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Resurgence in complex Chern-Simons theory at generic levels

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Abstract: In this note we study the resurgent structure of $sl(2,\mathbb{C})$ Chern-Simons state integral model on knot complements $S^3\backslash\mathbf{4}_1, S^3\backslash\mathbf{5}_2$ with generic discrete level $k \geq 1$ and with small boundary holonomy deformation. The coefficients of the saddle point expansions are in the trace field of the knot extended by the holonomy parameter. Despite increasing complication of the asymptotic series as the level k increases, the resurgent structure of the asymptotic series is universal: both the distribution of Borel plane singularities and the associated Stokes constants are independent of the level k.

Keywords: Chern-Simons Theories, Nonperturbative Effects, Topological Field Theories

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Contents

1 Introduction

Complex Chern-Simons theory remains an active field of research that links up many fields in both physics and mathematics. From the physics side, $sl(2,\mathbb{C})$ Chern-Simons theory was first considered in [1] as a means to describe three dimensional quantum gravity on Lorentzian spacetime with positive cosmological constant. Later a beautiful connection was revealed between $sl(N,\mathbb{C})$ Chern-Simons theory and 3d $N = 2$ superconformal field theories by the name of the 3d-3d correspondence [2–5] (see [6] for a review), which was used to define a large class of 3d SCFTs. Conversely, the BPS sector of the dual 3d SCFT was used in [7, 8] to define homological invariants $\widehat{Z}$ for the three manifold.

From the mathematical side, complex Chern-Simons theory plays an important role in quantum topology. It was found by Witten long time ago [9] that Jones polynomials of knots, which are Laurent polynomials in q, are vacuum expectation values of Wilson loops along the knot in SU(2) Chern-Simons theory with $q = \exp(2\pi\mathsf{i}/(k+2))$, where k is the discrete level. It is then natural to consider more general complex values of q, which translates to the complexification of the gauge group [10]. The complex Chern-Simons theory then provides a new topological invariant for 3d manifolds.

For instance, when M is a knot complement, the partition function of the $sl(2,\mathbb{C})$ Chern-Simons theory on M is generally believed to reduce to a finite-dimensional integral called the state integral, as proposed in [11–15] based on [16, 17], which is constructed based on an ideal triangulation of the manifold. The saddle points of the state integral in the weak coupling limit are non-Abelian $sl(2,\mathbb{C})$ flat connections on the three manifold [9], and the associated asymptotic series $\varphi^{(\sigma_j)}$ $(j \geq 1)$ can be obtained by Gaussian expansion [17]. The state integral and thus the asymptotic series are believed to be topological invariants as they do not change with a different triangulation scheme.

On the other hand, the state integral model has the disadvantage that it is incomplete in the sense that it misses the information of Abelian flat connections [18]. Instead, the asymptotic series $\varphi^{(\sigma_0)}(\tau)$ associated to the Abelian flat connection σ_0 can be determined from the colored Jones polynomials of the knot expanded in terms of $h = \log q$ [19–22].

It is natural to study the resurgent structure of these asymptotic series. On the one hand, complex Chern-Simon theory is a special quantum field theory with no renormalons, and therefore these asymptotic series of saddle points must transform to each other by Stokes automorphism, forming a so-called resurgent structure. On the other hand, the resurgence theory tells us that the Stokes constants that control all the Stokes automorphisms define necessarily new invariants which are non-perturbative in nature. The resurgence problem in Chern-Simons theory was first considered in [23], where it was found (and later emphasized again in [24]) that the Borel transform of $\varphi^{(\sigma_0)}(\tau)$ has poles at $\mathrm{Vol}(M) + \mathsf{i}CS(M) + 4\pi^2\mathbb{Z}\mathsf{i}$, which can be interpreted as a resurgent formulation of the Volume Conjecture [25, 26]. Later the resurgent problems of the series $\varphi^{(\sigma_j)}(\tau)$ $(j \neq 0)$ for non-Abelian saddle points were considered in [27–29]. See also [30] on related results on Faddeev's quantum dilogarithm which is a crucial ingredient of state integrals.

Recently the resurgent problem for the $sl(2,\mathbb{C})$ Chern-Simons theory at level $k = 1$ on the complement of the two simplest hyperbolic knots $\mathbf{4}_1$ and $\mathbf{5}_2$ were completely solved in [24, 31, 32]. It was demonstrated that $\varphi^{(\sigma_0)}(\tau)$ of the Abelian flat connection is related to $\varphi^{(\sigma_j)}(\tau)$ $(j \neq 0)$ of non-Abelian flat connections by Stokes automoprhisms but not the other way around, while $\varphi^{(\sigma_j)}(\tau)$ $(j \neq 0)$ transform to themselves. The Stokes constants of all these Stokes automorphisms were computed explicitly. Surprisingly the Stokes constants relating $\varphi^{(\sigma_j)}(\tau)$ $(j \neq 0)$ were found to coincide with the BPS invariants in the dual 3d $N = 2$ superconformal field theory on S^1 [31, 32], providing another interesting entry of dictionary in the 3d-3d correspondence. Furthermore, Stokes automorphisms were used to construct a new state integral for the knot $\mathbf{4}_1$, which now includes the contribution of the Abelian flat connection as well [24]. Finally, the statement that the series $\varphi^{(\sigma_0)}(\tau)$ Borel resums to the $\widehat{Z}$ invariant [33] was corrected to include additional contributions from Borel resummation of $\varphi^{(\sigma_j)}(\tau)$ [24].

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In this work we continue this line of research, and generalise the results of [31, 32] to $sl(2,\mathbb{C})$ Chern-Simons theories with levels $k \geq 1$ with small boundary holonomy deformation x. We first demonstrate that the asymptotic series $\varphi^{(k,\sigma_j)}(x;\tau)$ $(j \geq 1)$ associated to non-Abelian flat connections are such that their coefficients always live in the trace field of the knot extended by the holonomy parameter x. We then show that these asymptotic series enjoy universal resurgent structures: both the distribution of Borel plane singularities and the associated Stokes constants are independent of the level k, despite that the asymptotic series themselves depend highly non-trivally on the level.

The remainder of the paper is structured as follows. We summarise our results, both on the properties of the asymptotic series and their resurgent structure in section 2. We then give details in two example sections 3, 4. Finally we conclude and list open questions in section 5.

2 Resurgent structure of complex Chern-Simons theory at generic level

2.1 Perturbative complex Chern-Simons theory at generic level

The $sl(2,\mathbb{C})$ Chern-Simons theory on a three manifold M has action

$$I_{k,s} = tS_{\mathrm{CS}}(A) + \tilde{t}S_{\mathrm{CS}}(\overline{A}) \tag{2.1}$$

where $S_{\mathrm{CS}}(A)$ is the three dimensional Chern-Simons action

$$S_{\mathrm{CS}}(A) = \frac{1}{4\pi}\int_M \mathrm{Tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right), \tag{2.2}$$

A is a connection of the $sl(2,\mathbb{C})$ bundle over M, and $\overline{A}$ its complex conjugation. The couplings $t, \tilde{t}$ can be split as

$$t = \frac{1}{2}(k+\mathrm{i}s), \quad \tilde{t} = \frac{1}{2}(k-\mathrm{i}s). \tag{2.3}$$

In a consistent quantum theory, the level k must be an integer so that the partition function of the Chern-Simons theory is invariant under large gauge transformations. The other level s is constrained to be either real or imaginary in order for the CS theory to be unitary, which, however, we do not impose in this work.

Suppose the 3-manifold M is a knot complement $M = S^3\backslash K$ with torus boundary. The $sl(2,\mathbb{C})$ connection on the boundary torus is parametrised by the holonomies x, y along the meridian and the longitude. The condition that the $sl(2,\mathbb{C})$ connection can be extended to the interior of M constrains that x, y are not independent but are related by a polynomial equation

$$A(x,y) = 0 \tag{2.4}$$

called the A-polynomial. Furthermore, in a quantum theory, the holonomy x should be parametrised as

$$x = \exp\left(\frac{2\pi\mathsf{b}\mu}{k} - \frac{2\pi\mathrm{i}n}{k}\right), \tag{2.5}$$

where $\mu \in \mathbb{C}$, $n \in \mathbb{Z}_k$, and the parameter b is related to the levels k, s by

$$\mathsf{b}^2 = \frac{k - \mathsf{i}s}{k + \mathsf{i}s}. \tag{2.6}$$

By the 3d-3d correspondence, the $sl(2,\mathbb{C})$ Chern-Simons theory at discrete level $k \geq 1$ on M is dual to a 3d $N = 2$ SCFT denoted by $T_2[M]$ put on an orbifold of the squashed 3-sphere $S^3_{\mathsf{b}}/\mathbb{Z}_k$ [15],

$$S^3_{\mathsf{b}}/\mathbb{Z}_k = \left\{(z,w) \in \mathbb{C}^2 \,|\, \mathsf{b}^2|z|^2 + \mathsf{b}^{-2}|w|^2 = 1\right\}/(z,w) \sim (\mathrm{e}^{2\pi\mathsf{i}/k}z, \mathrm{e}^{-2\pi\mathsf{i}/k}w). \tag{2.7}$$

Here b has the geometric meaning of the squashing parameter. When M is a knot complement, the $T_2[M]$ theory has a $u(1)$ flavor symmetry. The parameters μ, n are then respectively the flavor mass, and the holonomy of the flavor vector field. The latter takes value in $\pi_1(S^3_{\mathsf{b}}/\mathbb{Z}_k) = \mathbb{Z}_k$.

When $M = S^3\backslash K$ is a hyperbolic manifold, it can be triangulated, namely decomposed to N ideal tetrahedra glued along their faces. In [4], the $T_2[M]$ theory was constructed based on the triangulation data from the $T_2[\Delta]$ associated with an ideal tetrahedron Δ, which is well understood. Based on this construction, the partition function of $T_2[M]$ on $S^3_{\mathsf{b}}/\mathbb{Z}_k$ was computed [15], and it was used to give the state integral model for the $sl(2,\mathbb{C})$ Chern-Simons theory at level k.

