Superconformal Index and Gravitational Path Integral

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Quantum Gravity

Quantum Gravity in asymptotically-AdS spacetime	AdS/CFT $\Longleftrightarrow$	Ordinary QFT living at conformal boundary

CONSISTENT AND NON-PERTURBATIVE
DIFINITION OF QUANTUM GRAVITY

Semiclassical Regime for Gravity

⋆ In AdS:

Gravity is weakly coupled

$\begin{pmatrix}\text{AdS much larger}\\ \text{than Planck scale}\end{pmatrix}$

and close to Einstein gravity

$\begin{pmatrix}\text{scale of higher-derivative corr.'s}\\ \text{much higher than AdS scale}\end{pmatrix}$

$\Longleftarrow$

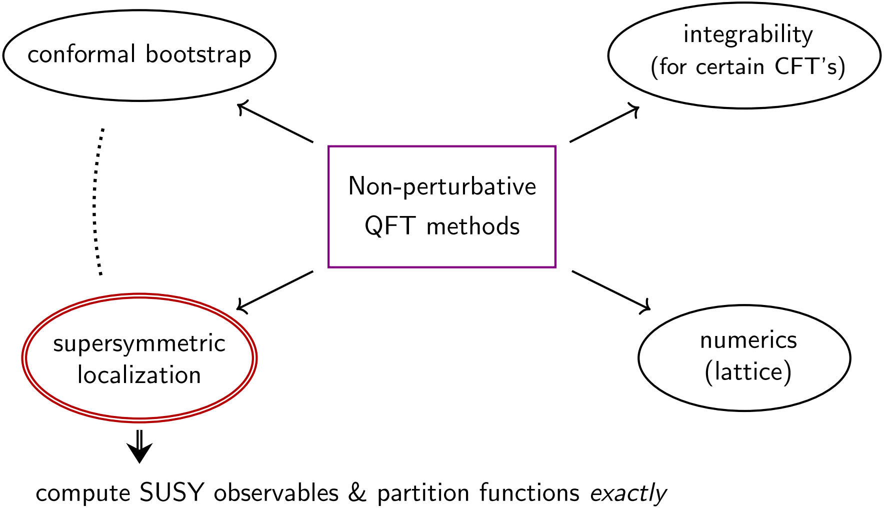
⋆ In QFT:

large
"central charge"
(large N)

QFT is
strongly coupled

! Take advantage of modern non-perturbative methods !

Quantum Gravity from Field Theory



Localization techniques $\longrightarrow$ QFT Euclidean partition functions

What can we learn about semi-classical expansion of "gravitational path integral"?

Black holes & Entropy

- Quantum corrections expected to play an important role
- Euclidean observables – *e.g.*, indices – capture Lorentzian physics

$$S_{\mathsf{BH}} = \frac{c^3}{G_N \hbar} \frac{\text{Area}}{4}$$

[Bekenstein 72, 73, 74; Hawking 74, 75]

Black hole = Ensemble of states in quantum gravity $\overset{\text{AdS/CFT}}{=}$ Ensemble of states in boundary QFT

$$S_{\mathsf{micro}} = \log N_{\mathsf{micro}} = \frac{\text{Area}}{4\, G_N} + \text{ perturbative \& non-perturbative corrections}$$

Black holes in AdS

⋆ String theory reproduces the Bekenstein-Hawking entropy of BPS black holes in asymptotically-flat spacetimes [Strominger, Vafa 96]

Since AdS/CFT grants us a fully non-perturbative definition of Quantum Gravity, it is interesting to study the black hole entropy in AdS

⋆ Strategy that proved to be effective: [FB, Hristov, Zaffaroni 15]

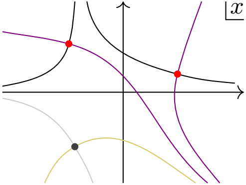
> Extract BPS black hole entropy in AdS from
> SUSY partition functions of boundary QFT at large N

Beyond Bekestein-Hawking

Saddle-point approximation is subtle: (*e.g.*, 1-dim integrals)

- Complex saddles play important role
- Not all of them contribute $\to$ steepest descent & Lefschetz thimbles

Does something similar happen in gravity?



