全息纠缠熵的比特流描述及其可能应用

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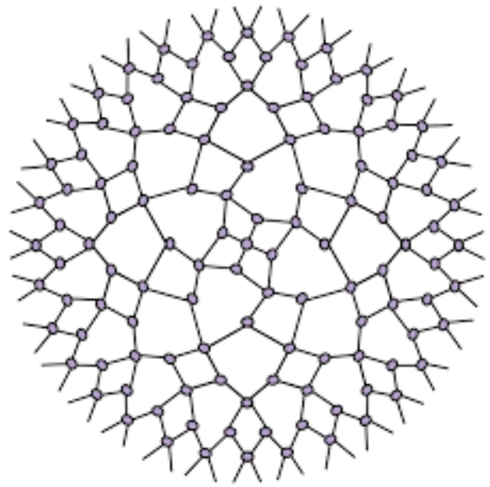
Based on 1804.00441 &1904.06871

2019.10.11 SEU



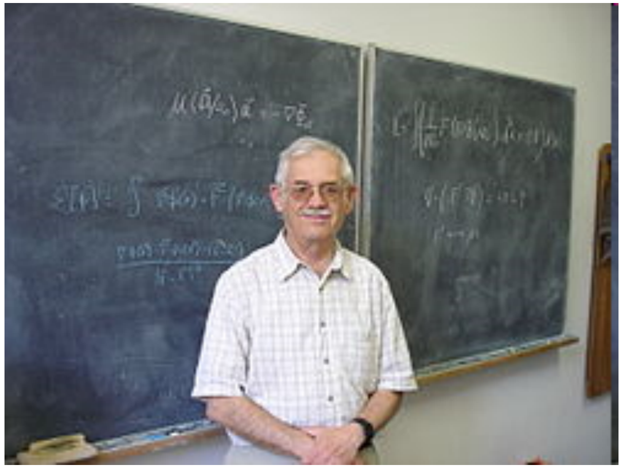
Outline

- Holographic entanglement entropy and tensor network: A review
- Quantum bit threads of MERA in large c
- Applications to HEoP
- Summary

Black hole entropy

From the first law we see BH has entropy, the Bekenstein-Hawking entropy(Bekenstein 1973)

$$S = \frac{Ac^3}{4G\hbar}$$



Usually entropy is the logarithm of phase space volume, but BH entropy scales like area

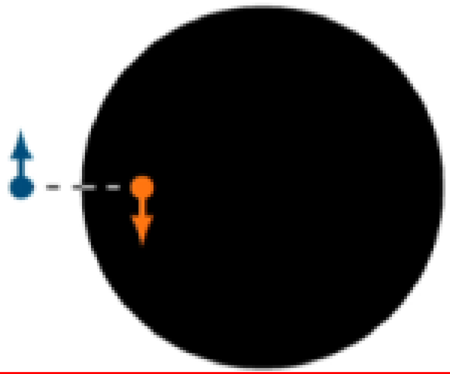
Holography?

D dimensional theory of gravity is equivalent to (D-1) dimensional quantum field theory without gravity. **All d.o.f. of BH are encoded into the surface of the BH**(93 ‘t Hooft) . One example is AdS/CFT correspondence, which states that d.o.f. of 5d AdS spacetime can be described by the ones of a CFT living on its 4d boundary (97 Maldacena)

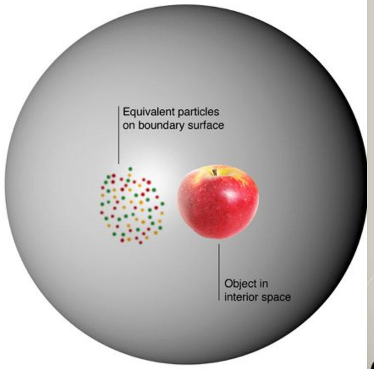
$$S = \frac{A c^3}{4 G \hbar}$$

Entanglement?

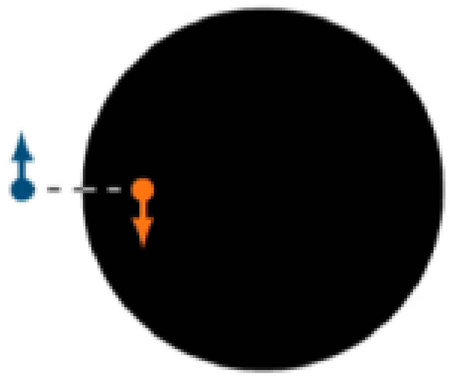
The black hole interior is entangled with its exterior radiations. **This naturally give the area scaling of the entropy** (Sorkin; Bombelli,Koul,Lee&Sorkin)