It is important to note that, as first pointed out in [18], the construction in [4] misses an entire sector related to the Abelian flat connection, and thus the partition function computed in [15] is also not complete. Rather, the state integral model should be regarded as the reduced partition function of the $sl(2,\mathbb{C})$ Chern-Simons theory that does not include the contribution from the Abelian flat connection.

Let us discuss in a bit detail the state integral model. The triangulation of M in terms of N tetrahedra is described by the small Neumann-Zagier data[1] that can be encoded in a tuple $\gamma = (\boldsymbol{A}, \boldsymbol{B}, \nu)$ consisting of two matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathrm{GL}(N,\mathbb{Z})$ and a vector $\nu \in \mathbb{Z}^N$. They encode the coefficients of Thurston's gluing equations for the triangulation including $N-1$ independent equations imposing condition on internal edges, and one equation describing holonomy on external edges. We order rows of $\boldsymbol{A}, \boldsymbol{B}$ and elements of ν so that the first rows of $\boldsymbol{A}, \boldsymbol{B}$ and the first element of ν corresponds to the external edge.

It was shown in [36] that $(\boldsymbol{A}\ \boldsymbol{B})$ forms the top half of a symplectic matrix, so that $\boldsymbol{A}\boldsymbol{B}^T$ is symmetric and $(\boldsymbol{A}\ \boldsymbol{B})$ is of full rank. On the other hand, giving a triangulation of M, the choice of Neumann-Zagier data is not unique. Following [34, 35] we always choose a set of Neumann-Zagier data where $\boldsymbol{B}$ is invertible over integers, and then $\boldsymbol{B}^{-1}\boldsymbol{A}$ must be symmetric.

Let us introduce in addition $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_N) \in \mathbb{C}^N$, $\boldsymbol{m} = (m_1, \ldots, m_N) \in (\mathbb{Z}_k)^N$ as well as $\boldsymbol{\mu} = (\mu, 0, \ldots, 0) \in \mathbb{C} \times 0^{N-1}$, $\boldsymbol{n} = (n, 0, \ldots, 0) \in \mathbb{Z}_k \times 0^{N-1}$. By following the procedure in [15] and extending slightly the formulas in [34], the state integral model can

[1]The complete Neumann-Zagier data introduced in [34, 35] also includes a choice of the solution to the Neumann-Zagier equation.

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be written as

$$
\begin{aligned}
&\mathcal{Z}_\gamma^{(k)}(\mu, n; \mathsf{b}) \\
&= \frac{1}{k^N\sqrt{\det \boldsymbol{B}}} \sum_{m\in(\mathbb{Z}_k)^N} \int \mathrm{d}^N\boldsymbol{\sigma}\, \mathrm{e}^{\frac{2\pi \mathrm{i}}{k}(-\boldsymbol{\sigma}\boldsymbol{B}^{-1}\boldsymbol{\mu}+\boldsymbol{m}\boldsymbol{B}^{-1}\boldsymbol{n})} \mathrm{e}^{\frac{\pi\mathrm{i}}{k}(-\boldsymbol{\sigma}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{\sigma}+\boldsymbol{m}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{m}+2c_{\mathsf{b}}\boldsymbol{\sigma}\boldsymbol{B}^{-1}\boldsymbol{\nu})} \\
&\quad (-1)^{\boldsymbol{m}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{m}} \prod_{i=1}^N \mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](\sigma_i, m_i).
\end{aligned}
\tag{2.8}
$$

The part in blue is due to non-vanishing boundary holonomy and thus new compared to the results in [34]. Here $\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](\sigma_i, m_i)$ is the partition function of the Chern-Simons theory on a tetrahedron, which can be expressed in terms of Faddeev's quantum dilogarithm $\Phi_{\mathsf{b}}(x)$

$$
\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](\mu, n) = \prod_{(r,s)\in\Gamma(k;n)} \Phi_{\mathsf{b}}\left(c_{\mathsf{b}} - \frac{1}{k}(\mu + \mathrm{i}\mathsf{b}r + \mathrm{i}\mathsf{b}^{-1}s)\right) \tag{2.9}
$$

where

$$
\Gamma(k;n) = \{(r,s) \in \mathbb{Z}^2 \mid 0 \le r, s < k, r - s \equiv n (\mathrm{mod}\ k)\} \tag{2.10}
$$

and $c_{\mathsf{b}} = \frac{\mathrm{i}}{2}(\mathsf{b} + \mathsf{b}^{-1})$. This defines a meromorphic function of $\mu \in \mathbb{C}$ for each $n \in \mathbb{Z}_k$, and it is defined for all values of b with b^2 in the cut plane $\mathbb{C}' = \mathbb{C}\backslash\mathbb{R}_{\le 0}$. When $\mathrm{Im}\,\mathsf{b} > 0$ or $\mathrm{Im}\,\mathsf{b} < 0$, it has the factorisation form

$$
\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](\mu, n) = (qx^{-1}; q)_\infty(\tilde{q}^{-1}\tilde{x}^{-1}; \tilde{q}^{-1})_\infty \tag{2.11}
$$

where we use the notation

$$
\begin{aligned}
q &= \exp\frac{2\pi\mathrm{i}}{k}(\mathsf{b}^2+1), & \tilde{q} &= \exp -\frac{2\pi\mathrm{i}}{k}(\mathsf{b}^{-2}+1), \\
x &= \exp\left(\frac{2\pi\mathsf{b}\mu}{k} - \frac{2\pi\mathrm{i}n}{k}\right), & \tilde{x} &= \exp\left(\frac{2\pi\mathsf{b}^{-1}\mu}{k} + \frac{2\pi\mathrm{i}n}{k}\right),
\end{aligned}
\tag{2.12}
$$

and $(a,q)_\infty$ is the q-Pochhammer symbol defined to be $\prod_{j=1}^\infty(1 - aq^j)$ if $|q| < 1$ or $1/(q^{-1}a; q^{-1})_\infty$ if $|q| > 1$.

One can give explicitly the contour of integral to make the N dimensional integral convergent [15]. But it is a bit complicated and we specify the contours in individual examples in sections 3, 4.

We are interested in the double scaling limit

$$
\mathsf{b} \to 0, \quad \mu \to \infty, \quad \mathsf{b}\mu \text{ fixed}, \tag{2.13}
$$

where the last condition guarantees that the holonomy parameter x remains finite. Let us introduce

$$
\zeta = \mathrm{e}^{2\pi\mathrm{i}/k}, \quad \tau = \mathsf{b}^2 \tag{2.14}
$$

By scaling the variables

$$
Z_i = 2\pi\mathsf{b}\sigma_i, \quad u_i = 2\pi\mathsf{b}\mu_i = \begin{cases} 2\pi\mathsf{b}\mu, & i = 1 \\ 0, & i \ge 2 \end{cases} \tag{2.15}
$$

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and using the asymptotic expansion (A.15) of the tetrahedron partition function, the integral (2.8) can be written as

$$\mathcal{Z}_\gamma^{(N)}(\mu,n;\tau) = \frac{1}{k^N\sqrt{\det \boldsymbol{B}}(4\pi^2\tau)^{N/2}} \sum_{\boldsymbol{m}\in(\mathbb{Z}_k)^N} (-1)^{\boldsymbol{m}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{m}}\zeta^{\frac{1}{2}\boldsymbol{m}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{m}+\boldsymbol{m}\boldsymbol{B}^{-1}\boldsymbol{n}} \int \mathrm{d}^N\boldsymbol{Z} \exp\sum_{\ell=0}^{\infty}(2\pi\mathsf{i}\tau)^{\ell-1}U_\ell^{(\boldsymbol{m})}(\boldsymbol{u},\boldsymbol{Z}) \tag{2.16}$$

where the coefficient functions are

$$U_0(\boldsymbol{u},\boldsymbol{Z}) = \frac{1}{k}\boldsymbol{Z}\boldsymbol{B}^{-1}\boldsymbol{u}+\frac{1}{2k}\boldsymbol{Z}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{Z} - \frac{\pi\mathsf{i}}{k}\boldsymbol{Z}\boldsymbol{B}^{-1}\boldsymbol{\nu} + \frac{1}{k}\sum_{i=1}^{N}\mathrm{Li}_2(\mathrm{e}^{-Z_i}), \tag{2.17a}$$

$$U_{\ell\geq1}^{(\boldsymbol{m})}(\boldsymbol{u},\boldsymbol{Z}) = -\frac{1}{2k}\boldsymbol{Z}\boldsymbol{B}^{-1}\boldsymbol{\nu}\delta_{\ell,1} + \sum_{i=1}^{N}\sum_{s=1}^{k}\frac{B_\ell(s/k)}{\ell!}\,\mathrm{Li}_{2-\ell}(\mathrm{e}^{\frac{2\pi\mathsf{i}}{k}(s+m_i)}\mathrm{e}^{-\frac{Z_i}{k}}). \tag{2.17b}$$

We drop the superscript $(\boldsymbol{m})$ for U_0 since it does not depend on the index $\boldsymbol{m}$. We have also simplified the expression of U_0 using the identity

$$\sum_{s=1}^{k}\mathrm{Li}_2(\mathrm{e}^{2\pi\mathsf{i}s/k}x) = \frac{1}{k}\,\mathrm{Li}_2(x^k), \tag{2.18}$$

which can be easily proved by the series expansion of $\mathrm{Li}_2(x)$. The saddle point equations, i.e. the critical point equations for U_0 are

$$u_j + \sum_{i=1}^{N}A_{ji}Z_i + \sum_{i=1}^{N}B_{ji}\log(1-\mathrm{e}^{-Z_i}) = \pi\mathsf{i}\nu_j, \quad j=1,\ldots,N, \tag{2.19}$$

or equivalently with

$$z_i = \mathrm{e}^{Z_i}, \quad w_i = \mathrm{e}^{u_i} \tag{2.20}$$

we find

$$w_j\prod_{i=1}^{N}z_i^{A_{ji}}(1-z_i^{-1})^{B_{ji}} = (-1)^{\nu_j}, \quad j=1,\ldots,N. \tag{2.21}$$

