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Article

Quasigroups, Braided Hopf (Co)quasigroups and Radford's Biproducts of Quasi-Diagonal Type

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Abstract: Given the Yetter–Drinfeld category over any quasigroup and a braided Hopf coquasigroup in this category, we first mainly study the Radford's biproduct corresponding to this braided Hopf coquasigroup. Then, we investigate Sweedler's duality of this braided Hopf coquasigroup and show that this duality is also a braided Hopf quasigroup in the Yetter–Drinfeld category, generalizing the main result in a Hopf algebra case of Ng and Taft's paper. Finally, as an application of our results, we show that the space of binary linearly recursive sequences is closed under the quantum convolution product of binary linearly recursive sequences.

Keywords: groups; Yetter–Drinfel's category; braided Hopf (co)quasigroups; symmetric category; binary linearly recursive sequence

MSC: 16T05; 16W99

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1. Introduction

The concept of a Hopf algebra contains a symmetry between its algebraic structure, and its coalgebraic structure and has many important applications (see [1–4]). A theory of linearly recursive sequence-related Hopf algebras was first studied in 1980 by Peterson and Taft (see, [5]) and later investigated in the papers [6–9].

The theory of braided Hopf algebras can be used to obtain a structure of Radford's biproduct, which has an important application in the classification of finite-dimensional pointed Hopf algebras (see [10]) and can provide a solution to the quantum Yang–Baxter equation (see [1,2]).

Over the last few years, there have been substantial developments in non-associative Hopf algebras (see [11,12]), non-coassociative Hopf algebras (see [11,13]), quantum quasigroups (see [14,15]), and so on. These have motivated some initial moves toward a unification of these two topics, non-associative and non-coassociative Hopf algebras, since none of the non-associative objects or the non-coassociative objects proposed up to now have been able to maintain the self-duality of the Hopf algebra concept. More recently, there have been developments in topics related to these Hopf algebras (see [16–21]).

The aim of the current paper is to study the Sweedlers' duality of a braided Hopf quasigroup in a symmetrical category and related Radford's biproducts. We also give an application of our theory to binary linearly recursive sequences. Our paper has three different settings from [6]: 1. we consider the Yetter–Drinfeld category over a quasigroup G, and not the left module category of G; 2. we consider binary linearly recursive sequences as an application, not linearly recursive sequences; and 3. we consider Hopf (co)quasigroups, not Hopf algebras. This article is organized as follows.

Background on quasigroups and loops, symmetric monoidal categories, the Yetter–Drinfel'd category and Hopf (co)quasigroups is provided in Section 2. In particular, we show that the Hopf (co)quasigroups are unital $\mathbb{H}$-bialgebras (see Proposition 1).

In Section 3, we study the notion of a braided Hopf coquasigroup H (see Definition 3) and the related Radford's biproduct $H \boxdot G$ (see Theorems 2 and 3).

In Section 4, we mainly study Sweedler's duality of the braided Hopf coquasigroup to obtain a new braided Hopf quasigroup, as investigated in [11]. The main result can be found in Theorem 5. Finally, we show in Section 5 that the space of binary linearly recursive sequences is closed under the quantum convolution product of binary linearly recursive sequences (see Theorems 6 and 7), generalizing the main result in [6] for the linear case.

Throughout this paper, let $\mathbb{F}$ be a fixed field. We will work over $\mathbb{F}$. Let C be a coalgebra with a coproduct Δ. Throughout, we will use the Heyneman–Sweedler's notation (see [4]), $\Delta(c) := \sum c_{(1)} \otimes c_{(2)}$ for all $c \in C$, for a coproduct, or, we will simply write $\Delta(c) := \sum c_1 \otimes c_2$.

2. Preliminaries

2.1. Symmetric Monoidal Categories

Recall from [22] that a monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, k, a, l, r)$ is a category $\mathcal{C}$ armed with a functor $\otimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$ (the tensor product), an object $k \in \mathcal{C}$ (the unit object) and natural isomorphisms $a = a_{U,V,W} : (U \otimes V) \otimes W \longrightarrow U \otimes (V \otimes W)$ for all $U, V, W \in \mathcal{C}$ (the associativity constraint), and invertible morphisms $l = l_U : k \otimes U \longrightarrow U$, $r = r_U : U \otimes K \longrightarrow U$ for any $U \in \mathcal{C}$ (the left unit constraint) and the right (unit constraint, respectively) such that the following two identities are satisfied for all $U, V, W, X \in \mathcal{C}$:

$$a_{U,V,W\otimes X} a_{U\otimes V,W,X} = (id_U \otimes a_{V,W,X}) a_{U,V\otimes W,X} (a_{U,V,W} \otimes id_X); \tag{1}$$

$$(r_U \otimes id_V) = (id_U \otimes l_V) a_{U,I,V}. \tag{2}$$

A monoidal category $\mathcal{C}$ is strict when all the constraints are identities. It is well known that each monoidal category is equivalent to a strict monoidal category. By $(\mathcal{C}, \otimes, k)$, we denote a strict monoidal category. For every object M in $\mathcal{C}$, there are two endofunctors: $M \otimes - : \mathcal{C} \longrightarrow \mathcal{C}$ and $- \otimes M : \mathcal{C} \longrightarrow \mathcal{C}$. The category $\mathcal{C}$ is braided if for every object M in $\mathcal{C}$ we have natural isomorphisms:

$$\mathbf{b}_{M,-} : M \otimes - \longrightarrow - \otimes M, \quad \mathbf{b}_{-,M} : - \otimes M \longrightarrow M \otimes -$$

which verify the following:

$$\begin{cases} \mathbf{b}_{-,V}(W) = \mathbf{b}_{-,W}(V), \\ \mathbf{b}_{V\otimes W,-} = (\mathbf{b}_{V,-} \otimes W) \circ (V \otimes \mathbf{b}_{W,-}), \\ \mathbf{b}_{-,V\otimes W} = (V \otimes \mathbf{b}_{-,W}) \circ (\mathbf{b}_{-,V} \otimes W). \end{cases}$$

As a consequence, it is easy to determine that $\mathbf{b}_{V,k} = \mathbf{b}_{k,V} = id_V$. The category $\mathcal{C}$ is called a symmetric braided monoidal category (simply, symmetric category) if

$$\mathbf{b}_{V,W} \circ \mathbf{b}_{W,V} = id, \qquad \forall\ V, W \in \mathcal{C}.$$

Throughout, $\mathcal{C}$ denotes a symmetric category $(\mathcal{C}, \otimes, k)$ with the braided $\mathbf{b}$. We will work on $\mathcal{C}$.

We denote $\mathbf{LS}_{\mathbb{F}}$ as the category of linear spaces and linear maps over $\mathbb{F}$. Then, $(\mathbf{LS}_{\mathbb{F}}, \otimes_{\mathbb{F}}, \mathbb{F})$ is a symmetric category.

2.2. Hopf (Co)quasigroups

The notions of (co)algebras in this subsection refer to the paper [12].

A coalgebra (C, Δ) is a vector space C equipped with a linear map $\Delta : C \longrightarrow C \otimes C$. The coalgebra (C, Δ) is called coassociative if $(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta$. A counital coalgebra (C, Δ, ε) is a vector space C equipped with two linear maps $\Delta : C \longrightarrow C \otimes C$ and $\varepsilon : C \longrightarrow \mathbb{F}$ such that $(id \otimes \varepsilon)\Delta = id = (\varepsilon \otimes id)\Delta$.

Recall from [11] that a Hopf quasigroup H is a counital coassociative coalgebra (H, Δ, ε) and unital algebra (H, ∇, μ) (not necessarily associative) armed with a linear map $S : H \longrightarrow H$ (called antipode) such that

$$\sum S(h_{(1)})(h_{(2)}g) = \varepsilon(h)g = \sum h_{(1)}(S(h_{(2)})g), \tag{3}$$
$$\sum (hg_{(1)})S(g_{(2)}) = h\varepsilon(g) = \sum (hS(g_{(1)}))g_{(2)} \tag{4}$$

for any $h, g \in H$.

Dually, an algebra (A, m) is a vector space A equipped with a linear map $m : A \otimes A \longrightarrow A$. The algebra (A, m) is called associative if $m(id \otimes m) = m(m \otimes id)$. It is customary to write $m(a \otimes b) = ab$, $\forall a, b \in A$. A unital algebra (A, m, μ) is a vector space A equipped with two linear maps $m : A \otimes A \longrightarrow A$ and $\mu : \mathbb{F} \longrightarrow A$ such that $m(id \otimes \mu) = id = m(\mu \otimes id)$. Generally, we write $1 \in A$ for $\mu(1_{\mathbb{F}})$.

Recall from [11] that a Hopf coquasigroup H is a unital associative algebra (H, m, μ) and a counital coalgebra (H, Δ, ε) (not necessarily coassociative) equipped with a linear map $S : H \longrightarrow H$ (called antipode), such that Δ is an algebra homomorphism and the following formulas hold:

$$\sum S(h_{(1)})h_{(2)(1)} \otimes h_{(2)(2)} = 1 \otimes h = \sum h_{(1)}S(h_{(2)(1)}) \otimes h_{(2)(2)}, \tag{5}$$
$$\sum h_{(1)(1)} \otimes S(h_{(1)(2)})h_{(2)} = h \otimes 1 = \sum h_{(1)(1)} \otimes h_{(1)(2)}S(h_{(2)}) \tag{6}$$

for all $h \in H$.

Remark 1. *A Hopf (co)quasigroup is a Hopf algebra if and only if its (co)product is (co)associative. There are two important sources for this generalized Hopf algebra, as follows.*

Definition 1 ([12], Definition 2)**.** *An $\mathbb{H}$-bialgebra $(H, m, \Delta, \varepsilon, \diagdown, \diagup)$ is a counital bialgebra $(H, m, \Delta, \varepsilon)$ with two extra bilinear operations, the left and right divisions:*

$$\diagdown : H \times H \longrightarrow H, (x, y) \mapsto x \diagdown y, \text{ and } \diagup : H \times H \longrightarrow H, (x, y) \mapsto x \diagup y$$

such that

$$\sum x_{(1)} \diagdown (x_{(2)}y) = \varepsilon(x)y = \sum x_{(1)}(x_{(2)} \diagdown y), \tag{7}$$
$$\sum (yx_{(1)}) \diagup x_{(2)} = \varepsilon(x)y = \sum (y \diagup x_{(1)})x_{(2)}. \tag{8}$$

A unital $\mathbb{H}$-bialgebra $(H, m, 1, \Delta, \varepsilon \diagdown, \diagup)$ is a unital counital bialgebra $(H, m, 1, \Delta, \varepsilon)$ such that $(H, m, \Delta, \varepsilon, \diagdown, \diagup)$ is an $\mathbb{H}$-bialgebra.

Proposition 1. *(1) Any Hopf quasigroup with antipode S is a unital coassociative $\mathbb{H}$-bialgebra.*

(2) Any Hopf coquasigroup with antipode S is a unital associative $\mathbb{H}$-bialgebra.

Moreover, in these two cases, $x\diagdown y = S(x)y$ and $x\diagup y = xS(y)$.

Proof. (1) The natural candidate for the left division is $x\diagdown y = S(x)y$. Actually, through Equation (3), we have Equation (7). Similarly, define a right division $x\diagup y = xS(y)$, then Equation (4) implies Equation (8).