⋆ **This talk:** analyze charged rotating BPS black holes in AdS_5

- very detailed computations are feasible

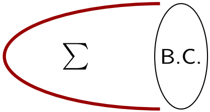
⋆ **Strategy:** count states in the boundary QFT

employing a grand canonical partition function

$$\mathcal{I}(y) = \sum_{\text{states}} y^Q$$

Difficult problem at strong coupling $\longrightarrow$ exploit SUSY

⋆ QFT partition function $\overset{\text{AdS/CFT}}{=}$ Euclidean "gravitational path integral" with fixed boundary conditions



[Witten 98][Dijkgraaf, Maldacena, Moore, E. Verlinde 00][Maloney, Witten 07]

⋆ Define gravitational path integral through QFT, computable with localization $\rightsquigarrow$ details analysis

Setup

Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ $\longleftrightarrow$ 4d $SU(N)$ $\mathcal{N} = 4$ Super-Yang-Mills

BPS black hole solutions in AdS_5 [Gutowski, Reall 04; . . .]

(use 5d gauged supergravity or uplift to 10d)

Kerr–Newman BPS black holes

Rotating & electrically-charged $\frac{1}{16}$-BPS black holes in AdS$_5$ [Gutowski, Reall 04]
[Chong, Cvetic, Lu, Pope 05][Kunduri, Lucietti, Reall 06]

- Angular momentum — Here: J_1, J_2
  Electric charges — Charges for $U(1)^3 \subset SO(6)$: R_1, R_2, R_3
- SUSY (1 cplx supercharge $\mathcal{Q}$)
  $\rightsquigarrow$ BPS linear relation: $2M = 2J_1 + 2J_2 + R_1 + R_2 + R_3$
  Extremal $(T = 0)$ $\rightsquigarrow$ non-linear relation among 5 charges $\to$ 4 parameters
  [Cabo-Bizet, Cassani, Martelli, Murthy 18; Cassani, Papini 19]
- Bekenstein-Hawking entropy (S^3 horizon):

$$S_{\text{BH}} = \frac{\text{Area}}{4G_N} = \pi\sqrt{R_1R_2 + R_1R_3 + R_2R_3 - 2N^2(J_1 + J_2)}$$

- Angular momenta, charges and entropy scale $\sim N^2$

Superconformal index

[Romelsberger 05; Kinney, Maldacena, Minwalla, Raju 05]

⋆ Counts (with sign) BPS states on S^3 $=$ protected operators on flat space

Index of $\mathcal{N} = 4$ SYM:

$$\mathcal{I}(p, q, y_1, y_2) = \mathrm{Tr}\,(-1)^F\, e^{-\beta\{\mathcal{Q},\mathcal{Q}^\dagger\}}\, p^{J_1 + \frac{1}{2}R_3}\, q^{J_2 + \frac{1}{2}R_3}\, y_1^{\frac{1}{2}(R_1 - R_3)}\, y_2^{\frac{1}{2}(R_2 - R_3)}$$

Write: $\quad p = e^{2\pi i \tau} \qquad q = e^{2\pi i \sigma} \qquad y_a = e^{2\pi i \Delta_a} \qquad F = R_3 = 2J_1 = 2J_2 \mod 2$

SUSY $\Rightarrow$ at most 4 independent fugacities

$\left(\begin{array}{c} \text{introduce } \Delta_3\text{:} \\ \Delta_1 + \Delta_2 + \Delta_3 - \tau - \sigma \in \mathbb{Z} \end{array} \right)$

⋆ *Exact* integral formula

[Aharony, Marsano, Minwalla, Papadodimas, Van Raamsdonk 03]
[Sundborg 99][Romelsberger 05][Kinney, Maldacena, Minwalla, Raju 05]

⋆ The index encodes (*weighted*) degeneracies:

$$\mathcal{I} = 1 \; + \; \#\, y \; + \; \#\, y^2 \; + \ldots + \; d(Q)\, y^Q \; + \ldots$$

To extract the degeneracies:

$$d(Q) = \frac{1}{2\pi i} \oint \frac{dy}{y^{Q+1}}\, \mathcal{I}(y) = \oint d\Delta \;\; e^{\log \mathcal{I}(\Delta) - 2\pi i Q \Delta}$$

Assuming large degeneracies, saddle-point approximation $\rightarrow$ Legendre transform

$$\text{entropy} = \log d(Q) \; \simeq \; \log \mathcal{I}(\Delta) - 2\pi i Q \Delta \,\Big|_{\Delta \,=\, \text{extremum}}$$