Note that at the end of the day, we must set $u_j=0$ or equivalently $w_j=1$ for $j=2,\ldots,N$. Since

$$x = \exp\frac{2\pi\mathsf{i}}{k}(-\mathsf{i}\mathsf{b}\mu - n) = \exp\left(\frac{u}{k}-\frac{2\pi\mathsf{i}n}{k}\right) = \zeta^{-n}w_1^{1/k}, \tag{2.22}$$

the critical equations can also be written as

$$x^k\prod_{i=1}^{N}z_i^{A_{ji}}(1-z_i^{-1})^{B_{ji}} = (-1)^{\nu_j}, \quad j=1, \tag{2.23a}$$

$$\prod_{i=1}^{N}z_i^{A_{ji}}(1-z_i^{-1})^{B_{ji}} = (-1)^{\nu_j}, \quad j=2,\ldots,N. \tag{2.23b}$$

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They can be regarded as a deformation of the Neumann-Zagier equations. Each solution $\boldsymbol{z}^* = \exp \boldsymbol{Z}^*$ to the critical equations corresponds to a non-Abelian $sl(2,\mathbb{C})$ flat connection σ_* over M.

We can choose one of the critical points (saddle points) $\boldsymbol{Z}^* = (Z_i^*)$ and expand around the critical point by $Z_j = Z_j^* + (2\pi\mathrm{i}\tau)^{1/2}\delta Z_j$. By an expansion first in terms of τ and then performing Gaussian integral order by order, we obtain a trans-series $\Phi_\gamma^{(k,\sigma_*)}(\mu, n;\tau)$. This was worked out in [34] when $\mu = 0$ and $n = 0$. In the generic case the calculation is similar, and we present the results here.

Let us introduce $z = \mathrm{e}^Z$, as well as

$$z' = (1-z)^{-1}, \quad z'' = 1 - z^{-1} \tag{2.24}$$

so that

$$zz'z'' = -1. \tag{2.25}$$

From (2.17b) it is necessary to choose a k-th root of Z^*. We introduce θ_i such that

$$(\theta_i)^k = z_i^*. \tag{2.26}$$

Let us also define the Average and Vev of a function g

$$\mathrm{Av}(g(\boldsymbol{m})) = \frac{\sum_{\boldsymbol{m}\in(\mathbb{Z}_k)^N} a_{\boldsymbol{m}}(x,\theta) g(\boldsymbol{m})}{\sum_{\boldsymbol{m}\in(\mathbb{Z}_k)^N} a_{\boldsymbol{m}}(x,\theta)} \tag{2.27}$$

with

$$a_{\boldsymbol{m}}(x,\theta) = x^{-\boldsymbol{m}\boldsymbol{B}^{-1}}(-1)^{\boldsymbol{m}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{m}}\zeta^{\frac{1}{2}(\boldsymbol{m}\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{m}+\boldsymbol{m}\boldsymbol{B}^{-1}\boldsymbol{\nu})}\theta^{-\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{m}}\prod_{i=1}^N(\zeta\theta_i^{-1};\zeta)_{m_i}^{-1}. \tag{2.28}$$

and

$$\langle g(\delta\boldsymbol{Z})\rangle = \frac{\int \mathrm{d}^N\delta\boldsymbol{Z}\,\mathrm{e}^{-\frac{1}{2}\delta\boldsymbol{Z}^T\boldsymbol{H}\delta\boldsymbol{Z}} g(\delta\boldsymbol{Z})}{\int \mathrm{d}^N\delta\boldsymbol{Z}\,\mathrm{e}^{-\frac{1}{2}\delta\boldsymbol{Z}^T\boldsymbol{H}\delta\boldsymbol{Z}}} \tag{2.29}$$

with

$$\boldsymbol{H} = \frac{1}{k}\left(-\boldsymbol{B}^{-1}\boldsymbol{A} + \Delta_{z^{*\prime}}\right), \tag{2.30}$$

and

$$\Delta_z = \mathrm{diag}(z_1,\ldots,z_N). \tag{2.31}$$

The perturbative expansion of the state integral (2.8) near the critical point $\boldsymbol{Z}^*$ then reads

$$\Phi_\gamma^{(k,\sigma_*)}(u,n;\tau) = \mathrm{e}^{U_{0,0}^{(k)}(\sigma_*)/(2\pi\mathrm{i}\tau)}\omega_\gamma^{(k,\sigma_*)}\varphi_\gamma^{(k,\sigma_*)}(u,n;\tau). \tag{2.32}$$

The 1-loop contribution is

$$\omega_\gamma^{(k,\sigma_*)} = \frac{1}{(\mathrm{i}k)^{N/2}\sqrt{\det(A\Delta_{z^{*\prime\prime}} + B\Delta_{z^*}^{-1})\boldsymbol{z}^{*\boldsymbol{B}^{-1}\boldsymbol{\nu}/k}}}\prod_{i=1}^N D_k^*(\theta_i^{-1})^{1/k}\sum_{\boldsymbol{m}\in(\mathbb{Z}_k)^N} a_{\boldsymbol{m}}(x,\theta). \tag{2.33}$$

Here recall the *cyclic quantum dilogarithms* are defined by

$$D_k(z)=\prod_{s=1}^{k-1}(1-\zeta^s z)^s,\quad D_k^*(z)=\prod_{s=1}^{k-1}(1-\zeta^{-s}z)^s. \tag{2.34}$$

Higher loop contributions are given in

$$\varphi_\gamma^{(k,\sigma_*)}(u,n;\tau)=\mathrm{Av}\left(\langle g(\delta\boldsymbol{Z},\boldsymbol{m};x,\theta)\rangle\right)=1+\tau\,\mathbb{C}[[\tau]] \tag{2.35}$$

and

$$g(\delta\boldsymbol{Z},\boldsymbol{m};x,\theta)=\exp\left(\sum_{\ell\geq 0}(2\pi\mathsf{i}\tau)^{\ell+d/2-1}\sum_{d\geq 0}{}' U_{k,(i^d)}^{(\ell,\boldsymbol{m})}\delta Z_i^d\right), \tag{2.36}$$

where $\sum_{d\geq 0}{}'$ means $d\geq 3$ for $\ell=0$, $d\geq 1$ for $\ell=1$, and $d\geq 0$ for $\ell\geq 2$. Besides, in each sum a summation over $i=1,\dots,N$ is implicit. The coefficients are

$$U_{0,0}^{(k)}=\frac{1}{k}\boldsymbol{Z}^*\boldsymbol{B}^{-1}\boldsymbol{u}-\frac{\pi\mathsf{i}}{k}\boldsymbol{Z}^*\boldsymbol{B}^{-1}\boldsymbol{\nu}+\frac{1}{2k}\boldsymbol{Z}^*\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{Z}^*+\frac{1}{k}\sum_{j=1}^{N}\mathrm{Li}_2(\mathrm{e}^{-Z_j^*}), \tag{2.37a}$$

$$U_{0,(i^d)}^{(k)}=\frac{(-1)^d}{kd!}\mathrm{Li}_{2-d}(\mathrm{e}^{-Z_i^*}),\quad d\geq 3, \tag{2.37b}$$

$$\begin{aligned}U_{\ell\geq 1,(i^d)}^{(k,\boldsymbol{m})}=&\,\delta_{\ell,1}\delta_{d,0}\left(-\frac{1}{2k}\boldsymbol{Z}^*\boldsymbol{B}^{-1}\boldsymbol{\nu}\right)+\delta_{\ell,1}\delta_{d,1}\left(-\frac{1}{2k}(\boldsymbol{B}^{-1}\boldsymbol{\nu})_i\right)\\&+\frac{(-1)^d}{k^d d!\ell!}\sum_{j=1}^{N}\sum_{s=1}^{k}B_\ell(s/k)\,\mathrm{Li}_{2-\ell-d}(\mathrm{e}^{\frac{2\pi\mathsf{i}}{k}(s+m_j)}\mathrm{e}^{-\frac{Z_j^*}{k}})(\delta_{ij})^d.\end{aligned} \tag{2.37c}$$

The trans-series $\Phi_\gamma^{(k,\sigma_*)}(u,n;\tau)$ has some very important properties. First of all, unlike the other coefficients, $U_{0,0}(\sigma_*)$ in (2.37a) depends on the level k only as an overall factor $1/k$,

$$U_{0,0}^{(k)}(\sigma_*)=\frac{1}{k}U_{0,0}(\sigma_*). \tag{2.38}$$

And $U_{0,0}(\sigma_*)$ only depends on the Neumann-Zagier data γ, the choice of solution σ_*, and the holonomy parameter $u=\log(x^k)$ (but not on the other holonomy parameter n). When $|u|\ll 1$, $U_{0,0}(\sigma_*)$ is the deformed complexified hyperbolic volume of the knot complement $S^3\backslash K$ [37].

