ preserves the antipode, i.e.,* $\varphi_\beta S_\alpha = S_{\beta\alpha\beta^{-1}}\varphi_\beta$ *for all* $\alpha, \beta \in \pi$;
(d) *if $\lambda = (\lambda_\alpha)_{\alpha\in\pi}$ is a left (resp. right) π-integral in H and $\beta \in \pi$, then $(\varphi_\beta(\lambda_{\beta^{-1}\alpha\beta}))_{\alpha\in\pi}$ is also a left (resp. right) π-integral on H;*
(e) *if $g = (g_\alpha)_{\alpha\in\pi}$ is a π-grouplike element of H and $\beta \in \pi$, then $(g_{\beta\alpha\beta^{-1}}\varphi_\beta)_{\alpha\in\pi}$ is also a π-grouplike element of H.*

Let H be a crossed Hopf non-coassociative π-algebra with cocrossing φ. If the antipode of H is bijective, then the opposite (resp. coopposite) coquasigroup Hopf π-algebra to H (see Example 2) is crossed with cocrossing given by

$$\varphi_\beta^{op}|H_\alpha^{op} = \varphi_\beta|H_{\alpha^{-1}} \qquad (resp.\ \varphi_\beta^{cop}|H_\alpha^{cop} = \varphi_\beta|H_\alpha)$$

for all $\alpha, \beta \in \pi$.

Let $H = (\{H_\alpha, m_\alpha, 1_\alpha\}, \Delta, \varepsilon, S, \varphi)$ be a crossed Hopf non-coassociative π-algebra. Similar to ([4], Section 11.6), its mirror $\bar{H}$ is defined by the following procedure: set $\bar{H}_\alpha = H_{\alpha^{-1}}$ as a coalgebra, $\bar{m}_{\alpha,\beta} = m_{\beta^{-1}\alpha^{-1}\beta,\beta^{-1}}(\varphi_{\beta^{-1}} \otimes id_{H_{\beta^{-1}}})$, $\bar{1} = 1$, $\bar{S_\alpha} = \varphi_\alpha S_{\alpha^{-1}}$, $\bar{\varphi}_\beta|\bar{H}_\alpha = \varphi_\beta|H_{\alpha^{-1}}$. It is also a crossed Hopf non-coassociative π-algebra.

6. Almost Cocommutative Hopf Non-Coassociative π-Algebras

The aim of this section is to discuss the definition and properties of an almost cocommutative Hopf non-coassociative π-algebra and to obtain its equivalent condition.

Definition 7. *A crossed Hopf non-coassociative π-algebra (H, φ) with a bijective antipode S is called almost cocommutative if there exists a family $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ of invertible elements (the R-matrix) such that, for any $\alpha, \beta, \gamma \in \pi$ and $x \in H_\gamma$,*

$$\Delta_\gamma^{cop}(x) \cdot (\varphi_{\gamma^{-1}} \otimes \varphi_{\gamma^{-1}})(R_{\alpha,\beta}) = R_{\alpha,\beta} \cdot \Delta_\gamma(x), \tag{28}$$

and the family R is invariant under the crossing, i.e., for any $\alpha, \beta, \gamma \in \pi$,

$$(\varphi_\gamma \otimes \varphi_\gamma)(R_{\alpha,\beta}) = R_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}}. \tag{29}$$

Note that $(H_1, R_{1,1})$ is an almost cocommutative Hopf coquasigroup. It is customary to write $R^{(1)}_{\alpha,\beta} \otimes R^{(2)}_{\alpha,\beta}$ for $R_{\alpha,\beta}$.

Equation (28) in Definition 7 can be written equivalently as:

$$\Delta_\gamma^{cop}(x) \cdot R_{\alpha,\beta} = (\varphi_\gamma \otimes \varphi_\gamma)(R_{\alpha,\beta}) \cdot \Delta_\gamma(x),$$

for any $\alpha, \beta, \gamma \in \pi$ and $x \in H_\gamma$.

It is obvious that, for any $\alpha, \beta, \gamma \in \pi$,

$$((\varphi_\gamma \otimes \varphi_\gamma)(R_{\alpha,\beta}))^{-1} = (\varphi_\gamma \otimes \varphi_\gamma)\left(R_{\alpha,\beta}^{-1}\right).$$

The family R^{-1} is therefore invariant under the crossing, i.e., for any $\alpha, \beta, \gamma \in \pi$,

$$(\varphi_\gamma \otimes \varphi_\gamma)\left(R_{\alpha,\beta}^{-1}\right) = R^{-1}_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}}.$$

Our first proposition generalizes the basic fact about almost cocommutative Hopf non-coassociative π-algebras.

Note that $(H_1, R_{1,1})$ is an almost cocommutative Hopf coquasigroup. It is customary to write $R^{(1)}_{\alpha,\beta} \otimes R^{(2)}_{\alpha,\beta}$ for $R_{\alpha,\beta}$.

Our first proposition generalizes the basic fact about almost cocommutative Hopf non-coassociative π-algebras.

Proposition 3. *Let H be a crossed Hopf non-coassociative π-algebra, and V, W left π-modules over H, then $V \otimes W = \{V_\alpha \otimes W_\alpha\}_{\alpha\in\pi}$ is also a left π-module over H. If H is almost cocommutative, then $V \otimes W \cong W \otimes V$ as left π-modules over H.*

Proof. Similar as in the Hopf coquasigroup case, we define

$$h \cdot (v \otimes w) = h_{(1)} \cdot v \otimes h_{(2)} \cdot w$$

for all $h \in H_\alpha$ and $v \in V_\beta$, $w \in W_\beta$. It is easy to see that $V \otimes W$ is a left π-module over H. If H is almost cocommutative with $R \in H \otimes H$. Then for all $v \in V_\alpha$, $w \in W_\alpha$, define

$$c^{R_{1,1}}_{V_\alpha,W_\alpha} : V_\alpha \otimes W_\beta \to W_\alpha \otimes V_\alpha, \quad c^{R_{1,1}}_{V_\alpha,W_\alpha}(v \otimes w) = R^{(2)}_{1,1} w \otimes R^{(1)}_{1,1} v$$

By Equation (28), $c^{R_{1,1}}_{V_\alpha,W_\alpha}$ is an isomorphism with inverse given by

$$\left(c^{R_{1,1}}_{V_\alpha,W_\alpha}\right)^{-1} : W_\alpha \otimes V_\alpha \to V_\alpha \otimes W_\alpha, \quad \left(c^{R_{1,1}}_{V_\alpha,W_\alpha}\right)^{-1}(w \otimes v) = U^{(1)}_{1,1} v \otimes U^{(2)}_{1,1} w$$

where $R^{-1}_{1,1} = U_{1,1} = U^{(1)}_{1,1} \otimes U^{(2)}_{1,1}$. □

Recall from Theorem 4 that if H be cocommutative, then $S^2 = id_H$. This fact can also be generalized.

Proposition 4. *Let H be an almost cocommutative Hopf non-coassociative π-algebra. Then $S^2 = \{S_{\alpha^{-1}} \circ S_\alpha\}_{\alpha\in\pi}$ is an inner automorphism of H. More precisely, let $u_{\beta^{-1}} = S_\beta\left(R^{(2)}_{1,\beta}\right)R^{(1)}_{1,\beta}$, where $R_{1,\beta} = R^{(1)}_{1,\beta} \otimes R^{(2)}_{1,\beta}$. Then, we have*

(1) u_α is invertible, $S_{\alpha^{-1}} \circ S_\alpha(h) = u_\alpha h u_\alpha^{-1} = (S_\alpha(u_\alpha))^{-1} h S_\alpha(u_\alpha)$, $S_\alpha \circ S_{\alpha^{-1}}(h) = u_\alpha h u_\alpha^{-1} = (S_\alpha(u_\alpha))^{-1} h S_\alpha(u_\alpha)$ and $S_1 \circ S_1(h) = u_\alpha h u_\alpha^{-1} = (S_\alpha(u_\alpha))^{-1} h S_\alpha(u_\alpha)$;

(2) $u_\alpha S_\alpha(u_\alpha)$ is relatively central for $H_\alpha \cup H_1 \cup H_{\alpha^{-1}}$;

(3) $\varphi_\beta(u_\alpha) = u_{\beta\alpha\beta^{-1}}$.

Proof. We first show that $u_\alpha h = S_1 \circ S_1(h) u_\alpha$, for all $h \in H_1$. Since H be almost cocommutative, we have

$$h_{(2)} \otimes R^{(1)}_{1,\alpha^{-1}} h_{(1)(1)} \otimes R^{(2)}_{1,\alpha^{-1}} h_{(1)(2)} = h_{(2)} \otimes h_{(1)(2)} R^{(1)}_{1,\alpha^{-1}} \otimes h_{(1)(1)} R^{(2)}_{1,\alpha^{-1}},$$

i.e.,

$$h_{(2)} \otimes R^{(2)}_{1,\alpha^{-1}} h_{(1)(2)} \otimes R^{(1)}_{1,\alpha^{-1}} h_{(1)(1)} = h_{(2)} \otimes h_{(1)(1)} R^{(2)}_{1,\alpha^{-1}} \otimes h_{(1)(2)} R^{(1)}_{1,\alpha^{-1}}.$$

Thus

$$S_1 \circ S_1\left(h_{(2)}\right) S_{\alpha^{-1}}\left(R^{(2)}_{1,\alpha^{-1}} h_{(1)(2)}\right) R^{(1)}_{1,\alpha^{-1}} h_{(1)(1)} = S_1 \circ S_1\left(h_{(2)}\right) S_{\alpha^{-1}}\left(h_{(1)(1)} R^{(2)}_{1,\alpha^{-1}}\right) h_{(1)(2)} R^{(1)}_{1,\alpha^{-1}}.$$

Since S is antimultiplicative, hence

$$\begin{aligned} &S_1 \circ S_1\left(h_{(2)}\right) S_1\left(h_{(1)(2)}\right) S_{\alpha^{-1}}\left(R^{(2)}_{1,\alpha^{-1}}\right) R^{(1)}_{1,\alpha^{-1}} h_{(1)(1)} \\ =&S_1 \circ S_1\left(h_{(2)}\right) S_{\alpha^{-1}}\left(R^{(2)}_{1,\alpha^{-1}}\right) S_1\left(h_{(1)(1)}\right) h_{(1)(2)} R^{(1)}_{1,\alpha^{-1}}, \end{aligned}$$

i.e.,

$$S_1\left(h_{(1)(2)} S_1\left(h_{(2)}\right)\right) u_\alpha h_{(1)(1)} = S_1 \circ S_1(h) u_\alpha.$$

Following the axiom (16) of Hopf non-coassociative π-algebra, we have

$$u_\alpha h = S_1 \circ S_1(h) u_\alpha, \text{ for all } h \in H_1. \tag{30}$$

The following two equalities can be verified in a similar way.

$$u_\alpha h = S_{\alpha^{-1}} \circ S_\alpha(h) u_\alpha, \text{ for all } h \in H_\alpha. \tag{31}$$

$$u_\alpha h = S_\alpha \circ S_{\alpha^{-1}}(h) u_\alpha, \text{ for all } h \in H_{\alpha^{-1}}. \tag{32}$$

We next show that u_α is invertible. Write $R^{-1}_{1,\alpha^{-1}} = U_{1,\alpha} = U^{(1)}_{1,\alpha} \otimes U^{(2)}_{1,\alpha}$. Applying $m_{1,1} \circ \sigma_{H_1,H_1} \circ (\mathrm{id}_{H_1} \otimes S_1)$ to both sides of $R^{(1)}_{1,\alpha^{-1}} U^{(1)}_{1,\alpha} \otimes R^{(2)}_{1,\alpha^{-1}} U^{(2)}_{1,\alpha} = 1 \otimes 1$ yields $S_\alpha\left(U^{(2)}_{1,\alpha}\right) u_\alpha U^{(1)}_{1,\alpha} = 1$ from which $S_\alpha\left(U^{(2)}_{1,\alpha}\right) S_1 \circ S_1\left(U^{(1)}_{1,\alpha}\right) u_\alpha = 1$ follows by Equation (30). Observe that we have not used the fact that S is bijective at this point. Since S is bijective we can use Equation (32) to calculate $1 = S_\alpha\left(U^{(2)}_{1,\alpha}\right) u_\alpha U^{(1)}_{1,\alpha} = S_\alpha \circ S_{\alpha^{-1}} \circ S^{-1}_{\alpha^{-1}}\left(U^{(2)}_{1,\alpha}\right) u_\alpha U^{(1)}_{1,\alpha} = u_\alpha S^{-1}_{\alpha^{-1}}\left(U^{(2)}_{1,\alpha}\right) U^{(1)}_{1,\alpha}$. We have shown that u_α has a left inverse and a right inverse. u_α is therefore invertible. By Equations (30)–(32), the three equations below can be therefore deduced:

$$S_1 \circ S_1(h) = u_\alpha h u_\alpha^{-1}, \text{ for all } h \in H_1. \tag{33}$$

$$S_{\alpha^{-1}} \circ S_\alpha(h) = u_\alpha h u_\alpha^{-1}, \text{ for all } h \in H_\alpha. \tag{34}$$

$$S_\alpha \circ S_{\alpha^{-1}}(h) = u_\alpha h u_\alpha^{-1}, \text{ for all } h \in H_{\alpha^{-1}}. \tag{35}$$

Applying S_1 to Equation (33) and replacing h by $S_1^{-1}(h)$ yields the formula $S_1 \circ S_1(h) = (S_\alpha(u_\alpha))^{-1} h S_\alpha(u_\alpha)$.

Applying S_α to Equation (34) and replacing h by $S_\alpha^{-1}(h)$ gives rise to the formula $S_\alpha \circ S_{\alpha^{-1}}(h) = (S_\alpha(u_\alpha))^{-1} h S_\alpha(u_\alpha)$.

Applying $S_{\alpha^{-1}}$ to Equation (35) and replacing h by $S^{-1}_{\alpha^{-1}}(h)$ gives birth to the formula $S_{\alpha^{-1}} \circ S_\alpha(h) = (S_\alpha(u_\alpha))^{-1} h S_\alpha(u_\alpha)$.

To check that $u_\alpha S_\alpha(u_\alpha)$ is relatively central for H_α, we will prove that for all $g \in H_\alpha$, $g u_\alpha S_\alpha(u_\alpha) = u_\alpha S_\alpha(u_\alpha) g$.