Remarks:

- We are interested in $Q \sim N^2$ in the large N limit
- One can prove that, at least at leading order in N,
  the index captures the full entropy [Sen 09; FB, Hristov, Zaffaroni 16]

Many approaches to large N matrix model:

- direct saddle-point approx
- Cardy limit $\tau \to 0$
- saddle-point approx for non-analytic extension
- Gross-Witten-Wadia-like expansion
- giant graviton expansion

⋆ Here: Bethe Ansatz formulation

Bethe Ansatz formula for the superconformal index

Alternative formula: (set $\tau = \sigma$)

[Closset, Kim, Willett 17]
[FB, Milan 18]
[FB, Rizi 21]

$$\mathcal{I} = \sum_{u \,\in\, \mathfrak{M}_{\text{BAE}}} \mathcal{Z}(u;\Delta,\tau,\tau)\; H(u;\Delta,\tau)^{-1}$$

1. $\mathfrak{M}_{\text{BAE}}$ are solutions to "Bethe Ansatz Equations" for $\mathrm{rk}(G)$ complexified holonomies $[u_i]$ living on a complex torus T^2_τ of modular parameter τ:

   $\mathfrak{M}_{\text{BAE}}$:
   $SU(N)$ $\mathcal{N}=4$ SYM

   $$Q_i(u) = \prod_{a=1}^{3}\prod_{j=1}^{N} \frac{\theta(\Delta_a - u_{ij};\tau)}{\theta(\Delta_a + u_{ij};\tau)} = 1$$

   $u_{ij} = u_i - u_j \neq 0$

   Equations are defined on T^2_τ and are invariant under $SL(2,\mathbb{Z})$

2. $\mathcal{Z}$ is the same integrand as in the integral formula

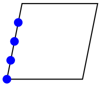
3. H is a Jacobian: $H = \det_{ij} \partial Q_i / \partial u_j$

★ *Discrete* family of exact solutions [Hosseini, Nedelin, Zaffaroni 16; Hong, Liu 18]

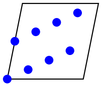
Classified by subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$ of order N

Labelled by $\{m, r\}$ with $m \cdot n = N$ and $r \in \mathbb{Z}_n$

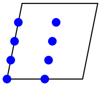
- Basic solution $\{1, 0\}$: $u_j \sim \frac{\tau}{N}\, j$



- $SL(2,\mathbb{Z})$-transformed sol's *e.g.*

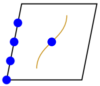


- More general $SL(2,\mathbb{Z})$ orbits: $m > 1$ $\quad \gcd(m,n,r) > 1$



★ *Continuous* families of solutions

(conjectured to correspond to vacua of $\mathcal{N} = 1^*$ theory) *e.g.*



[Ardehali, Hong, Liu 19; Lezcano, Hong, Liu, Pando Zayas 21; FB, Rizi 21]

Contribution of BASIC SOLUTION at large N

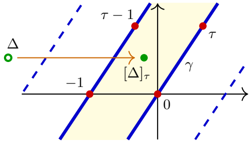
Does the index reproduce the Bekenstein-Hawking entropy?

- Contribution of BASIC SOLUTION $\{1, 0\}$ at large N:

$$\lim_{N\to\infty} \mathcal{I}(\tau, \Delta_1, \Delta_2)\Big|_{\substack{\text{BASIC}\\ \text{SOLUTION}}} \simeq \exp\left(-i\pi N^2 \frac{[\Delta_1]_\tau\,[\Delta_2]_\tau\,[\Delta_3]_\tau}{\tau^2}\right)$$

Large N limit is a *discontinuous* analytic function: Stokes phenomenon

$$[\Delta]_\tau \equiv \Delta + n \quad \text{s.t.} \in \text{STRIP}$$



Black hole entropy

Extract Bekenstein-Hawking entropy from $\mathcal{I}\big|_{\text{BASIC SOLUTION}}$

⋆ Set $X_1 = [\Delta_1]_\tau \quad X_2 = [\Delta_2]_\tau$. Obtain "entropy function":

$$\boxed{\log \mathcal{I} = -i\pi N^2 \, \frac{X_1 X_2 X_3}{\tau^2}} \qquad \text{with} \qquad \sum_{a=1}^{3} X_a = 2\tau - 1$$