Next, even though the expression of the trans-series $\Phi_\gamma^{(k,\sigma_*)}(u,n;\tau)$ depends explicitly on θ_i^*, a choice of the k-th root of z_i^*, one can show that a different choice of θ_i^* merely amounts to an overall constant factor. The exponential $\exp U_{0,0}/(2\pi\mathsf{i}\tau)$ manifestly depends on z_i^* instead of θ_i^*. Besides, one can prove that *both* $(\omega_\gamma^{(k,\sigma_*)})^{2k}$ *and* $\varphi_\gamma^{(k,\sigma_*)}(\tau)$ *are invariant under the transformation* $\theta_i\to\zeta\theta_i$. This has already been proved when the holonomy x is turned off in [34], and the coefficients of the power series $\varphi_\gamma^{(k,\sigma_*)}(\tau)$ are said to live in the trace field of the knot. When the holonomy x is turned on, a similar proof following closely [34] can be written down with all the necessary ingredients provided in (2.33), (2.35), and we do not repeat the proof here.

In addition, the dependence on the holonomy of the trans-series $\Phi_\gamma^{(k,\sigma_*)}(u,n;\tau)$ is quite elegant. *Both* $(\omega_\gamma^{(k,\sigma_*)})^{2k}$ *and* $\varphi_\gamma^{(k,\sigma_*)}(\tau)$ *depend on the combination* $x = \exp(u/k - 2\pi i n/k)$ *but not on* u *and* n *individually.* This is because the NZ solution z^* depends on u and n implicitly through the modified NZ equations (2.23a), (2.23b), which only depend on x^k. On top of that, $(\omega_\gamma^{(k,\sigma_*)})^{2k}$ and $\varphi_\gamma^{(k,\sigma_*)}(\tau)$ depend on u,n through the NZ solution z^* as well as $a_{\boldsymbol{m}}(x,\theta)$, which is also a function of x. Combining the discussion of these two paragraphs, we thus claim that the coefficients of the power series $\varphi_\gamma^{(k,\sigma_*)}(u,n;\tau)$ are understood to live in $F_{k,x}$, the trace field of the knot extended by holonomy x, defined by

$$F_{k,x} = F_k(x), \quad F_k = F(\zeta), \quad F = \mathbb{Q}(z_1,\ldots,z_N). \tag{2.39}$$

Finally, although a universal expression (2.33) of the one-loop contribution $\omega_\gamma^{(k,\sigma_*)}$ at any level is possible, the power series $\varphi_\gamma^{(k,\sigma_*)}$ depend on the level in a very complicated way, as we will see in example sections 3.1, 4.1.

We also comment that the trans-series (2.32) can be computed from radial asymptotics of q-series that come from the evaluation of the state integral model for q at complex roots of unity, similar to the discussion in [38]. Examples of the q-series, also known as holomorphic blocks, are shown in sections 3.2 and 4.2. They can be regarded as x-feformation of the Nahm sums.

Even though the form of trans-series (2.32) is elegant and it allows us to infer the above universal properties, it is computationally inefficient as it involves an N-dimensional integral. Often the state integral (2.8) can be simplified. For example in the case of complements of knots $\mathbf{4}_1$ and $\mathbf{5}_2$, in Thurston's scheme of triangulation the number N of ideal tetrahedra is respectively 2 and 3, and their state integral models involve a two- and a three-dimensional integral respectively. However using integral identities of the tetrahedron partition function $\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](\sigma,m)$, both of them can be reduced to one-dimensional integrals [13, 15], and their perturbative expansion in terms of Gaussian integral is much simpler. This will be demonstrated in sections 3, 4. Nevertheless the universal properties of the asymptotic series discussed in this section still hold.

2.2 Resurgent structure of complex Chern-Simons theory

The power series $\varphi_\gamma^{(k,\sigma_*)}(u,n;\tau)$ are asymptotic. To make sense of them, we need to apply the resurgence theory [39, 40] (see reviews [41–43] by physicists). Generically in physics perturbation series are asymptotic. The coefficients grow factorially due to the proliferation of Feynman diagrams,[2]

$$f(\tau) = \sum_{\ell=0}^{\infty} a_n \tau^n, \quad a_n \sim \mathcal{O}(A^n n!). \tag{2.40}$$

The series is divergent and it cannot be summed to an exact finite value in the traditional sense. Nevertheless, we can convert it into an analytic function of τ by the means of Borel resummation.

[2]Renormalons can also contribute to factorial growth of coefficients in a general QFT, but they are absent in complex Chern-Simons theory.

We first construct the Borel transform of the original series

$$\mathcal{B}[f](t) = \sum_{\ell=0}^{\infty} \frac{a_n}{n!} t^n, \tag{2.41}$$

which is now a convergent series with a finite radius of convergence $1/A$. The boundary of the disk of convergence is punctuated by singular points. If the singular points are sparse, the convergent series can be analytically continued to the entire complex plane, known as the Borel plane. The Borel transform is said to be *resurgent* if its singular points are isolated and it can be analytically continued to infinity in all directions bypassing its singular points. In this case, we can perform the Laplace transform and define the Borel resummation

$$s[f](\tau) = \int_0^\infty \mathcal{B}[f](\tau t) \mathrm{e}^{-t} \mathrm{d}t. \tag{2.42}$$

This definition involves integration along a ray ρ_θ of angle $\theta = \arg \tau$. If the Borel transform has no singular points along the ray, the integral is well-defined and it gives us a function of τ. If, however, the Borel transform does have a singular point w on the ray ρ_θ, the integration is obstructed, and we are missing non-perturbative corrections of the order $\mathrm{e}^{-w/\tau}$ coming from another saddle point, and the action of the new saddle point is $V_w = V_0 + w$, where V_0 is the action at the original saddle point associated to the series f we start with. We can define a pair of lateral Borel resummations

$$s_\pm[f](\tau) = \int_0^{\mathrm{e}^{\pm \mathrm{i}0}\infty} \mathcal{B}[f](\tau t) \mathrm{e}^{-t} \mathrm{d}t, \tag{2.43}$$

and their difference, called the Stokes discontinuity

$$\mathrm{disc}_\theta[f](\tau) = s_+[f](\tau) - s_-[f](\tau), \tag{2.44}$$

is of the order $\mathrm{e}^{-w/\tau}$.

The Borel transform $\mathcal{B}[f]$ is in addition called a *simple resurgent function* if all its singular points are simple poles or branch points. In this paper, we assume there are only logarithmic branch points. Then at the vicinity of a singular point at w, the resurgent function $\mathcal{B}[f](t)$ has the form

$$\mathcal{B}[f](t+w) = -\frac{\mathcal{S}_w}{2\pi\mathrm{i}} \mathcal{B}[g](t) \log(t) + \mathrm{reg}(t). \tag{2.45}$$

where both $\mathcal{B}[g](t)$ and $\mathrm{reg}(t)$ are regular at $t = 0$, and the constant $\mathcal{S}_w$ is called the Stokes constant. Here we make it manifest that the regular function $\mathcal{B}[g](t)$ can be regarded as the Borel transform of another asymptotic series $g(\tau)$, which in fact is the perturbation series at the new saddle point w. Suppose no other singular points of $\mathcal{B}[f](t)$ share the same argument $\theta = \arg w$, the Stokes discontinuity of the original Borel resummation across the Stokes ray ρ_θ is

$$\mathrm{disc}_\theta[f](\tau) = \mathcal{S}_w \mathrm{e}^{-w/\tau} s_-[g](\tau). \tag{2.46}$$

To put it in a more democratic form, we can introduce elementary trans-series

$$F(\tau) = \mathrm{e}^{-V_0/\tau} f(\tau), \quad G(\tau) = \mathrm{e}^{-V_w/\tau} g(\tau). \tag{2.47}$$

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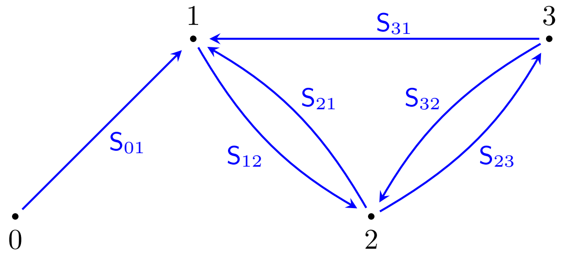

Figure 1. Demonstration of resurgent structure.

Then the formula of Stokes discontinuity reads

$$\mathrm{disc}_\theta[F](\tau) = s_+[F](\tau) - s_-[F](\tau) = \mathcal{S}_w s_-[G](\tau). \tag{2.48}$$

Abstractly, it can also be represented as a linear transformation of trans-series, whose coefficient is the Stokes constant,

$$\mathfrak{S}_\theta F = F + \mathcal{S}_w G. \tag{2.49}$$

This is called the Stokes automorphism.

In summary, starting from a single perturbation series φ_0 in a theory, one can find new saddle points and their actions by looking for singular points of the Borel transform $\mathcal{B}[\varphi_0]$, and furthermore explore the asymptotic series associated to the new saddle points by computing the Stokes discontinuity (or expansion of Borel transform at the singular point) of φ_0. The same procedure can be repeated on the new asymptotic series to uncover additional saddle points. In general, all the saddle points of the theory are interconnected in this way, which is called the resurgent structure of the theory, cf. figure 1. These relationships are completely controlled by Stokes constants, which define for us new invariants of the theory. We note that Stokes constants are not necessarily symmetric, i.e. $\mathcal{S}_{ij} \neq \mathcal{S}_{ji}$.

In practise we usually do not have the entire asymptotic series $f(\tau)$ but only the truncated power series up to certain (relatively high) order. The Borel transform of the truncated power series is only a polynomial which has no poles. In order to mimic the singularity structure of the Borel transform of the full series, we make use of the Padé approximant to approximate $\mathcal{B}[f](t)$,

$$\mathcal{P}_N[f](t) = \frac{p_0 + p_1 t + \cdots + p_N t^N}{1 + q_1 t + \cdots + q_N t^N}\,. \tag{2.50}$$

p_i and q_i are determined by requiring that the power series expansion of $\mathcal{P}_N[f](t)$ in t coincides with $\mathcal{B}[f](t)$ up to order $2N$. Thus the poles of $\mathcal{P}_N[f](t)$ can be used to infer the singularities of the Borel transform. Then we numerically Laplace transform the Padé approximant to obtain the Borel resummation. The precision of the Padé approximant can be improved by utilizing, say, the conformal mapping method. One can find more numerical techniques in [44].