Let $h = S_\alpha(u_\alpha) g S_{\alpha^{-1}}(u_\alpha^{-1})$, then

$$S_{\alpha^{-1}} \circ S_\alpha(h) = u_\alpha h u_\alpha^{-1} = u_\alpha S_\alpha(u_\alpha) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right) u_\alpha^{-1}$$

and

$$S_{\alpha^{-1}} \circ S_\alpha(h) = (S_\alpha(u_\alpha))^{-1} h S_\alpha(u_\alpha) = (S_\alpha(u_\alpha))^{-1} S_\alpha(u_\alpha) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right) S_\alpha(u_\alpha) = g.$$

So

$$g = u_\alpha S_\alpha(u_\alpha) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right) u_\alpha^{-1}$$

i.e., $g u_\alpha S_\alpha(u_\alpha) = u_\alpha S_\alpha(u_\alpha) g$ for all $g \in H_\alpha$.

To check that $u_\alpha S_\alpha(u_\alpha)$ is relatively central for H_1, we will prove that for all $g \in H_1$, $g u_\alpha S_\alpha(u_\alpha) = u_\alpha S_\alpha(u_\alpha) g$.

Let $h = S_\alpha(u_\alpha) g S_{\alpha^{-1}}(u_\alpha^{-1})$, then

$$S_1 \circ S_1(h) = u_\alpha h u_\alpha^{-1} = u_\alpha S_\alpha(u_\alpha) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right) u_\alpha^{-1}$$

and

$$S_1 \circ S_1(h) = (S_\alpha(u_\alpha))^{-1} h S_\alpha(u_\alpha) = (S_\alpha(u_\alpha))^{-1} S_\alpha(u_\alpha) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right) S_\alpha(u_\alpha) = g.$$

So

$$g = u_\alpha S_\alpha(u_\alpha) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right) u_\alpha^{-1}$$

i.e., $g u_\alpha S_\alpha(u_\alpha) = u_\alpha S_\alpha(u_\alpha) g$ for all $g \in H_1$.

To check that $u_\alpha S_\alpha(u_\alpha)$ is relatively central for $H_{\alpha^{-1}}$, we will prove that for all $g \in H_{\alpha^{-1}}$, $g u_\alpha S_\alpha(u_\alpha) = u_\alpha S_\alpha(u_\alpha) g$.

Let $h = S_\alpha(u_\alpha) g S_{\alpha^{-1}}(u_\alpha^{-1})$, then

$$S_\alpha \circ S_{\alpha^{-1}}(h) = u_\alpha h u_\alpha^{-1} = u_\alpha S_\alpha(u_\alpha) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right) u_\alpha^{-1}$$

and

$$S_\alpha \circ S_{\alpha^{-1}}(h) = (S_\alpha(u_\alpha))^{-1} h S_\alpha(u_\alpha) = (S_\alpha(u_\alpha))^{-1} S_\alpha(u_\alpha) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right) S_\alpha(u_\alpha) = g.$$

So

$$g = u_\alpha S_\alpha(u_\alpha) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right) u_\alpha^{-1}$$

i.e., $g u_\alpha S_\alpha(u_\alpha) = u_\alpha S_\alpha(u_\alpha) g$ for all $g \in H_{\alpha^{-1}}$.

$$\begin{aligned}\varphi_\beta(u_\alpha) &= \varphi_\beta\left(S_{\alpha^{-1}}\left(R^{(2)}_{1,\alpha^{-1}}\right) R^{(1)}_{1,\alpha^{-1}}\right) = \varphi_\beta\left(S_{\alpha^{-1}}\left(R^{(2)}_{1,\alpha^{-1}}\right)\right) \varphi_\beta\left(R^{(1)}_{1,\alpha^{-1}}\right) \\ &= S_{\beta\alpha^{-1}\beta^{-1}}\left(\varphi_\beta\left(R^{(2)}_{1,\alpha^{-1}}\right)\right) \varphi_\beta\left(R^{(1)}_{1,\alpha^{-1}}\right) = S_{\beta\alpha^{-1}\beta^{-1}}\left(R^{(2)}_{1,\beta\alpha^{-1}\beta^{-1}}\right) R^{(1)}_{1,\beta\alpha^{-1}\beta^{-1}} = u_{\beta\alpha\beta^{-1}}.\end{aligned}$$

This completes the proof. □

Corollary 8.

(1) $S_{\alpha^{-1}} \circ S_\alpha(u_\alpha) = u_\alpha$;

(2) $S_\alpha \circ S_{\alpha^{-1}}(u_\alpha^{-1}) = u_\alpha^{-1}$;

(3) $S_\alpha(u_\alpha) = u_\alpha S_\alpha(u_\alpha) u_\alpha^{-1}$. *In particular,* u_α *and* $S_\alpha(u_\alpha)$ *commute;*

(4) $S_{\alpha^{-1}}(u_\alpha^{-1}) = u_\alpha S_{\alpha^{-1}}(u_\alpha^{-1}) u_\alpha^{-1}$. *In particular,* u_α *and* $S_{\alpha^{-1}}(u_\alpha^{-1})$ *commute;*

(5) $\varphi_\beta(u_\alpha^{-1}) = u_{\beta\alpha\beta^{-1}}^{-1}$.

Proof. Part (1) is straightforward from Proposition 4. Apropos of part (2), we calculate as follows:

$$S_{\alpha^{-1}} \circ S_\alpha(u_\alpha) S_\alpha \circ S_{\alpha^{-1}}(u_\alpha^{-1}) = S_1 \circ S_1(u_\alpha u_\alpha^{-1}) = S_1 \circ S_1(1) = 1,$$

and also

$$S_\alpha \circ S_{\alpha^{-1}}(u_\alpha^{-1}) S_{\alpha^{-1}} \circ S_\alpha(u_\alpha) = S_1 \circ S_1(u_\alpha^{-1} u_\alpha) = S_1 \circ S_1(1) = 1.$$

Thus $S_{\alpha^{-1}} \circ S_\alpha(u_\alpha)$ and $S_\alpha \circ S_{\alpha^{-1}}(u_\alpha^{-1})$ are inverses, from which $S_\alpha \circ S_{\alpha^{-1}}(u_\alpha^{-1}) = u_\alpha^{-1}$.

To show part (3), we use part (1) and Proposition 4 to calculate

$$S_\alpha(u_\alpha) = S_\alpha \circ S_{\alpha^{-1}} \circ S_\alpha(u_\alpha) = u_\alpha S_\alpha(u_\alpha) u_\alpha^{-1},$$

whereby u_α and $S_\alpha(u_\alpha)$ commute.

To establish part (4), we use part (3) to make the following calculation:

$$S_{\alpha^{-1}}(u_\alpha^{-1}) = (S_\alpha(u_\alpha))^{-1} = \left(u_\alpha S_\alpha(u_\alpha) u_\alpha^{-1}\right)^{-1} = u_\alpha (S_\alpha(u_\alpha))^{-1} u_\alpha^{-1} = u_\alpha S_{\alpha^{-1}}(u_\alpha^{-1}) u_\alpha^{-1},$$

whereby u_α and $S_{\alpha^{-1}}(u_\alpha^{-1})$ commute.

It remains to check part (5). Observe that

$$\varphi_\beta(u_\alpha^{-1})\varphi_\beta(u_\alpha) = \varphi_\beta(u_\alpha^{-1}u_\alpha) = \varphi_\beta(1) = 1$$

and also that

$$\varphi_\beta(u_\alpha)\varphi_\beta(u_\alpha^{-1}) = \varphi_\beta(u_\alpha u_\alpha^{-1}) = \varphi_\beta(1) = 1.$$

Thus $\varphi_\beta(u_\alpha)$ and $\varphi_\beta(u_\alpha^{-1})$ are inverses.

It follows from Proposition 4 that

$$\varphi_\beta(u_\alpha^{-1}) = \left(\varphi_\beta(u_\alpha)\right)^{-1} = u_{\beta\alpha\beta^{-1}}^{-1}. \quad \square$$

Corollary 9. $S_{\beta^{-1}} \circ S_\beta(h) = u_\alpha h \varphi_{\beta^{-1}}(u_\alpha^{-1}) = \varphi_\beta\big((S_\alpha(u_\alpha))^{-1}\big) h S_\alpha(u_\alpha)$.

Proof. We first show that $u_\alpha h = S_{\beta^{-1}} \circ S_\beta(h)\varphi_{\beta^{-1}}(u_\alpha)$, for all $h \in H_\beta$. Since H is almost cocommutative, we have

$$h_{(2)} \otimes R^{(1)}_{1,\alpha^{-1}} h_{(1)(1)} \otimes R^{(2)}_{1,\alpha^{-1}} h_{(1)(2)} = h_{(2)} \otimes h_{(1)(2)} \varphi_{\beta^{-1}}(R^{(1)}_{1,\alpha^{-1}}) \otimes h_{(1)(1)} \varphi_{\beta^{-1}}(R^{(2)}_{1,\alpha^{-1}}),$$

i.e.,

$$h_{(2)} \otimes R^{(2)}_{1,\alpha^{-1}} h_{(1)(2)} \otimes R^{(1)}_{1,\alpha^{-1}} h_{(1)(1)} = h_{(2)} \otimes h_{(1)(1)} R^{(2)}_{1,\beta^{-1}\alpha^{-1}\beta} \otimes h_{(1)(2)} R^{(1)}_{1,\beta^{-1}\alpha^{-1}\beta}.$$

Thus

$$\begin{aligned} &S_{\beta^{-1}} \circ S_\beta\left(h_{(2)}\right) S_{\alpha^{-1}\beta}\left(R^{(2)}_{1,\alpha^{-1}} h_{(1)(2)}\right) R^{(1)}_{1,\alpha^{-1}} h_{(1)(1)} \\ &= S_{\beta^{-1}} \circ S_\beta\left(h_{(2)}\right) S_{\alpha^{-1}\beta}\left(h_{(1)(1)} R^{(2)}_{1,\beta^{-1}\alpha^{-1}\beta}\right) h_{(1)(2)} R^{(1)}_{1,\beta^{-1}\alpha^{-1}\beta}. \end{aligned}$$

Using that S is antimultiplicative we have

$$\begin{aligned} &S_{\beta^{-1}} \circ S_\beta\left(h_{(2)}\right) S_\beta\left(h_{(1)(2)}\right) S_{\alpha^{-1}}\left(R^{(2)}_{1,\alpha^{-1}}\right) R^{(1)}_{1,\alpha^{-1}} h_{(1)(1)} \\ =&S_{\beta^{-1}} \circ S_\beta\left(h_{(2)}\right) S_{\beta^{-1}\alpha^{-1}\beta}\left(R^{(2)}_{1,\beta^{-1}\alpha^{-1}\beta}\right) S_\beta\left(h_{(1)(1)}\right) h_{(1)(2)} R^{(1)}_{1,\beta^{-1}\alpha^{-1}\beta} \end{aligned}$$

i.e.,

$$S_1\left(h_{(1)(2)} S_\beta\left(h_{(2)}\right)\right) u_\alpha h_{(1)(1)} = S_{\beta^{-1}} \circ S_\beta(h) u_{\beta^{-1}\alpha\beta}.$$

Following the axiom (16) of coquasigroup Hopf π-algebra, we have

$$u_\alpha h = S_{\beta^{-1}} \circ S_\beta(h) u_{\beta^{-1}\alpha\beta}, \text{ for all } h \in H_\beta.$$

It follows that

$$S_{\beta^{-1}} \circ S_\beta(h) = u_\alpha h u^{-1}_{\beta^{-1}\alpha\beta} = u_\alpha h \left(\varphi_{\beta^{-1}}(u_\alpha)\right)^{-1} = u_\alpha h \varphi_{\beta^{-1}}(u_\alpha^{-1}), \text{ for all } h \in H_\beta.$$

Applying S_β to this expression and replacing h by $S_\beta^{-1}(h)$ yields the following calculation:

$$\begin{aligned}S_\beta \circ S_{\beta^{-1}}(h) &= \left(S_{\beta^{-1}\alpha\beta}(u_{\beta^{-1}\alpha\beta})\right)^{-1} h S_\alpha(u_\alpha) = \left(S_{\beta^{-1}\alpha\beta}(\varphi_{\beta^{-1}}(u_\alpha))\right)^{-1} h S_\alpha(u_\alpha) \\ &= \left(\varphi_{\beta^{-1}}(S_\alpha(u_\alpha))\right)^{-1} h S_\alpha(u_\alpha) = \varphi_{\beta^{-1}}\left((S_\alpha(u_\alpha))^{-1}\right) h S_\alpha(u_\alpha),\end{aligned}$$

or equivalently, $S_{\beta^{-1}} \circ S_\beta(h) = \varphi_\beta\big((S_\alpha(u_\alpha))^{-1}\big) h S_\alpha(u_\alpha)$. □

Corollary 10. *For any $\alpha \in \pi$, $g\varphi_{\beta^{-1}}(u_\alpha)S_\alpha(u_\alpha) = u_\alpha\varphi_\beta(S_\alpha(u_\alpha))g$ for all $g \in H_\beta$. In particular, $u_1 S_1(u_1)$ is a central element of H.*

Proof. Let $h = \varphi_\beta(S_\alpha(u_\alpha))gS_{\alpha^{-1}}(u_\alpha^{-1})$, then

$$S_{\beta^{-1}} \circ S_\beta(h) = u_\alpha h \varphi_{\beta^{-1}}\left(u_\alpha^{-1}\right) = u_\alpha \varphi_\beta(S_\alpha(u_\alpha)) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right)\varphi_{\beta^{-1}}\left(u_\alpha^{-1}\right)$$

and

$$\begin{aligned}S_{\beta^{-1}} \circ S_\beta(h) =& \varphi_\beta\left((S_\alpha(u_\alpha))^{-1}\right) h S_\alpha(u_\alpha) \\ =& \varphi_\beta\left((S_\alpha(u_\alpha))^{-1}\right)\varphi_\beta(S_\alpha(u_\alpha)) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right) S_\alpha(u_\alpha) \\ =& \varphi_\beta\left((S_\alpha(u_\alpha))^{-1} S_\alpha(u_\alpha)\right) g S_1\left(u_\alpha u_\alpha^{-1}\right) = \varphi_\beta(1) g S_1(1) = g.\end{aligned}$$

So

$$g = u_\alpha \varphi_\beta(S_\alpha(u_\alpha)) g S_{\alpha^{-1}}\left(u_\alpha^{-1}\right)\varphi_{\beta^{-1}}\left(u_\alpha^{-1}\right)$$

i.e., $g\varphi_{\beta^{-1}}(u_\alpha)S_\alpha(u_\alpha) = u_\alpha\varphi_\beta(S_\alpha(u_\alpha))g$ for all $g \in H_\beta$. □

It is well-known that the two equivalent conditions for a Hopf coquasigroup to be almost cocommutative have been obtained in [17]. Next in a similar way we will prove one equivalent condition for a Hopf non-coassociative π-algebra to be almost cocommutative.