(2) Similar to (1), one lets $x\diagdown y = S(x)y$ and $x\diagup y = xS(y)$. By applying $(id \otimes \varepsilon)$ to both sides of Equation (5), we have that

$$\sum S(h_{(1)})h_{(2)} = \varepsilon(h) = \sum h_{(1)}S(h_{(2)})$$

and by the associativity, we have that

$$\sum S(h_{(1)})(h_{(2)}a) = \varepsilon(h)a = \sum h_{(1)}(S(h_{(2)}a),$$

and so, Equation (7) holds. The same applies to Equation (8).

Recall from [23] that a quasigroup is a non-empty set G with a product, identity e and with the property that for each $g \in G$ there is $g^{-1} \in G$ such that

$$g^{-1}(gh) = h, \qquad (hg)g^{-1} = h, \qquad \text{for all } h \in G.$$

A quasigroup is flexible if $g(hg) = (gh)g$ for any $g, h \in G$ and alternative if also $g(gh) = (gg)h, g(hh) = (gh)h$ for all $g, h \in G$. It is called Moufang if $g(h(gl)) = ((gh)g)l$ for all $g, h, l \in G$. It is easy to see that in any quasigroup G, one has unique inverses and

$$(g^{-1})^{-1} = g, \qquad (gh)^{-1} = h^{-1}g^{-1}, \qquad \text{for all } g, h \in G.$$

□

Example 1. *(i) Given that $G_5 = \{1, 2, 3, 4, 5\}$, then G_5 is a quasigroup with product $\cdot$ given by the following Cayley table Table 1).*

Table 1. Cayley table of a quasigroup G_5.

·	1	2	3	4	5
1	1	2	3	4	5
2	2	1	4	5	3
3	3	5	1	2	4
4	4	3	5	1	2
5	5	4	3	2	1

(ii) Let G be a quasigroup. Then, it follows from ([11], Proposition 4.7) that $H = \mathbb{F}[G]$ is a Hopf quasigroup with a linear extension of the product and $\Delta(h) = h \otimes h, \varepsilon(h) = 1$ and $S(h) = h^{-1}$ on the basis of element $h \in G$. Moreover, H is Moufang if G is.

For example, consider G_5 in the item (i), where we have a Hopf quasigroup $\mathbb{F}[G_5]$ with $\Delta(i) = i \otimes i, \varepsilon(i) = 1$ and $S(i) = i$ with $i \in G_5$.

(iii) In (ii), if G is a finite quasigroup, then $(\mathbb{F}[G])^*$ is a Hopf coquasigroup (see [11]). Explicitly, a basis of $(\mathbb{F}[G])^*$ is the set of projections $\{p_g \mid g \in G\}$, that is, for any $g \in G$ and $x = \sum_{h\in G} \alpha_h h \in \mathbb{F}[G]$, $p_g(x) = \alpha_g \in \mathbb{F}$. The set $\{p_g\}$ consists of orthogonal idempotents whose

sum is 1. *The coproduct of* $(\mathbb{F}[G])^*$ *is given by* $\Delta(p_g) = \sum_{h\in G} p_{gh^{-1}} \otimes p_h$, *and the counit is given by* $\varepsilon(p_g) = \delta_{1,g}$ *(where* δ *denotes the Kronecker delta) (see [19]).*

(iv) For $(L, [,])$, *a Maltsev algebra over* k *is not of characteristic* 2, 3, *whereby the enveloping algebra* $U(L)$ *in [19] is a Moufang Hopf quasigroup with the structure maps* $\Delta : U(L) \longrightarrow U(L) \otimes U(L), \varepsilon : U(L) \longrightarrow k$ *defined by* $\Delta(x) = x \otimes 1 + 1 \otimes x$ *and* $\varepsilon(x) = 0$ *for all* $x \in L$ *extended to* $U(L)$ *as algebra homomorphisms, and* $S : U(L) \longrightarrow U(L)$ *defined by* $S(x) = -x$ *that is extended as an antialgebra homomorphism (see [11], Propositions 4.8 and 4.9).*

2.3. Yetter–Drinfeld Modules over Quasigroups

Let H be a Hopf quasigroup. Recall from [13] that we say that $(M, \cdot)$ is a left H-quasimodule if M is a vector space and $\cdot : H \otimes M \longrightarrow M$ is a linear map (called the quasi-action) satisfying

$$1 \cdot m = m, \quad \text{and} \quad \sum h_1 \cdot (S(h_2) \cdot m) = \sum S(h_1) \cdot (h_2 \cdot m) = \varepsilon(h)m \tag{9}$$

for all $h \in H$ and $m \in M$.

Given two left H-quasimodules $(M, \cdot)$ and $(N, \cdot)$, a linear map $f : M \longrightarrow N$ is a morphism of left H-quasimodules if $f(h \cdot m) = h \cdot f(m)$ for all $h \in H$ and $m \in M$.

The notion of a left H-comodule is exactly the same as for ordinary Hopf algebras since it only depends on the coalgebraic structure of H. That is, we say that (M, ρ_M) is a left H-comodule if M is a vector space and $\rho_M : M \longrightarrow H \otimes M$ $(m \mapsto \sum m_{(-1)} \otimes m_0)$ is a linear map (called the coaction) satisfying the comodule conditions (see [4]).

We shall denote by ${}_H\mathcal{MQ}$ the category of left H-quasimodules and we will denote by ${}^H\mathcal{M}$ the category of left H-comodules.

Let H be a Hopf quasigroup. Recall from [13] that we say that M is a left-left Yetter–Drinfeld quasimodule over H if M is an object in ${}_H\mathcal{MQ}$ with the action $\cdot$ and an object in ${}^H\mathcal{M}$ with the coaction ρ, which satisfies the following equalities:

$$\sum (a_1 \cdot m)_{(-1)} a_2 \otimes (a_1 \cdot m)_0 = \sum a_1 m_{(-1)} \otimes a_2 \cdot m_0, \tag{10}$$
$$\sum m_{(-1)}(ab) \otimes m_0 = \sum (m_{(-1)}a)b \otimes m_0, \tag{11}$$
$$\sum a(m_{(-1)}b) \otimes m_0 = \sum (am_{(-1)})b \otimes m_0 \tag{12}$$

for all $a, b \in H$ and $m \in M$. The first equation in the above three equations is equivalent to the following equation:

$$\sum (a \cdot m)_{(-1)} \otimes (a \cdot m)_0 = \sum (a_1 m_{(-1)})S(a_3) \otimes a_2 \cdot m_0$$

for all $a, b \in H$ and $m \in M$. In fact, if Equation (10) holds, then we have

$$\begin{aligned} \sum (a \cdot m)_{(-1)} \otimes (a \cdot m)_0 &= \sum (a_1 \cdot m)_{(-1)} \varepsilon(a_2) \otimes (a_1 \cdot m)_0 \\ &\overset{(1.4)}{=} \sum [(a_1 \cdot m)_{(-1)} a_2] S(a_3) \otimes (a_1 \cdot m)_0 \\ &\overset{(1.10)}{=} \sum (a_1 m_{(-1)}) S(a_3) \otimes a_2 \cdot m_0 \end{aligned}$$

and so, we obtain the result. Conversely, it is also true.

Let M and N be two left-left Yetter–Drinfeld quasimodules over H. We say that $f : M \longrightarrow N$ is a morphism of left-left Yetter–Drinfeld quasimodules if f is a morphism of H-quasimodules and H-comodules.

We shall denote by ${}^{H}_{H}\mathcal{YDQ}$ the category of left-left Yetter–Drinfeld quasimodules over H. Moreover, if we assume that M is a left H-module, we say that M is a left-left Yetter–Drinfeld module over H. Obviously, left-left Yetter–Drinfeld modules with the obvious morphisms is a subcategory of ${}^{H}_{H}\mathcal{YDQ}$. This subcategory will be denoted by ${}^{H}_{H}\mathcal{YD}$.

Theorem 1 ([13], Proposition 1.8)**.** *If H is a Hopf quasigroup over $\mathbb{F}$ with a bijective antipode, then ${}^{H}_{H}\mathcal{YDQ}$ is a braided monoidal category with braiding given by a linear map $\mathbf{b}_{M,N} : M \otimes N \longrightarrow N \otimes M$, defined by*

$$\mathbf{b}_{M,N}(m \otimes n) = \sum m_{(-1)} \cdot n \otimes m_0 \tag{13}$$

for all $m \in M$ and $n \in N$.

Let G be a quasigroup. By Example 1 (ii), $\mathbb{F}[G]$ is a Hopf quasigroup. Then, the category ${}^{\mathbb{F}[G]}_{\mathbb{F}[G]}\mathcal{YDQ}$ of left-left Yetter–Drinfeld quasimodules over $\mathbb{F}[G]$ is the category of left $\mathbb{F}[G]$-quasimodules (denoted by ${}_{G}\mathcal{MQ}$), which are G-graded vector spaces $V = \bigoplus_{g\in G} V_g$ such that each V_g is stable under the quasi-action of G, i.e., $h \cdot v \in V_g$ for all $h \in G, v \in V_g$. The G-grading gives rise to a left $k[G]$-comodule structure on V via $\rho : V \longrightarrow \mathbb{F}[G] \otimes V$ given by $\rho(v) = g \otimes v$ for any $v \in V_g$. This forms a category of left $\mathbb{F}[G]$-comodules (denoted by ${}^{G}\mathcal{M}$). The morphisms of ${}^{\mathbb{F}[G]}_{\mathbb{F}[G]}\mathcal{YDQ}$ are the G-linear maps $f : V \longrightarrow W$ with $f(V_g) \subset W_g$ for all $g \in G$. We denote the category ${}^{\mathbb{F}[G]}_{\mathbb{F}[G]}\mathcal{YDQ}$ simply by ${}^{G}_{G}\mathcal{YDQ}$.

As a corollary of Theorem 1, we have the following proposition.

Proposition 2. *Let G be a quasigroup. Then, ${}^{G}_{G}\mathcal{YDQ}$ is a symmetric category with the following monoidal structures:*

$$\begin{cases} \textit{QMC:} & g\cdot(g^{-1}\cdot v) = g^{-1}\cdot(g\cdot v) = v,\ \forall v \in V;\\ \textit{YDC1:} & xg\otimes g\cdot v = gx\otimes g\cdot v,\ \forall v\in V_x;\\ \textit{YDC2:} & (ug)y\otimes v = u(gy)\otimes v,\ \forall v\in V_u;\\ \textit{YDC2:} & g(uy)\otimes v = (gu)y\otimes v,\ \forall v\in V_u;\\ \textit{MC1:} & g\cdot(v\otimes w) := (g\cdot v)\otimes(g\cdot w),\ \forall v\in V, w\in W;\\ \textit{MC2:} & (V\otimes W)_g := \bigoplus_{xy=g} V_x\otimes W_y;\\ \textit{BC:} & \mathbf{b} : V\otimes W\longrightarrow W\otimes V,\ \mathbf{b}(v\otimes w) := (g\cdot w)\otimes v,\ \forall v\in V_g, w\in W \end{cases}$$

for any $V, W \in {}^{G}_{G}\mathcal{YDQ}$ and $g, x, y, u \in G$.

Remark 2 ([13], Example 2.13)**.** *Let G be a quasigroup. Define*

$$A(G)_l = \{u \in G \mid u(gy) = (ug)y \text{ for all } g, y \in G\},$$
$$A(G)_m = \{u \in G \mid g(uy) = (gu)y \text{ for all } g, y \in G\},$$
$$A(G)_r = \{u \in G \mid (gy)u = g(yu) \text{ for all } g, y \in G\}.$$

The sets $A(G)_l$, $A(G)_m$ and $A(G)_r$ are called the left-, middle- and right-associators (nuclei) of G, respectively (see [14]). The intersection of these three sets is called the associator (nucleus) of G and will be denoted by $A(G)$.