Suppose that the theory in question has finitely many saddle points σ_i, with the associated trans-series

$$F_i(\tau) = \mathrm{e}^{-V_i/\tau} f_i(\tau), \quad i = 1, \ldots, r. \tag{2.51}$$

We represent all the trans-series at once by an r-dimensional vector $F(\tau)$

$$F(\tau) = \begin{pmatrix} F_1(\tau) \\ \vdots \\ F_r(\tau) \end{pmatrix}. \tag{2.52}$$

If saddle point j appears as a singular point at $V_j - V_i$ of the Borel transform of $F_i(\tau)$ and the Stokes constant is $\mathcal{S}_{ij}$, and it is the only singular point on the Stokes ray $\rho_{\theta_{ij}}$ where $\theta_{ij} = \arg(V_j - V_i)$, then the Borel resummation above and below $\rho_{\theta_{ij}}$ are related by the Stokes automorphism

$$s_+[F](\tau) = \mathfrak{S}_{\theta_{ij}} s_-[F](\tau), \tag{2.53}$$

where $\mathfrak{S}_{\theta_{ij}}$ is an $r \times r$ matrix called the Stokes matrix, and it has the form

$$\mathfrak{S}_{\theta_{ij}} = I + \mathcal{S}_{ij} E_{ij} \tag{2.54}$$

with I the identity matrix, and $E_{i,j}$ the elementary matrix with (i,j)-entry 1 and all other entries zero.

If there are multiple singular points on the ray ρ_θ, we assume the *locality condition*: the Stokes matrices of any two Borel plane singular points commute if they are on the same Stokes ray. In this case, (2.53) should be generalised to

$$s_+[F](\tau) = \mathfrak{S}_\theta s_-[F](\tau), \quad \mathfrak{S}_\theta = \prod_{\arg \iota = \theta} \mathfrak{S}_\iota. \tag{2.55}$$

The order of the product of the local Stokes matrices is irrelevant due to the locality condition. Furthermore, given two rays ρ_{θ^+} and ρ_{θ^-} whose arguments satisfy $0 < \theta^+ - \theta^- \leq \pi$, we define the *global Stokes automorphism*

$$s_{\theta^+}[F](\tau) = \mathfrak{S}_{\theta^- \to \theta^+} s_{\theta^-}[F](\tau) \tag{2.56}$$

where both sides should be analytically continued to the same value of τ. The global Stokes automorphism $\mathfrak{S}_{\theta^- \to \theta^+}$ has the property of unique factorisation

$$\mathfrak{S}_{\theta^- \to \theta^+} = \overleftarrow{\prod_{\theta^- < \theta < \theta^+}} \mathfrak{S}_\theta. \tag{2.57}$$

The superscript $\leftarrow$ indicates that the product is ordered so that θ increases from right to left.

We can apply this theory to the trans-series $\Phi_\gamma^{(k,\sigma_i)}(u,n;\tau)$ with $(i = 1, \ldots, r)$ in (2.32) coming from the state integral of $sl(2,\mathbb{C})$ Chern-Simons theory. We also define the vector of trans-series

$$\Phi_\gamma^{(k)}(u,n;\tau) = \begin{pmatrix} \Phi_\gamma^{(k,\sigma_1)} \\ \vdots \\ \Phi_\gamma^{(k,\sigma_r)} \end{pmatrix} (u,n;\tau). \tag{2.58}$$

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We will be concerned with the regime where $|u| \ll 1$, which is equivalent to $|x| \sim 1$. We note that in this regime both holonomy parameters (u, n) can be read off uniquely from x: u is given by

$$u = \log(x^k), \tag{2.59}$$

after which $n \in \mathbb{Z}_k$ is chosen such that

$$\zeta^n x = \mathrm{e}^u. \tag{2.60}$$

The Borel transform $\mathcal{B}[\Phi_\gamma^{(k,\sigma_i)}](u, n; t)$ has singularities at

$$\Lambda_i^{(k),0} = \{\iota_{i,j}/k \mid j = 1, \dots, r,\ j \neq i\} \tag{2.61}$$

where

$$\iota_{i,j} = \frac{U_{0,0}(\sigma_i) - U_{0,0}(\sigma_j)}{2\pi\mathsf{i}}. \tag{2.62}$$

They correspond to the flat connections $j \neq i$ as predicted by the resurgence theory. Furthermore, there are additional singularities on top of and slightly away from these singular points and on the imaginary axis as well; the full set of singular points is

$$\Lambda_i^{(k)} = \left\{ \frac{\iota_{i,j} + 2\pi\mathsf{i}\ell + \log(x^k)m}{k} \;\middle|\; j = 1, \dots, r,\ \ell \in \begin{cases} \mathbb{Z} & j \neq i \\ \mathbb{Z}_{\neq 0} & j = i \end{cases},\ m \in I(i,j,\ell) \right\}. \tag{2.63}$$

Here $I(i, j, \ell)$ are finite and continuous sets of integers centered on 0. These singularities form infinite towers in the imaginary direction (see figures 2, 5 for illustrations), and the Stokes rays passing through them are said to form "peacock patterns" [31]. This type of distribution of Borel plane singularities is in fact quite univeral. Similar patterns have already been seen in large N expansion of Chern-Simons matrix integral [45, 46], in closedly related topological string free energies [47–50], in earlier studies of complex Chern-Simons theory [23], and in other related works [30]. In the context of complex Chern-Simons theory, this reflects the fact that due to ambiguity of the Chern-Simons action,[3] each trans-series $\Phi_\gamma^{(k,\sigma_i)}(u, n; \tau)$ should be upgraded to a family of trans-series with the same power series but with shifted instanton action. Recall the definition

$$\tilde{q} = \exp\left(-\frac{2\pi\mathsf{i}}{k\tau} - \frac{2\pi\mathsf{i}}{k}\right), \quad \tilde{x} = \exp\left(\frac{\log(x^k)}{k\tau} + \frac{2\pi\mathsf{i}n}{k}\right). \tag{2.64}$$

It is convenient to parametrise the trans-series in each family as

$$\Phi_\gamma^{(k,\sigma_i)_{\ell,m}}(u, n; \tau) = \tilde{x}^m \tilde{q}^\ell \Phi_\gamma^{(k,\sigma_i)}(u, n; \tau), \quad \ell, m \in \mathbb{Z}. \tag{2.65}$$

On the other hand, the Stokes constant that relates two trans-series labelled by $(\sigma_i)_{\ell,m}$ and $(\sigma_j)_{\ell',m'}$ only depends on σ_i, σ_j and the differences $\ell - \ell'$, $m - m'$, and we denote it by $\mathcal{S}_{i,j;\ell-\ell',m-m'}$. As a consequence, we only need to study the resurgent properties of the

[3]For instance by a large gauge transformation the Chern-Simons action changes by 2π. This will change the complexified hyperbolic volume $U_{0,0}/(2\pi\mathsf{i})$ in (2.38), whose imaginary part is the Chern-Simons action.

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vector of trans-series $\Phi_\gamma^{(k)}(u,n;\tau)$ defined in (2.58). The Stokes automorphism associated to the Borel plane singularity $\iota_{i,j;\ell,m}/k = (\iota_{i,j} + 2\pi \mathrm{i}\ell + \log(x^k)m)/k$ is given by

$$s_+[\Phi_\gamma^{(k)}](u,n;\tau) = \mathfrak{S}_{\iota_{i,j;\ell,m}/k}(\tilde{x},\tilde{q})s_-[\Phi_\gamma^{(k)}](u,n;\tau) \tag{2.66}$$

where the Stokes matrix is

$$\mathfrak{S}_{\iota_{i,j;\ell,m}/k}(\tilde{x},\tilde{q}) = I + \mathcal{S}_{i,j,\ell,m}\tilde{x}^m\tilde{q}^\ell E_{i,j}. \tag{2.67}$$

Due to the factorisation property (2.57), in order to compute all the Stokes constants, it suffices to compute a suitable choice of finite number of global Stokes automorphisms, and extract the Stoks constants from their factorised form. For instance we can choose to compute

$$\mathfrak{S}_+^{(k)}(\tilde{x},\tilde{q}) = \mathfrak{S}^{(k)}_{0_-\to\pi+0_-}(\tilde{x},\tilde{q}), \quad \mathfrak{S}_-^{(k)}(\tilde{x},\tilde{q}) = \mathfrak{S}^{(k)}_{\pi+0_-\to 2\pi+0_-}(\tilde{x},\tilde{q}). \tag{2.68}$$

The entries of the global Stokes automorphisms are no longer constants, but elements in $\mathbb{Z}[\tilde{x}^{\pm1}][[\tilde{q}]]$ (upper half plane) or $\mathbb{Z}[\tilde{x}^{\pm1}][[1/\tilde{q}]]$(lower half plane). See [24, 31] for concrete examples.

In the case of the two simplest hyperbolic knots $\mathbf{4}_1, \mathbf{5}_2$, at level $k=1$ and for $|u| \ll 1$, the global Stokes automorphisms $\mathfrak{S}_\pm^{(1)}(\tilde{x},\tilde{q})$ have been conjectured in [31, 32]. In this paper, we generalise the studies of these two knots to generic level $k \geq 1$, and the main result of this paper is that the Stokes automorphisms are *independent* of the level k

$$\mathfrak{S}_\pm^{(k)}(\tilde{x},\tilde{q}) = \mathfrak{S}_\pm^{(1)}(\tilde{x},\tilde{q}). \tag{2.69}$$

This is rather surprising because the power series $\varphi_\gamma^{(k,\sigma_*)}(u,n;\tau)$ are very different at different levels. We will demonstrate the complexity of power series at higher levels and provide numerical evidence of (2.69) in sections 3, 4. But before ending this section, we give some philosophical argument for (2.69). The following subsection is more speculative.