Holography

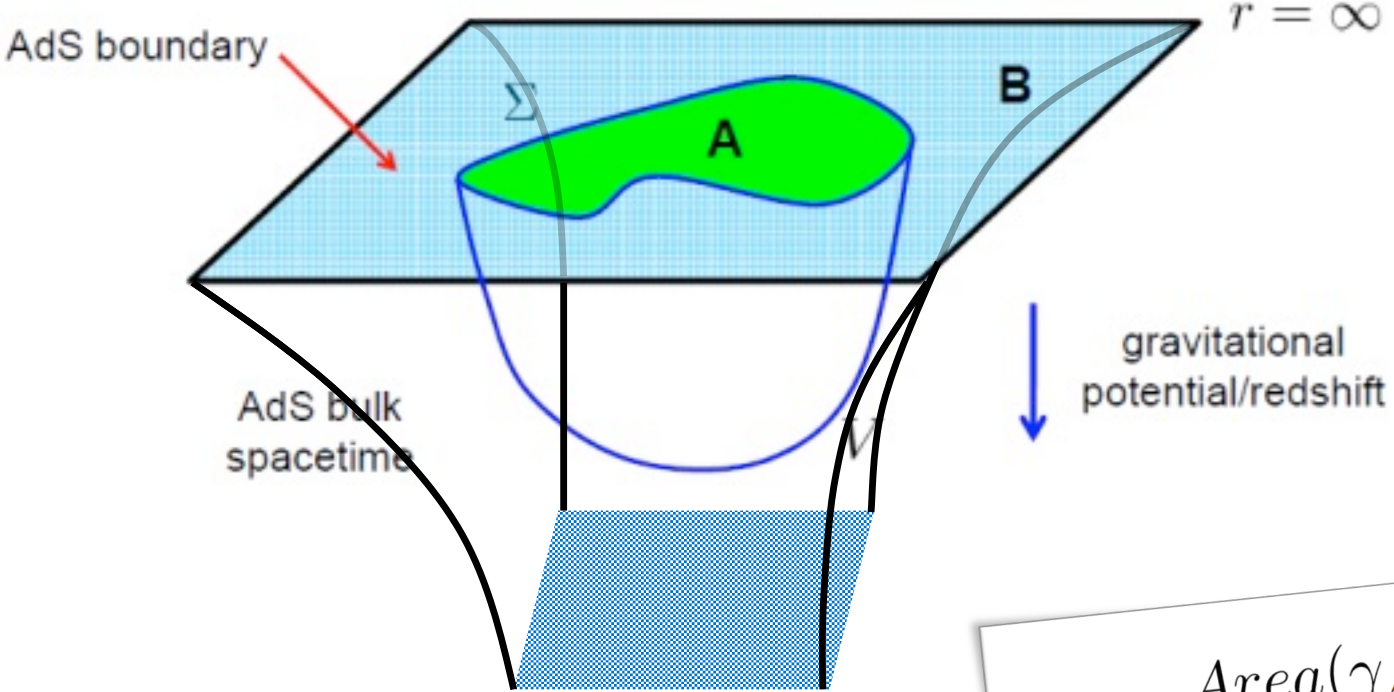


Entanglement



(Ryu & Takayanagi `06)

Holographic Entanglement Entropy:

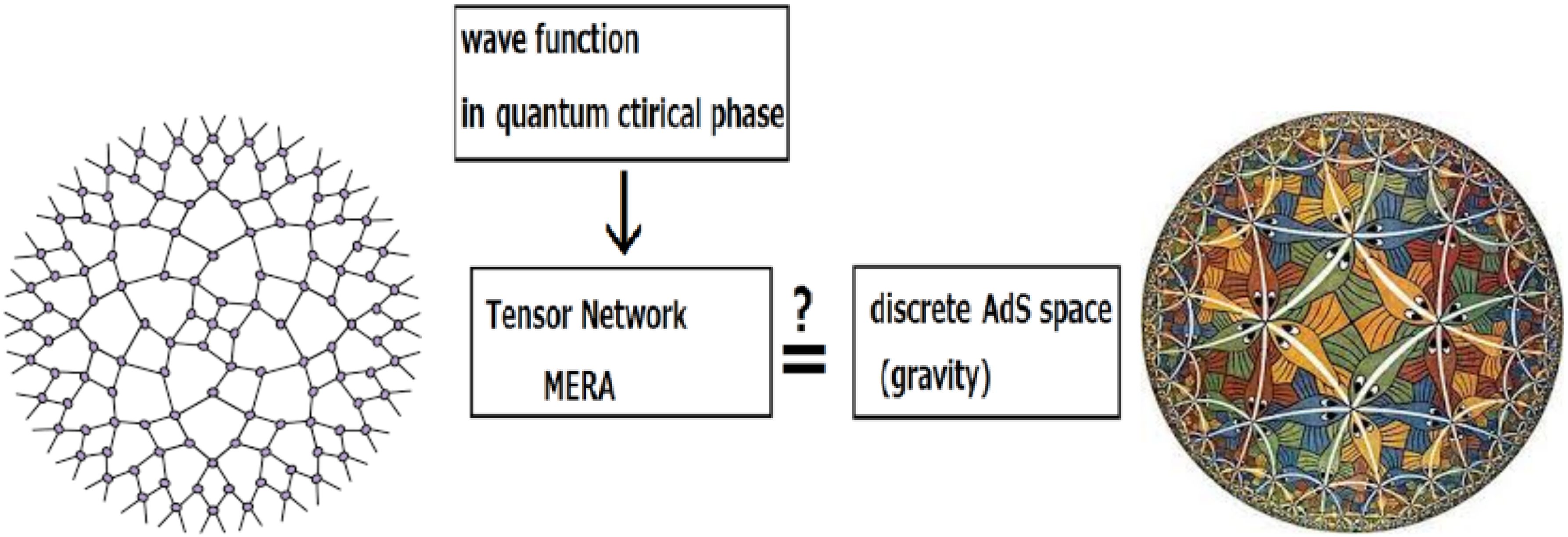


Entanglement entropy between A & B on a time slice is given by

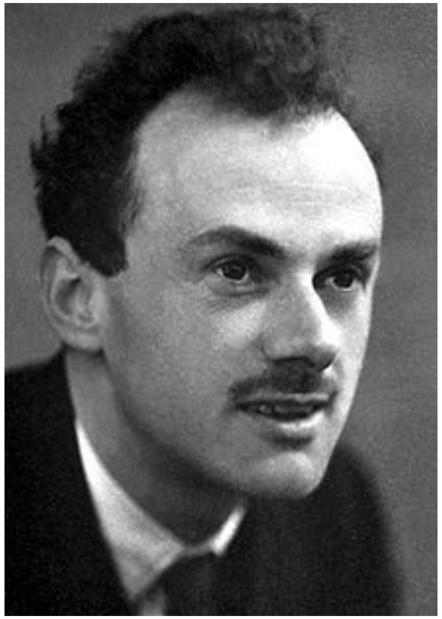
$$S_A = \frac{Area(\gamma_A)}{4G_N^{d+2}}$$

Holography & tensor network

Recently the relation between **Tensor Network of wave function in quantum critical phase** and discrete **Anti de Sitter (AdS) space** has been suggested. *(B.Swingle '2009)*



Why tensor network?