Its (constrained) Legendre transform *exactly* gives the BH black hole entropy:

$$S_{\mathsf{BH}} = \log \mathcal{I} - 2\pi i \left(\sum X_a \tfrac{R_a}{2} + 2\tau J \right) \Big|_{\substack{\text{constrained} \\ \text{extremum}}}$$

Similar procedures work in other setups and dimensions, from AdS_4 to AdS_7

Bekenstein-Hawking entropy from various types of indices:

BPS rotating black holes	BPS black holes with R-symmetry magnetic charge
(possibly with electric and magnetic flavor charges)	(possibly rotating and with electric/magnetic flavor charges)
↓	↓
superconformal indices	topologically twisted indices

[Azzurli, Bobev, Choi, Crichigno, Fluder, Gang, Hosseini, Hristov, Hwang, Jain, Kantor, Kim, Min, Nedelin, Nian, Pando Zayas, Papageorgakis, Passias, Richmond, Suh, Uhlemann, Willett, Yaakov, Zaffaroni, . . .]

Beyond the leading order . . .

Expansion of the index at large N:

$$\mathcal{I} = \sum_{\text{solutions} \in \mathfrak{M}_{\text{BAE}}} e^{\mathcal{O}(N^2) + \dots}$$

It looks like a semiclassical expansion

⋆ Large N contribution of $\{m, r\}$ solutions (with fixed m, r):

$$\log \mathcal{I}_{\{m,r\}} = -\frac{i\pi N^2}{m} \frac{[m\Delta_1]_{\check{\tau}}[m\Delta_2]_{\check{\tau}}[m\Delta_3]_{\check{\tau}}}{(m\tau + r)^2} + \log N + \mathcal{O}(1)$$
$$+ \sum e^{\frac{2\pi i N}{m} \frac{[m\Delta_a]_{\check{\tau}}}{\check{\tau}} + \dots} + \dots$$

where $\sum_a [m\Delta_a]_{\check{\tau}} = 2\check{\tau} - 1$ and $\check{\tau} = m\tau + r$

- Is there anything to learn from this QFT data?

Gravitational path-integral

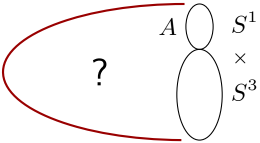
- Superconformal index is computed by Euclidean partition function in QFT

$$\mathcal{I}_{\mathsf{SCFT}} = Z_{S^3\times S^1} \qquad \text{(with suitable regularization)}$$

- Holographically: $Z_{S^3\times S^1} =$ string theory path-integral $\simeq$ classical saddles $+$ corrections

Fill-in bulk geometry
for given boundary conditions

[Witten 98; Dijkgraaf, Maldacena, Moore, Verlinde 00]
[Maloney, Witten 07]



- *Only* SUSY configurations contribute to SUSY observables (localization)

- Euclidean rotation of Lorentzian BPS black hole
  has $\beta = \infty$ (extremal, $T = 0$)

$\Rightarrow$ Look for complex Euclidean SUSY solutions

Complex Euclidean solutions

[Cabo-Bizet, Cassani, Martelli, Murthy 18]

⋆ Consider full family of
of non-SUSY black hole solutions (here 6-dim)

[Chong, Cvetic, Lu, Pope 05]
[Cvetic, Gibbons, Lu, Pope 05]
[Wu 11]

Generic *complex* values of parameters
$\Rightarrow$ *complex* metric and gauge fields

Impose SUSY but *not* extremality

Impose the boundary conditions

- As for the saddle-point approximation to one-dimensional integrals, we are let to include complex saddles in Euclidean semi-classical expansion of gravity.

- Boundary metric: $$ds^2_{\text{bdy}} = \underbrace{dt_{\text{E}}^2}_{S^1} + \underbrace{d\hat{\theta}^2 + \sin^2\hat{\theta}\, d\phi^2 + \cos^2\hat{\theta}\,\psi^2}_{S^3}$$

with $(t_{\text{E}}, \phi, \psi) \cong (t_{\text{E}} + \beta,\ \phi + 2\pi\tau^{\text{g}} - i\beta,\ \psi + 2\pi\sigma^{\text{g}} - i\beta)$ (from regularity at the horizon)

ϕ, ψ defined mod 2π $\Rightarrow$ all $\tau^{\text{g}}, \sigma^{\text{g}} + \mathbb{Z}$ give *same* boundary metric

- Boundary gauge field: $$\exp\left\{-i \oint_{S^1(\text{bdy})} A_a\right\} = \exp\left\{2\pi i \Delta_a^{\text{g}} + \beta\right\}$$

Holonomy is gauge inv. $\Rightarrow$ all $\Delta_a^{\text{g}} + \mathbb{Z}$ give *same* boundary gauge bundle

⋆ B.C.'s only fix (constrained) complex potentials up to $\mathbb{Z}$ shifts!