2.3 Stokes constants and BPS invariants

An interesting discovery concerning the Stokes constants extracted from the Stokes automorphisms $\mathfrak{S}_\pm^{(k=1)}(\tilde{x},\tilde{q})$ was made in [31, 32]: they were identified with the BPS invariants of the 3d SCFT $T_2[M]$ related to the $sl(2,\mathbb{C})$ Chern-Simons theory on $M = S^3\backslash K$ by the 3d-3d correspondence [4]. In particular, the entries of $\mathfrak{S}_+^{(k=1)}(\tilde{x},\tilde{q})$ (respectively $\mathfrak{S}_-^{(k=1)}(\tilde{x},\tilde{q})$), which are power series in $\mathbb{Z}[\tilde{x}^{\pm1}][[\tilde{q}]]$ (respectively $\mathbb{Z}[\tilde{x}^{\pm1}][[\tilde{q}^{-1}]]$), were identified with linear combinations of the rotated superconformal indices $\mathrm{Ind}_K^{\mathrm{rot}}(m,\zeta;q)$ [14, 51] with different magnetic fluxes m. The identification is most clean in the $(1,1)$ entry associated to the geometry flat connection σ_1, where one found

$$\mathfrak{S}_+^{(k=1)}(x,q)_{(1,1)} = \mathrm{Ind}_K^{\mathrm{rot}}(0,x;q). \tag{2.70}$$

The original motivation of this work was to generalise this relationship.

The identification (2.70) has not been proved. Nevertheless, it should certainly be understood in the framework of 3d-3d correspondence, which in particular claims that the supersymmetric vacua of the dual 3d SCFT are given by $sl(2,\mathbb{C})$ flat connections on M [2], which are precisely the saddle points of the complex Chern-Simons theory. At a generic level $k \geq 1$, on the one hand, the Hilbert space of the quantised Chern-Simons theory is merely [15]

$$\mathcal{H}^{(k)} = \mathcal{H}^{(k=1)} \otimes_{\mathbb{C}} \mathbb{C}^k. \tag{2.71}$$

On the other hand, there are no more BPS states on the side of the SCFT.[4] It then seems natural that the Stokes automorphism at generic level $k \geq 1$ should still be identified with the 3d BPS invariants, and therefore be independent of the level k of the Chern-Simons theory.

3 Figure eight

3.1 Asymptotic series

The state integral of figure eight knot at level k is already given as a one-dimensional integral in [13]

$$\begin{aligned}\chi^{(k)}_{\mathbf{4}_1}(\mu,n;\tau) =& \frac{\eta_k}{k}\sum_{m\in\mathbb{Z}_k}\int_{\mathbb{R}+\mathrm{i}\,\mathrm{Im}(c_{\mathsf{b}})-\mathrm{i}|\mu|-\mathrm{i}0} \mathrm{d}\sigma \mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma-\mu,m-n)\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma,m)\\ &\times(-1)^m \mathrm{e}^{\frac{\pi\mathrm{i}}{k}((\sigma-c_{\mathsf{b}})^2-2(\mu-\sigma+c_{\mathsf{b}})^2-m^2+2(n-m)^2)},\end{aligned} \tag{3.1}$$

where $\eta_k = \exp\frac{\pi\mathrm{i}}{6}(k+2c_{\mathsf{b}}^2/k)$. This can be derived from the generic formula (2.8). We can choose the Neumann-Zagier data

$$\boldsymbol{A} = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix},\quad \boldsymbol{B} = \begin{pmatrix}0 & -1\\ -1 & -1\end{pmatrix},\quad \boldsymbol{\nu} = \begin{pmatrix}0\\0\end{pmatrix}, \tag{3.2}$$

with which (2.8) yields

$$\begin{aligned}&\mathcal{Z}^{(k)}_{\gamma}(\mu,n;\mathsf{b})\\ =&\frac{1}{\mathrm{i}k^2}\sum_{m\in(\mathbb{Z}_k)^2}\int \mathrm{d}^2\sigma \mathrm{e}^{\frac{2\pi\mathrm{i}}{k}(-\sigma_1\mu_1+\sigma_2\mu_1+m_1n_1-m_2n_1)}\mathrm{e}^{\frac{\pi\mathrm{i}}{k}(2\sigma_1\sigma_2-2m_1m_2)}\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma_1,m_1)\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma_2,m_2)\\ =&\frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{12}(k+\frac{8c_{\mathsf{b}}^2}{k})}}{\mathrm{i}k}\sum_{m\in\mathbb{Z}_k}\int\mathrm{d}\sigma\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma,m)\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma+\mu,m+n)\\ &\qquad(-1)^{m+n}\exp\frac{\pi\mathrm{i}}{k}\left(-(\sigma-c_{\mathsf{b}})^2-4\mu\sigma-\mu^2+2c_{\mathsf{b}}\mu+m^2+4mn+n^2\right).\end{aligned} \tag{3.3}$$

In the last step, we have integrated over σ_2 and summed over m_2 using the Fourier transformation (A.13). We then relabeled (σ_1,m_1) to (σ,m) as well as (μ_1,n_1) to (μ,n). After changing the signs of μ,n we find the same expression as (3.1) up to an overall irrelevant factor

$$\chi^{(k)}_{\mathbf{4}_1}(\mu,n;\tau) = \mathrm{i}\mathrm{e}^{\frac{\pi\mathrm{i}}{12}(k-\frac{4c_{\mathsf{b}}^2}{k})}(-1)^n\mathrm{e}^{\frac{\pi\mathrm{i}}{k}(-\mu^2-2c_{\mathsf{b}}\mu+n^2)}\mathcal{Z}^{(k)}_{\gamma}(-\mu,-n;\hbar). \tag{3.4}$$

[4] There are missing BPS states concerning the Abelian flat connection σ_0. But they are related to the Stokes automorphisms of the asymptotic series associated to σ_0, which is invisible in the state integral model. See some discussion in section 5.

As a one-dimensional integral, it is much easier to perform saddle point analysis on (3.1). We first introduce an alternative representation of the tetrahedron partition function[5]

$$\mathrm{D}_{\mathsf{b}}^{(k)}(u,m) = \mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](c_{\mathsf{b}} - \sqrt{k}u, m), \tag{3.5}$$

in terms of which, the state integral (3.1) reads

$$\begin{aligned}\chi_{\mathbf{4}_1}^{(k)}(\mu,n;\tau) = \frac{\eta_k}{k}\sum_{m\in\mathbb{Z}_k}&(-1)^m \mathrm{e}^{\frac{\pi\mathrm{i}}{k}(2n^2-4nm+m^2)}\\ &\times\int_{\mathbb{R}+\mathrm{i}0}\mathrm{d}\sigma\, \mathrm{D}_{\mathsf{b}}^{(k)}(\frac{1}{\sqrt{k}}(\sigma+\mu),m-n)\,\mathrm{D}_{\mathsf{b}}^{(k)}(\frac{1}{\sqrt{k}}\sigma,m)\mathrm{e}^{\frac{\pi\mathrm{i}}{k}(-2\mu^2-4\mu\sigma-\sigma^2)}.\end{aligned} \tag{3.6}$$

We scale the variables

$$Z = 2\pi\mathsf{b}\sigma,\quad u = 2\pi\mathsf{b}\mu \tag{3.7}$$

and then perform asymptotic expansion of the tetrahedron partition functions in the small b limit using (A.15),

$$\chi_{\mathbf{4}_1}^{(k)}(\mu,n;\mathsf{b}) \sim \frac{\eta_k}{2\pi\mathsf{b}k}\sum_{m\in\mathbb{Z}_k}(-1)^m\mathrm{e}^{\frac{\pi\mathrm{i}}{k}(2n^2-4nm+m^2)}\int\mathrm{d}Z\exp\sum_{\ell\geq0}(2\pi\mathrm{i}\mathsf{b}^2)^{\ell-1}U_\ell(u,Z) \tag{3.8}$$

where the potential functions are

$$U_0(u,Z) = \frac{1}{k}\left(u^2+2uZ+\frac{Z^2}{2}+\mathrm{Li}_2(-\mathrm{e}^Z)+\mathrm{Li}_2(-\mathrm{e}^{u+Z})\right), \tag{3.9a}$$

$$U_{\ell\geq1}(u,Z) = \frac{1}{\ell!}\sum_{j\in\mathbb{Z}_k}B_\ell(1-1/(2k)-j/k)\left(\mathrm{Li}_{2-\ell}(\zeta^{m-j-1/2}\mathrm{e}^{Z/k})+\mathrm{Li}_{2-\ell}(\zeta^{m-j-1/2}\mathrm{e}^{Z/k}x)\right). \tag{3.9b}$$

The critical equation of the leading order potential function reads

$$0 = \frac{\partial}{\partial Z}U_0(u,Z) = \frac{1}{k}\left(2u+Z-\log(1+\mathrm{e}^Z)-\log(1+\mathrm{e}^{u+Z})\right), \tag{3.10}$$

which can be simplified to the algebraic equation

$$-X^2Y = (1-XY)(1-Y), \tag{3.11}$$

with

$$X = \mathrm{e}^u = x^k,\quad Y = -\mathrm{e}^Z. \tag{3.12}$$

We also introduce the variable θ such that

$$Y = -\theta^k. \tag{3.13}$$

The critical point equation (3.11) has two solutions in terms of Y, corresponding to the geometric and conjugate flat connections σ_1,σ_2 on the knot complement. Numerically, they

[5] This version of tetrahedron partition function was introduced in [13].