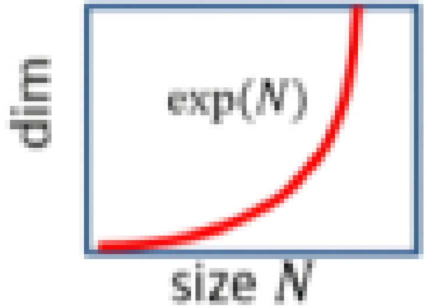


"The fundamental laws necessary for the mathematical treatment of a large part of physics and the whole of chemistry are thus completely known, and the difficulty lies only in the fact that application of these laws leads to equations that are too **complex** to be solved."

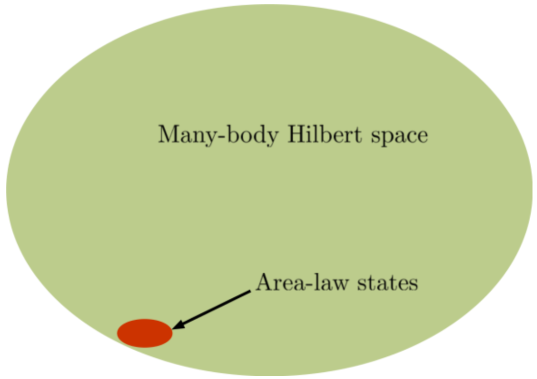
—**Paul Dirac**

$$i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$$

Schrodinger equation



Tensor network



For many body system, only a tiny part of the Hilbert space is relavant. Many d.o.f. are irrelavant to the ground state in question.

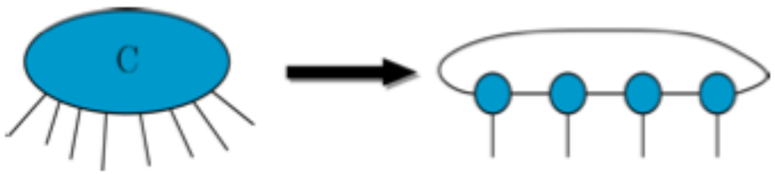
We can reduce the Hilbert space

Decompose big tensor into a set of small ones

For a ground state

$$|\psi\rangle = \sum_{i_1 i_2 \ldots i_N} C_{i_1 i_2 \ldots i_N} |i_1 i_2 \ldots i_N\rangle \longrightarrow \sum_{i_1 i_2 \ldots i_N} \mathrm{Tr}\left[A_{i_1}^{(1)} A_{i_2}^{(2)} \ldots A_{i_N}^{(N)}\right] |i_1 i_2 \ldots i_N\rangle$$

MPS decomposition



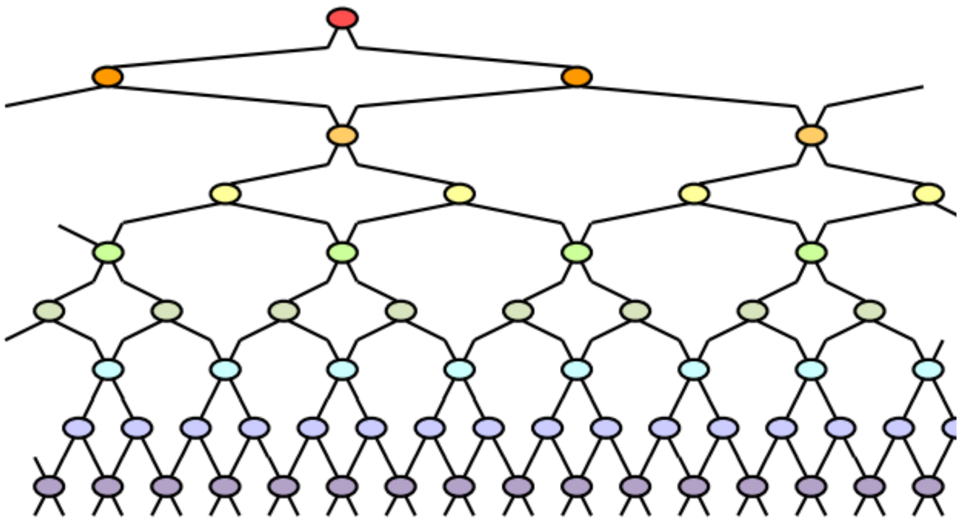
$O(d^N)$ $\qquad$ $O(Nd\chi^2)$

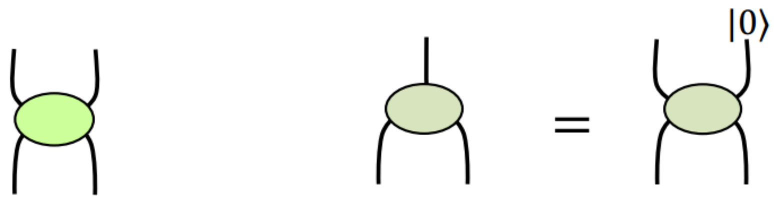
MERA

G.Vidal PRL, 2007, 99: 220405,
PRL, 2008, 101: 110501.

Multi-scale entanglement renormalization ansatz

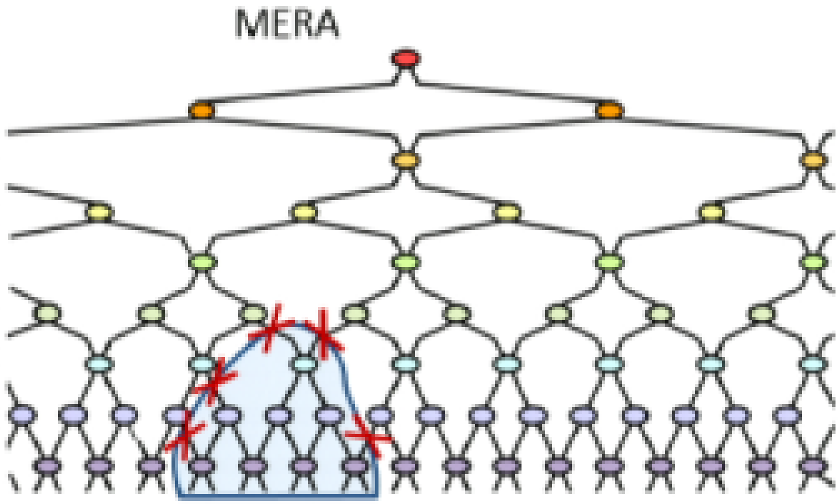
(MERA)