3. Braided Hopf Coquasigroups

In this section, G denotes a quasigroup. We will study the notion of a Hopf coquasigroup in ${}^{G}_{G}\mathcal{YDQ}$.

We have the following important example: We denote by $\widehat{G}$ the character quasigroup of all quasigroup homomorphisms from G to the multiplicative group $\mathbb{F}^* = \mathbb{F}\backslash\{0\}$.

Definition 2. *Let $V \in {}^G_G\mathcal{YDQ}$. If there is a basis $x_i, i \in I$, of V and $g_i \in G, \chi_i \in \widehat{G}$ for all $i \in I$ such that*

$$g \cdot x_i = \chi_i(g)x_i \quad \text{and} \quad x_i \in V_{g_i},$$

then we say V is of quasi-diagonal type.

Example 2. *(i) Note that if $\mathbb{F}$ is algebraically closed of characteristic 0 and G is finite, then any finite-dimensional $V \in {}^G_G\mathcal{YDQ}$ is of diagonal type. For the braiding, we have $\mathbf{b}(x_i \otimes x_j) = \chi_j(g_i)x_j \otimes x_i$ for $1 \leq i, j \leq \theta$. Hence, the braiding is determined by the so-called braiding matrix of V*

$$(b_{ij})_{1\leq i,j\leq\theta} := (\chi_j(g_i))_{1\leq i,j\leq\theta}.$$

(ii) We use $\mathbb{F}^\chi_g$ to denote the vector space $\mathbb{F}$ with coaction $x \mapsto g \otimes x$ and action $h \cdot x = \chi(h)x$ for $x \in \mathbb{F}, h \in G$ and $\chi \in \widehat{G}$. Then, $\mathbb{F}^\chi_g \in {}^G_G\mathcal{YDQ}$ if and only if

$$\chi(h)gh = hg\chi(h)$$

for $h, g \in G$. Conversely, any one-dimensional Yetter–Drinfeld module over G arises in this way. If $V \in {}^G_G\mathcal{YDQ}$, then V^χ_g denotes the isotypic component of V of type $\mathbb{F}^\chi_g$.

Similar to ([13], Definition 1.1), we recall the monoidal version of the notion of a Hopf coquasigroup introduced in ([11], Definition 4.1).

Note that a counital coalgebra (H, Δ, ε) in ${}^G_G\mathcal{YDQ}$ means that $\Delta = \{\Delta_g : H_g \longrightarrow H_g \otimes H_g\}_{g\in G}$ and $\varepsilon : H_1 \longrightarrow \mathbb{F}$ such that ε is counital and Δ is not necessarily coassociative.

Definition 3. *Let G be a quasigroup, $H \in {}^G_G\mathcal{YDQ}$, and let H be of quasi-diagonal type. We say that H is a braided Hopf coquasigroup if it is a unital associative algebra $(H, m, 1)$ and a counital coalgebra (H, Δ, ε) such that the following axioms hold:*

(1) H is G-graded vector spaces $H = \bigoplus_{g\in G} H_g$ such that H_g is an algebra and $H_x H_g = 0$ with $x \neq g$.

(2) The morphisms Δ_H and ε_H are algebraic morphisms, i.e.,

$$\varepsilon(1) = 1, \quad \Delta(1) = 1 \otimes 1, \quad \varepsilon(ab) = \varepsilon(a)\varepsilon(b), \tag{14}$$

$$\Delta(xy) = \sum x_1\chi_i(g)y_1 \otimes x_2y_2 \tag{15}$$

for any $a, b \in H_1$ and $x \in H_g, y \in H_{g_i}$ with $g, g_i \in G$.

(3) There exists a morphism $S : H \longrightarrow H$ in ${}^G_G\mathcal{YDQ}$ (called the antipode of H) such that

$$S = \{S_g : H_g \longrightarrow H_g\}, \tag{16}$$

$$\sum S(h_1)h_{21} \otimes h_{22} = 1 \otimes h = \sum h_1S(h_{21}) \otimes h_{22}, \tag{17}$$

$$\sum h_{11} \otimes S(h_{12})h_2 = h \otimes 1 = \sum h_{11} \otimes h_{12}S(h_2) \tag{18}$$

for all $h \in H_g$ with $g \in G$.

A morphism between braided Hopf coquasigroups H and B is a morphism $f : H \longrightarrow B$ which is both an algebraic and coalgebraic morphism. Note that a braided Hopf coquasigroup is coassociative if and only if it is a braided Hopf algebra (see [2]).

Remark 3. *(i) Let H be a Hopf quasigroup in ${}^G_G\mathcal{YDQ}$. Then the antipode S is unique, antimultiplicative, anticomultiplicative and leaves the unit and the counit invariable.*

(ii) If $H = (H, m, 1, \Delta, \varepsilon, S)$ is a Hopf coquasigroup in ${}^G_G\mathcal{YDQ}$, then so are $H = (H, m^{op}, 1, \Delta, \varepsilon, S^{op})$ and $H = (H, m, 1, \Delta^{cop}, \varepsilon, S^{cop})$.

(iii) Given two Hopf coquasigroups H and B in ${}^G_G\mathcal{YDQ}$, one has the algebra $H\underline{\otimes}B$ with the following multiplication:

$$(x\underline{\otimes}y)(a\underline{\otimes}b) = x(g \cdot a)\underline{\otimes}yb$$

for $x, a \in H$ and $y, b \in B_g$.

Proposition 3. *Let G be a quasigroup and H an algebra. If G quasi-acts on H, i.e., $g \cdot (ab) = (g \cdot a)(g \cdot b)$ and $g \cdot 1_H = 1_H$ with $a, b \in H$ and $g \in G$, then there is a unital non-associative algebra (called skew quasigroup algebra) $H * G = H \otimes \mathbb{F}[G]$ as a vector space with a product given by*

$$(a * x)(b * y) = a(x \cdot b) * xy \tag{19}$$

for any $a \in H, b \in H_g$ and $x, y \in G$.

Proof. Since G is a quasigroup, the product given by Equation (19) is also non-associative. For any $b \in H$ and $y \in G$, we compute

$$(1_H * 1_G)(b * y) = (1_G \cdot b) * y = b * y = b(y \cdot 1_H) * y = (b * y)(1_H * 1_G).$$

This ends the proof. □

For example, when we consider the polynomial algebra $A = \mathbb{F}[x]$ in one variable x and the quasigroup $G_5 = \{1, 2, 3, 4, 5\}$ given in Example 1, we define a quasi-action of G_5 on A as follows: $i \cdot x^m = x^{im}$ with $i \in G_5$ and $m \in \mathbb{N}$. Thus, we have the skew quasigroup algebra $\mathbb{F}[x] * G_5 = \mathbb{F}[x] \otimes \mathbb{F}[G_5]$ with a product given by

$$(x^m * i)(x^n * j) = x^{m+in} * ij$$

for any $i, j \in G_5$ and and $m, n \in \mathbb{N}$.

Proposition 4. *Let G be a quasigroup and let $H = \bigoplus_{g \in G} H_g$ be a G-graded counital non-coassociative coalgebra with a coproduct $\Delta = \{\Delta_g : H_g \longrightarrow H_g \otimes H_g\}_{g \in G}$ and counit $\varepsilon : H_1 \longrightarrow \mathbb{F}$. Then, there is a counital non-coassociative coassociative coalgebra $H \diamond G = H \otimes \mathbb{F}[G] = \bigoplus_{g \in G}(H_g \otimes \mathbb{F}[< g >])$ as a vector space with a counit $\varepsilon = \varepsilon_H \otimes \varepsilon_G : H_1 \diamond G \longrightarrow k$, $a \otimes g \mapsto \varepsilon_H(a)$ and with a coproduct given by*

$$\Delta(a \diamond x) = \sum (a_1 \diamond gx) \otimes (a_2 \diamond x) \tag{20}$$

for any $x, g \in G$ and $a \in H_g$.

Proof. It is obvious that ε is a counital for Δ given by Equation (20). In fact, for any $x \in G$ and $a \in H_1$, we have

$$\begin{aligned} (\varepsilon \otimes id)\Delta(a \diamond x) &= \sum (\varepsilon_H \otimes \varepsilon_G)(a_1 \diamond x) \otimes (a_2 \diamond x) \\ &= \varepsilon_H(a_1)\varepsilon_G(x)(a_2 \diamond x) \\ &= \varepsilon_G(x)(a \diamond x) = a \diamond x. \end{aligned}$$

and

$$(id \otimes \varepsilon)\Delta(a \diamond x) = \sum (a_1 \diamond x) \otimes (\varepsilon_H \otimes \varepsilon_G)(a_2 \diamond x)$$
$$= \sum (a_1 \diamond x)\varepsilon_H(a_2)\varepsilon_G(x) = a \diamond x.$$

This finishes the proof. □

Recall from [12] that a bialgebra (A, Δ) is an algebra (A, m) and a coalgebra (A, Δ) such that $\Delta(ab) = \Delta(a)\Delta(b)$ for all $a, b \in A$. A unital bialgebra (A, m, μ, Δ) is a coalgebra (A, Δ) and a unital (A, m, μ) such that $\Delta(ab) = \Delta(a)\Delta(b)$ and $\Delta(1) = 1$ for all $x, y \in A$. A counital bialgebra $(A, \nabla, \Delta, \varepsilon)$ is a counital coalgebra (A, Δ, ε) and an algebra (A, m) such that $\Delta(ab) = \Delta(a)\Delta(b)$ and $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ for all $a, b \in A$. A unital counital bialgebra $(A, \Delta, \varepsilon, m, \mu)$ is both a unital bialgebra (A, Δ, m, μ) and a counital bialgebra $(A, \Delta, \varepsilon, m)$ such that $\varepsilon(1) = 1$.

A Hopf algebra always means a unital counital associative coassociative bialgebra with an antipode ([4]). A Hopf quasigroup as introduced in Section 2.2 always means a unital counital non-associative coassociative bialgebra with an antipode; and similarly, a Hopf coquasigroup always means a unital counital associative non-coassociative bialgebra with an antipode ([11]).

With the conditions given in Propositions 3 and 4, define $H \odot G = H \otimes \mathbb{F}[G]$ as a vector space with the product given by Equation (19) and the coproduct given by Equation (20).

Theorem 2. *Let G be a quasigroup. Suppose $H = \bigoplus_{g \in G} H_g$ is a unital associative algebra in ${}_G\mathcal{M}\mathcal{Q}$ such that H_g is an algebra and $H_x H_g = 0$ with $x \neq g$, and a counital non-coassociative coalgebra in ${}^G\mathcal{M}$. Then, the following are equivalent:*

(a) $H \odot G$ is a unital counital non-associative non-coassociative bialgebra.

(b) H is a unital associative algebra in ${}^G\mathcal{M}$ and a counital non-coassociative coalgebra in ${}_G\mathcal{M}\mathcal{Q}$, ε_H is an algebra map, $\Delta_H(1) = 1 \otimes 1$, and the identity

$$\Delta(ab) = \sum a_1(g \cdot b_1) \otimes a_2 b_2, \tag{21}$$
$$(g \cdot (x \cdot b)) \diamond (xy) = (gx \cdot b) \diamond (gx)(gy), \tag{22}$$

for any $g, x, y \in G$ and $a \in H_g, b \in H$.