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are the ones that in the limit $X \to 1$ reduce respectively to the solutions $\mathrm{e}^{\pi \mathsf{i}/3}$ and $\mathrm{e}^{-\pi \mathsf{i}/3}$ of (3.11). Near each critical point σ_j, the integrand is approximated by the exponential

$$\exp\left(\frac{V^{(k)}(\sigma_j)}{2\pi \mathsf{i}\tau}\right), \tag{3.14}$$

where

$$V^{(k)}(\sigma_j) = U_0(\log(X), \log(-Y_j)) + \frac{\pi^2}{6k}. \tag{3.15}$$

The extra factor $\pi^2/(6k)$ comes from the factor η_k. Note the interesting relation $V^{(k)}(X,Y) = V^{(1)}(X,Y)/k$.

In addition, by expanding near the critical point and performing the Gaussian integration order by order, for each critical point σ_j we can find an asymptotic power series $\varphi^{(k,\sigma_j)}(x;\tau)$, which, together with the exponential factor (3.14), forms the trans-series

$$\Phi^{(k,\sigma_j)}_{\mathbf{4}_1}(x;\tau) = \mathrm{e}^{V^{(k)}(\sigma_j)/(2\pi\mathsf{i}\tau)}\varphi^{(k,\sigma_j)}_{\mathbf{4}_1}(x;\tau), \quad j=1,2. \tag{3.16}$$

The one-loop contribution $\varphi^{(k,\sigma_j)}_{\mathbf{4}_1}(x;0) = \omega^{(k,\sigma_j)}_{\mathbf{4}_1}(x)$ has a universal expression

$$\begin{aligned}\omega^{(k,\sigma_j)}_{\mathbf{4}_1}(x) =& \frac{\mathrm{e}^{\frac{\pi\mathsf{i}}{4}+\frac{\pi\mathsf{i}k}{6}+\frac{2\pi\mathsf{i}}{k}(n^2-1/12)}}{\sqrt{k(x^k\theta^{2k}-1)}}\theta^{1/2}xD^*_k(\zeta^{-1/2}\theta)^{1/k}D^*_k(\zeta^{-1/2}\theta x)^{1/k}\\ &\times\sum_{m\in\mathbb{Z}_k}(-1)^m\zeta^{m^2/2}\theta^m x^{2m}(\zeta^{1/2}\theta;\zeta)^{-1}_m(\zeta^{1/2}\theta x;\zeta)^{-1}_m.\end{aligned} \tag{3.17}$$

The power series $\varphi^{(k,\sigma_j)}_{\mathbf{4}_1}(x;\tau)/\omega^{(k,\sigma_j)}_{\mathbf{4}_1}(x)$ on the other hand, depends in a very complicated way on the level k.

The case of $k=1$ is particularly simple. The dependence on $n \in \mathbb{Z}_1$ is trivial, and the state integral reduces to [31]

$$\chi_{\mathbf{4}_1}(\mu;\tau) = \chi^{(1)}_{\mathbf{4}_1}(\mu;\tau) = \Phi_{\mathsf{b}}(0)^{-2}\int_{\mathbb{R}+\mathsf{i}0}\mathrm{d}v\,\Phi_{\mathsf{b}}(\mu+v)\Phi_{\mathsf{b}}(v)\mathrm{e}^{\pi\mathsf{i}(v^2-2(\mu+v)^2)}. \tag{3.18}$$

where we have used $\eta_1 = \Phi_{\mathsf{b}}(0)^{-2}$. The power series $\varphi^{(\sigma_i)}_{\mathbf{4}_1}(x;\tau) = \varphi^{(k=1,\sigma_i)}_{\mathbf{4}_1}(x;\tau)$ are given by [31]

$$\begin{aligned}(\omega^{(\sigma_1)}_{\mathbf{4}_1})^{-1}\varphi^{(\sigma_1)}_{\mathbf{4}_1}(x;\frac{\tau}{2\pi\mathsf{i}}) =& 1 - \frac{1}{24\gamma(x)^3}\left(x^{-3}-x^{-2}-2x^{-1}+15-2x-x^2+x^3\right)\tau\\ &+\frac{1}{1152\gamma^6(x)}\left(x^{-6}-2x^{-5}-3x^{-4}+610x^{-3}-606x^{-2}-1210x^{-1}\right.\\ &\left.+3117-1210x-606x^2+610x^3-3x^4-2x^5+x^6\right)\tau^2+\mathcal{O}(\tau^3).\end{aligned} \tag{3.19}$$

with

$$\gamma(x) = \sqrt{x^{-2}-2x^{-1}-1-2x+x^2}. \tag{3.20}$$

while $\varphi_2^{(\sigma_2)}(x;\tau) = \mathsf{i}\varphi_1^{(\sigma_2)}(x;-\tau)$. At level $k=2$, the power series are

$$\begin{aligned}&(\omega_{\mathbf{4}_1}^{(k=2,\sigma_j)})^{-1}\varphi_{\mathbf{4}_1}^{(k=2,\sigma_j)}(x;\tau)\\ =&1-((1+6x+5x^2-30x^3+10x^4+30x^5-3x^6+30x^7+10x^8-30x^9+5x^{10}+6x^{11}+x^{12})\\ &(-1-x^2+x^4+2x^2Y_j))/(48(-1-x+x^2)^2(1-x+x^2)^2(-1+x+x^2)^2(1+x+x^2)^2)\tau+\mathcal{O}(\tau^2).\end{aligned} \tag{3.21}$$

Additional terms as well as first few terms of the power series at level $k=3$ can be found in appendix B.1.

Several conclusions can be drawn from these data. It is clear that the coefficients of the power series depend only on the solution Y_j to the NZ equation, but not on a k-th root of Y_j. Furthermore, they depend on the deformation parameters u,n only through the holonomy parameter x. Finally, it is evident that the power series become increasingly more complicated at higher levels k.

3.2 Holomorphic blocks

One happy fact of state integral model of knot $\mathbf{4}_1$ at level 1 is that when $\operatorname{Im}\mathsf{b}\neq 0$, it can be evaluated by summing over residues and it factorises to products of q- and $\tilde{q}$-series [52], i.e., the holomorphic blocks [51]. In addition, variants known as the descendants of the state integral can be written, which also enjoys factorisation. This fact turns out to be very useful in the resurgence analysis, as the Stokes matrices can be written in terms of them [31, 32]. We find here at generic level k, we can also write down descendants of the state integral and they also factorise to holomorphic blocks. In fact, the holomorphic blocks are independent of k (only implicitly through $x,\tilde{x},q,\tilde{q}$, cf. [15]), while the factorisation formulas are not.

We introduce the following descendant state integral for knot $\mathbf{4}_1$

$$\begin{aligned}\chi^{(k)}_{\mathbf{4}_1;r,s}(\mu,n;\tau) = &\frac{\eta_k}{k}\sum_{m\in\mathbb{Z}_k}\int_{\mathcal{C}}\mathrm{d}\sigma\,\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma-\mu,m-n)\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma,m)\\ &\times(-1)^m\mathrm{e}^{\frac{\pi\mathsf{i}}{k}((\sigma-c_{\mathsf{b}})^2-2(\mu-\sigma+c_{\mathsf{b}})^2-m^2+2(n-m)^2)}\\ &\times\mathrm{e}^{\frac{2\pi\mathsf{i}}{k}(-\mathsf{i}(r\mathsf{b}+s\mathsf{b}^{-1})(c_{\mathsf{b}}-\sigma)+(r-s)m)}.\end{aligned} \tag{3.22}$$

The motivation for the particular form of the descendant state integral is that both the tetrahedron partition function $\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma,m)$ and the extra factor colored in blue

$$f(\sigma,m)=\exp\left(\frac{2\pi}{k}(r\mathsf{b}+s\mathsf{b}^{-1})(c_{\mathsf{b}}-\sigma)+\frac{2\pi\mathsf{i}}{k}(r-s)m\right) \tag{3.23}$$

in the integrand enjoys nice quasi-periodicity property under the shift

$$(\sigma,m)\to(\sigma-\mathsf{i}\mathsf{b}^{\pm1},m\pm1),\quad (\sigma,m)\to(\sigma+\mathsf{i}\mathsf{b}^{\pm1},m\mp1). \tag{3.24}$$

The former is given in (A.8), and the latter is simply

$$f(\sigma\mp\mathsf{i}\mathsf{b},m\pm1)=f(\sigma,m)q^{\pm r}, \tag{3.25a}$$

$$f(\sigma\pm\mathsf{i}\mathsf{b}^{-1},m\pm1)=f(\sigma,m)\tilde{q}^{\pm s}. \tag{3.25b}$$

In addition, the form of the descendant state integral may be related to insertion of a light knot from the Chern-Simons theory point of view [53], or to turning on FI parameters from the 3d SCFT point of view [54].