$\tau^{\text{g}},\ \sigma^{\text{g}},\ \Delta_a^{\text{g}}$ parametrize gravity solutions [Cabo-Bizet, Cassani, Martelli, Murthy 18]

SUSY: $\sum_a \Delta_a^{\text{g}} = \tau^{\text{g}} + \sigma^{\text{g}} \mp 1$

Match with Bethe Ansatz formula?

⋆ On-shell action of complex Euclidean SUSY solutions: (for $\tau = \sigma$)

$$I_{\text{grav}} = -i\pi N^2 \frac{(\Delta_1 - n_1)(\Delta_2 - n_2)(\Delta_3 - n_3)}{(\tau + n_4)(\tau + n_5)}$$

with $\sum_a \Delta_a = 2\tau - 1$ and $\sum_{\alpha=1}^{5} n_\alpha = 0$ $\rightsquigarrow \sum_{n_1,n_2,n_3,n_4}$

⋆ Large N index contribution of $m = 1$ subfamily $\{1, r\}$:

$$\log \mathcal{I}_{\{m,r\}} = -i\pi N^2 \frac{[\Delta_1]_{\check{\tau}}[\Delta_2]_{\check{\tau}}[\Delta_3]_{\check{\tau}}}{(\tau + r)^2} + \ldots$$

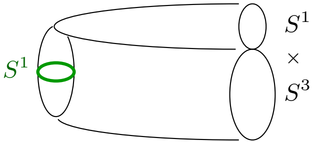
where $\sum_a [\Delta_a]_{\check{\tau}} = 2\check{\tau} - 1$ and $\check{\tau} = \tau + r$ $\rightsquigarrow \sum_r$

- Matching contributions but ... gravity has *too many* solutions!

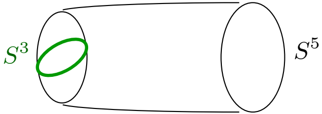
Euclidean D3-branes

Non-perturbative corrections from Euclidean SUSY D3-branes wrapped on 10d geometry at the horizon

- Two possible $S^1 \subset S^3$



- Three possible $S^3 \subset S^5$



On-shell action:

$$S_{\text{D3}} = 2\pi N \, \frac{\Delta_a^{\text{g}}}{\tau^{\text{g}}} \qquad \text{or} \qquad S_{\text{D3}} = 2\pi N \, \frac{\Delta_a^{\text{g}}}{\sigma^{\text{g}}}$$

Non-perturbative corrections: generic *positive* integer linear combinations of those

⋆ Effect of D3-brane corrections:

$$\mathcal{I} = Z_{S^3\times S^1} \simeq e^{I_{\text{grav}}} + \sum_k e^{I_{\text{grav}}}\, e^{ikS_{\text{D3}}} \simeq \exp\Big\{ \underbrace{I_{\text{grav}}}_{\mathcal{O}(N^2)} + \sum_k \underbrace{e^{ikS_{\text{D3}}}}_{\mathcal{O}(e^{-N})} \Big\}$$

Criterium to retain a complex saddle:

$$\boxed{\mathbb{I}\text{m}\, S_{\text{D3}} > 0 \qquad \text{for all (SUSY) D3-brane embeddings}}$$

Violation implies "D3-brane condensation" towards some other saddle point.
Expected to signal that complex saddle point does *not* contribute to integral.