(c) The counit ε_H and the left $\mathbb{F}[G]$-comodule structure map ρ on H are algebra maps; the module structure map $\cdot_H : \mathbb{F}[G] \otimes H \longrightarrow H$ is a coalgebra map, $\Delta_H(1) = 1 \otimes 1$; and Equations (21) and (22) hold.

Proof. (i) We claim that $\varepsilon = \varepsilon_H \otimes \varepsilon_G$ is an algebra map if and only if ε_G is multiplicative and ε_H is an algebra map and $\varepsilon_H(g \cdot a) = \varepsilon_H(a)$ holds for $g \in G$ and $a \in H$. In fact, if $\varepsilon[(a * x)(b * y)] = \varepsilon(a * x)\varepsilon(b * y)$ for any $a, b \in H_1$ and $x, y \in G$, we have $\varepsilon_H[a(x \cdot b)]\varepsilon_G(xy) = \varepsilon_H(a)\varepsilon_H(b)\varepsilon_G(x)\varepsilon_G(y)$ which proves the claim.

(ii) We have that $\Delta(1 \diamond 1) = (1 \diamond 1) \otimes (1 \diamond 1)$ if and only if $\rho(1) = 1 \otimes 1$ and $\Delta_H(1) = 1 \otimes 1$.

(iii) Assuming that $\rho(1) = 1 \otimes 1$. It is straightforward to check that Δ is multiplicative if and only if

$$\sum (a_1(g \cdot (x \cdot b_1)) \diamond (xy)) \otimes (a_2(x \cdot b_2) \diamond xy)$$
$$= \sum [a_1(gx \cdot b_1) \diamond (gx)(gy)] \otimes [a_2(x \cdot b_2) \diamond xy]$$

from which it follows Equations (21) and (22).

(b) $\Longleftrightarrow$ (c) is clear. (a) $\Longrightarrow$ (b) follows from the preceding calculations, so it remains to show that (b) $\Longrightarrow$ (a). Assume that (b) holds. Then, the equations of (i), (ii) and (iii) are valid. By (i), ε is an algebra map, and by (iii), to show that Δ is an algebra map, we only need to show that Δ is multiplicative. But for this, it suffices to show by (iii) that

$$\begin{aligned}
\Delta[(a\diamond x)(b\diamond y)] &= \sum(((a(x\cdot b))_1\diamond(xy))\otimes(((a(x\cdot b))_2\diamond xy) \\
&= \sum(a_1(g\cdot(x\cdot b)_1)\diamond(xy))\otimes(a_2(x\cdot b)_2\diamond xy) \\
&= \sum(a_1(g\cdot(x\cdot b_1))\diamond(xy))\otimes(a_2(x\cdot b_2)\diamond xy) \\
&= \sum[a_1(gx\cdot b_1)\diamond(gx)(gy)]\otimes[a_2(x\cdot b_2)\diamond xy] \\
&= \sum[(a_1\diamond gx)(b_1\diamond gy)]\otimes[(a_2\diamond x)(b_2\diamond y)] \\
&= [\sum(a_1\diamond gx)\otimes(a_2\diamond x)][\sum(b_1\diamond gy)\otimes(b_2\diamond y)] \\
&= \Delta(a\diamond x)\Delta(b\diamond y),
\end{aligned}$$

for any $a,b\in H_g$ and $x,y,g\in G$. This completes the proof of the theorem. □

In the above theorem, we have derived necessary and sufficient conditions for $H\otimes G$ to be a unital counital non-associative non-coassociative bialgebra with the algebra structure of $H*G$ and the coalgebra structure of $H\diamond G$. In case $H\odot G$ is a unital counital non-associative non-coassociative bialgebra, we say that the pair (G,H) is quasi-admissible and denote this a unital counital non-associative non-coassociative bialgebra by $H\diamond G$.

Remark 4. *If (G,H) is a quasi-admissible pair, then Δ_H is not necessarily multiplicative.*

In what follows, for a quasi-admissible pair (G,H), we show that the mapping system $H \leftrightarrows^{\Pi_H}_{j_H} H\diamond G \rightleftarrows^{\pi_G}_{i_G} G$ characterizes $H\odot G$, where G means $k[G]$.

Definition 4. *Let (G,H) be a quasi-admissible pair and suppose that A is a unital counital non-associative non-coassociative bialgebra. Then, $H \leftrightarrows^{\Pi}_{j} A \rightleftarrows^{\pi}_{i} G$ is a quasi-admissible mapping system if the following conditions are satisfied:*

(QAP1) $\Pi\circ j = id_H$ and $\pi\circ i = id_G$.

(QAP2) i and π are bialgebra maps, j is an algebra map, and Π is a coalgebra map.

(QAP3) Π is a G-bimodule map (A is given the G-bimodule structure via pullback along i, and H is given the trivial right G-module structure).

(QAP4) $j(H)$ is a sub-G-bicomodule of A and $\Pi|_{j(H)}$ is a bicomodule map (A is given the G-bicomodule structure via pushout along π, and H is given the trivial right G-comodule structure).

(QAP5) $(j\circ\Pi)*(i\circ\pi) = id$.

Our next result gives two mapping descriptions of $H\diamond G$.

Theorem 3. *Let (G,H) be a quasi-admissible pair.*

(a) $H \leftrightarrows^{\Pi_H}_{j_H} H\diamond G \rightleftarrows^{\pi_G}_{i_G} G$ is a quasi-admissible mapping system.

Let A be a unital counital non-associative non-coassociative bialgebra and let $H \leftrightarrows^{\Pi}_{j} A \rightleftarrows^{\pi}_{i} G$ be a quasi-admissible mapping system.

(b) There exists a unique unital non-associative algebra map $\Theta : H\diamond G \longrightarrow A$ such that

(i) $\Theta\circ j_H = j$ and $\Theta\circ i_G = i$;

(ii) $\Pi\circ\Theta = \Pi_H$ and $\pi\circ\Theta = \pi_G$, and Θ is a unital counital non-associative non-coassociative bialgebra isomorphism.

(c) There exists a unique unital non-coassociative coalgebra map $\Upsilon : A \longrightarrow H\diamond G$ such that

(iii) $\Pi_H \circ \Upsilon = \Pi$ *and* $\pi_G \circ \Upsilon = \pi$;

(iv) $\Upsilon \circ j = j_H$ *and* $\Upsilon \circ i = i_G$, *and* Υ *is a unital counital non-associative non-coassociative bialgebra isomorphism.*

Proof. (a) is straightforward. In fact, in $H \leftrightarrows^{\Pi_H}_{j_H} H \diamond G \rightleftarrows^{\pi_G}_{i_G} G$, we have $\Pi_H(a \diamond g) = a$; $j_H(a) = a \diamond 1$; $\pi_G(b \diamond g) = \varepsilon_H(b)g$; $i_G(g) = 1_H \diamond g$ for all $a \in H, b \in H_1$ and $g \in G$. We will check all conditions (QAP1)–(QAP5) in Definition 4 as follows:

For (QAP1), we compute $(\Pi_H \circ j_H)(a) = \Pi_H(a \diamond 1) = a = id_H(a)$ and $(\pi_G \circ i_G)(g) = \pi_G(1_H \diamond g) = \varepsilon_H(1_H)g = id_G(g)$.

For (QAP2) $i_G(gx) = 1_H \diamond gx = (1_H\varepsilon_G(g)1_H \diamond gx) = (1_H(g \cdot 1_H) \diamond gx) = (1_H \diamond g)(1_H \diamond x) = i_G(g)i_G(x)$ with $g, x \in G$, and $(\Delta \circ i_G)(g) = \Delta(1_H \diamond g) = (1_H \diamond (1 \cdot g)) \otimes (1_H \diamond g)$ (since $1_H \in H_1$) $= (i_G \otimes i_G)\Delta_G(g)$, and so, i_G is a bialgebra map.

Similarly, $\pi_G((a \diamond g)(b \diamond x)) = \pi_G(a(g \cdot b) \diamond gx) = \varepsilon_H(a(g \cdot b))gx = \varepsilon_H(a)\varepsilon_H(g \cdot b)gx = \varepsilon_H(a)\varepsilon_H(b)gx = \pi_G(a \diamond g)\pi_G(b \diamond x)$ with $g, x \in G$ and $a, b \in H_g$, and $(\Delta_G \circ \pi_G)(a \diamond g) = \varepsilon_H(a)\Delta_G(g) = \varepsilon_H(a)g \otimes g = \sum \pi_G(a_1)\varepsilon_H(a_2)g \otimes g = \sum \pi_G(a_1 \diamond g)\pi_G(a_2 \diamond g)) = \sum(\pi_G \otimes \pi_G)((a_1 \diamond g)) \otimes (a_2 \diamond g)) = (\pi_G \otimes \pi_G)\Delta(a \diamond g)$ for any $g \in G$ and $a \in H_1$, and π_G is a bialgebra map.

Furthermore, $j_H(ab) = ab \diamond 1 = (a(1 \cdot b) \diamond 1) = (a \diamond 1)(b \diamond 1) = j_H(a)j_H(b)$ with $a, b \in H$, and so, j_H is an algebra map. We also have $(\Delta_H \circ \Pi_H)(a \diamond g) = \sum a_1 \otimes a_2 = \sum(\Pi_H \otimes \Pi_H)[(a_1 \diamond xg) \otimes (a_2 \diamond g)] = (\Pi_H \otimes \Pi_H)\Delta(a \diamond g)$ with $x, g \in G$ and $a \in H_x$, and so, Π_H is a coalgebra map.

AS for (QAP3), we check that Π_H is a G-bimodule map. We note that $H \diamond G$ is given the G-bimodule structure via pullback along i_G, i.e., $g \cdot (a \diamond x) = (1_H \diamond g)(a \diamond x) = g \cdot a \diamond gx$ and $(a \diamond x) \cdot g = (a \diamond x)(1_H \diamond g) = a \diamond xg$. In fact, we have $\Pi_H[g \cdot (a \diamond x)] = \Pi_H(g \cdot a \diamond gx = g \cdot a = g \cdot \Pi_H(a \diamond x)$ and $\Pi_H[(a \diamond x) \cdot g] = \Pi_H(a \diamond xg) = a = a \cdot g = \Pi_H(a \diamond x) \cdot g$ since H is given the trivial right G-module structure.

As for (QAP4), we check that $j_H(H)$ is a sub-G-bicomodule of $H \diamond G$ and $\Pi_H|_{j_H(H)}$ is a bicomodule map. One notes that $H \diamond G$ is given the G-bicomodule structure via pushout along π_G, i.e., $\rho^r(a \diamond g) = (a \diamond g) \otimes 1$ and $\rho^l(a \diamond g) = xg \otimes (a \diamond g)$ with $a \in H_x$ and $g, x \in G$. Actually, $(\rho^r \circ j_H)(a) = \rho^r(a \diamond 1) = (a \diamond 1) \otimes 1 = (j_H \otimes id)(a \otimes 1) = (j_H \otimes id)\rho^r(a)$ since H is given the trivial right G-comodule structure, and $(\rho^l \circ j_H)(a) = \rho^l(a \diamond 1) = x \otimes (a \diamond 1) = (id \otimes j_H)(x \otimes a) = (id \otimes j_H)\rho^l(a)$.