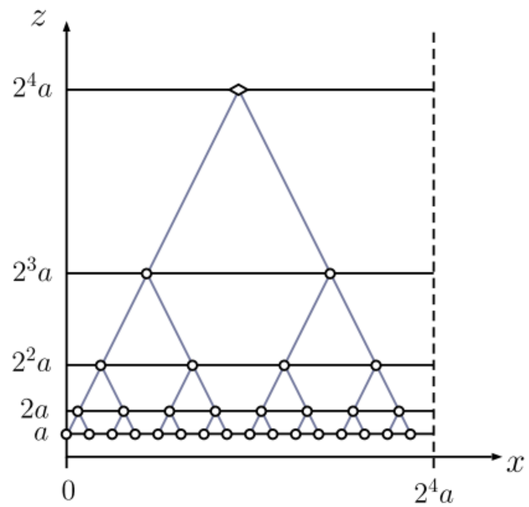


MERA as AdS/CFT

B. Swingle,PRD, 2012, 86(6): 065007.
B. Swingle, arXiv:1209.3304



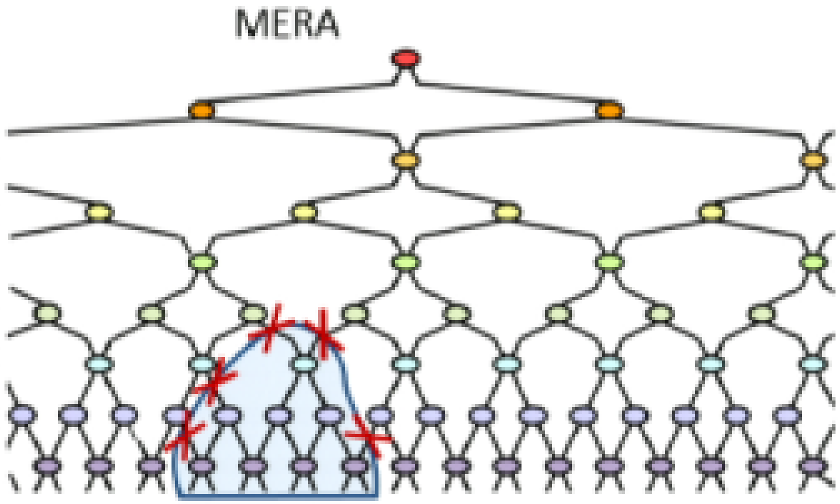
$S(A) \approx \log L$



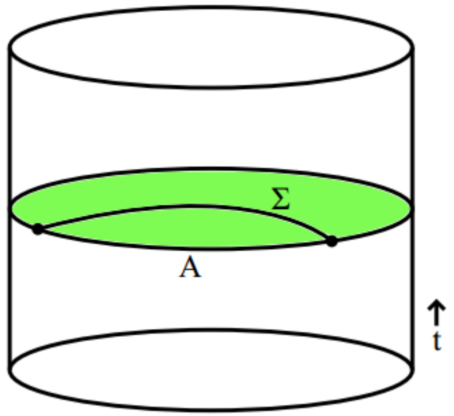
(b)

MERA as AdS/CFT

B. Swingle,PRD, 2012, 86(6): 065007.
B. Swingle, arXiv:1209.3304



$S(A) \approx \log L$



$$S_A = \frac{Area(\gamma_A)}{4G_N^{d+2}} \propto \log L$$

◆ **cMERA ⟹ emergent geometry:**

zero temperature	**AdS time slice**
thermal	**BTZ time slice**

Nozaki *et al* JHEP10(2012)193

W.C. Gan(甘文聪),FWS, M.H.Wu(吴孟和), PLB 760 (2016) 796 , 772 (2017) 464–470

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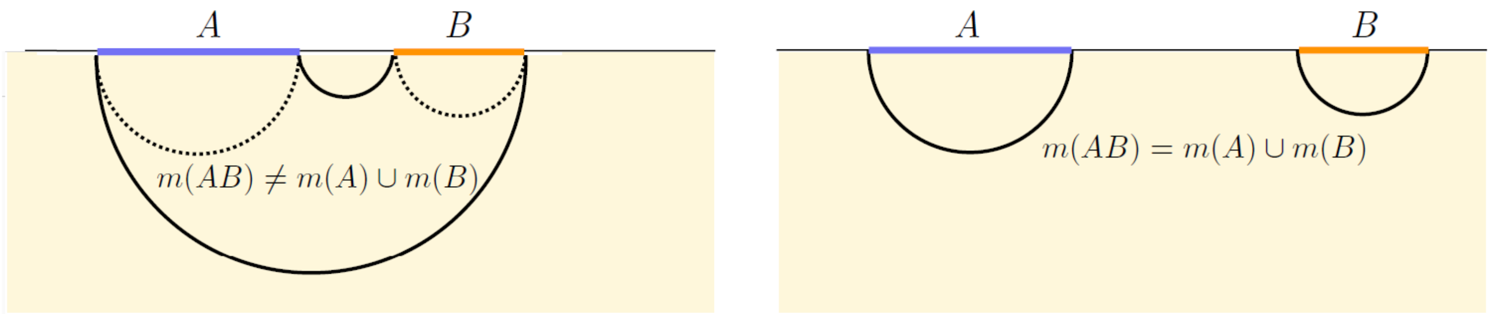
Bit threads