⋆ Apply criterium (for $\tau = \sigma$ in QFT) $\Rightarrow$

$$\begin{cases} \tau^{\text{g}} = \sigma^{\text{g}} = \tau + r & \text{for any } r \\ \Delta_a^{\text{g}} = [\Delta_a]_{\tau+r} & \rightsquigarrow \sum_r \end{cases}$$

Precise match between cplx gravitational saddles and $\{1, r\}$ subfamily

Exponents of non-perturbative corrections match:

$$e^{iS_{\mathsf{D3}}} \;=\; e^{2\pi i N\,\frac{\Delta_a^{\mathrm{g}}}{\tau^{\mathrm{g}}}} \quad \text{or} \quad e^{2\pi i N\,\frac{\Delta_a^{\mathrm{g}}}{\sigma^{\mathrm{g}}}}$$

$$\log \mathcal{I}_{\{1,r\}} = \;\dots\; + \sum \#\, e^{2\pi i N\,\frac{[\Delta_a]_{\check{\tau}}}{\check{\tau}}} \;\;\dots$$

Exponentially small $\mathcal{O}(e^{-N})$ corrections when criterium is satisfied

- Interesting to compute prefactor # and compare with D3-brane quantization

Orbifold geometries: $m > 1$

The $\{m, r\}$ solutions with $m > 1$ correspond to
SUSY orbifolds of 10d lift of the previous solutions

- Take a SUSY complex solution with $\quad \widetilde{\beta} = m\,\beta\,, \quad \widetilde{\tau}^{\mathrm{g}}\,, \widetilde{\sigma}^{\mathrm{g}}\,, \quad \widetilde{\Delta}_a^{\mathrm{g}}$

  Orbifold:

$$(t_{\mathrm{E}},\ \hat{\phi},\ \hat{\psi},\ \phi_a) \ \cong\ \Big(t_{\mathrm{E}} + \tfrac{\widetilde{\beta}}{m}\,,\ \ \hat{\phi} - \tfrac{2\pi r_1}{m}\,,\ \ \hat{\psi} - \tfrac{2\pi r_2}{m}\,,\ \ \phi_a - \tfrac{2\pi s_a}{m}\Big)$$

"Stability" of Euclidean D3-branes $\quad \Rightarrow \quad \begin{cases} \widetilde{\tau}^{\mathrm{g}} = \widetilde{\sigma}^{\mathrm{g}} = m\tau + r \equiv \check{\tau} \\ \widetilde{\Delta}_a^{\mathrm{g}} = [m\Delta_a]_{\check{\tau}} \end{cases}$

On-shell action reduced by $\frac{1}{m}$ $\quad \rightsquigarrow \quad$ Match with $\log \mathcal{I}_{\{m,r\}}$

$$\log \mathcal{I}_{\{m,r\}} = -\frac{i\pi N^2}{m}\,\frac{[m\Delta_1]_{\check{\tau}}[m\Delta_2]_{\check{\tau}}[m\Delta_3]_{\check{\tau}}}{\check{\tau}^2} \ +\ \ldots\ +\ \sum \#\, e^{\frac{2\pi i N}{m}\,\frac{[m\Delta_a]_{\check{\tau}}}{\check{\tau}}} \ +\ \ldots$$

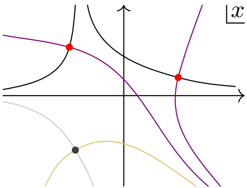
We expect our criterium

on the sign of the imaginary part of the exponent

in non-perturbative corrections

to play the role of a proxy for

steepest descent and Lefschetz-thimble analysis in gravity



Hints of new physics?

[in progress]

⋆ In expansion of the superconformal index,
there are other contributions we have not yet evaluated:

- $\{m, r\}$ discrete solutions with different scaling with N
- continuous families of solutions

They might capture interesting gravity solutions

⋆ There are *other* Euclidean SUSY D3-branes.

They destabilize even the solutions that match with the index,
in certain regions of parameter space.

What does this destabilization represent? Where does it lead to?

Conclusions

Summary:

- Careful analysis of superconformal index of $\mathcal{N} = 4$ SYM, using an alternative Bethe Ansatz formulation.
  Large N: each Bethe Ansatz solution represents a complex saddle point.
- One solution exactly reproduces the Bekenstein-Hawking entropy of BPS black holes in $\text{AdS}_5 \times S^5$.
- Other solutions give corrections from complex gravitational saddles.
  Criterium: discard complex saddles with diverging D3-instanton corrections.

Some open questions:

- Consequences for Lorentzian physics? Which phases / phase transitions?
- Can we compute corrections more precisely?
- Other Bethe Ansatz solutions? Continuous families?
- Multi-center black holes? [We have found probe branes]