Finally, for (QAP5), we need to check $(j_H \circ \Pi_H) * (i_G \circ \pi_G) = id$. We compute

$$
\begin{aligned}
& [(j_H \circ \Pi_H) * (i_G \circ \pi_G)](a \diamond g) \\
= & \sum (j_H \circ \Pi_H)(a_1 \diamond xg)(i_G \circ \pi_G)(a_2 \diamond g) \\
= & \sum j_H(a_1)i_G(\varepsilon_H(a_2)g) \\
= & (a \diamond 1)(1 \diamond g) = (a \diamond g)
\end{aligned}
$$

for any $a \in H_x$ and $g, x \in G$.

(b) If $\Theta : H \diamond G \longrightarrow A$ is an algebra map, then (i) holds if and only if $\Theta(a \diamond g) = \Theta(a \diamond 1)\Theta(1 \diamond g) = \Theta(a \diamond 1)\Theta(1 \diamond g) \overset{(i)}{=} (\Theta(j_H(a)))(\Theta(i_G(g))) = j(b)i(g)$ holds for any $a \in H$ and $g \in G$.

If $\Upsilon : A \longrightarrow H \diamond G$ is a coalgebra map, then (iii) holds if and only if $\Upsilon(p) = \sum(j_H \circ \Pi_H)(\Upsilon(p_1))(i_G \circ \pi_G)(\Upsilon(p_2))$ (by (QAP5) of (a))$= \sum(\Pi_H(\Upsilon(p_1)) \diamond 1)(1 \diamond \pi_G(\Upsilon(p_2))) \overset{(iii)}{=} \sum \Pi(p_1) \diamond \pi(p_2)$ holds for $p \in A$. Therefore, we have the uniqueness of Θ and Υ.

Let Θ and Υ be defined as above. Then, through calculations, we can show that Θ and Υ are inverses. Thus, the proof will be complete once we show that Θ is an algebra map and Υ is a coalgebra map. These checks are similar to those of proofing (a).

The proof of the remaining is straightforward and is left to the reader.

This completes the proof. □

The remainder of this section is devoted to studying basic properties of $H \odot G$; in particular, we derive necessary and sufficient conditions for $H \odot G$ to be a unital counital non-associative coassociative Hopf coquasigroup.

Proposition 5. *Suppose that (G, H) is a quasi-admissible pair.*

(a) $H \diamond G$ is commutative if and only if H and G are commutative and the module structure map is trivial.

(b) $H \diamond G$ is co-commutative if and only if H and G are co-commutative and the comodule structure map is trivial.

Proof. The proof of this proposition is straightforward. □

Proposition 6. *Suppose that (G, H) is a quasi-admissible pair.*

(a) If $H \diamond G$ is a unital counital non-associative coassociative Hopf coquasigroup with antipode S, then S satisfies Equations (17) and (18). Furthermore, the identity id_H has an inverse in the convolution algebra $End(H)$.

(b) If S_H satisfies Equations (5) and (6), then $H \diamond G$ is a unital counital non-associative coassociative Hopf coquasigroup with antipode λ described by

$$\lambda(a \diamond g) = \begin{cases} (1 \diamond g^{-1})(S_H(a) \diamond 1), & \text{when } a \in H_1; \\ 0, & \text{when } a \neq H_1. \end{cases}$$

Proof. (a) is left to the reader since it is a straightforward calculation.

(b) We need to check that Equations (17) and (18) hold. For Equation (17), in fact, we have

$$\begin{aligned}
& \sum \lambda((a \diamond g)_{(1)})(a \diamond g)_{(2)(1)} \otimes (a \diamond g)_{(2)(2)} \\
= & \sum \lambda(a_{(1)} \diamond g)(a_{(2)} \diamond g)_{(1)} \otimes (a_{(2)} \diamond g)_{(2)} \\
= & \sum \lambda(a_{(1)} \diamond g)(a_{(2)(1)} \diamond g) \otimes (a_{(2)(2)} \diamond g) \\
= & \sum [(1 \diamond g^{-1})(S_H(a_{(1)}) \diamond 1)](a_{(2)(1)} \diamond g) \otimes (a_{(2)(2)} \diamond g) \\
= & \sum (g^{-1} \cdot S_H(a_{(1)}) \diamond g^{-1})(a_{(2)(1)} \diamond g) \otimes (a_{(2)(2)} \diamond g) \\
= & \sum [(g^{-1} \cdot S_H(a_{(1)}))(g^{-1} \cdot (a_{(2)(1)}))] \diamond g^{-1}g) \otimes (a_{(2)(2)} \diamond g) \\
= & \sum [(g^{-1} \cdot (S_H(a_{(1)})a_{(2)(1)})) \diamond 1)] \otimes (a_{(2)(2)} \diamond g) \\
= & \sum [(g^{-1} \cdot 1) \diamond 1)] \otimes (a \diamond g) \\
= & (1 \diamond 1) \otimes (a \diamond g)
\end{aligned}$$

for any $a \in H_1$ and $g \in G$, and

$$\begin{aligned}
& \sum (a \diamond g)_{(1)} \lambda (a \diamond g)_{(2)(1)} \otimes (a \diamond g)_{(2)(2)} \\
= & \sum (a_{(1)} \diamond g) \lambda (a_{(2)} \diamond g)_{(1)} \otimes (a_{(2)} \diamond g)_{(2)} \\
= & \sum (a_{(1)} \diamond g) \lambda (a_{(2)(1)} \diamond g) \otimes (a_{(2)(2)} \diamond g) \\
= & \sum (a_{(1)} \diamond g)((1 \diamond g^{-1})(S_H(a_{(2)(1)}) \diamond 1) \otimes (a_{(2)(2)} \diamond g) \\
= & \sum (a_{(1)} \diamond g)((g^{-1} \cdot S_H(a_{(2)(1)})) \diamond g^{-1}) \otimes (a_{(2)(2)} \diamond g) \\
= & \sum (a_{(1)} [g \cdot (g^{-1} \cdot S_H(a_{(2)(1)}))] \diamond g g^{-1}) \otimes (a_{(2)(2)} \diamond g) \\
= & \sum (a_{(1)} S_H(a_{(2)(1)}) \diamond 1) \otimes (a_{(2)(2)} \diamond g) \\
= & (1 \diamond 1) \otimes (a \diamond g).
\end{aligned}$$

Equation (18) can be proven in a similar way.

This completes the proof. □

Corollary 1. *Let G be a quasigroup. Let $B \in {}^G_G\mathcal{YDQ}$ and let B be Hopf algebra of quasi-diagonal type with an antipode S_B. Then, we have a unital counital non-associative coassociative Hopf algebra $B \odot G$ with the Hopf algebraic structures as follows:*

$$\begin{cases}
(a \odot x)(b \odot y) = a\chi_g(g) b \odot xy, \text{ for } a, b \in B_g, x, y, g \in G; \\
\Delta(a \odot x) = \sum (a_1 \odot gx) \otimes (a_2 \odot x), \text{ for } a \in B_g, x, y, g \in G; \\
\varepsilon = \varepsilon_B \otimes \varepsilon_G, ; \\
\lambda(a \diamond g) = \begin{cases} (1 \diamond g^{-1})(S_H(a) \diamond 1), & \text{when } a \in B_1; \\ 0, & \text{when } a \neq B_1. \end{cases}
\end{cases}$$

4. Duality of Braided Hopf Coquasigroups of Quasi-Diagonal Type

Recall form [4] the notion of the Sweedler's duality of an associative algebra. Let (A, m_A, μ_A) be a unital associative algebra. Then, we have the counital coalgebra A^0 given by Sweedler, as follows:

$$A^0 = \{ f \in A^* \mid \operatorname{Ker} f \text{ contains a cofinite ideal} \}$$

where A^* is the linear dual space of A, and a cofinite ideal is an ideal J in A, wherein A/J is finite-dimensional.

For $f \in A^*$ and $a, b \in A$, one sets $(a \rightharpoonup f)(b) = f(ba)$; similarly, $(f \leftharpoonup a)(b) = f(ab)$. Then, one obtains that A^* is an A-A-bimodule. The following lemma follows from Sw.

Lemma 1. *With the above notations, the following states are equivalent for any $f \in A^*$:*
(1) $f \in A^0 = (m_A^*)^{-1}(A^* \otimes A^*)$;
(2) $dim(A \rightharpoonup f) < \infty$;
(3) $dim(f \leftharpoonup A) < \infty$;
(4) $dim(A \rightharpoonup f \leftharpoonup A) < \infty$.

Since the duality of an algebra is not generally a coalgebra, the duality of a Hopf algebra is not usually a Hopf algebra. But, if A is a Hopf algebra, A^0 has a natural Hopf algebraic structure, which was described in [4] (Section 6.2). We can describe the Sweedler's duality in the setting of Hopf coquasigroups as follows:

Theorem 4. *Let A be a Hopf quasigroup. Then, A^0 forms a braided Hopf coquasigroup in $\mathbf{LS}_{\Bbbk}$. Conversely, if A is a Hopf coquasigroup, then A^0 is a braided Hopf quasigroup in $\mathbf{LS}_{\Bbbk}$.*

Proof. Let A be a Hopf quasigroup. By the dual theory in [4] (Chapter 6), we only check Equations (5) and (6) from Equations (3) and (4). For instance, for Equation (5), we have

$$\begin{aligned} & < \sum S^0(h^0_{(1)})h^0_{(2)(1)} \otimes h^0_{(2)(2)}, a \otimes b > \\ = \; & \sum < S^0(h^0_{(1)})h^0_{(2)(1)}, a >< h^0_{(2)(2)}, b > \\ = \; & \sum < S^0(h^0_{(1)}), a_1 >< h^0_{(2)(1)}, a_2 >< h^0_{(2)(2)}, b > \\ = \; & \sum < h^0_{(1)}, S(a_1) >< h^0_{(2)}, a_2 b > \\ = \; & \sum < h^0, S(a_1)(a_2 b) > \\ = \; & < h^0, b > \varepsilon(a) =< \varepsilon \otimes h^0, a \otimes b >, \end{aligned}$$

for any $h^0 \in A^0$ and $a, b \in A$, and so, the first equation in Equation (5) holds. The same applies to other equations in Equations (5) and (6).

Conversely, the proof is similar. □

Let A be a Hopf coquasigroup in ${}^G_G\mathcal{YDQ}$.

Lemma 2. *With the above notation,*

(1) The $\rightharpoonup$ is a left G-linear;

(2) The $\leftharpoonup$ is a right G-linear.

Proof. Straightforward. □

Lemma 3. *A^0 is a G-submodule of A^*.*

Proof. For any $g \in G$ and $f \in A^0, a \in A$, we notice that

$$(g \cdot f)(a) = f(g^{-1} \cdot a).$$

From this formula, we can finish the proof. □

Proposition 7. *A^0 is a subalgebra of A^*.*

Proof. Let $f, g \in A^*$. For $a \in A_x, b \in A$ with $x \in G$, we compute

$$\begin{aligned} & ((fg) \leftharpoonup a)(b) = (fg)(ab) = (f \otimes g)\Delta(ab) \\ = \; & f(a_1(x \cdot b_1)g(a_2 b_2) = f(x \cdot [(S^{-1}(x \cdot a_1)b_1])g(a_2 b_2) \\ = \; & (x \cdot f)[(x \cdot a_1)b_1]g(a_2 b_2) \\ = \; & [(x \cdot f) \leftharpoonup (x \cdot a_1)](b_1)(g \leftharpoonup a_2)(b_2) \\ = \; & \Delta^*[x \cdot (f \leftharpoonup a_1) \otimes g \leftharpoonup a_2)](b) \end{aligned}$$

where we apply that A is a G-module algebra to the third equation above and use Lemma 2 to get the final equation.