M. Freedman and M. Headrick, Commun.Math. Phys. 352, 407 (2017)

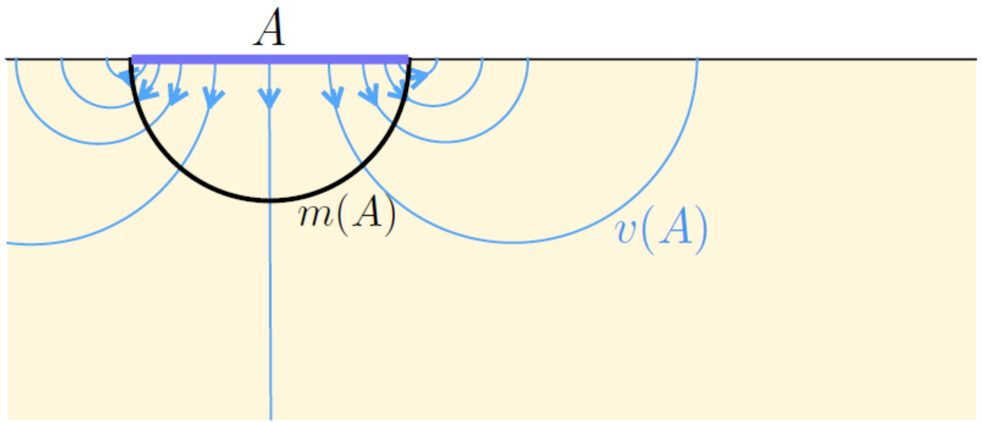
Subtlety: **A strangely discontinuous transition of the bulk minimal surface under continuous deformations of *A*.**



The bits encoding the microstate of A somehow "live on" the minimal surface and strangely jump from one place to another!

Max-Flow/Min-Cut

M. Freedman and M. Headrick, Commun.Math. Phys. 352, 407 (2017)



$$\max_{v} \int_A v = C \min_{m \sim A} \text{area}(m) \, .$$

The area of minimal surface can be replaced by a maximal flux.
Information is encoded in the flow, not the surface.

- Bits information is quantum, but network is classical
- Flow picture implies threads live in a Lorentzian manifold

Quantum bit threads

C-B Chen(陈崇斌), FWS, M-H Wu(吴孟和), arXiv:1804.00441

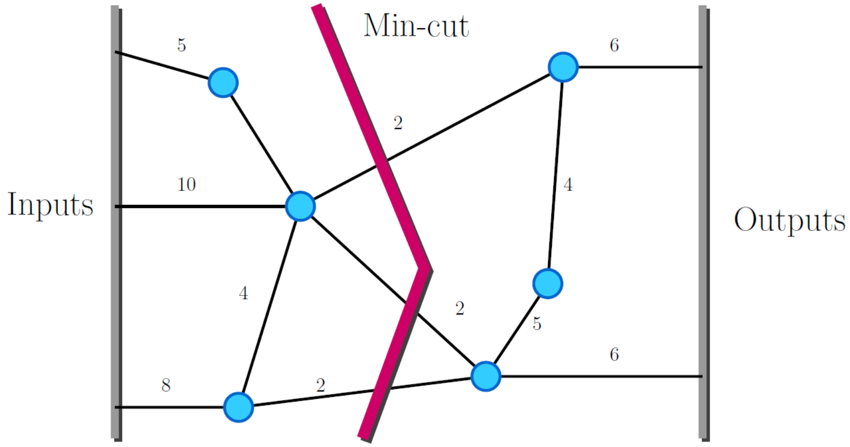
- **A classical network is incomplete for quantum bits!**

Classical Network ⟶ **Tensor Network**

- **AdS time slice is a spacelike hypersurface and don't have causal structure**

AdS time slice ⟶ **Kinematic space**

Quantum Max-flow/Min-cut



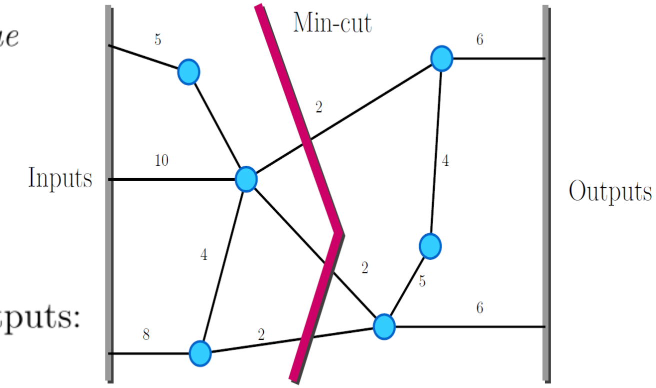
For fixing basic of inputs S and outputs T, each tensor assignments determine a state of this tensor network:

$$|\alpha(G, a; \mathcal{T})\rangle := \sum_{\substack{i_1,\cdots,i_{|S|} \\ j_1,\cdots,j_{|T|}}} C_{\, i_1,\cdots,i_{|S|},j_1,\cdots,j_{|T|}} \times |i_1, \cdots, i_{|S|}\rangle_S |j_1, \cdots, j_{|T|}\rangle_T,$$

S. X. Cui, M. H. Freedman et.al, J.Math. Phys. **57** 062206 (2016)

Definition 1 (Quantum Min-cut). *The quantum min-cut $QMC(G,a)$ is the minimum value of product of capacities over all edge cut sets, i.e.*

$$QMC(G,a) := \min_{A} \prod_{e \in C} a_e.$$



There is a linear map $\beta(G,a;\mathcal{T}) \in V_S^* \bigotimes V_T = Hom(V_S, V_T)$ from inputs to outputs: $V_S \mapsto V_T$ acting on the inputs state:

$$\beta(G,a;\mathcal{T})|i_1,\cdots,i_{|S|}\rangle_S := \sum_{j_1,\cdots,j_{|T|}} C_{i_1,\cdots,i_{|S|},j_1,\cdots,j_{|T|}} \times |j_1,\cdots,j_{|T|}\rangle_T.$$

Briefly $\beta(G,a;\mathcal{T})$ is the map of big tensor consist of tensors from S to T .