Set $\overline{H}_\alpha = H_{\alpha^{-1}}$, $\overline{m}_{\alpha,\beta} = m^{op}_{\alpha^{-1},\beta^{-1}} = m_{\beta^{-1},\alpha^{-1}} \circ \sigma_{H_{\alpha^{-1}},H_{\beta^{-1}}}$, $\overline{\Delta}_\alpha = \Delta^{cop}_{\alpha^{-1}}$, $\overline{\varepsilon}_\alpha = \varepsilon_{\alpha^{-1}}$ and $\overline{S}_\alpha = S_{\alpha^{-1}}$. Recall from the statement (3) in Example 2 that

$$\overline{H} = (\{\overline{H}_\alpha\}_{\alpha\in\pi}, \overline{m} = \{\overline{m}_{\alpha,\beta}\}_{\alpha,\beta\in\pi}, 1, \overline{\Delta} = \{\overline{\Delta}_\alpha\}_{\alpha\in\pi}, \overline{\varepsilon}_\alpha, \overline{S} = \{\overline{S}_\alpha\}_{\alpha\in\pi})$$

is again a Hopf non-coassociative π-algebra where we write $\overline{m}_{\alpha,\beta}(a \otimes b) = a \cdot b = ba$.

We can now define π-module actions of $\overline{H} = \{\overline{H}_\alpha\}_{\alpha\in\pi} = \{H_{\alpha^{-1}}\}_{\alpha\in\pi}$ on H^* by

$$(h \rightharpoonup p)(g) = p(g \cdot h) \quad \text{and} \quad (q \leftharpoonup h)(g) = q(h \cdot g)$$

for all $g \in \overline{H}_{\alpha^{-1}} = H_\alpha$, $h \in \overline{H}_{\beta^{-1}} = H_\beta$ and $p \in H^*_{\beta\alpha}$, $q \in H^*_{\alpha\beta}$.

Fix $\gamma \in \pi$, and define π-module actions of H on $\{\mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_{\alpha\gamma})\}_{\alpha\in\pi}$ by

$$(h \xrightarrow{\gamma} f)(p) = f\left(h_{(1)} \rightharpoonup p\right) \cdot h_{(2)} \text{ and } (f \xleftarrow{\gamma} h)(q) = \varphi_{\gamma^{-1}}\left(h_{(1)} \cdot \varphi_\gamma \circ f\left(q \leftharpoonup h_{(2)}\right)\right)$$

for all $h \in H_\beta$, $p \in H^*_{\beta\alpha}$, $q \in H^*_{\alpha\beta}$ and $f \in \mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_{\alpha\gamma})$.

It is easy to check that

$$h \rightharpoonup (p \leftharpoonup g) = (h \rightharpoonup p) \leftharpoonup g$$

whereby

$$h \xrightarrow{\gamma} (f \xleftarrow{\gamma} g) = (h \xrightarrow{\gamma} f) \xleftarrow{\gamma} g$$

for all $p \in H^*_\alpha$ and $f \in \mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_{\alpha\gamma})$.

Therefore, we can define

$$h \xrightharpoonup{\gamma} f = h_{(1)} \xrightarrow{\gamma} f \xleftarrow{\gamma} S_\beta\Big(h_{(2)}\Big) \in \mathrm{Hom}_{\Bbbk}\Big(H^*_{\beta\alpha\beta^{-1}}, H_{\beta\alpha\beta^{-1}\gamma}\Big)$$

for all $h \in H_\beta$ and $f \in \mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_{\alpha\gamma})$. It is obvious that

$$g \xrightharpoonup{\gamma} (h \xrightharpoonup{\gamma} f) = (gh) \xrightharpoonup{\gamma} f \text{ and } 1 \xrightharpoonup{\gamma} f = f.$$

Next we will prove that there is a close relationship between the π-module actions $\xrightarrow{\gamma}$ and $\xleftarrow{\gamma}$ of H on $\{\mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_{\alpha\gamma})\}_{\alpha\in\pi}$.

Lemma 14. *We have $h \xrightarrow{\gamma} f = \Big(h_{(1)} \xrightharpoonup{\gamma} f\Big) \xleftarrow{\gamma} h_{(2)}$, for all $h \in H_\beta$ and $f \in \mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_{\alpha\gamma})$.*

Proof. Let $h \in H_\beta$ and $f \in \mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_{\alpha\gamma})$, then

$$\begin{aligned}\Big(h_{(1)} \xrightharpoonup{\gamma} f\Big) \xleftarrow{\gamma} h_{(2)} &= \Big(h_{(1)(1)} \xrightarrow{\gamma} f \xleftarrow{\gamma} S_\beta\Big(h_{(1)(2)}\Big)\Big) \xleftarrow{\gamma} h_{(2)} \\ &= \Big(h_{(1)(1)} \xrightarrow{\gamma} f\Big) \xleftarrow{\gamma} \Big(S_\beta\Big(h_{(1)(2)}\Big)h_{(2)}\Big) = h \xrightarrow{\gamma} f \xleftarrow{\gamma} 1 = h \xrightarrow{\gamma} f.\end{aligned}$$

The third equality follows from the axioms of a coquasigroup Hopf π-algebra. □

Now, we give an equivalent condition for a Hopf non-coassociative π-algebra to be almost cocommutative, provided the family R is invariant under the crossing.

Proposition 5. *Let H be a Hopf non-coassociative π-algebra and $R = \{R_{\alpha,\beta} = R^{(1)}_{\alpha,\beta} \otimes R^{(2)}_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$. Define $f \in \mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_{\alpha\gamma})$ by $f(p) = p\Big(R^{(2)}_{\gamma\alpha,\alpha}\Big)\varphi_{\gamma^{-1}}\Big(R^{(1)}_{\gamma\alpha,\alpha}\Big)$, $\forall p \in H^*_\alpha$. Give $\{\mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_\beta)\}_{\alpha\in\pi}$ the π-module structures over H described above. Then the following are equivalent:*

(1) for all $\alpha, \beta, \gamma \in \pi$ and $h \in H_\gamma$, we have

$$h_{(2)}\varphi_{\gamma^{-1}}\Big(R^{(1)}_{\alpha,\beta}\Big) \otimes h_{(1)}\varphi_{\gamma^{-1}}\Big(R^{(2)}_{\alpha,\beta}\Big) = R^{(1)}_{\alpha,\beta}h_{(1)} \otimes R^{(2)}_{\alpha,\beta}h_{(2)};$$

(2) for all $h \in H_\beta$ and $f \in \mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_{\alpha\gamma})$, we have

$$(f \xleftarrow{\gamma} h) = \varphi_{\gamma^{-1}} \circ (h \underset{\gamma}{\rightarrow} \widetilde{f}),$$

where $\underset{\gamma}{\rightarrow}$ is formally similar to the π-module action $\xrightarrow{\gamma}$ and $\widetilde{f}$ is an associated function defined by $\widetilde{f}(p) = p\Big(\varphi_{\beta^{-1}}\Big(R^{(2)}_{\gamma\alpha,\alpha}\Big)\Big)\varphi_{\beta^{-1}}\Big(R^{(1)}_{\gamma\alpha,\alpha}\Big)$, for any $p \in H^*_{\beta^{-1}\alpha\beta}$.

Proof. (1) ⇒ (2) For all $h \in H_\beta$, $p \in H^*_{\alpha\beta}$ and $f \in \mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_{\alpha\gamma})$,

$$\begin{aligned}(f \xleftarrow{\gamma} h)(p) &= \varphi_{\gamma^{-1}}\Big(h_{(1)} \cdot \varphi_\gamma \circ f\Big(p \leftharpoonup h_{(2)}\Big)\Big) \\ &= \varphi_{\gamma^{-1}}\Big(h_{(1)} \cdot \varphi_\gamma\Big(\Big(p \leftharpoonup h_{(2)}\Big)\Big(R^{(2)}_{\gamma\alpha,\alpha}\Big)\varphi_{\gamma^{-1}}\Big(R^{(1)}_{\gamma\alpha,\alpha}\Big)\Big)\Big) \\ &= \varphi_{\gamma^{-1}}\Big(h_{(1)} \cdot \varphi_\gamma\Big(p\Big(h_{(2)} \cdot R^{(2)}_{\gamma\alpha,\alpha}\Big)\varphi_{\gamma^{-1}}\Big(R^{(1)}_{\gamma\alpha,\alpha}\Big)\Big)\Big) \\ &= p\Big(h_{(2)} \cdot R^{(2)}_{\gamma\alpha,\alpha}\Big)\varphi_{\gamma^{-1}}\Big(h_{(1)} \cdot \varphi_\gamma\Big(\varphi_{\gamma^{-1}}\Big(R^{(1)}_{\gamma\alpha,\alpha}\Big)\Big)\Big) \\ &= p\Big(h_{(2)} \cdot R^{(2)}_{\gamma\alpha,\alpha}\Big)\varphi_{\gamma^{-1}}\Big(h_{(1)} \cdot R^{(1)}_{\gamma\alpha,\alpha}\Big) \\ &= p\Big(R^{(2)}_{\gamma\alpha,\alpha}h_{(2)}\Big)\varphi_{\gamma^{-1}}\Big(R^{(1)}_{\gamma\alpha,\alpha}h_{(1)}\Big).\end{aligned}$$

Since $h_{(2)}\varphi_{\beta^{-1}}\left(R^{(1)}_{\gamma\alpha,\alpha}\right)\otimes h_{(1)}\varphi_{\beta^{-1}}\left(R^{(2)}_{\gamma\alpha,\alpha}\right) = R^{(1)}_{\gamma\alpha,\alpha}h_{(1)}\otimes R^{(2)}_{\gamma\alpha,\alpha}h_{(2)}$, thus

$$\begin{aligned}&p\left(R^{(2)}_{\gamma\alpha,\alpha}h_{(2)}\right)\varphi_{\gamma^{-1}}\left(R^{(1)}_{\gamma\alpha,\alpha}h_{(1)}\right)\\ =&p\left(h_{(1)}\varphi_{\beta^{-1}}\left(R^{(2)}_{\gamma\alpha,\alpha}\right)\right)\varphi_{\gamma^{-1}}\left(h_{(2)}\varphi_{\beta^{-1}}\left(R^{(1)}_{\gamma\alpha,\alpha}\right)\right).\end{aligned}$$

It follows that

$$\begin{aligned}(f\overset{\gamma}{\leftarrow}h)(p) &= p\left(h_{(1)}\varphi_{\beta^{-1}}\left(R^{(2)}_{\gamma\alpha,\alpha}\right)\right)\varphi_{\gamma^{-1}}\left(h_{(2)}\varphi_{\beta^{-1}}\left(R^{(1)}_{\gamma\alpha,\alpha}\right)\right)\\ &= p\left(\varphi_{\beta^{-1}}\left(R^{(2)}_{\gamma\alpha,\alpha}\right)\cdot h_{(1)}\right)\varphi_{\gamma^{-1}}\left(\varphi_{\beta^{-1}}\left(R^{(1)}_{\gamma\alpha,\alpha}\right)\cdot h_{(2)}\right)\\ &= \varphi_{\gamma^{-1}}\left(p\left(\varphi_{\beta^{-1}}\left(R^{(2)}_{\gamma\alpha,\alpha}\right)\cdot h_{(1)}\right)\varphi_{\beta^{-1}}\left(R^{(1)}_{\gamma\alpha,\alpha}\right)\cdot h_{(2)}\right)\\ &= \varphi_{\gamma^{-1}}\left(\left(h_{(1)}\rightharpoonup p\right)\left(\varphi_{\beta^{-1}}\left(R^{(2)}_{\gamma\alpha,\alpha}\right)\right)\varphi_{\beta^{-1}}\left(R^{(1)}_{\gamma\alpha,\alpha}\right)\cdot h_{(2)}\right)\\ &= \varphi_{\gamma^{-1}}\left(\widetilde{f}\left(h_{(1)}\rightharpoonup p\right)\cdot h_{(2)}\right)\\ &= \varphi_{\gamma^{-1}}\circ(h\underset{\gamma}{\rightarrow}\widetilde{f})(p),\end{aligned}$$

for all $h\in H_\beta$, $p\in H^*_{\alpha\beta}$ and $\widetilde{f}\in \mathrm{Hom}_{\Bbbk}\left(H^*_{\beta^{-1}\alpha\beta}, H_{\beta^{-1}\gamma\alpha\beta}\right)$.

(2) $\Rightarrow$ (1) For all $h\in H_\beta$, $p\in H^*_{\alpha\beta}$ and $f\in \mathrm{Hom}_{\Bbbk}(H^*_\alpha, H_{\alpha\gamma})$, we have

$$(f\overset{\gamma}{\leftarrow}h)(p) = \varphi_{\gamma^{-1}}\circ(h\underset{\gamma}{\rightarrow}\widetilde{f})(p).$$

Thus

$$p\left(R^{(2)}_{\gamma\alpha,\alpha}h_{(2)}\right)\varphi_{\gamma^{-1}}\left(R^{(1)}_{\gamma\alpha,\alpha}h_{(1)}\right) = p\left(h_{(1)}\varphi_{\beta^{-1}}\left(R^{(2)}_{\gamma\alpha,\alpha}\right)\right)\varphi_{\gamma^{-1}}\left(h_{(2)}\varphi_{\beta^{-1}}\left(R^{(1)}_{\gamma\alpha,\alpha}\right)\right),$$

i.e.,

$$(p\otimes\varphi_{\gamma^{-1}})\left(R^{(2)}_{\gamma\alpha,\alpha}h_{(2)}\otimes R^{(1)}_{\gamma\alpha,\alpha}h_{(1)}\right) = (p\otimes\varphi_{\gamma^{-1}})\left(h_{(1)}\varphi_{\beta^{-1}}\left(R^{(2)}_{\gamma\alpha,\alpha}\right)\otimes h_{(2)}\varphi_{\beta^{-1}}\left(R^{(1)}_{\gamma\alpha,\alpha}\right)\right).$$

Then we have

$$R^{(2)}_{\gamma\alpha,\alpha}h_{(2)}\otimes R^{(1)}_{\gamma\alpha,\alpha}h_{(1)} = h_{(1)}\varphi_{\beta^{-1}}\left(R^{(2)}_{\gamma\alpha,\alpha}\right)\otimes h_{(2)}\varphi_{\beta^{-1}}\left(R^{(1)}_{\gamma\alpha,\alpha}\right)$$

This completes the proof. □

The following corollary is a direct conclusion.

Corollary 11. *Let H be an almost cocommutative Hopf non-coassociative π-algebra with an invertible antipode S. Then $h\overset{1}{\rightarrow} f = \varepsilon_1(h)f$, for all $h\in H_1$.*

7. Quasitriangular Hopf Non-Coassociative π-Algebras

In the current section, we will introduce and discuss the definition of a quasitriangular Hopf non-coassociative π-algebra and study its main properties. We construct a new Turaev's braided monoidal category $Rep_\pi(H)$ over a quasitriangular Hopf non-coassociative π-algebra H.