Therefore, we have

$$\begin{aligned}(fg) \leftharpoondown A &\subseteq \Delta^*[G \cdot (f \leftharpoondown A) \otimes g \leftharpoondown A] \\ &\subseteq \Delta^*[(G \cdot f) \leftharpoondown A) \otimes g \leftharpoondown A].\end{aligned}$$

By $f \in A^0$ and Lemma 3, one obtains $G \cdot f \in A^0$. Applying $f, g \in A^0$, and the left-hand side of the above containment is finite-dimensional, so, $fg \in A^0$. Finally, it is straightforward to show that $\varepsilon_A^*(1) \in A^0$.

This concludes the proof. □

Lemma 4. *$i \circ \tau : (A^*)^{op} \underline{\otimes} (A^*)^{op} \longrightarrow (A \underline{\otimes} A)^{*op}$ is a homomorphism as an algebra in ${}^G_G\mathcal{YDQ}$.*

Proof. It follows the definition of the coaction of G on A^*. □

Theorem 5. *Assume that $(A, m_A, \mu_A, \Delta_A, \varepsilon_A, S_a)$ is a Hopf coquasigroup in ${}^G_G\mathcal{YDQ}$. Then, $(A^0, (m_{A^0})^{op}, \varepsilon_A^*, (\Delta_{A^0})^{op}, \mu_A^*, S_A^*)$ is a braided Hopf quasigroup in ${}^G_G\mathcal{YDQ}$.*

Proof. Following [6], Theorem 3.4, we have to finish checking the following steps:

(Step 1) A^0 is a G-subcomodule of A^*.

(Step 2) Observe that $(m_{A^0})^{op} = \Delta_A^* \circ i \circ \tau : A^0 \otimes A^0 \longrightarrow A^*$. It is a morphism in ${}^G_G\mathcal{YDQ}$. Obviously, $\varepsilon_A^* : \mathbb{F} \longrightarrow A^0$ is. Thus, $(A^0, (m_{A^0})^{op}, \varepsilon_A^*)$ is a unital non-associative algebra in the category ${}_G\mathcal{MQ}$.

(Step 3) Note that $(\Delta_{A^0})^{op}$ is the composite map $A^0 \xrightarrow{m_A^*} i(A^0 \otimes A^0) \xrightarrow{(i\circ\tau)^{-1}} A^0 \otimes A^0$. This is a morphism in ${}^G_G\mathcal{YDQ}$. $\mu_A^* : A^0 \longrightarrow \mathbb{F}$. So, $(A^0, (\Delta_{A^0})^{op}, \mu_A^*)$ is a counital coassociative coalgebra in the category ${}_G\mathcal{MQ}$.

(Step 4) $(\Delta_{A^0})^{op} : (A^0)^{op} \longrightarrow (A^0)^{op} \underline{\otimes} (A^0)^{op}$ as an algebra map.

(Step 5) $S_A^*(A^0) \subseteq A^0$.

(Step 6) $(m_{A^0})^{op}(S_A^* \otimes id_{A^0})(\Delta_{A^0})^{op} = \varepsilon_A^* \mu_A^*$ and $(m_{A^0})^{op}(id_{A^0} \otimes S_A^*)(\Delta_{A^0})^{op} = \varepsilon_A^* \mu_A^*$.

These checks are straightforward. We omit them here and leave the readers. □

As a straightforward result of Theorem 5, we have the following.

Corollary 2 ([6], Theorem 3.4)**.** *Let G be a group. Given a quasitriangular Hopf algebra $(\mathbb{F}[G], R)$ with a bijective antipode S, when $(A, m_A, \mu_A, \Delta_A, \varepsilon_A, S_A)$ is a braided Hopf algebra ${}_G\mathcal{M}$, $(A^0, (m_{A^0})^{op}, \varepsilon_A^*, (\Delta_{A^0})^{op}, \mu_A^*, S_A^*)$ is also a braided Hopf algebra in ${}_G\mathcal{M}$.*

Corollary 3. *Let G be a group. Given a coquasitriangular Hopf algebra $(\mathbb{F}[G], |)$ with a bijective antipode S, when $(A, m_A, \mu_A, \Delta_A, \varepsilon_A, S_A)$ is a braided Hopf algebra ${}^G\mathcal{M}$, $(A^0, (m_{A^0})^{op}, \varepsilon_A^*, (\Delta_{A^0})^{op}, \mu_A^*, S_A^*)$ is also a braided Hopf algebra in ${}^G\mathcal{M}$.*

Finally, as an application, let $A = \mathbb{F}[x, y]$ be the bialgebra with an x group-like element and with y $(x, 1)$-primitive. Consider a cyclic group $G =< g >$ of order n. One has a Hopf algebra $H = \mathbb{F}[G]$ with a g group-like element for any $g \in G$. Moreover, H is quasitriangular with $R = (1/n) \sum_{i,j=0}^{n-1} p^{-ij}(g^i \otimes g^j)$, where p is a primitive nth root of unity in $\mathbb{F}$ (see [6] or [2]). Thus, ${}_H\mathcal{M}$ is a braided monoidal subcategory of ${}^H_H\mathcal{YD}$. We can study Sweedler's duality $A^0 = \mathbb{F}[x, y]^0$.

5. Binary Linearly Recursive Sequences

Consider the polynomial algebra $A = \mathbb{F}[x]$ in one variable x. It has a bialgebraic structure given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\varepsilon(x) = 0$. On the one hand, we can identify an element f in the dual space A^* with the sequences $(f_n)_{n\geq 0} = (f_0, f_1, f_2, \cdots)$, where $f_n = f(x^n)$ for $n \geq 0$. On the other hand, A has a dual coalgebra $A^0 = \{f \in A^* \mid f(J) = 0$ for some cofinite ideal J of A, i.e., A/J is finite-dimensional. Since a cofinite ideal J of $A = \mathbb{F}[x]$ is just a nonzero ideal generated by a monic polynomial $h(x) = x^r - h_1 x^{r-1} - \cdots - h_r$, the condition $f(J) = 0$ means that $f_n = h_1 f_{n-1} + \cdots + h_r f_{n-r}$ for all $n > r$. This means that f is linearly recursive, satisfying the recursive relation $h(x)$. Thus, the space of linearly recursive sequences has a Hopf algebraic structure.

Let $q \neq 0$ in $\mathbb{F}$. In 1997, Ng and Taft [6] showed that the space of linearly recursive sequences is closed under the quantum convolution product $(f_n) *_q (g_n) = (h_n)$; here, $h_n = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right)_q f_i g_{n-i}$ when q is a root of unity.

We now consider the bialgebra $A = \mathbb{F}[x, y]$ with x group-like element and with y $(x, 1)$-primitive. $A = \mathbb{F}[x] \otimes \mathbb{F}[y]$ as an algebra, and thus $A^0 = \mathbb{F}[x]^0 \otimes \mathbb{F}[y]^0$ as a coalgebra. We identify each f in A^* as a binary-sequence $(f_{i,j})$ for $i, j > 0$, where $f_{i,j} = f(x^i y^j)$. A row of such a binary-sequence is a sequence $\{f_{i,p} \mid p \geq 0\}$ for a fixed $i \geq 0$, which we say is parallel to the y-axis, or a sequence $\{f_{p,j} \mid p \geq 0\}$ for a fixed $j \geq 0$, which we say is parallel to the x-axis.

Let f be in A^0, $f(J) = 0$ for a cofinite ideal J of A. For each i, j, the powers of x (y) span a finite-dimensional space in A/J, so there is a minimal monic $h_i(x)$ ($h_j(x)$) in $\mathbb{F}[x]$ such that each row of f parallel to the y (x)-axis satisfies $h_i(x)$ ($h_j(x)$). Thus, J contains the cofinite elementary ideal Γ generated by $h_i(x)h_j(y)$.

Given a $q \neq 0$ in $\mathbb{k}$ and an integer $n > 0$, one knows

$$(n)_q = q^n - 1/q - 1 = 1 + q + \cdots + q^{n-1}.$$

The q-factorial of n is given by $(0)!_q = 1$ and

$$(n)!_q = (1)_q (2)_q \cdots (n)_q = \frac{(q-1)(q^2-1)\cdots(q^n-1)}{(q-1)^n}$$

if $n > 0$. It is a polynomial in q with coefficients in $\mathbb{Z}$. Moreover, it has value at $q = 1$ equal to $n!$.

The Gaussian polynomials is given by for $0 \leq i \leq n$

$$\left(\begin{array}{c} n \\ i \end{array} \right)_q = \frac{(n)_q!}{(i)_q!(n-i)_q!}.$$

Let x and y be variables subject to the quantum plane relation $yx = qxy$. Then, for any $n > 0$, we have

$$(x+y)^n = \sum_{0 \leq i \leq n} \binom{n}{i}_q x^i y^{n-i}. \tag{23}$$

Let $q = 1$. Then, we have that $\binom{n}{k}$ is the ordinary binomial coefficient.

Proposition 8 ([6], Lemma 5.2). *Let q be a primitive nth root of 1. For integers $a \geq b \geq 0$, write $a = a'n + r, b = b'n + s$ for $0 \leq r, s < n$. Then,*

$$\begin{pmatrix} a \\ b \end{pmatrix}_q = \begin{pmatrix} a' \\ b' \end{pmatrix}\begin{pmatrix} r \\ s \end{pmatrix}_q,$$

where $\begin{pmatrix} r \\ s \end{pmatrix}_q = 0$ if $r < s$.

We will let $A = \mathbb{F}_q[x, y]$ with $yx = qxy$; here, $0 \neq q \in \mathbb{F}$. Then, $f \in A^*$ is regarded as the binary sequences $(f_{m,n})_{m,n \geq 0} = (f_{0,0}, f_{0,1}, \cdots, f_{0,n}, f_{1,0}, \cdots, f_{1,n}, \cdots, f_{m,0}, \cdots, f_{m,n}, \cdots)$ where

$$f_{m,n} = f(x^m y^n) = q^{-mn} f(y^n x^m) = q^{-mn} f_{n,m},$$

for all $m, n \geq 0$. We call them the q-binary sequences. A has a bialgebra structure with a group-like element x and with an $(x, 1)$-primitive y, i.e., we have a comultiplication Δ given by the following:

$$\begin{aligned} &\Delta(x) = x \otimes x, \quad \Delta(y) = x \otimes y + y \otimes 1 \\ &\varepsilon(x) = 1, \quad \varepsilon(y) = 0, \end{aligned}$$

requiring Δ to be an algebra homomorphism from A to $A \otimes A$. Thus, one has

$$\Delta(x^m y^n) = \sum_{0 \leq k \leq n} \begin{pmatrix} n \\ k \end{pmatrix}_q x^{m+k} y^{n-k} \otimes x^m y^k, \tag{24}$$

for any $m, n \geq 0$. Therefore, the quantum convolution product on A^* is given by $f_{m,n} *_q g_{m,n} = h_{m,n}$, where $h_{m,n} = \sum_{0 \leq k \leq n} \begin{pmatrix} n \\ k \end{pmatrix}_q f_{m+k,n-k} \otimes g_{m,k}$ for $m, n \geq 0$.