Definition 2 (Quantum Max-flow). *For over all tensor assignments, there exists a maximal value of the rank of map $\beta(G,a;\mathcal{T})$ and we define this maximal value as the quantum max-flow:*

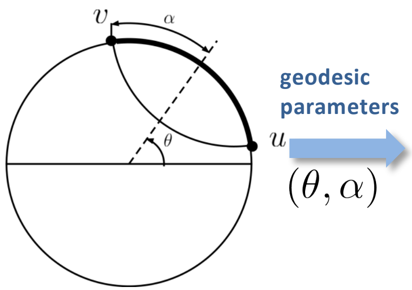
$$QMF(G,a) := \max_{\mathcal{T}} \ rank(\beta(G,a;\mathcal{T})).$$

maximum for all tensor assignments

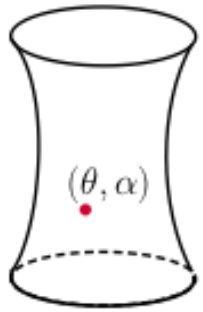
Kinematic space

B. Czech *et al* JHEP **1607**, 100(2016)

Kinematic space is defined as a collection of geodesics in AdS time slice



A point (θ, α) in Kinematic space



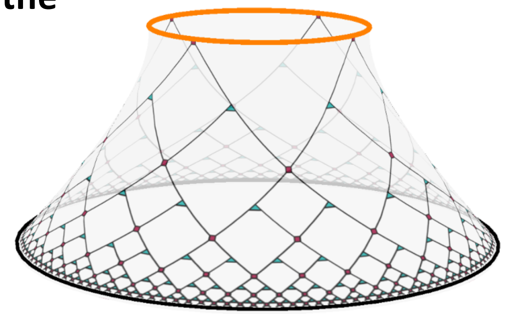
The length of a curve can be measured by the number of the geodesics which intersect it

$$\text{length of } \gamma = \frac{1}{4}\int_K \omega(\theta,\alpha) n_\gamma(\theta,\alpha)\,,$$

$\omega(\theta, \alpha)$**; the measure of the Kinematic space， determining the structure of the space**

$$u = \theta - \alpha$$
$$v = \theta + \alpha$$

$$\omega(\theta,\alpha) = \frac{\partial^2 S(u,v)}{\partial u \partial v} du \wedge dv = \frac{1}{2\sin^2(\frac{v-u}{2})} du \wedge dv$$



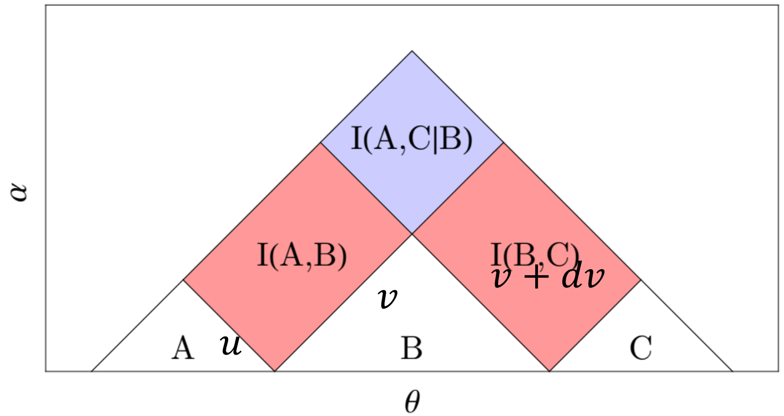
Kinematic space is a de Sitter space! A Lorentzian manifold with causal structure.

Kinematic space VS MERA

B. Czech *et al* JHEP **1607**, 100(2016)

For causal structure consideration, kinematic space should be viewed as the corresponding geometry of the MERA

The conditional mutual information can be obtained by counting the number of edges which is the net reduction of edges through a causal diamond from bottom up



$$I(A, C|B) = S(AB) + S(BC) - S(B) - S(ABC)$$
$$\approx \frac{\partial^2 S(u,v)}{\partial u \partial v} du dv$$

- **The volume of a Kinematic space is determined by # of isometries**

$$\mathcal{D}(\text{isometry}) = I(A, C|B)$$

- **Metric of MERA is given by the one of Kinematic：**

$$ds^2_{MERA} = I(\Delta u, \Delta v|B) \xrightarrow{MERA} (\#\text{ of isometry}) \Delta u \Delta v$$

QMF/QMC

C-B Chen(陈崇斌), FWS, M-H Wu(吴孟和), arXiv:1804.00441

Now question is if QMF=QMC?

In generaly, QMF/QMC theorem cannot be satisfied, instead

$$QMF(G,a) \leq QMC(G,a)$$

which is different from the classical case. This implies some quantum effects.

When QMF=QMC?

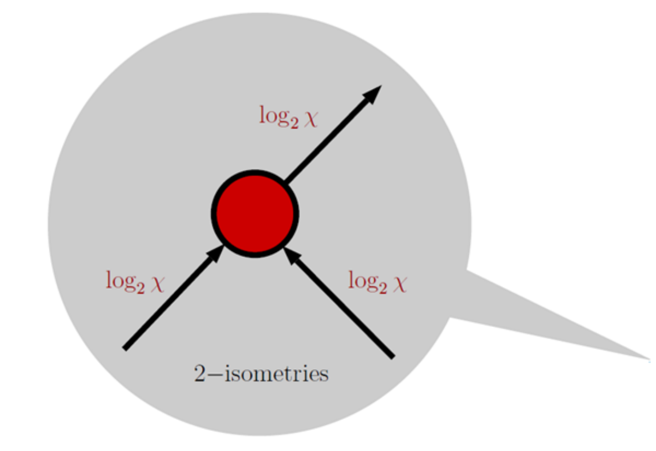
(i) If the type of the tensors is not constraint, then we have
For a given graph G(V,E), if the capacity a of each edge is a power of d, where d is an positive integer, then QMF=QMC.