Definition 8. *A quasitriangular Hopf non-coassociative π-algebra is a crossed Hopf non-coassociative π-algebra (H, φ) with a family $\mathcal{R} = \{\mathcal{R}_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ of elements (the R-matrix) satisfying Equations (28) and (29) such that, for any $\alpha, \beta, \gamma \in \pi$,*

$$(id_{H_{\alpha\beta}} \otimes \Delta_\gamma)(\mathcal{R}_{\alpha\beta,\gamma}) = (\mathcal{R}_{\alpha,\gamma})_{13} \cdot (\mathcal{R}_{\beta,\gamma})_{12}, \tag{36}$$

$$(\Delta_\alpha \otimes id_{H_{\beta\gamma}})(\mathcal{R}_{\alpha,\beta\gamma}) = (\mathcal{R}_{\alpha,\beta})_{13} \cdot (\mathcal{R}_{\alpha,\gamma})_{23}, \tag{37}$$

$$(\varepsilon_\alpha \otimes \mathrm{id}_{H_1})\mathcal{R}_{\alpha,1} = 1, \tag{38}$$

$$(\mathrm{id}_{H_1} \otimes \varepsilon_\alpha)\mathcal{R}_{1,\alpha} = 1 \tag{39}$$

where, for $\Bbbk$-spaces P, Q and $r = \Sigma_j p_j \otimes q_j \in P \otimes Q$, we set $r_{12} = r \otimes 1 \in P \otimes Q \otimes H_1$, $r_{23} = 1 \otimes r \in H_1 \otimes P \otimes Q$ and $r_{13} = \Sigma_j p_j \otimes 1 \otimes q_j \in P \otimes H_1 \otimes Q$.

Note that $\mathcal{R}_{1,1}$ is a (classical) R-matrix for the Hopf coquasigroup H_1.

We find that a quasitriangular Hopf non-coassociative π-algebra also constructs a solution to the generalized quantum Yang–Baxter equation and a much stronger property of its antipode holds which are similar as a quasitriangular Hopf coquasigroup in [23].

Example 3. *Let H be a quasitriangular Hopf non-coassociative π-algebra with R-matrix $R = \{R_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$.*

(1) We can consider the coopposite crossed Hopf non-coassociative π-algebra H^{op} to H . It is quasitriangular by setting $\mathcal{R}^{op}_{\alpha,\beta} = (S_\alpha \otimes id_{H_{\beta^{-1}}})(\mathcal{R}_{\alpha,\beta^{-1}})$.

(2) Consider again the coopposite crossed Hopf non-coassociative π-algebra H^{op} to H . It is quasitriangular by setting $\mathcal{R}^{op}_{\alpha,\beta} = \sigma_{\beta^{-1},\alpha^{-1}}(\mathcal{R}_{\beta^{-1},\alpha^{-1}})$.

Lemma 15. *If $(H, \mathcal{R})$ is quasitriangular, then the following additional properties hold:*

(1) $\left(1 \otimes (\varepsilon_\alpha \otimes \mathrm{id}_{H_\beta})\mathcal{R}_{\alpha,\beta}\right) \cdot \mathcal{R}_{\alpha,\gamma} = \mathcal{R}_{\alpha,\beta\gamma}$;

(2) $\mathcal{R}_{\alpha,\beta} \cdot \left(1 \otimes (\varepsilon_\alpha \otimes \mathrm{id}_{H_\gamma})\mathcal{R}_{\alpha,\gamma}\right) = \mathcal{R}_{\alpha,\beta\gamma}$;

(3) $((\mathrm{id}_{H_\alpha} \otimes \varepsilon_\gamma)\mathcal{R}_{\alpha,\gamma} \otimes 1) \cdot \mathcal{R}_{\beta,\gamma} = \mathcal{R}_{\alpha\beta,\gamma}$;

(4) $\mathcal{R}_{\alpha,\gamma} \cdot \left((\mathrm{id}_{H_\beta} \otimes \varepsilon_\gamma)\mathcal{R}_{\beta,\gamma} \otimes 1\right) = \mathcal{R}_{\alpha\beta,\gamma}$.

Proof. We only need to show part (1) since the proof of other parts is similar. Applying $\varepsilon_\alpha \otimes \mathrm{id}_{H_\alpha} \otimes \mathrm{id}_{H_{\beta\gamma}}$ to both sides of Equation (37), we obtain $\mathcal{R}_{\alpha,\beta\gamma} = (\varepsilon_\alpha \otimes \mathrm{id}_{H_\alpha} \otimes \mathrm{id}_{H_{\beta\gamma}})(\Delta_\alpha \otimes \mathrm{id}_{H_{\beta\gamma}})(\mathcal{R}_{\alpha,\beta\gamma}) = (\varepsilon_\alpha \otimes \mathrm{id}_{H_\alpha} \otimes \mathrm{id}_{H_{\beta\gamma}})\left((\mathcal{R}_{\alpha,\beta})_{13} \cdot (\mathcal{R}_{\alpha,\gamma})_{23}\right)$ whence $\left(1 \otimes (\varepsilon_\alpha \otimes \mathrm{id}_{H_\beta})\mathcal{R}_{\alpha,\beta}\right) \cdot \mathcal{R}_{\alpha,\gamma} = \mathcal{R}_{\alpha,\beta\gamma}$. □

Lemma 16. *Let $(H, \mathcal{R})$ be a quasitriangular Hopf non-coassociative π-algebra, and write $\mathcal{R}_{\alpha,\beta} = \mathcal{R}^{(1)}_{\alpha,\beta} \otimes \mathcal{R}^{(2)}_{\alpha,\beta}$. Then, for any $\alpha \in \pi$, $\mathcal{R}_{\alpha,1}$ is invertible. More precisely, we have $\mathcal{R}^{-1}_{\alpha,1} = (S_\alpha \otimes \mathrm{id}_{H_1})\mathcal{R}_{\alpha,1}$.*

Proof. Using Equation (38) and applying $(m_{\alpha^{-1},\alpha} \otimes \mathrm{id}_{H_1}) \circ (S_\alpha \otimes \mathrm{id}_{H_\alpha} \otimes \mathrm{id}_{H_1})$ and $(m_{\alpha,\alpha^{-1}} \otimes \mathrm{id}_{H_1}) \circ (\mathrm{id}_{H_\alpha} \otimes S_\alpha \otimes \mathrm{id}_{H_1})$ to both sides of Equation (37), we obtain

$$1 \otimes 1 = 1\varepsilon_\alpha\left(\mathcal{R}^{(1)}_{\alpha,1}\right) \otimes \mathcal{R}^{(2)}_{\alpha,1} = S_\alpha\left(\mathcal{R}^{(1)}_{\alpha,1}\right)\widehat{\mathcal{R}}^{(1)}_{\alpha,1} \otimes \mathcal{R}^{(2)}_{\alpha,1}\widehat{\mathcal{R}}^{(2)}_{\alpha,1}$$

and

$$1 \otimes 1 = 1\varepsilon_\alpha\left(\mathcal{R}^{(1)}_{\alpha,1}\right) \otimes \mathcal{R}^{(2)}_{\alpha,1} = \mathcal{R}^{(1)}_{\alpha,1}S_\alpha\left(\widehat{\mathcal{R}}^{(1)}_{\alpha,1}\right) \otimes \mathcal{R}^{(2)}_{\alpha,1}\widehat{\mathcal{R}}^{(2)}_{\alpha,1}$$

where $\widehat{\mathcal{R}}_{\alpha,1} = \mathcal{R}_{\alpha,1}$. Thus $\mathcal{R}_{\alpha,1}$ and $(S_\alpha \otimes \mathrm{id}_{H_1})\mathcal{R}_{\alpha,1}$ are inverses. □

Theorem 10. *Let $(H,\mathcal{R})$ be a quasitriangular Hopf non-coassociative π-algebra, and write $\mathcal{R}_{\alpha,\beta} = \mathcal{R}^{(1)}_{\alpha,\beta} \otimes \mathcal{R}^{(2)}_{\alpha,\beta}$. Then $\mathcal{R}$ is invertible and $\mathcal{R}^{-1}_{\alpha,\beta^{-1}} = (S_\alpha \otimes \mathrm{id}_{H_\beta})\mathcal{R}_{\alpha,\beta}$.*

Proof. Applying $(m_{\alpha^{-1},\alpha} \otimes \mathrm{id}_{H_\beta}) \circ (S_\alpha \otimes \mathrm{id}_{H_\alpha} \otimes \mathrm{id}_{H_\beta})$ to both sides of Equation (37) yields:

$$1 \otimes \varepsilon_\alpha(\mathcal{R}^{(1)}_{\alpha,\beta})\mathcal{R}^{(2)}_{\alpha,\beta} = S_\alpha\left(\mathcal{R}^{(1)}_{\alpha,\beta}\right)\mathcal{R}^{(1)}_{\alpha,1} \otimes \mathcal{R}^{(2)}_{\alpha,\beta}\mathcal{R}^{(2)}_{\alpha,1},$$

or equivalently,

$$1 \otimes (\varepsilon_\alpha \otimes \mathrm{id}_{H_\beta})\mathcal{R}_{\alpha,\beta} = (S_\alpha \otimes \mathrm{id}_{H_\beta})(\mathcal{R}_{\alpha,\beta}) \cdot \mathcal{R}_{\alpha,1}.$$

Multiplying both sides on the left by $\mathcal{R}_{\alpha,\beta^{-1}}$, by using Lemma 15, we obtain

$$\mathcal{R}_{\alpha,1} = \mathcal{R}_{\alpha,\beta^{-1}} \cdot (S_\alpha \otimes \mathrm{id}_{H_\beta})(\mathcal{R}_{\alpha,\beta}) \cdot \mathcal{R}_{\alpha,1}.$$

Hence, $\mathcal{R}_{\alpha,\beta^{-1}} \cdot (S_\alpha \otimes \mathrm{id}_{H_\beta})(\mathcal{R}_{\alpha,\beta}) = 1$ follows by the invertiblity of $\mathcal{R}_{\alpha,1}$.

Applying $(m_{\alpha,\alpha^{-1}} \otimes \mathrm{id}_{H_\beta}) \circ (\mathrm{id}_{H_\alpha} \otimes S_\alpha \otimes \mathrm{id}_{H_\beta})$ to both sides of Equation (37) yields:

$$1 \otimes \varepsilon_\alpha(\mathcal{R}^{(1)}_{\alpha,\beta})\mathcal{R}^{(2)}_{\alpha,\beta} = \mathcal{R}^{(1)}_{\alpha,1}S_\alpha\left(\mathcal{R}^{(1)}_{\alpha,\beta}\right) \otimes \mathcal{R}^{(2)}_{\alpha,1}\mathcal{R}^{(2)}_{\alpha,\beta},$$

or equivalently,

$$1 \otimes (\varepsilon_\alpha \otimes \mathrm{id}_{H_\beta})\mathcal{R}_{\alpha,\beta} = \mathcal{R}_{\alpha,1} \cdot (S_\alpha \otimes \mathrm{id}_{H_\beta})(\mathcal{R}_{\alpha,\beta}).$$

Multiplying both sides on the right by $\mathcal{R}_{\alpha,\beta^{-1}}$, by using Lemma 15, we obtain

$$\mathcal{R}_{\alpha,1} = \mathcal{R}_{\alpha,1} \cdot (S_\alpha \otimes \mathrm{id}_{H_\beta})(\mathcal{R}_{\alpha,\beta}) \cdot \mathcal{R}_{\alpha,\beta^{-1}}.$$

Hence, $(S_\alpha \otimes \mathrm{id}_{H_\beta})(\mathcal{R}_{\alpha,\beta}) \cdot \mathcal{R}_{\alpha,\beta^{-1}} = 1$ follows by the invertiblity of $\mathcal{R}_{\alpha,1}$.

Therefore $\mathcal{R}_{\alpha,\beta^{-1}}$ is invertible and $\mathcal{R}^{-1}_{\alpha,\beta^{-1}} = (S_\alpha \otimes \mathrm{id}_{H_\beta})(\mathcal{R}_{\alpha,\beta})$. □

Theorem 11. *If $(H,\mathcal{R})$ is quasitriangular, then the following additional properties hold:*

(1) $(S_\beta \otimes S_\gamma)\mathcal{R}_{\beta,\gamma} = \mathcal{R}_{\beta^{-1},\gamma^{-1}}$;

(2) $\mathcal{R}_{1,\alpha} = \left(\mathrm{id}_{H_1} \otimes S_{\alpha^{-1}}\right)\mathcal{R}^{-1}_{1,\alpha}$;

(3) $\mathcal{R}_{\alpha^{-1},\beta} = \left(\mathrm{id}_{H_{\alpha^{-1}}} \otimes S_{\beta^{-1}}\right)\mathcal{R}^{-1}_{\alpha,\beta}$;

(4) *$\mathcal{R}$ satisfies the generalized quantum Yang–Baxter equation:*

$$(\mathcal{R}_{\delta,\lambda})_{12}(\mathcal{R}_{\alpha,\beta})_{13}(\mathcal{R}_{\alpha,\gamma})_{23} = (\mathcal{R}_{\alpha,\beta})_{23}(\mathcal{R}_{\alpha,\gamma})_{13}((\varphi_{\alpha^{-1}} \otimes \varphi_{\alpha^{-1}})\mathcal{R}_{\delta,\lambda})_{12}.$$

Proof. We first establish part (1). Using Lemma 15 and Theorem 10, we apply $(S_{\alpha\beta} \otimes \mathrm{id}_{H_1}) \circ (\mathrm{id}_{H_{\alpha\beta}} \otimes m_{\gamma^{-1},\gamma}) \circ (\mathrm{id}_{H_{\alpha\beta}} \otimes S_\gamma \otimes \mathrm{id}_{H_\gamma})$ to both sides of Equation (36) to obtain

$$\begin{aligned}
&(S_{\alpha\beta} \otimes S_1)\left(\mathcal{R}_{\alpha,\gamma} \cdot \mathcal{R}^{-1}_{\beta^{-1},\gamma}\right) = (S_{\alpha\beta} \otimes S_1)\left(\mathcal{R}^{(1)}_{\alpha\beta,\gamma}\varepsilon_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma}\right) \otimes 1\right)\\
=&S_{\alpha\beta}\left(\mathcal{R}^{(1)}_{\alpha\beta,\gamma}\varepsilon_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma}\right)\right) \otimes 1 = S_{\alpha\beta}\left(\mathcal{R}^{(1)}_{\alpha\beta,\gamma}\right) \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma(1)}\right)\mathcal{R}^{(2)}_{\alpha\beta,\gamma(2)}\\
=&S_\beta\left(\mathcal{R}^{(1)}_{\beta,\gamma}\right)S_\alpha\left(\mathcal{R}^{(1)}_{\alpha,\gamma}\right) \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\beta,\gamma}\right)\mathcal{R}^{(2)}_{\alpha,\gamma} = (S_\beta \otimes S_\gamma)\mathcal{R}_{\beta,\gamma} \cdot \left(S_\alpha \otimes \mathrm{id}_{H_\gamma}\right)\mathcal{R}_{\alpha,\gamma}\\
=&(S_\beta \otimes S_\gamma)\mathcal{R}_{\beta,\gamma} \cdot \mathcal{R}^{-1}_{\alpha,\gamma^{-1}},
\end{aligned}$$

i.e., $(S_\beta \otimes S_\gamma)\mathcal{R}_{\beta,\gamma} = (S_{\alpha\beta} \otimes S_1)\left(\mathcal{R}_{\alpha,\gamma} \cdot \mathcal{R}^{-1}_{\beta^{-1},\gamma}\right) \cdot \mathcal{R}_{\alpha,\gamma^{-1}}$. Thus part (1) follows by setting $\alpha = \beta^{-1}$.