By a cofinite ideal J of $A = \mathbb{F}_q[x, y]$ we mean a nonzero ideal generated by a monic binary polynomial:

$$\begin{aligned} h(x,y) \quad = \quad & x^r y^s - h_{1,0} x^{r-1} y^s - \cdots - h_{r,0} y^s \\ & - h_{0,1} x^r y^{s-1} - h_{1,1} x^{r-1} y^{s-1} - \cdots - h_{r,1} y^{s-1} \\ & - h_{0,2} x^r y^{s-2} - h_{1,2} x^{r-1} y^{s-2} - \cdots - h_{r,2} y^{s-2} \\ & \cdots \quad \cdots \quad \cdots \quad \cdots \\ & - h_{0,s} x^r - h_{1,s} x^{r-1} - \cdots - h_{r,s}. \end{aligned}$$

By the condition $f(J) = 0$, we have the following cases:

Case 1: If $f(x^{m-r} h(x,y) y^{n-s}) = 0$, then we have a binary linearly recursive sequence $f = (f_{m,n})_{m \geq r, n \geq s}$ satisfying the recursive relation $h(x, y)$, where

$$\begin{aligned} f_{m,n} \quad = \quad & h_{1,0} f_{m-1,n} + h_{2,0} f_{m-2,n} + \cdots + h_{r,0} f_{m-r,n} \\ & + h_{0,1} f_{m,n-1} + h_{1,1} f_{m-1,n-1} + \cdots + h_{r,1} f_{m-r,n-1} \\ & + h_{0,2} f_{m,n-2} + h_{1,2} f_{m-1,n-2} + \cdots + h_{r,2} f_{m-r,n-2} \\ & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ & + h_{0,s} f_{m,n-s} + h_{1,s} f_{m-1,n-s} + \cdots + h_{r,s} f_{m-r,n-s}. \end{aligned}$$

Case 2: If $f(x^{m-r}y^{n-s}h(x,y)) = 0$, then we have a parameterized binary linearly recursive sequence $f = (f_{m,n})_{m\geq r, n\geq s}$ satisfying the recursive relation $h(x,y)$, where

$$\begin{array}{rcl} f_{m,n} & = & q^{-(n-s)}h_{1,0}f_{m-1,n} + q^{-2(n-s)}h_{2,0}f_{m-2,n} + \cdots + q^{-r(n-s)}h_{r,0}f_{m-r,n} \\ & + & h_{0,1}f_{m,n-1} + q^{-(n-s)}h_{1,1}f_{m-1,n-1} + q^{-2(n-s)}h_{2,1}f_{m-2,n-1} \\ & & \quad + \cdots + q^{-r(n-s)}h_{r,1}f_{m-r,n-1} \\ & + & h_{0,2}f_{m,n-2} + q^{-(n-s)}h_{1,2}f_{m-1,n-2} + q^{-2(n-s)}h_{2,2}f_{m-2,n-2} \\ & & \quad + \cdots + q^{-r(n-s)}h_{r,2}f_{m-r,n-2} \\ & & \cdots \quad \cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ & + & h_{0,s}f_{m,n-s} + q^{-(n-s)}h_{1,s}f_{m-1,n-s} + q^{-2(n-s)}h_{2,s}f_{m-2,n-2} \\ & & \quad + \cdots + q^{-r(n-s)}h_{r,s}f_{m-r,n-s}. \end{array}$$

Case 3: If $f(h(x,y)x^{m-r}y^{n-s}) = 0$, then we have a parameterized binary linearly recursive sequence $f = (f_{m,n})_{m\geq r, n\geq s}$ satisfying the recursive relation $h(x,y)$, where

$$\begin{array}{rcl} f_{m,n} & = & h_{1,0}f_{m-1,n} + h_{2,0}f_{m-2,n} + \cdots + h_{r,0}f_{m-r,n} \\ & + & q^{-(m-r)}h_{0,1}f_{m,n-1} + q^{-(m-r)}h_{1,1}f_{m-1,n-1} + q^{-(m-r)}h_{2,1}f_{m-2,n-1} \\ & & \quad + \cdots + q^{-(m-r)}h_{r,1}f_{m-r,n-1} \\ & + & q^{-2(m-r)}h_{0,2}f_{m,n-2} + q^{-2(m-r)}h_{1,2}f_{m-1,n-2} + q^{-2(m-r)}h_{2,2}f_{m-2,n-2} \\ & & \quad + \cdots + q^{-2(m-r)}h_{r,2}f_{m-r,n-2} \\ & & \cdots \quad \cdots \quad \cdots \quad \cdots\cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ & + & q^{-s(m-r)}h_{0,s}f_{m,n-s} + q^{-s(m-r)}h_{1,s}f_{m-1,n-s} + \cdots + q^{-s(m-r)}h_{r,s}f_{m-r,n-s}. \end{array}$$

Remark 5. *(1) Set $h'_{i,j} = q^{-i(n-s)}h_{i,j}$ in Case 1. We then have a new monic binary polynomial $h'(x,y)$ as follows:*

$$\begin{array}{rcl} h'(x,y) & = & x^r y^s - q^{n-s}h'_{1,0}x^{r-1}y^s - \cdots - q^{r(n-s)}h'_{r,0}y^s \\ & & -h'_{0,1}x^r y^{s-1} - q^{n-s}h'_{1,1}x^{r-1}y^{s-1} - \cdots - q^{r(n-s)}h'_{r,1}y^{s-1} \\ & & -h'_{0,2}x^r y^{s-2} - q^{n-s}h'_{1,2}x^{r-1}y^{s-2} - \cdots - q^{r(n-s)}h'_{r,2}y^{s-2} \\ & & \cdots \quad \cdots \quad \cdots \quad \cdots \\ & & -h'_{0,s}x^r - q^{n-s}h'_{1,s}x^{r-1} - \cdots - q^{r(n-s)}h'_{r,s}. \end{array}$$

In this situation, we obtain a new binary linearly recursive sequence $f' = (f'_{m,n})_{m\geq r, n\geq s}$ satisfying the recursive relation $h'(x,y)$, where

$$\begin{array}{rcl} f'_{m,n} & = & h'_{1,0}f_{m-1,n} + h'_{2,0}f_{m-2,n} + \cdots + h'_{r,0}f_{m-r,n} \\ & & +h'_{0,1}f_{m,n-1} + h'_{1,1}f_{m-1,n-1} + \cdots + h'_{r,1}f_{m-r,n-1} \\ & & +h'_{0,2}f_{m,n-2} + h'_{1,2}f_{m-1,n-2} + \cdots + h'_{r,2}f_{m-r,n-2} \\ & & \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad \cdots \\ & & +h'_{0,s}f_{m,n-s} + h'_{1,s}f_{m-1,n-s} + \cdots + h'_{r,s}f_{m-r,n-s}, \end{array}$$

which satisfies the relation $h'(x,y)$.

(2) Similarly, set $h''_{i,j} = q^{-j(m-r)}h_{i,j}$ in Case 2. We then have a new monic binary polynomial $h''(x,y)$; hence, we can obtain a new binary linearly recursive sequence $f'' = (f''_{m,n})_{m\geq r, n\geq s}$ satisfying the recursive relation $h''(x,y)$.

Example 3. *Let $h(x,y) = (x-a)(y-b)$ for any $a, b \in k$. Then, we have a binary linearly recursive sequence*

$$f_{m,n} = bf_{m,n-1} + af_{m-1,n} - abf_{m-1,n-1},$$

satisfying $h(x,y)$, for all $m \geq 1$ and $n \geq 1$. If we take $f_{0,0} = 1$, $f_{0,i} = 0$ and $f_{i,0} = 0$ for all $i \in \mathbb{N}$, then one obtains a binary linearly recursive sequence: $(1, 0, 0, \cdots, 0, -ab, -ab^2, -ab^3, \cdots, 0, -a^2b, -a^2b^2, -a^2b^3, \cdots)$.

Remark 6. *In the case of $q = 1$, Case 1, Case 2 and Case 3 are the same.*

Let $q = 1$. Then, we have $f_{m,n} = f_{n,m}$ for any $m, n \geq 0$ and $\begin{pmatrix} n \\ k \end{pmatrix}$ is the ordinary binomial coefficient. In this section, we will study the binary linearly recursive sequence in Case 1 satisfying the recursive relation $h(x,y)$.

In what follows, by a method similar to that in [6], we can consider $A = \mathbb{F}[x,y]$ in ${}_H\mathcal{M}$ via

$$g^i \cdot (x^k y^l) = p^{i(k+l)} x^k y^l, \quad \text{for all } x^k, y^l \in A,\ g^i \in H. \tag{25}$$

It is not difficult to verify that A is an algebra in ${}_H\mathcal{M}$. By (2) and (5), we have

$$\begin{aligned}
& \tau_{A,A}(x^k y^l \otimes x^s y^t) \\
= & \frac{1}{n} \sum_{i,j=0}^{n-1} p^{-ij} (g^j \cdot (x^s y^t) \otimes g^i \cdot (x^k y^l)) \\
= & (\frac{1}{n} \sum_{i,j=0}^{n-1} p^{-ij} p^{j(s+t)} p^{i(k+l)} (x^s y^t \otimes x^k y^l) \\
= & \frac{1}{n} \sum_{i,j=0}^{n-1} p^{-ij} p^{jr} p^{iu} (x^s y^t \otimes x^k y^l) \\
= & \frac{1}{n} \sum_{i,j=0}^{n-1} p^{iu} (\sum_{j=0}^{n-1} p^{j(r-i)}) (x^s y^t \otimes x^k y^l),
\end{aligned}$$

where we write $s + t = an + r, k + l = bn + u$ for some $0 \leq r, u < n$.

If $i \neq r$, then $\sum_{j=0}^{n-1} p^{j(r-i)} = 0$. Therefore, $\tau_{A,A}(x^k y^l \otimes x^s y^t) = p^{(s+t)(k+l)}(x^s y^t \otimes x^k y^l)$. Thus, we have a braided algebra $A\#A$ with

$$(x^{k_1} y^{l_1} \# x^{s_1} y^{t_1})(x^{k_2} y^{l_2} \# x^{s_2} y^{t_2}) = p^{(s_1+t_1)(k_2+l_2)} (x^{k_1+k_2} y^{l_1+l_2} \# x^{s_1+s_2} y^{t_1+t_2}), \tag{26}$$

for all $x^{k_1} y^{l_1} \# x^{s_1} y^{t_1}, x^{k_2} y^{l_2} \# x^{s_2} y^{t_2} \in A\#A$.

Note that $q = 1$, so $xy = yx$. It follows from (6) that $(x\#y)(y\#1) = p(xy\#y) = p(y\#1)(x\#y)$. If we regard A as a bialgebra in the category ${}_H\mathcal{M}$, then by Majid's bosonization, this requires Δ to be an algebraic morphism in ${}_H\mathcal{M}$. Note that $(x\#y)^l = p^{\frac{1}{2}l(l+1)}(x^l \# y^l)$, and so, by (1) and (6), we have

$$\Delta(x^m y^l) = \sum_{0 \leq k \leq l} \begin{pmatrix} l \\ k \end{pmatrix}_p p^{\frac{1}{2}[(l-k)(l-k+1)+m(m+2l+1)]} x^{m+l-k} y^k \# x^m y^{l-k}, \tag{27}$$

for $m, l \geq 0$.