(ii) If the type of the tensors is fixed, then:
QMF=QMC is "asymptotically" true in the large c($\log \chi \simeq \frac{c}{6}$) limit

$$QMF(G,\chi,O) = QMC(G,\chi) \cdot (1 - o(1))$$

M. B. Hastings,arXiv:1603.03717

Entanglement entropy flow in MERA tensor network



C-B Chen(陈崇斌), FWS, M-H Wu(吴孟和), arXiv:1804.00441

There are $\log \chi$ (bits) entanglement entropy flow in each edge of tensor network.

From bottom up, $2\log \chi$ (bits) entropy flow become $\log \chi$ (bits) entropy flow

By introducing a density of the isometry for flow (bits) conservation:

$$|\rho| \leq \rho_M \,, \qquad \text{(Finite)}$$

$$\nabla_\mu f^\mu = -\rho \qquad \text{(conservation)}$$

ρ: density of isometries, the sources of sinks of the flow

HEE from quantum bit threads

- **Subadditivity**

$$S(A) + S(B) \geq \int_{C_A} f(AB) + \int_{C_B} f(AB)$$
$$\geq \int_{C_{AB}} f(AB) = S(AB).$$

- **Strong subadditivity**

$$I(A:C|B) = \int_{C_{AB}} f(A,B,C) + \int_{C_{BC}} f(A,B,C)$$
$$- \int_{C_B} f(A,B,C) - \int_{C_{ABC}} f(A,B,C)$$
$$= S(AB) + S(BC) - S(B) - S(ABC)$$
$$= -\int_D \rho \geq 0.$$

- **Araki-Lieb inequality**

$$S(AB) + S(A) \geq \int_{AB} f(B,A',A) + \int_{D_{AB}} \rho$$
$$+ \int_{A'} f(B,A',A) + \int_{D_{A'}} \rho$$
$$\geq \int_{A'AB} f(B,A',A) + \int_{D_{A'AB}} \rho$$
$$\geq \int_{D_B} \rho + \int_B f(B,A',A) = S(B).$$

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Entanglement of purification

- **S_A does not measure entanglement when ρ_{AB} is a mixed state**
- **Long distance quantum teleportation or global quantum key distribution need to distribute a certain supply of pairs of particles in a maximally entangled state to two distant users**

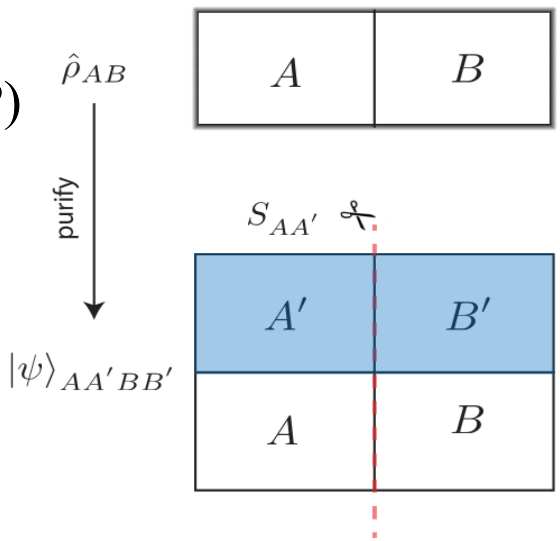
→ **Entanglement of purification**

B. M. Terhal *et al*, J. Math. Phys. 43 (2002) 4286

Let $|\psi\rangle \in H_{AA'} \otimes H_{BB'}$ be a purification of ρ_{AB}, so that $\mathrm{Tr}_{A'B'}|\psi\rangle\langle\psi| = \rho_{AB}$, the entanglement of purification (EoP) is

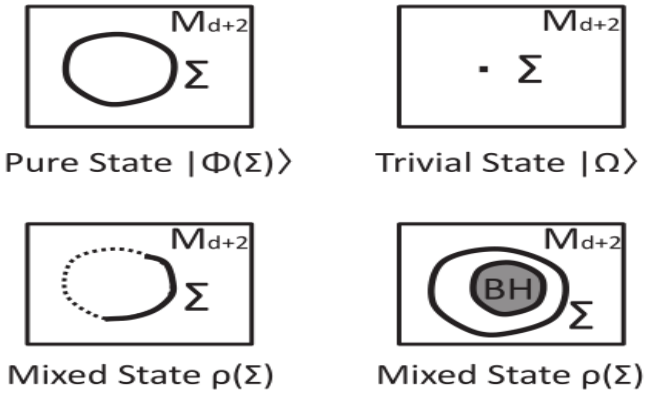
$$E_P(A:B) = \min_{\psi, A'B'} S_{AA'}$$

where $S_{AA'}$ is Von Neumann entropy.



Holographic entanglement of purification

◆ Surface/state correspondence

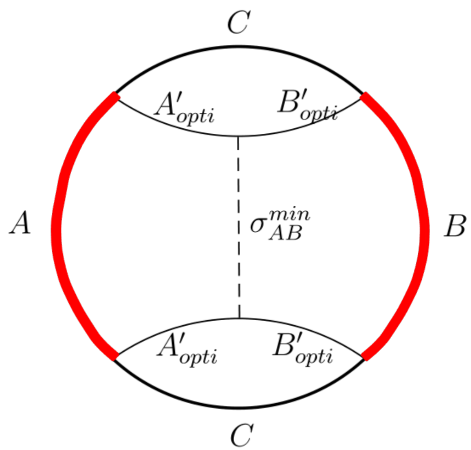


M. Miyaji *et al*, PTEP 2015(2015)7, 073B03, PRL 115, no.17, 171602 (2015)

◆ In AdS/CFT, the HEoP is conjectured to be given by

$$E_P(A:B) = \frac{\mathrm{area}(\sigma_{AB}^{\min})}{4G_N} \equiv E_W(A:B)$$

where $\sigma_{AB}^{\min}$ is the minimal cross section on the entanglement wedge r_{AB} .