Parts (2) and (3) follow directly from part (1) and Theorem 10.

To show part (4), we use Equation (37) to calculate

$$
\begin{aligned}
&(\mathcal{R}_{\delta,\lambda})_{12}(\mathcal{R}_{\alpha,\beta})_{13}(\mathcal{R}_{\alpha,\gamma})_{23} = (\mathcal{R}_{\delta,\lambda})_{12}(\Delta_\alpha \otimes id_{H_{\beta\gamma}})(\mathcal{R}_{\alpha,\beta\gamma}) \\
=&\mathcal{R}_{\delta,\lambda}\Delta_\alpha(\mathcal{R}^{(1)}_{\alpha,\beta\gamma}) \otimes \mathcal{R}^{(2)}_{\alpha,\beta\gamma} = \Delta^{cop}_\alpha(\mathcal{R}^{(1)}_{\alpha,\beta\gamma})(\varphi_{\alpha^{-1}} \otimes \varphi_{\alpha^{-1}})\mathcal{R}_{\delta,\lambda} \otimes \mathcal{R}^{(2)}_{\alpha,\beta\gamma} \\
=&(\mathcal{R}_{\alpha,\beta})_{23}(\mathcal{R}_{\alpha,\gamma})_{13}((\varphi_{\alpha^{-1}} \otimes \varphi_{\alpha^{-1}})\mathcal{R}_{\delta,\lambda})_{12}.
\end{aligned}
$$

Thus $\mathcal{R}$ satisfies the generalized quantum Yang–Baxter equation. □

Proposition 6. *Let $(H, \mathcal{R})$ be a quasitriangular Hopf non-coassociative π-algebra. For any $\alpha \in \pi$, set $u_\alpha = S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\right)\mathcal{R}^{(1)}_{1,\alpha^{-1}}$. Then u_α is invertible, $u_\alpha^{-1} = \mathcal{R}^{(2)}_{1,\alpha^{-1}} S_1 \circ S_1\left(\mathcal{R}^{(1)}_{1,\alpha^{-1}}\right)$, $S_{\beta^{-1}} \circ S_\beta(h) = u_\alpha h u^{-1}_{\beta^{-1}\alpha\beta} = u_\alpha h\left(\varphi_{\beta^{-1}}(u_\alpha)\right)^{-1} = u_\alpha h\left(\varphi_{\beta^{-1}}(u_\alpha^{-1})\right)$, $\varphi_\beta(u_\alpha) = u_{\beta\alpha\beta^{-1}}$ and $\varphi_\beta(u_\alpha^{-1}) = u^{-1}_{\beta\alpha\beta^{-1}}$, for all $h \in H_\beta$.*

Proof. The calculations in the proof of Corollary 9 and Proposition 4 showed that $u_\alpha h = S_{\beta^{-1}} \circ S_\beta(h) u_{\beta^{-1}\alpha\beta}$ holds and $S_\alpha\left(U^{(2)}_{1,\alpha}\right) S_1 \circ S_1\left(U^{(1)}_{1,\alpha}\right) u_\alpha = 1$ where $\mathcal{R}^{-1}_{1,\alpha^{-1}} = U^{(1)}_{1,\alpha} \otimes U^{(2)}_{1,\alpha}$ as well as $\varphi_\beta(u_\alpha) = u_{\beta\alpha\beta^{-1}}$. Let $v_{\alpha^{-1}} = S_\alpha\left(U^{(2)}_{1,\alpha}\right) S_1 \circ S_1\left(U^{(1)}_{1,\alpha}\right)$. Then $v_{\alpha^{-1}} u_\alpha = 1$ and $v_{\alpha^{-1}} = S_\alpha\left(U^{(2)}_{1,\alpha}\right) S_1 \circ S_1\left(U^{(1)}_{1,\alpha}\right) = S_\alpha\left(\mathcal{R}^{(2)}_{1,\alpha}\right) S_1 \circ S_1 \circ S_1\left(\mathcal{R}^{(1)}_{1,\alpha}\right) = \mathcal{R}^{(2)}_{1,\alpha^{-1}} S_1 \circ S_1\left(\mathcal{R}^{(1)}_{1,\alpha^{-1}}\right)$ by Theorems 10 and 11. Let $h_\alpha = S_1\left(U^{(1)}_{1,\alpha}\right) U^{(2)}_{1,\alpha}$. Then $v_{\alpha^{-1}} = S_\alpha(h_\alpha)$. Now $v_{\alpha^{-1}} = S_\alpha \circ S_{\alpha^{-1}}(v_{\alpha^{-1}})$ by Theorem 11. Therefore, $v_{\alpha^{-1}} = S_\alpha \circ S_{\alpha^{-1}} \circ S_\alpha(h_\alpha)$. Since

$$
\begin{aligned}
S_\alpha \circ S_{\alpha^{-1}} \circ S_\alpha(h_\alpha) u_\alpha S_\alpha(u_\alpha) &= u_\alpha S_\alpha(h_\alpha) S_\alpha(u_\alpha) = u_\alpha S_{\alpha^2}(u_\alpha h_\alpha) \\
&= u_\alpha S_{\alpha^2}(S_{\alpha^{-1}} \circ S_\alpha(h_\alpha) u_\alpha) = u_\alpha S_\alpha(u_\alpha) S_\alpha \circ S_{\alpha^{-1}} \circ S_\alpha(h_\alpha),
\end{aligned}
$$

it follows that $v_{\alpha^{-1}}$ and $u_\alpha S(u_\alpha)$ commute. Consequently

$$
\begin{aligned}
u_\alpha(S_\alpha(u_\alpha) v_{\alpha^{-1}} S_{\alpha^{-1}}(v_{\alpha^{-1}})) &= ((u_\alpha S_\alpha(u_\alpha)) v_{\alpha^{-1}}) S_{\alpha^{-1}}(v_{\alpha^{-1}}) \\
&= (v_{\alpha^{-1}}(u_\alpha S_\alpha(u_\alpha))) S_{\alpha^{-1}}(v_{\alpha^{-1}}) = (v_{\alpha^{-1}} u_\alpha)(S_\alpha(u_\alpha) S_{\alpha^{-1}}(v_{\alpha^{-1}})) \\
&= (v_{\alpha^{-1}} u_\alpha) S_1(v_{\alpha^{-1}} u_\alpha) = 1 S_1(1) = 1.
\end{aligned}
$$

We have shown that u_α has a left inverse $v_{\alpha^{-1}}$ and also has a right inverse. Therefore u_α is invertible. As $u_\alpha h = S_{\beta^{-1}} \circ S_\beta(h) u_{\beta^{-1}\alpha\beta}$ and $\varphi_\beta(u_\alpha) = u_{\beta\alpha\beta^{-1}}$ hold, our proof is complete. □

Definition 9. *Let $(H, \mathcal{R})$ be a quasitriangular Hopf non-coassociative π-algebra over $\Bbbk$. The Drinfel'd element of $(H, \mathcal{R})$ is the element $u = \{u_\alpha\}_{\alpha \in \pi}$ of Proposition 6. The quantum Casimir element of H is the family $\{u_\alpha S_\alpha(u_\alpha)\}_{\alpha \in \pi}$ of products.*

Theorem 12. *Let $(H, \mathcal{R})$ be a quasitriangular Hopf non-coassociative π-algebra with the antipode S. Then S is bijective; thus H is almost cocommutative.*

Proof. We set $T_\alpha(h) = u_\alpha^{-1} S_{\alpha^{-1}}(h) u_\alpha$. Using Proposition 6 we have

$$
\begin{aligned}
T_\alpha(h_{(1)}) \cdot h_{(2)(1)} \otimes h_{(2)(2)} &= h_{(2)(1)} T_\alpha(h_{(1)}) \otimes h_{(2)(2)} = h_{(2)(1)} u_\alpha^{-1} S_{\alpha^{-1}}(h_{(1)}) u_\alpha \otimes h_{(2)(2)} \\
=& u_\alpha^{-1} S_\alpha \circ S_{\alpha^{-1}}(h_{(2)(1)}) S_{\alpha^{-1}}(h_{(1)}) u_\alpha \otimes h_{(2)(2)} \\
=& u_\alpha^{-1} S_1\left(h_{(1)} S_{\alpha^{-1}}(h_{(2)(1)})\right) u_\alpha \otimes h_{(2)(2)} = u_\alpha^{-1} S_1(1) u_\alpha \otimes h = u_\alpha^{-1} 1 u_\alpha \otimes h = 1 \otimes h,
\end{aligned}
$$

and similarly for $h_{(1)} \cdot T_\alpha(h_{(2)(1)}) \otimes h_{(2)(2)} = 1 \otimes h$, $h_{(1)(1)} \otimes T_\alpha(h_{(1)(2)}) \cdot h_{(2)} = h \otimes 1$, and $h_{(1)(1)} \otimes h_{(1)(2)} \cdot T_\alpha(h_{(2)}) = h \otimes 1$.

This means that $T = \{T_\alpha\}_{\alpha\in\pi}$ is an antipode on H^{op} and hence the inverse of the antipode S on H according to Proposition 2. $\square$

The following reconciles the original definition of quasitriangular Hopf non-coassociative π-algebra with the one given here.

Proposition 7. *Let H be a crossed Hopf non-coassociative π-algebra over $\Bbbk$ and $\mathcal{R} = \{\mathcal{R}_{\alpha,\beta}\}_{\alpha,\beta\in\pi}$. Then the following are equivalent:*

(a) $(H,\mathcal{R})$ *is quasitriangular.*

(b) *H is almost cocommutative, where $\mathcal{R}$ is invertible and satisfies Equations (36) and (37).*

Proof. Part (a) implies part (b) by definition and Theorem 12. Suppose that the hypothesis of part (b) holds. We only need to show that Equations (38) and (39) hold. Applying $\varepsilon_\alpha \otimes \mathrm{id}_{H_\alpha} \otimes \mathrm{id}_{H_1}$ to both sides of Equation (37), we obtain $\mathcal{R}_{\alpha,1} = (\varepsilon_\alpha \otimes \mathrm{id}_{H_\alpha} \otimes \mathrm{id}_{H_1})(\Delta_\alpha \otimes \mathrm{id}_{H_1})(\mathcal{R}_{\alpha,1}) = (\varepsilon_\alpha \otimes \mathrm{id}_{H_\alpha} \otimes \mathrm{id}_{H_1})((\mathcal{R}_{\alpha,1})_{13}\cdot(\mathcal{R}_{\alpha,1})_{23})$ whence $(\varepsilon_\alpha \otimes \mathrm{id}_{H_1})\mathcal{R}_{\alpha,1} = 1$ since $\mathcal{R}_{\alpha,1}$ is invertible. Similarly for $(\mathrm{id}_{H_1} \otimes \varepsilon_\alpha)\mathcal{R}_{1,\alpha} = 1$. $\square$

What the entire preceding discussion illustrates is the following equivalent characterization for a quasitriangular Hopf non-coassociative π-algebra:

Definition 10. *A quasitriangular Hopf non-coassociative π-algebra is a crossed Hopf non-coassociative π-algebra (H,φ) with a family $\mathcal{R} = \{\mathcal{R}_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ of invertible elements (the R-matrix) satisfying Equations (28), (29), (36) and (37).*

Corollary 12. *Let $(H,\mathcal{R})$ be a quasitriangular Hopf non-coassociative π-algebra with a bijective antipode S, then*

(1) $u_\alpha^{-1} = S_{\alpha^{-1}}^{-1} S_\alpha^{-1}\left(\mathcal{R}_{1,\alpha^{-1}}^{(2)}\right)\mathcal{R}_{1,\alpha^{-1}}^{(1)}$;

(2) $(S_{\alpha^{-1}} \circ S_\alpha)^2(h) = g_\alpha h g_\alpha^{-1}$ *for all* $h \in H_\alpha$, *where* $u_\alpha = S_{\alpha^{-1}}\left(\mathcal{R}_{1,\alpha^{-1}}^{(2)}\right)\mathcal{R}_{1,\alpha^{-1}}^{(1)}$ *and* $g_\alpha = u_\alpha(S_\alpha(u_\alpha))^{-1}$;

(3) $(S_\alpha \circ S_{\alpha^{-1}})^2(h) = g_\alpha h g_\alpha^{-1}$ *for all* $h \in H_{\alpha^{-1}}$, *where* $u_\alpha = S_{\alpha^{-1}}\left(\mathcal{R}_{1,\alpha^{-1}}^{(2)}\right)\mathcal{R}_{1,\alpha^{-1}}^{(1)}$ *and* $g_\alpha = u_\alpha(S_\alpha(u_\alpha))^{-1}$;

(4) $S_1^4(h) = g_\alpha h g_\alpha^{-1}$ *for all* $h \in H_1$, *where* $u_\alpha = S_{\alpha^{-1}}\left(\mathcal{R}_{1,\alpha^{-1}}^{(2)}\right)\mathcal{R}_{1,\alpha^{-1}}^{(1)}$ *and* $g_\alpha = u_\alpha(S_\alpha(u_\alpha))^{-1}$;

(5) $\varepsilon_{\alpha^2}(g_\alpha) = 1$ *and* $\varphi_\beta(g_\alpha) = g_{\beta\alpha\beta^{-1}}$.