Notice that x^m is not a group-like element since $\Delta(x^m) = p^{\frac{1}{2}m(m+1)}(x^m \# y^m)$, and y^n is $(x^n, 1)$-primitive, i.e., $\Delta(y^n) = x^n \# y^n + y^n \# 1$ since $\begin{pmatrix} n \\ k \end{pmatrix}_p = 0$ for $1 \leq k \leq n-1$. The counit ε of A is given as usual by $\varepsilon(x^m) = 1, \varepsilon(y^k) = \delta_{0,k}$. It is easy to check that Δ and ε are morphisms in ${}_H\mathcal{M}$. But, A is not a Hopf algebra in ${}_H\mathcal{M}$ unless $x^2 = 1$ with $S(x) = x$ and $S(y) = -xy$.

In what follows, Theorem 6 shows that the space A^0 of binary linearly recursive sequences is a bialgebra in ${}_H\mathcal{M}$. The quantum convolution product in A^* is given as $(f_{m,l}) *_p (g_{m,l}) = (h_{m,l})$, where

$$h_{m,l} = \sum_{0\le k\le l} \begin{pmatrix} l \\ k \end{pmatrix}_p p^{\frac{1}{2}[(l-k)(l-k+1)+m(m+2l+1)]} f_{m+l-k,k} g_{m,l-k}. \tag{28}$$

Thus, we have the following.

Theorem 6. *Let p be a root of unity in k. Then, the binary linearly recursive sequences are closed under the quantum convolution product $(f_{m,l}) *_p (g_{m,l}) = (h_{m,l})$, where $h_{m,l}$ is given by (8).*

Remark 7. *(1) If $x = 1$ in $A = k[x, y]$, then by (7), we have*

$$\Delta(y^l) = \sum_{0\le k\le l} \begin{pmatrix} l \\ k \end{pmatrix}_p y^k \# y^{l-k},$$

for all $l \ge 0$. Hence, we can obtain the result in [6], Theorem 4.1.

(2) If $y = 1$ in $A = k[x, y]$, then by (7), one has

$$\Delta(x^m) = p^{\frac{1}{2}m(m+1)}(x^m \# y^m).$$

In this case, we have

$$h_m = p^{\frac{1}{2}m(m+1)} f_{m+l-k} g_m. \tag{29}$$

It should be noted the algebraic structure of linearly recursive sequences under the Hadamard product was described in [24]. We call the product given by (9) a quantum Hadamard product. Then, we have the following corollary.

Corollary 4. *The linearly recursive sequences are closed under the quantum Hadamard product $(f_m) *_p (g_m) = (h_m)$, where h_m is given by (9).*

Next, we give a direct proof of Theorem 6. First, we have the following lemma.

Lemma 5. *Let $f = (f_{m,l})$ be the binary linearly recursive sequence over k satisfying the relation $h(x, y)$. Let $\alpha_1, \cdots, \alpha_k; \beta_1, \cdots, \beta_u$ be the roots of $h(x, y)$ in $\bar{k}$, the algebraic closure of k. Then, for any $i, j \ge 0, s, t \ge 0$, the subsequence $f^{(i,j)(s,t)}$ given by*

$$\begin{aligned} f^{(i,j)(s,t)} = \quad & (f_{i,s}, f_{i,s+t}, f_{i,s+2t}, \cdots \\ & f_{i+j,s}, f_{i+j,s+t}, f_{i+j,s+2t}, \cdots \\ & f_{i+2j,s}, f_{i+2j,s+t}, f_{i+2j,s+2t}, \cdots \\ & \cdots\cdots \quad \cdots\cdots \quad \cdots\cdots \quad \cdots) \end{aligned}$$

is a binary linearly recursive sequence over k, satisfying the relation $h_{j,t}(x, y) = (x - \alpha_1^j) \cdots (x - \alpha_k^j)(y - \beta_1^t) \cdots (y - \beta_u^t)$.

Proof. It suffices to show $h_{j,t}(x,y) \rightharpoonup f^{(i,j)(s,t)} = 0$ in order to verify that $f^{(i,j)(s,t)}$ satisfies $h_{j,t}(x,y)$; here, $\rightharpoonup$ is defined by $(a \rightharpoonup f)(b) = f(ba)$ for $a, b \in A$, $f \in A^*$, because A is a commutative algebra.

Note that $f^{(i,j)(s,t)}(x^v y^w) = f_{i+vj,s+wt} = f(x^{i+vj}y^{s+wt})$ for $v, w \geq 0$. So, for all $g(x,y) \in k[x,y]$, $f^{(i,j)(s,t)}(g(x,y)) = f(x^i y^s g(x^j y^t))$. Thus, we have

$$\begin{aligned} & h_{j,t}(x,y) \rightharpoonup f^{(i,j)(s,t)}(x^v y^w) \\ = \; & f^{(i,j)(s,t)}(x^v y^w h_{j,t}(x,y)) \\ = \; & f(x^i y^s x^{vj} y^{wt} h_{j,t}(x^j, y^t)) \\ = \; & f(x^{i+vj} y^{s+wt} h_{j,t}(x^j, y^t)). \end{aligned}$$

For $h_{j,t}(x^j, y^t) = I(x,y)h(x,y)$ for some $I(x,y) \in k[x,y]$, and $h(x,y) \rightharpoonup f = 0$, one has $f(x^{i+vj}y^{s+wt}h_{j,t}(x^j,y^t)) = (h(x,y) \rightharpoonup f)(x^{i+vj}y^{s+wt}I(x,y)) = 0$, concluding the proof. □

Example 4. *In Example 2, we have the subsequence*

$$\begin{aligned} f^{(1,2)(3,4)} \quad = \quad & (f_{1,3}, f_{1,7}, f_{1,11}, \cdot \\ & f_{3,3}, f_{3,7}, f_{3,11}, \cdots \\ & f_{5,3}, f_{5,7}, f_{5,11}, \cdots\cdots), \end{aligned}$$

satisfying $h_{2,4}(x,y) = (x - a^2)(y - b^4)$. Explicitly, one has a binary linearly recursive sequence $g_{m,n} = b^4 g_{m,n-1} + a^2 g_{m-1,n} - a^2 b^4 g_{m-1,n-1}$ satisfying $h_{2,4}(x,y)$, and $(g_{m,n})_{m \geq 1, n \geq 1}$ is the same as $f^{(1,2)(3,4)}$. For example, $g_{0,0} = f_{1,3} = -ab^3$, $g_{0,1} = f_{1,7} = -ab^7$, $g_{1,0} = f_{3,3} = -a^3b^3$, and it follows from the formula $g_{m,n}$ that

$$\begin{aligned} g_{1,1} &= b^4 g_{1,0} + a^2 g_{0,1} - a^2 b^4 g_{0,0} \\ &= b^4(-a^3b^3) + a^2(-ab^7) - a^2b^4(-ab^3) \\ &= -a^3b^7 - a^3b^7 + a^3b^7 = -a^3b^7, \end{aligned}$$

which equals to $f_{3,7}$.

Theorem 7. *Let p be a primitive nth root of unity. Let $f = (f_{m,l})$ and $g = (g_{m,l})$ be binary linearly recursive sequences, satisfying $h(x,y) = \prod_{i,j}(x - \alpha_i)^{m_i}(y - \beta_j)^{n_j}$ and $J(x,y) = \prod_{k,r}(x - \gamma_k)^{s_k}(y - \lambda_r)^{t_r}$, respectively. Then, the quantum product $(f_{m,l}) *_p (g_{m,l}) = (h_{m,l})$ where $h_{m,l}$ is given by (8) is a binary linearly recursive sequence, satisfying $W(x^n, y^n)$; here, $W(x,y) = \prod_{i,k;j,r}[x - (\alpha_i^n + \gamma_k^n)^{m_i+s_k-1}][y - (\beta_j^n + \lambda_r^n)^{n_j+t_r-1}]$.*

Proof. For $0 \leq a, b \leq n$ and $c, d \geq 0$, we have

$$\begin{aligned}
& h^{(a,n)(b,n)}(x^c y^d) = h_{a+cn,b+dn} \\
= & \sum_{u=0}^{b+dn} \binom{b+dn}{u}_p p^{\frac{1}{2}[(b+dn-u)(b+dn-u+1)+(a+cn)(a+2b+(c+2d)n+1)]} \\
& f_{a+b+(c+d)n-u,u} g_{a+cn,b+dn-u} \quad \text{(by (8))} \\
= & \sum_{s=0}^{b}\sum_{j=0}^{d} \binom{b+dn}{s+jn}_p p^{\frac{1}{2}\{[(b-s)+(d-j)n][(b-s)+(d-j)n+1]+(a+cn)[a+2b+(c+2d)n+1]\}} \\
& f_{a+b-s+(c+d-j)n,s+jn} g_{a+cn,b-s+(d-j)n} \\
= & \sum_{s=0}^{b}\sum_{j=0}^{d} \binom{d}{j}\binom{b}{s}_p p^{\frac{1}{2}\{[(b-s)+(d-j)n][(b-s)+(d-j)n+1]+(a+cn)[a+2b+(c+2d)n+1]\}} \\
& f_{a+b-s+(c+d-j)n,s+jn} g_{a+cn,b-s+(d-j)n} \\
= & \sum_{s=0}^{b} \binom{b}{s}_p p^{\frac{1}{2}[(b-s)(b-s+1)+a(a+2b+1)]} \\
& (\sum_{j=0}^{d} \binom{d}{j} (f^{(a+b-s,n)(s,n)})_{(c+d-j,j)} (g^{(a,n)(b-s,n)})_{(c,d-j)}) \\
= & \sum_{s=0}^{b} \binom{b}{s}_p p^{\frac{1}{2}[(b-s)(b-s+1)+a(a+2b+1)]} \\
& (\sum_{i=0}^{d} \binom{d}{i} (f^{(a+b-s,n)(s,n)})_{(c+i,d-i)} (g^{(a,n)(b-s,n)})_{(c,i)}).
\end{aligned}$$

Therefore,

$$h^{(a,n)(b,n)} = \sum_{s=0}^{b} \binom{b}{s}_p p^{\frac{1}{2}[(b-s)(b-s+1)+a(a+2b+1)} f^{(a+b-s,n)(s,n)} * g^{(a,n)(b-s,n)},$$

where $*$ is the usual convolution product (refer to Formula (4) for $q = 1$). It follows from Lemma 2 that $f^{(a+b-s,n)(s,n)}$ and $g^{(a,n)(b-s,n)}$ are binary linearly recursive sequences, satisfying $h_{n,n}(x,y) = \prod_{i,j}(x-\alpha_i^n)^{m_i}(y-\beta_j^n)^{n_j}$ and $J_{n,n}(x,y) = \prod_{k,r}(x-\gamma_k^n)^{s_k}(y-\lambda_r^n)^{t_r}$, respectively. Hence, Theorem holds. In fact, h is the interlacing of the sequences $h^{(0,n)(0,n)}$, $h^{(0,n)(1,n)}, \cdot, h^{(0,n)(n-1,n)}, \cdots, h^{(1,n)(0,n)}, \cdots, h^{(1,n)(n-1,n)}, \cdots, h^{(n-1,n)(0,n)}, \cdots, h^{(n-1,n)(n-1,n)}$.

This finishes the proof. □

Remark 8. *This paper studied the quantum convolution product of binary linearly recursive sequences. But what about the case of multi-linearly recursive sequences? We are sure that this topic is related to the polynomial algebra $A = \mathbb{F}[x_1, x_2, \cdots, x_n]$ in n-variable $x_1, x_2, \cdots, x_n$.*

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