K. Umemoto and T. Takayanagi, Nature Phys. 14, no. 6, 573 (2018)

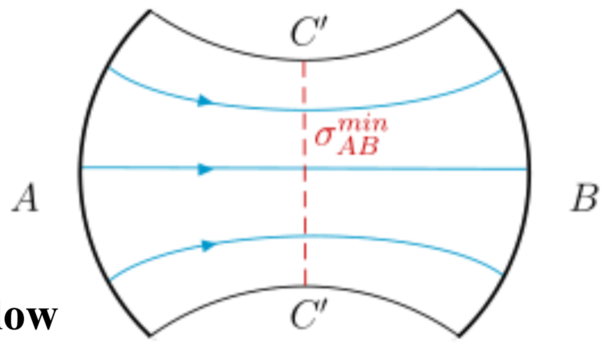
Bit thread formulation of HEoP

D-H Du(杜东辉), C-B Chen (陈崇斌), FWS, 1904.06871, JHEP 08(2019)140

- **Define a flow from A to its complement, a vector field on the entanglement wedge r_{AB}, satisifying**

$$\nabla\cdot\upsilon_{\mathrm{AB}}=0, |\upsilon_{\mathrm{AB}}|\le\frac{1}{4G_N},$$

$$\upsilon_{AB}=-\upsilon_{AB}, \hat{\mathrm{n}}\cdot\upsilon_{AB}=0 \quad \text{on } m_{\mathrm{AB}}$$



- **We asume that the EoP can be written as the flux of a max flow from A to B**

$$\mathrm{E_P}(A:B)=\max_{\substack{\upsilon_{AB}:\\ \hat{n}\cdot\upsilon_{AB}|_{m_{AB}}=0}}\int_A\upsilon_{AB}$$

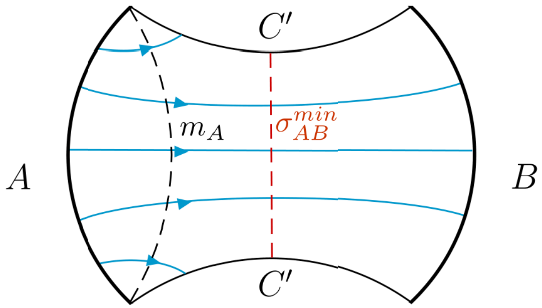
M. Ghodrati, X.-M. Kuang, B. Wang, C.-Y. Chen, Y.-T. Zhou, arXiv:1902.02475

- **Using the generalized Riemannian MFMC theorem**

$$\max_{\substack{v_{AB}:\\ \hat{n}\cdot v_{AB}|m_{AB}=0}}\int_A v_{AB}=\min_{\substack{\sigma_{AB}\sim A\\ rel\ m_{AB}\ on\ \partial r_{AB}}}\frac{\mathrm{area}(\sigma_{AB})}{4G_N}\equiv E_W(A:B)$$

M. Headrick and V. E. Hubeny, Class. Quant. Grav. 35, no. 10, 10 (2018)

Bit thread formulation of the quantum advantage of dense code (QAoDC)



$\tilde{v}_{A(B,C')}$

- Choose a flow on r_{AB} that simultaneously maximizes the flux from A to (BC') and the flux from A to B, satisfying

$$\nabla \cdot \tilde{\upsilon}_{A(B,C')} = 0, \left|\tilde{\upsilon}_{A(B,C')}\right| \le \frac{1}{4G_N}, \tilde{\upsilon}_{A(B,C')} = -\tilde{\upsilon}_{(B,C')A}$$

For this flow configuration, we have

$$S(A) = \int_A \tilde{\upsilon}_{A(B,C')} = \int_B \tilde{\upsilon}_{(B,C')A} + \int_{C'} \tilde{\upsilon}_{(B,C')A} = E_P(A:B) + \int_{C'} \tilde{\upsilon}_{(B,C')A}$$

- Recalling that for pure tripartite state, we have $S(A) = E_P(A:B) + \Delta(C'>A)$[1]

The QAoDC can be written as $\Delta(B>A) \equiv S(A) - inf_{\Lambda_B} S[(I_A \otimes \Lambda_B)\rho_{AB}] = sup_{\Lambda_B} I'(B\rangle A).$

$$\Delta(C'>A) = \int_{C'} \tilde{\upsilon}_{(B,C')A}$$

1. M. Horodecki and M. Piani, J. Phys. A: Math. Theor. 45 (2012) 105306

Flow-based proofs of the properties of EoP

(i) The E_P is bounded above by the entanglement entropy

$$E_P(A:B) \leq \min(S(A), S(B))$$

(ii) The E_P is monotonic

$$E_P(A:BC) \geq E_P(A:B)$$

(iii) The E_P is bounded below by half the mutual information

$$E_P(A:B) \geq \frac{I(A:B)}{2}$$

(iv) The E_P is polygamous for a tripartite pure state

$$E_P(A:B) + E_P(A:C) \geq E_P(A:BC)$$

(v) For a tripartite system,

$$E_P(A:BC) \geq \frac{I(A:B)}{2} + \frac{I(A:C)}{2}$$

And more

(i) Flow based proof of the monogamy relation of QAoDC with the EoP for the tripartite state:

$$\mathrm{S(A)} \geq E_P(A:B) + \Delta(C > A)$$

(ii) A new lower bound for S(AB) in terms of QAoDC, which is tighter than the one given by the Araki-Lieb inequality: $|S(A) - S(B)| \leq S(AB)$

$$\Delta(\mathrm{C} > A) + \Delta(C > B) \leq S(AB)$$

(iii) A new inequality for EoP: $E_P(A:BC) \leq E_P(B:AC) + E_P(C:AB)$

Summary

- **Tensor network provide a possible perspective on the relation between holographic space and entanglement**
- **HEE admits a bit-thread interpretation**
- **Classical bit-thread picture is incomplete, quantum version should be introduced.**
- **Bit-thread interpretation can be applied to some fields, such as holographic entanglement of purification.**

Thank you

Some pictures and slides come from other author's slides, acknowledgements are also given to them.