Proof. Apropos of part (1). Write $\mathcal{R}_{1,\alpha^{-1}}^{-1} = U_{1,\alpha}^{(1)} \otimes U_{1,\alpha}^{(2)}$. Consider the calculation:

$$\begin{aligned}
u_\alpha S_{\alpha^{-1}}^{-1}\left(U_{1,\alpha}^{(2)}\right)U_{1,\alpha}^{(1)} &= S_\alpha \circ S_{\alpha^{-1}}\left(S_{\alpha^{-1}}^{-1}\left(U_{1,\alpha}^{(2)}\right)\right)u_\alpha U_{1,\alpha}^{(1)} \\
&= S_\alpha \circ S_{\alpha^{-1}}\left(S_{\alpha^{-1}}^{-1}\left(U_{1,\alpha}^{(2)}\right)\right)S_{\alpha^{-1}}\left(R_{1,\alpha^{-1}}^{(2)}\right)R_{1,\alpha^{-1}}^{(1)}U_{1,\alpha}^{(1)} \\
&= S_\alpha\left(U_{1,\alpha}^{(2)}\right)S_{\alpha^{-1}}\left(R_{1,\alpha^{-1}}^{(2)}\right)R_{1,\alpha^{-1}}^{(1)}U_{1,\alpha}^{(1)} \\
&= S_1\left(R_{1,\alpha^{-1}}^{(2)}U_{1,\alpha}^{(2)}\right)R_{1,\alpha^{-1}}^{(1)}U_{1,\alpha}^{(1)} = 1,
\end{aligned}$$

from which we obtain $u_\alpha^{-1} = S_{\alpha^{-1}}^{-1}\left(U_{1,\alpha}^{(2)}\right)U_{1,\alpha}^{(1)}$. We use Theorem 11 to obtain $\mathcal{R}_{1,\alpha^{-1}} = (\mathrm{id}_{H_1} \otimes S_\alpha)\mathcal{R}_{1,\alpha^{-1}}^{-1} = U_{1,\alpha}^{(1)} \otimes S_\alpha\left(U_{1,\alpha}^{(2)}\right)$, or equivalently $(\mathrm{id}_{H_1} \otimes S_\alpha^{-1})\mathcal{R}_{1,\alpha^{-1}} = U_{1,\alpha}^{(1)} \otimes U_{1,\alpha}^{(2)}$ by the bijectivity of S, thus leading to the formula:

$$u_\alpha^{-1} = S_{\alpha^{-1}}^{-1} S_\alpha^{-1}\left(\mathcal{R}_{1,\alpha^{-1}}^{(2)}\right)\mathcal{R}_{1,\alpha^{-1}}^{(1)}.$$

To establish part (2), observe from Proposition 4 that $u_\alpha \in H_\alpha$ is invertible and $S_{\alpha^{-1}} \circ S_\alpha(h) = u_\alpha h u_\alpha^{-1} = (S_\alpha(u_\alpha))^{-1} h S_\alpha(u_\alpha)$ for all $h \in H_\alpha$, then

$$\begin{aligned} S_{\alpha^{-1}} \circ S_\alpha \circ S_{\alpha^{-1}} \circ S_\alpha(h) =& S_{\alpha^{-1}} \circ S_\alpha\left(u_\alpha h u_\alpha^{-1}\right) \\ =& S_{\alpha^{-1}} \circ S_\alpha(u_\alpha) S_{\alpha^{-1}} \circ S_\alpha(h) S_\alpha \circ S_{\alpha^{-1}}\left(u_\alpha^{-1}\right) \\ =& u_\alpha (S_\alpha(u_\alpha))^{-1} h S_\alpha(u_\alpha) u_\alpha^{-1}, \end{aligned}$$

or equivalently $S_{\alpha^{-1}} \circ S_\alpha \circ S_{\alpha^{-1}} \circ S_\alpha(h) = g_\alpha h g_\alpha^{-1}$ for all $h \in H_\alpha$. Similarly for parts (3) and (4). Part (5) follows from the calculations below:

$$\begin{aligned} \varepsilon_{\alpha^2}(g_\alpha) &= \varepsilon_{\alpha^2}(u_\alpha (S_\alpha(u_\alpha))^{-1}) = \varepsilon_{\alpha^2}\left(u_\alpha S_{\alpha^{-1}}(u_\alpha^{-1})\right) \\ &= \varepsilon_\alpha(u_\alpha)\varepsilon_\alpha\left(S_{\alpha^{-1}}(u_\alpha^{-1})\right) = \varepsilon_\alpha(u_\alpha)\varepsilon_{\alpha^{-1}}(u_\alpha^{-1}) = \varepsilon_1(u_\alpha u_\alpha^{-1}) = \varepsilon_1(1) = 1_{\mathbb{k}} \end{aligned}$$

and

$$\begin{aligned} \varphi_\beta(g_\alpha) &= \varphi_\beta(u_\alpha (S_\alpha(u_\alpha))^{-1}) = \varphi_\beta(u_\alpha)\varphi_\beta((S_\alpha(u_\alpha))^{-1}) \\ &= \varphi_\beta(u_\alpha)\varphi_\beta\left(S_{\alpha^{-1}}(u_\alpha^{-1})\right) = \varphi_\beta(u_\alpha) S_{\beta\alpha^{-1}\beta^{-1}}\left(\varphi_\beta(u_\alpha^{-1})\right) \\ &= u_{\beta\alpha\beta^{-1}} S_{\beta\alpha^{-1}\beta^{-1}}\left(u_{\beta\alpha\beta^{-1}}^{-1}\right) = u_{\beta\alpha\beta^{-1}}\left(S_{\beta\alpha\beta^{-1}}\left(u_{\beta\alpha\beta^{-1}}\right)\right)^{-1} = g_{\beta\alpha\beta^{-1}}. \quad \square \end{aligned}$$

Corollary 13. *$S_{\beta^{-1}} \circ S_\beta \circ S_{\beta^{-1}} \circ S_\beta(h) = u_\alpha \varphi_\beta((S_\alpha(u_\alpha))^{-1}) h \left(\varphi_{\beta^{-1}}(u_\alpha)(S_\alpha(u_\alpha))^{-1}\right)^{-1}$ for all $\beta \in \pi$ and $h \in H_\beta$, where $u_\alpha = S_{\alpha^{-1}}\left(\mathcal{R}_{1,\alpha^{-1}}^{(2)}\right)\mathcal{R}_{1,\alpha^{-1}}^{(1)}$.*

Proof. Observe from Propositon 4 and Corollary 9 that $u_\alpha \in H_\alpha$ is invertible and $S_{\beta^{-1}} \circ S_\beta(h) = u_\alpha h \varphi_{\beta^{-1}}(u_\alpha^{-1}) = \varphi_\beta((S_\alpha(u_\alpha))^{-1}) h S_\alpha(u_\alpha)$ for all $h \in H_\beta$, then

$$\begin{aligned} S_{\beta^{-1}} \circ S_\beta \circ S_{\beta^{-1}} \circ S_\beta(h) &= S_{\beta^{-1}} \circ S_\beta\left(u_\alpha h \varphi_{\beta^{-1}}(u_\alpha^{-1})\right) \\ =& S_{\alpha^{-1}} \circ S_\alpha(u_\alpha) S_{\beta^{-1}} \circ S_\beta(h) S_{\beta^{-1}\alpha\beta} \circ S_{\beta^{-1}\alpha^{-1}\beta}\left(\varphi_{\beta^{-1}}(u_\alpha^{-1})\right) \\ =& S_{\alpha^{-1}} \circ S_\alpha(u_\alpha) S_{\beta^{-1}} \circ S_\beta(h) S_{\beta^{-1}\alpha\beta} \circ S_{\beta^{-1}\alpha^{-1}\beta}\left(u_{\beta^{-1}\alpha\beta}^{-1}\right) \\ =& u_\alpha \varphi_\beta\left((S_\alpha(u_\alpha))^{-1}\right) h S_\alpha(u_\alpha) u_{\beta^{-1}\alpha\beta}^{-1} \\ =& u_\alpha \varphi_\beta\left((S_\alpha(u_\alpha))^{-1}\right) h \left(u_{\beta^{-1}\alpha\beta}(S_\alpha(u_\alpha))^{-1}\right)^{-1} \\ =& u_\alpha \varphi_\beta\left((S_\alpha(u_\alpha))^{-1}\right) h \left(\varphi_{\beta^{-1}}(u_\alpha)(S_\alpha(u_\alpha))^{-1}\right)^{-1}, \end{aligned}$$

or equivalently $S_{\beta^{-1}} \circ S_\beta \circ S_{\beta^{-1}} \circ S_\beta(h) = u_\alpha \varphi_\beta((S_\alpha(u_\alpha))^{-1}) h \left(\varphi_{\beta^{-1}}(u_\alpha)(S_\alpha(u_\alpha))^{-1}\right)^{-1}$ for all $h \in H_\beta$. $\square$

Proposition 8. *Let $(H, \mathcal{R})$ be a quasitriangular Hopf non-coassociative π-algebra with antipode S over $\mathbb{k}$ and let $u = \{u_\alpha\}_{\alpha\in\pi}$ be the Drinfel'd element of $(H, \mathcal{R})$. If the second tensor factor of $\mathcal{R}_{1,1}$ is coassociative, then the following hold:*

(a) $\Delta_\alpha(u_\alpha) = \left(\sigma_{H_1,H_1}(\mathcal{R}_{1,1})\mathcal{R}_{1,\alpha^{-1}}\right)^{-1}(u_\alpha \otimes u_1) = (u_\alpha \otimes u_1)\left(\sigma_{H_1,H_1}(\mathcal{R}_{1,1})\mathcal{R}_{1,\alpha^{-1}}\right)^{-1}$ *and* $\varepsilon_\alpha(u_\alpha) = 1_{\mathbb{k}}$.
(b) $\Delta_{\alpha^{-1}} S_\alpha(u_\alpha) = (\sigma_{H_1,H_\alpha}(\mathcal{R}_{1,\alpha})\mathcal{R}_{1,1})^{-1}(S_1(u_1) \otimes S_\alpha(u_\alpha))$.
(c) $\Delta_\alpha S_{\alpha^{-1}}(u_\alpha^{-1}) = \sigma_{H_1,H_\alpha}(\mathcal{R}_{1,\alpha})\mathcal{R}_{1,1}(S_1(u_1^{-1}) \otimes S_{\alpha^{-1}}(u_\alpha^{-1}))$.
(d) $g_1 = u_1(S_1(u_1))^{-1}$ *is a group-like element of* H_1.

Proof. To show part (a), we write $\mathcal{R}_{\alpha,\beta} = \mathcal{R}^{(1)}_{\alpha,\beta} \otimes \mathcal{R}^{(2)}_{\alpha,\beta}$. Therefore $u_\alpha = S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\right)\mathcal{R}^{(1)}_{1,\alpha^{-1}}$.

Applying $\Delta_1 \otimes \mathrm{id}_{H_{\alpha^{-1}}} \otimes \mathrm{id}_{H_{\alpha^{-1}}}$ and $\mathrm{id}_{H_1} \otimes \mathrm{id}_{H_1} \otimes \Delta_{\alpha^{-1}}$ to both sides of Equation (36), respectively, we obtain

$$\begin{aligned}
&\mathcal{R}^{(1)}_{1,\alpha^{-1}(1)} \otimes \mathcal{R}^{(1)}_{1,\alpha^{-1}(2)} \otimes \mathcal{R}^{(2)}_{1,\alpha^{-1}(1)} \otimes \mathcal{R}^{(2)}_{1,\alpha^{-1}(2)} \\
&= \mathcal{R}^{(1)}_{1,\alpha^{-1}(1)}\widehat{\mathcal{R}}^{(1)}_{1,\alpha^{-1}(1)} \otimes \mathcal{R}^{(1)}_{1,\alpha^{-1}(2)}\widehat{\mathcal{R}}^{(1)}_{1,\alpha^{-1}(2)} \otimes \widehat{\mathcal{R}}^{(2)}_{1,\alpha^{-1}} \otimes \mathcal{R}^{(2)}_{1,\alpha^{-1}} \\
&= \mathcal{R}^{(1)}_{1,\alpha^{-1}}\widehat{\mathcal{R}}^{(1)}_{1,\alpha^{-1}} \otimes \mathcal{R}^{(1)}_{1,1}\widehat{\mathcal{R}}^{(1)}_{1,1} \otimes \widehat{\mathcal{R}}^{(2)}_{1,\alpha^{-1}}\widehat{\mathcal{R}}^{(2)}_{1,1} \otimes \mathcal{R}^{(2)}_{1,\alpha^{-1}}\mathcal{R}^{(2)}_{1,1}.
\end{aligned}$$

Using Proposition 6 and part (1) of Theoerm 11, we calculate

$$\begin{aligned}
\Delta_\alpha(u_\alpha) &= \Delta_\alpha(S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\right)\mathcal{R}^{(1)}_{1,\alpha^{-1}}) = \Delta_\alpha(S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\right))\Delta_1(\mathcal{R}^{(1)}_{1,\alpha^{-1}}) \\
&= (S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\right)_{(1)} \otimes S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\right)_{(2)})(\mathcal{R}^{(1)}_{1,\alpha^{-1}(1)} \otimes \mathcal{R}^{(1)}_{1,\alpha^{-1}(2)}) \\
&= (S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}(2)}\right) \otimes S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}(1)}\right))(\mathcal{R}^{(1)}_{1,\alpha^{-1}(1)} \otimes \mathcal{R}^{(1)}_{1,\alpha^{-1}(2)}) \\
&= S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}(2)}\right)\mathcal{R}^{(1)}_{1,\alpha^{-1}(1)} \otimes S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}(1)}\right)\mathcal{R}^{(1)}_{1,\alpha^{-1}(2)} \\
&= S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\mathcal{R}^{(2)}_{1,1}\right)\mathcal{R}^{(1)}_{1,\alpha^{-1}}\widehat{\mathcal{R}}^{(1)}_{1,\alpha^{-1}} \otimes S_{\alpha^{-1}}\left(\widehat{\mathcal{R}}^{(2)}_{1,\alpha^{-1}}\widehat{\mathcal{R}}^{(2)}_{1,1}\right)\mathcal{R}^{(1)}_{1,1}\widehat{\mathcal{R}}^{(1)}_{1,1} \\
&= S_1\left(\mathcal{R}^{(2)}_{1,1}\right)S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\right)\mathcal{R}^{(1)}_{1,\alpha^{-1}}\widehat{\mathcal{R}}^{(1)}_{1,\alpha^{-1}} \otimes S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1}\right)S_{\alpha^{-1}}\left(\widehat{\mathcal{R}}^{(2)}_{1,\alpha^{-1}}\right)\mathcal{R}^{(1)}_{1,1}\widehat{\mathcal{R}}^{(1)}_{1,1} \\
&= S_1\left(\mathcal{R}^{(2)}_{1,1}\right)u_\alpha\widehat{\mathcal{R}}^{(1)}_{1,\alpha^{-1}} \otimes S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1}\right)S_{\alpha^{-1}}\left(\widehat{\mathcal{R}}^{(2)}_{1,\alpha^{-1}}\right)\mathcal{R}^{(1)}_{1,1}\widehat{\mathcal{R}}^{(1)}_{1,1} \\
&= S_1\left(\mathcal{R}^{(2)}_{1,1}\right)S_1 \circ S_1\left(\widehat{\mathcal{R}}^{(1)}_{1,\alpha^{-1}}\right)u_\alpha \otimes S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1}\right)S_{\alpha^{-1}}\left(\widehat{\mathcal{R}}^{(2)}_{1,\alpha^{-1}}\right)\mathcal{R}^{(1)}_{1,1}\widehat{\mathcal{R}}^{(1)}_{1,1} \\
&= S_1\left(\mathcal{R}^{(2)}_{1,1}\right)S_1\left(\widehat{\mathcal{R}}^{(1)}_{1,\alpha}\right)u_\alpha \otimes S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1}\right)\widehat{\mathcal{R}}^{(2)}_{1,\alpha}\mathcal{R}^{(1)}_{1,1}\widehat{\mathcal{R}}^{(1)}_{1,1}
\end{aligned}$$

and thus

$$\Delta_\alpha(u_\alpha) = S_1\left(\widehat{\mathcal{R}}^{(1)}_{1,\alpha}\mathcal{R}^{(2)}_{1,1}\right)u_\alpha \otimes S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1}\right)\widehat{\mathcal{R}}^{(2)}_{1,\alpha}\mathcal{R}^{(1)}_{1,1}\widehat{\mathcal{R}}^{(1)}_{1,1}.$$

Since H is quasitriangular, S is bijective by Theorem 12. Write $\mathcal{R}_{\alpha,\beta} = \mathcal{R}^{(1)}_{\alpha,\beta} \otimes \mathcal{R}^{(2)}_{\alpha,\beta}$. By Equations (28) and (29), we have

$$\begin{aligned}
&h_{(2)} \otimes \Delta^{cop}_\gamma(h_{(1)})(\varphi_{\gamma^{-1}} \otimes \varphi_{\gamma^{-1}})(\mathcal{R}_{\alpha,\beta}) = h_{(2)} \otimes \mathcal{R}_{\alpha,\beta}\Delta_\gamma(h_{(1)}) \\
\Longrightarrow\ & S^{-1}_{\gamma^{-1}}(h_{(2)}) \otimes \Delta^{cop}_\gamma(h_{(1)})\mathcal{R}_{\gamma^{-1}\alpha\gamma,\gamma^{-1}\beta\gamma} = S^{-1}_{\gamma^{-1}}(h_{(2)}) \otimes \mathcal{R}_{\alpha,\beta}\Delta_\gamma(h_{(1)}) \\
\Longrightarrow\ & S^{-1}_{\gamma^{-1}}(h_{(2)})h_{(1)(2)}\mathcal{R}^{(1)}_{\gamma^{-1}\alpha\gamma,\gamma^{-1}\beta\gamma} \otimes h_{(1)(1)}\mathcal{R}^{(2)}_{\gamma^{-1}\alpha\gamma,\gamma^{-1}\beta\gamma} = S^{-1}_{\gamma^{-1}}(h_{(2)})\mathcal{R}^{(1)}_{\alpha,\beta}h_{(1)(1)} \otimes \mathcal{R}^{(2)}_{\alpha,\beta}h_{(1)(2)} \\
\Longrightarrow\ & \mathcal{R}^{(1)}_{\gamma^{-1}\alpha\gamma,\gamma^{-1}\beta\gamma} \otimes h\mathcal{R}^{(2)}_{\gamma^{-1}\alpha\gamma,\gamma^{-1}\beta\gamma} = S^{-1}_{\gamma^{-1}}(h_{(2)})\mathcal{R}^{(1)}_{\alpha,\beta}h_{(1)(1)} \otimes \mathcal{R}^{(2)}_{\alpha,\beta}h_{(1)(2)} \\
\Longrightarrow\ & \mathcal{R}^{(1)}_{\gamma^{-1}\alpha\gamma,\gamma^{-1}\beta\gamma}\mathcal{R}^{(2)}_{\gamma^{-1}\xi\gamma,\gamma^{-1}\zeta\gamma} \otimes h\mathcal{R}^{(2)}_{\gamma^{-1}\alpha\gamma,\gamma^{-1}\beta\gamma}\mathcal{R}^{(1)}_{\gamma^{-1}\xi\gamma,\gamma^{-1}\zeta\gamma} \\
&= S^{-1}_{\gamma^{-1}}(h_{(2)})\mathcal{R}^{(1)}_{\alpha,\beta}h_{(1)(1)}\mathcal{R}^{(2)}_{\gamma^{-1}\xi\gamma,\gamma^{-1}\zeta\gamma} \otimes \mathcal{R}^{(2)}_{\alpha,\beta}h_{(1)(2)}\mathcal{R}^{(1)}_{\gamma^{-1}\xi\gamma,\gamma^{-1}\zeta\gamma},
\end{aligned}$$

from which we derive the commutation relation:

$$\begin{aligned}
&\mathcal{R}^{(1)}_{\gamma^{-1}\alpha\gamma,\gamma^{-1}\beta\gamma}\mathcal{R}^{(2)}_{\gamma^{-1}\xi\gamma,\gamma^{-1}\zeta\gamma} \otimes h\mathcal{R}^{(2)}_{\gamma^{-1}\alpha\gamma,\gamma^{-1}\beta\gamma}\mathcal{R}^{(1)}_{\gamma^{-1}\xi\gamma,\gamma^{-1}\zeta\gamma} \\
&\qquad = S^{-1}_{\gamma^{-1}}(h_{(2)})\mathcal{R}^{(1)}_{\alpha,\beta}\mathcal{R}^{(2)}_{\xi,\zeta}h_{(1)(2)} \otimes \mathcal{R}^{(2)}_{\alpha,\beta}\mathcal{R}^{(1)}_{\xi,\zeta}h_{(1)(1)}.
\end{aligned} \quad (40)$$

Applying $\mathrm{id}_{H_{\alpha\beta}} \otimes \Delta_\gamma \otimes \mathrm{id}_{H_\gamma}$ to both sides of the equation of Equation (36) we obtain

$$\begin{aligned}\mathcal{R}^{(1)}_{\alpha\beta,\gamma} \otimes \mathcal{R}^{(2)}_{\alpha\beta,\gamma(1)(1)} \otimes \mathcal{R}^{(2)}_{\alpha\beta,\gamma(1)(2)} \otimes \mathcal{R}^{(2)}_{\alpha\beta,\gamma(2)} &= \left(\mathcal{R}^{(1)}_{\alpha\beta,\gamma} \otimes \Delta_\gamma \mathcal{R}^{(2)}_{\alpha\beta,\gamma(1)} \otimes \mathcal{R}^{(2)}_{\alpha\beta,\gamma(2)}\right) \\ &= \left(\mathcal{R}^{(1)}_{\alpha,\gamma}\mathcal{R}^{(1)}_{\beta,\gamma} \otimes \Delta_\gamma \mathcal{R}^{(2)}_{\beta,\gamma} \otimes \mathcal{R}^{(2)}_{\alpha,\gamma}\right) = \mathcal{R}^{(1)}_{\alpha,\gamma}\mathcal{R}^{(1)}_{\beta,\gamma} \otimes \mathcal{R}^{(2)}_{\beta,\gamma(1)} \otimes \mathcal{R}^{(2)}_{\beta,\gamma(2)} \otimes \mathcal{R}^{(2)}_{\alpha,\gamma} \\ &= \mathcal{R}^{(1)}_{\alpha,\gamma}\mathcal{R}^{(1)}_{\beta,\gamma}\mathcal{R}^{(1)}_{1,\gamma} \otimes \mathcal{R}^{(2)}_{1,\gamma} \otimes \mathcal{R}^{(2)}_{\beta,\gamma} \otimes \mathcal{R}^{(2)}_{\alpha,\gamma} = \mathcal{R}^{(1)}_{\alpha,\gamma}\mathcal{R}^{(1)}_{1,\gamma}\mathcal{R}^{(1)}_{\beta,\gamma} \otimes \mathcal{R}^{(2)}_{\beta,\gamma} \otimes \mathcal{R}^{(2)}_{1,\gamma} \otimes \mathcal{R}^{(2)}_{\alpha,\gamma},\end{aligned}$$

hence

$$\begin{aligned}&\mathcal{R}^{(1)}_{\alpha\beta,\gamma} \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma}\right)_{(1)} \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma}\right)_{(2)(1)} \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma}\right)_{(2)(2)} \\ &= \mathcal{R}^{(1)}_{\alpha\beta,\gamma} \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma(2)}\right) \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma(1)}\right)_{(1)} \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma(1)}\right)_{(2)} \\ &= \mathcal{R}^{(1)}_{\alpha\beta,\gamma} \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma(2)}\right) \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma(1)(2)}\right) \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\alpha\beta,\gamma(1)(1)}\right) \\ &= \mathcal{R}^{(1)}_{\alpha,\gamma}\mathcal{R}^{(1)}_{1,\gamma}\mathcal{R}^{(1)}_{\beta,\gamma} \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\alpha,\gamma}\right) \otimes S_\gamma\left(\mathcal{R}^{(2)}_{1,\gamma}\right) \otimes S_\gamma\left(\mathcal{R}^{(2)}_{\beta,\gamma}\right).\end{aligned} \tag{41}$$

Applying $\mathrm{id}_{H_1} \otimes S_1$ to both sides of $\mathcal{R}^{(1)}_{\alpha,\beta} S_\alpha\left(\mathcal{R}^{(1)}_{\alpha,\beta^{-1}}\right) \otimes \mathcal{R}^{(2)}_{\alpha,\beta}\mathcal{R}^{(2)}_{\alpha,\beta^{-1}} = 1 \otimes 1$, which follows from Theorem 10, and using part (1) of Theorem 11, we obtain

$$\begin{aligned}&\mathcal{R}^{(1)}_{\alpha,\beta} S_\alpha\left(\mathcal{R}^{(1)}_{\alpha,\beta^{-1}}\right) \otimes S_1\left(\mathcal{R}^{(2)}_{\alpha,\beta}\mathcal{R}^{(2)}_{\alpha,\beta^{-1}}\right) = 1 \otimes S_1(1) \\ &\Longrightarrow \mathcal{R}^{(1)}_{\alpha,\beta} S_\alpha\left(\mathcal{R}^{(1)}_{\alpha,\beta^{-1}}\right) \otimes S_{\beta^{-1}}\left(\mathcal{R}^{(2)}_{\alpha,\beta^{-1}}\right) S_\beta\left(\mathcal{R}^{(2)}_{\alpha,\beta}\right) = 1 \otimes 1 \\ &\Longrightarrow \mathcal{R}^{(1)}_{\alpha,\beta}\mathcal{R}^{(1)}_{\alpha^{-1},\beta} \otimes \mathcal{R}^{(2)}_{\alpha^{-1},\beta} S_\beta\left(\mathcal{R}^{(2)}_{\alpha,\beta}\right) = 1 \otimes 1.\end{aligned} \tag{42}$$

Using Equations (40)–(42) as well as part (1) of Theorem 11 again, we continue our calculation of

$$\begin{aligned}\Delta_\alpha(u_\alpha) &= S_1\left(\mathcal{R}^{(1)}_{1,\alpha}\mathcal{R}^{(2)}_{1,1}\right) u_\alpha \otimes S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1}\right)\mathcal{R}^{(2)}_{1,\alpha}\mathcal{R}^{(1)}_{1,1}\widehat{\mathcal{R}}^{(1)}_{1,1} \\ &= S_1\left(S_1^{-1}(S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1}\right)_{(2)})\mathcal{R}^{(1)}_{1,\alpha}\mathcal{R}^{(2)}_{1,1} S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1}\right)_{(1)(2)}\right) u_\alpha \otimes \mathcal{R}^{(2)}_{1,\alpha}\mathcal{R}^{(1)}_{1,1} S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1}\right)_{(1)(1)}\widehat{\mathcal{R}}^{(1)}_{1,1} \\ &= S_1\left(S_1^{-1}(S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1\,(1)}\right))\mathcal{R}^{(1)}_{1,\alpha}\mathcal{R}^{(2)}_{1,1} S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1\,(2)}\right)_{(2)}\right) u_\alpha \otimes \mathcal{R}^{(2)}_{1,\alpha}\mathcal{R}^{(1)}_{1,1} S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1\,(2)}\right)_{(1)}\widehat{\mathcal{R}}^{(1)}_{1,1} \\ &= S_1\left(S_1^{-1}(S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1\,(1)}\right))\mathcal{R}^{(1)}_{1,\alpha}\mathcal{R}^{(2)}_{1,1} S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1\,(2)(1)}\right)\right) u_\alpha \otimes \mathcal{R}^{(2)}_{1,\alpha}\mathcal{R}^{(1)}_{1,1} S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1\,(2)(2)}\right)\widehat{\mathcal{R}}^{(1)}_{1,1} \\ &= S_1\left(S_1^{-1}(S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1\,(1)(1)}\right))\mathcal{R}^{(1)}_{1,\alpha}\mathcal{R}^{(2)}_{1,1} S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1\,(1)(2)}\right)\right) u_\alpha \otimes \mathcal{R}^{(2)}_{1,\alpha}\mathcal{R}^{(1)}_{1,1} S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1\,(2)}\right)\widehat{\mathcal{R}}^{(1)}_{1,1} \\ &= S_1\left(S_1^{-1}(S_1\left(\widehat{\widehat{\widehat{\mathcal{R}}}}^{(2)}_{1,1}\right))\mathcal{R}^{(1)}_{1,\alpha}\mathcal{R}^{(2)}_{1,1} S_1\left(\widehat{\widehat{\mathcal{R}}}^{(2)}_{1,1}\right)\right) u_\alpha \otimes \mathcal{R}^{(2)}_{1,\alpha}\mathcal{R}^{(1)}_{1,1} S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1}\right)\widehat{\mathcal{R}}^{(1)}_{1,1}\widehat{\widehat{\mathcal{R}}}^{(1)}_{1,1}\widehat{\widehat{\widehat{\mathcal{R}}}}^{(1)}_{1,1} \\ &= S_1\left(\widehat{\widehat{\widehat{\mathcal{R}}}}^{(2)}_{1,1}\mathcal{R}^{(1)}_{1,\alpha}\mathcal{R}^{(2)}_{1,1} S_1\left(\widehat{\widehat{\mathcal{R}}}^{(2)}_{1,1}\right)\right) u_\alpha \otimes \mathcal{R}^{(2)}_{1,\alpha}\mathcal{R}^{(1)}_{1,1} S_1\left(\widehat{\mathcal{R}}^{(2)}_{1,1}\right)\widehat{\mathcal{R}}^{(1)}_{1,1}\widehat{\widehat{\mathcal{R}}}^{(1)}_{1,1}\widehat{\widehat{\widehat{\mathcal{R}}}}^{(1)}_{1,1} \\ &= S_1 \circ S_1\left(\widehat{\widehat{\mathcal{R}}}^{(2)}_{1,1}\right) S_1\left(\mathcal{R}^{(2)}_{1,1}\right) S_1\left(\mathcal{R}^{(1)}_{1,\alpha}\right) S_1\left(\widehat{\widehat{\widehat{\mathcal{R}}}}^{(2)}_{1,1}\right) u_\alpha \otimes \mathcal{R}^{(2)}_{1,\alpha}\mathcal{R}^{(1)}_{1,1} u_1 \widehat{\widehat{\mathcal{R}}}^{(1)}_{1,1}\widehat{\widehat{\widehat{\mathcal{R}}}}^{(1)}_{1,1} \\ &= S_1 \circ S_1\left(\widehat{\widehat{\mathcal{R}}}^{(2)}_{1,1}\right) S_1\left(\mathcal{R}^{(2)}_{1,1}\right) S_1\left(\mathcal{R}^{(1)}_{1,\alpha}\right) S_1\left(\widehat{\widehat{\widehat{\mathcal{R}}}}^{(2)}_{1,1}\right) u_\alpha \\ &\quad \otimes \mathcal{R}^{(2)}_{1,\alpha}\mathcal{R}^{(1)}_{1,1} S_1 \circ S_{(1)}\left(\widehat{\widehat{\mathcal{R}}}^{(1)}_{1,1}\right) S_1 \circ S_1\left(\widehat{\widehat{\widehat{\mathcal{R}}}}^{(1)}_{1,1}\right) u_1\end{aligned}$$

$$
\begin{aligned}
&= \widehat{\widehat{\mathcal{R}}}^{(2)}_{1,1} S_1\left(\mathcal{R}^{(2)}_{1,1}\right) S_1\left(\mathcal{R}^{(1)}_{1,\alpha}\right) \widehat{\widehat{\widehat{\mathcal{R}}}}^{(2)}_{1,1} u_\alpha \otimes \mathcal{R}^{(2)}_{1,\alpha} \mathcal{R}^{(1)}_{1,1} \widehat{\widehat{\mathcal{R}}}^{(1)}_{1,1} S_1\left(\widehat{\widehat{\widehat{\mathcal{R}}}}^{(1)}_{1,1}\right) u_1 \\
&= S_1\left(\mathcal{R}^{(1)}_{1,\alpha}\right) \widehat{\widehat{\widehat{\mathcal{R}}}}^{(2)}_{1,1} u_\alpha \otimes \mathcal{R}^{(2)}_{1,\alpha} S_1\left(\widehat{\widehat{\widehat{\mathcal{R}}}}^{(1)}_{1,1}\right) u_1 = \mathcal{R}^{-1(1)}_{1,\alpha^{-1}} \mathcal{R}^{-1(2)}_{1,1} u_\alpha \otimes \mathcal{R}^{-1(2)}_{1,\alpha^{-1}} S_1\left(\mathcal{R}^{-1(1)}_{1,1}\right) u_1 \\
&= \mathcal{R}^{-1}_{1,\alpha^{-1}} \sigma_{H_1,H_1}\left(\mathcal{R}^{-1}_{1,1}\right)(u_\alpha \otimes u_1) = \left(\sigma_{H_1,H_1}(\mathcal{R}_{1,1})\mathcal{R}_{1,\alpha^{-1}}\right)^{-1}(u_\alpha \otimes u_1),
\end{aligned}
$$

from which we also have

$$\Delta_\alpha(u_\alpha) = (u_\alpha \otimes u_1)\left(\sigma_{H_1,H_1}(\mathcal{R}_{1,1})\mathcal{R}_{1,\alpha^{-1}}\right)^{-1}.$$

Moreover,

$$
\begin{aligned}
\varepsilon_\alpha(u_\alpha) &= \varepsilon_\alpha(S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\right)\mathcal{R}^{(1)}_{1,\alpha^{-1}}) = \varepsilon_\alpha(S_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\right))\varepsilon_1(\mathcal{R}^{(1)}_{1,\alpha^{-1}}) \\
&= \varepsilon_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\right)\varepsilon_1(\mathcal{R}^{(1)}_{1,\alpha^{-1}}) = \varepsilon_1(\mathcal{R}^{(1)}_{1,\alpha^{-1}}\varepsilon_{\alpha^{-1}}\left(\mathcal{R}^{(2)}_{1,\alpha^{-1}}\right)) = \varepsilon_1(1) = 1_{\Bbbk}.
\end{aligned}
$$

We have established part (a).

To see parts (b) and (c), we deduce from part (a) that

$$
\begin{aligned}
\Delta_{\alpha^{-1}} S_\alpha(u_\alpha) &= \sigma_{H_{\alpha^{-1}},H_{\alpha^{-1}}}(S_\alpha \otimes S_\alpha)\Delta_\alpha(u_\alpha) \\
&= S_\alpha(u_1 \mathcal{R}^{(2)}_{1,\alpha} S_1\left(\mathcal{R}^{(1)}_{1,1}\right)) \otimes S_\alpha(u_\alpha S_1\left(\mathcal{R}^{(1)}_{1,\alpha}\right)\mathcal{R}^{(2)}_{1,1}) \\
&= S_1 S_1\left(\mathcal{R}^{(1)}_{1,1}\right) S_\alpha(\mathcal{R}^{(2)}_{1,\alpha}) S_1(u_1) \otimes S_1(\mathcal{R}^{(2)}_{1,1}) S_1 S_1\left(\mathcal{R}^{(1)}_{1,\alpha}\right) S_\alpha(u_\alpha) \\
&= S_1\left(\mathcal{R}^{(1)}_{1,1}\right)\mathcal{R}^{(2)}_{1,\alpha^{-1}} S_1(u_1) \otimes \mathcal{R}^{(2)}_{1,1} S_1\left(\mathcal{R}^{(1)}_{1,\alpha^{-1}}\right) S_\alpha(u_\alpha) \\
&= \mathcal{R}^{-1(1)}_{1,1}\mathcal{R}^{-1(2)}_{1,\alpha} S_1(u_1) \otimes \mathcal{R}^{-1(2)}_{1,1}\mathcal{R}^{-1(1)}_{1,\alpha} S_\alpha(u_\alpha) \\
&= (\mathcal{R}^{-1(1)}_{1,1}\mathcal{R}^{-1(2)}_{1,\alpha} \otimes \mathcal{R}^{-1(2)}_{1,1}\mathcal{R}^{-1(1)}_{1,\alpha})(S_1(u_1) \otimes S_\alpha(u_\alpha)) \\
&= \mathcal{R}^{-1}_{1,1}\sigma_{H_1,H_{\alpha^{-1}}}(\mathcal{R}^{-1}_{1,\alpha})(S_1(u_1) \otimes S_\alpha(u_\alpha)) \\
&= (\sigma_{H_1,H_\alpha}(\mathcal{R}_{1,\alpha})\mathcal{R}_{1,1})^{-1}(S_1(u_1) \otimes S_\alpha(u_\alpha))
\end{aligned}
$$

and the two factors commute; thus

$$\Delta_\alpha S_{\alpha^{-1}}(u_\alpha^{-1}) = \sigma_{H_1,H_\alpha}(\mathcal{R}_{1,\alpha})\mathcal{R}_{1,1}(S_1(u_1^{-1}) \otimes S_{\alpha^{-1}}(u_\alpha^{-1}))$$

and the two factors commute.

It remains to establish part (d). Consider the following calculation:

$$
\begin{aligned}
\Delta_1(g_1) &= \Delta_1(u_1(S_1(u_1))^{-1}) = \Delta_1(u_1)\Delta_1((S_1(u_1))^{-1}) = \Delta_1(u_1)\Delta_1\left(S_1(u_1^{-1})\right) \\
&= (u_1 \otimes u_1)\left(\sigma_{H_1,H_1}(\mathcal{R}_{1,1})\mathcal{R}_{1,1}\right)^{-1}\sigma_{H_1,H_1}(\mathcal{R}_{1,1})\mathcal{R}_{1,1}(S_1(u_1^{-1}) \otimes S_1(u_1^{-1})) \\
&= (u_1 \otimes u_1)(S_1(u_1^{-1}) \otimes S_1(u_1^{-1})) = u_1 S_1(u_1^{-1}) \otimes u_1 S_1(u_1^{-1})) = g_1 \otimes g_1. \quad \square
\end{aligned}
$$

In [28], the twisting theory for quasitriangular Hopf algebras was studied by a 2-cocycle. By using the dual of cocycle (called a 2-cocycle), multiplication alteration for bialgebras was investigated in [29,30]. In what follows, we will introduce the definition of 2-cocycle for Hopf non-coassociative π-algebra.

Definition 11. *Let (H, φ) be a crossed Hopf non-coassociative π-algebra. If there exists a family $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ of invertible elements (the R-matrix) such that, the family R is invariant under the crossing, i.e., for any $\alpha, \beta, \gamma \in \pi$,*

$$(\varphi_\gamma \otimes \varphi_\gamma)(R_{\alpha,\beta}) = R_{\gamma\alpha\gamma^{-1},\gamma\beta\gamma^{-1}},$$

and, for any $\alpha, \beta, \gamma, \delta \in \pi$,

$$((\varphi_\alpha \otimes \varphi_\alpha)R_{\delta,\gamma})_{12}(\Delta_\alpha \otimes \mathrm{id}_{H_{\beta\gamma}})(R_{\alpha,\beta\gamma}) = (R_{\alpha,\beta})_{23}\big(\mathrm{id}_{H_{\alpha\delta}} \otimes \Delta_\gamma\big)(R_{\alpha\delta,\gamma}), \tag{43}$$

i.e.,

$$R^{(1)}_{\alpha\delta\alpha^{-1},\alpha\gamma\alpha^{-1}}R^{(1)}_{\alpha,\beta\gamma(1)} \otimes R^{(2)}_{\alpha\delta\alpha^{-1},\alpha\gamma\alpha^{-1}}R^{(1)}_{\alpha,\beta\gamma(2)} \otimes R^{(2)}_{\alpha,\beta\gamma} = R^{(1)}_{\alpha\delta,\gamma} \otimes R^{(1)}_{\alpha,\beta}R^{(2)}_{\alpha\delta,\gamma(1)} \otimes R^{(2)}_{\alpha,\beta}R^{(2)}_{\alpha\delta,\gamma(2)}$$

Then R is called a 2-cocycle.

From Theorem 11, it is easy to see that a quasitriangular Hopf non-coassociative π-algebra is a crossed Hopf non-coassociative π-algebra with a 2-cocycle.

Definition 12. *Let H be a Hopf non-coassociative π-algebra. We say that a family of $M = \{M_\alpha\}_{\alpha\in\pi}$ is a π-*representation *over H if M has a right π-module structure, it means that there is a family*

$$\psi = \{\psi_{\alpha,\beta} : M_\alpha \otimes H_\beta \longrightarrow M_{\alpha\beta}\}_{\alpha,\beta\in\pi}$$

of $\Bbbk$-linear maps (the π-action), such that ψ is associative in the sense that, for any $\alpha, \beta, \gamma \in \pi$,

$$\psi_{\alpha\beta,\gamma}(m_{\alpha,\beta} \otimes id_{A_\gamma}) = \psi_{\alpha,\beta\gamma}(id_{A_\alpha} \otimes \psi_{\beta,\gamma}); \tag{44}$$

$$\psi_{\alpha,1}(id_{H_\alpha} \otimes 1) = id_{H_\alpha}. \tag{45}$$

We shall associate with every Hopf non-coassociative π-algebra $H = (\{H_\alpha, m, 1_\alpha\}, \Delta_\alpha, \varepsilon, S)$ a category of π-representations $Rep_\pi(H)$ which has a natural structure of a π-category.

Explicitly, for any $\alpha \in \pi$, by an object M_α in the category $Rep_\alpha(H)$ we mean a vector space M_α is a right H-module with a structure:

$$\psi_\alpha = \{\psi_{\alpha,\beta} : M_\alpha \otimes H_\beta \longrightarrow M_{\alpha\beta}\}_{\alpha,\beta\in\pi}.$$

The category $Rep_\pi(H)$ is the disjoint union of the categories $\{Rep_\alpha\}_{\alpha\in\pi}$ where $Rep_\alpha(H)$ is the category of H-modules and H-linear homomorphisms. By Proposition 3, the tensor product and the unit object in $Rep_\pi(H)$ are defined in the usual way using the comultiplication Δ_H and the unit 1. That is,

$$h_\alpha \cdot (m \otimes n) = \sum h_{\alpha(1)} \cdot m \otimes h_{\alpha(2)} \cdot n$$

for any $m \in M_\beta$ and $n \in N_\gamma$.

The associativity morphisms are the standard identification isomorphisms.

Furthermore, let $H = (\{H_\alpha, m, 1_\alpha\}, \Delta_\alpha, \varepsilon, S, \varphi, R)$ be a quasitriangular Hopf π-quasialgebra. The automorphism φ_α of H defines an automorphism, Φ_α of $Rep_\pi(H)$.

If M_β is in $Rep(H)_\beta$, then $\Phi_\alpha(M)$ has the same underlying vector space as M and each $x \in H_{\alpha\beta\alpha^{-1}}$ acts as multiplication by $\varphi_\alpha^{-1}(x) \in H_\beta$. Every H_β-homomorphism $M \longrightarrow N$ is mapped to itself considered as a $H_{\alpha\beta\alpha^{-1}}$-homomorphism. It is easy to check that $Rep_\pi(H)$ is a crossed π-category (see [4]).

A universal R-matrix $R = \{R_{\alpha,\beta} \in H_\alpha \otimes H_\beta\}_{\alpha,\beta\in\pi}$ in H induces a braiding in $Rep_\pi(H)$ as follows. For $M \in Rep(H)_\alpha$) and $N \in Rep(H)_\beta$), the braiding

$$c_{M,N} : M \otimes N \longrightarrow {}^{M}N \otimes M$$

is the composition of multiplication by $R_{\alpha,\beta}$, permutation $M \otimes N \longrightarrow N \otimes M$. The conditions defining a universal R-matrix ensure that $\{c_{M,N}\}_{M,N}$ is a braiding.

We now obtain

Theorem 13. *Let H be any quasitriangular Hopf non-coassociative π-algebra. Then the category $Rep_{\pi}(H)$ of π-representations is a braided T-category.*

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