Note that compared to the state integral $\chi^{(k)}_{\mathbf{4}_1}(\mu, n; \tau)$ defined in (3.1), here the contour of integral $\mathcal{C}$ has to be modified such that it asymptotes at both $\pm\infty$ to the horizontal line $\operatorname{Im}\sigma = \sigma_0$ with $\sigma_0 < \operatorname{Im}(c_{\mathsf{b}}) - |\operatorname{Im}(\mu)| - |\operatorname{Re}(r\mathsf{b} + s\mathsf{b}^{-1})|$ but is deformed near the origin so that all the poles of the integrand are below the contour. The integrand only has poles coming from the two tetrahedron partition functions $\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma - \mu, m - n)$ and $\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma, m)$, and they are located respectively at

$$v = \mu - \mathsf{i}\mathsf{b}\alpha - \mathsf{i}\mathsf{b}^{-1}, \qquad \alpha - \beta \equiv m - n \pmod k, \tag{3.26}$$

$$\sigma = -\mathsf{i}\mathsf{b}\,\alpha - \mathsf{i}\mathsf{b}^{-1}\beta, \qquad \alpha - \beta \equiv m \pmod k. \tag{3.27}$$

On the other hand, since the integrand decays exponentially fast towards infinity below the contour $\mathcal{C}$, we can evaluate the descendant state integrand by completing the contour from below and adding up residues from these poles. Then we find in this way that

$$\begin{aligned} \chi^{(k)}_{\mathbf{4}_1;r,s}(\mu, n; \tau) =& \, \mathrm{e}^{-\frac{\pi \mathrm{i} k}{4} - \frac{\pi\mathrm{i}}{2k} - \frac{\pi\mathrm{i}}{4k}(\tau+\tau^{-1})} A_r(x,q) A_{-s}(\tilde{x}, \tilde{q}^{-1}) \\ &+ \mathrm{e}^{\frac{\pi \mathrm{i} k}{4} + \frac{\pi\mathrm{i}}{2k} + \frac{\pi\mathrm{i}}{4k}(\tau+\tau^{-1})} B_r(x,q) B_{-s}(\tilde{x}, \tilde{q}^{-1}). \end{aligned} \tag{3.28}$$

Here $A_r(x,q), B_r(x,q)$ are descendant holomorphic blocks defined by

$$A_r(x,q) = \theta(-q^{1/2}x; q)^{-2} x^{2r} J(q^r x^2, x; q), \tag{3.29}$$

$$B_r(x,q) = \theta(-q^{-1/2}x; q) x^r J(q^r x, x^{-1}; q), \tag{3.30}$$

where $J(x, y; q^\varepsilon) := J^\varepsilon(x, y; q)$ for $|q| < 1$ and $\varepsilon = \pm$ is the q-Hahn Bessel function

$$J^+(x,y;q) = (qy;q)_\infty \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} x^n}{(q;q)_n (qy;q)_n} \tag{3.31a}$$

$$J^-(x,y;q) = \frac{1}{(y;q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)} x^n y^{-n}}{(q;q)_n (qy^{-1};q)_n}. \tag{3.31b}$$

Note that the definition of descendant holomorphic blocks as function of x, q are independent of k. They enjoy the same properties as discussed in [31]. If we define the descendant Wronskian matrix

$$W_r(x;q) = \begin{pmatrix} A_r(x;q) & B_r(x;q) \\ A_{r+1}(x;q) & B_{r+1}(x;q) \end{pmatrix}, \tag{3.32}$$

it enjoys the inversion property that

$$W_m(x;q) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_\ell(x;q^{-1})^T \in \mathrm{GL}(2, \mathbb{Z}[q^{\pm 1} x^{\pm 1}]). \tag{3.33}$$

In particular

$$W_{-1}(x;q) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_{-1}(x;q^{-1})^T = \begin{pmatrix} x^{-2} + x^{-1} - 1 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.34}$$

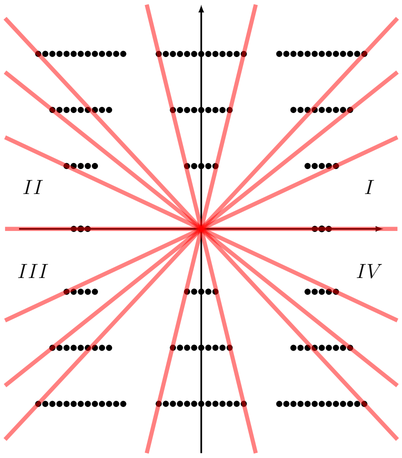


Figure 2. Singular points of the Borel transform of the vector of trans-series of the $sl(2,\mathbb{C})$ Chern-Simons theory at level $k=1$ on the complement of the knot $\mathbf{4}_1$.

3.3 Resurgent structure

We collect the trans-series associated to the geometric and conjugate flat connections in a vector

$$\Phi^{(k)}_{\mathbf{4}_1}(x;\tau) = \begin{pmatrix} \Phi^{(k,\sigma_1)}_{\mathbf{4}_1} \\ \Phi^{(k,\sigma_2)}_{\mathbf{4}_1} \end{pmatrix} (x;\tau). \tag{3.35}$$

The singularities of the Borel transform of each component trans-series are located in $\Lambda^{(k)}_i$ given in (2.63). In the case of level $k=1$, we superimpose the Borel singularities of the components of $\Phi^{(1)}_{\mathbf{4}_1}(x;\tau)$ and plot them in figure 2. At a generic level $k\geq 1$, the distrubtion of the Borel plane singularities is the same and their actual locations are reduced by a factor of $1/k$ according to (2.63).

As explained in section 2, Stokes constants of individual Borel plane singularities can be extracted from global Stokes automorphisms. The entire Borel plane is divided by Stokes rays of components of $\Phi^{(k)}_{\mathbf{4}_1}(x;\tau)$ into infinitely many cones. We label the four cones above and below the positive and negative real axes by I, II, III, IV, ordered in counter-clockwise direction, as illustrated in figure 2. We choose to calculate the global Stokes automorphism $\mathfrak{S}^{(k)}_+(\tilde{x};\tilde{q}) = \mathfrak{S}^{(k)}_{IV\to II}(\tilde{x};\tilde{q})$ and $\mathfrak{S}^{(k)}_-(\tilde{x};\tilde{q}) = \mathfrak{S}^{(k)}_{II\to IV}(\tilde{x};\tilde{q})$, both in the anti-clockwise direction, which encode all the Stokes constants.

In order to derive the Stokes automorphisms, we first calculate the Borel resummation of the trans-series. Following [31] and with numerical evidence presented shortly after, We claim that they can be written as bilinear products of holomorphic blocks in q and $\tilde{q}$. Concretely,

$$s_I(\Phi^{(k)}_{\mathbf{4}_1})(x;\tau) = \begin{pmatrix} -\tilde{x} & 1+\tilde{x}^{-1} \\ 0 & 1 \end{pmatrix} W_{-1}(\tilde{x};\tilde{q}^{-1})\Delta^{(k)}(\tau)B(x;q), \quad |\tilde{q}|<1, \tag{3.36a}$$

$$s_{II}(\Phi^{(k)}_{\mathbf{4}_1})(x;\tau) = \begin{pmatrix} 0 & -\tilde{x} \\ 1 & -\tilde{x}-\tilde{x}^2 \end{pmatrix} W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Delta^{(k)}(\tau)B(x;q), \quad |\tilde{q}|<1, \tag{3.36b}$$

$$s_{III}(\Phi^{(k)}_{\mathbf{4}_1})(x;\tau) = \begin{pmatrix} 0 & -\tilde{x} \\ 1 & 1 \end{pmatrix} W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1}) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Delta^{(k)}(\tau) B(x;q), \quad |\tilde{q}| > 1, \tag{3.36c}$$

$$s_{IV}(\Phi^{(k)}_{\mathbf{4}_1})(x;\tau) = \begin{pmatrix} -\tilde{x} & -\tilde{x} \\ 0 & 1 \end{pmatrix} W_{-1}(\tilde{x};\tilde{q}^{-1}) \Delta^{(k)}(\tau) B(x;q), \quad |\tilde{q}| > 1. \tag{3.36d}$$

Here $W_{-1}(x,q)$ is the Wronskian of holomorphic blocks defined in (3.32), and

$$B(x;q) = \begin{pmatrix} A_0(x;q) \\ B_0(x;q) \end{pmatrix} \tag{3.37}$$

$\Delta^{(k)}(\tau)$ is the diagonal matrix defined by

$$\Delta^{(k)}(\tau) = \mathrm{diag}(\mathrm{e}^{-\frac{\pi i k}{4} - \frac{\pi i}{2k} - \frac{\pi i}{4k}(\tau+\tau^{-1})}, \mathrm{e}^{\frac{\pi i k}{4} + \frac{\pi i}{2k} + \frac{\pi i}{4k}(\tau+\tau^{-1})}). \tag{3.38}$$

From (3.36a) and (3.36b), we can derive the Stokes automorphism $\mathfrak{S}^{(k)}_{I\to II}(\tilde{q})$ which encompasses all the Stokes constants in the upper half plane. Concretely it is done by analytically continuing the right hand side of (3.36a) to the cone II and compare with the right hand side of (3.36b) after substituting both of them in the definition of the Stokes automorphism

$$s_{II}(\Phi^{(k)}_{\mathbf{4}_1})(x;\tau) = \mathfrak{S}^{(k)}_{I\to II}(\tilde{q}) s_I(\Phi^{(k)}_{\mathbf{4}_1})(x;\tau). \tag{3.39}$$

This step involves only multiplication of $\tilde{q}$-series. On the other hand, from (3.36a) and (3.36d) we can derive the Stokes constants associated to the three Borel plane singularities on the positive real axis, and this step involves analytic continuation of the Wronskian $W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1})$ from inside to outside the unit circle with the help of (3.34). We can combine the two results to write down the global Stokes automorphism $\mathfrak{S}^{(k)}_{+}(\tilde{x};\tilde{q})$ from the ray ρ_{0_-} to $\rho_{\pi+0_-}$. Likewise, we can compute the global Stokes automorphism $\mathfrak{S}^{(k)}_{-}(\tilde{x};\tilde{q})$ from the ray $\rho_{\pi+0_-}$ to $\rho_{2\pi+0_-}$. Together they encompass the Stokes constants of all the Borel plane singularities.

$$\mathfrak{S}^{(k)}_{+}(\tilde{x};\tilde{q}) = \begin{pmatrix} 0 & -1 \\ \tilde{x}^{-1} & -1-\tilde{x} \end{pmatrix} W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1}) W_{-1}(\tilde{x};\tilde{q})^T \begin{pmatrix} 0 & \tilde{x} \\ -1 & -1-\tilde{x}^{-1} \end{pmatrix}, \quad |\tilde{q}| < 1 \tag{3.40}$$

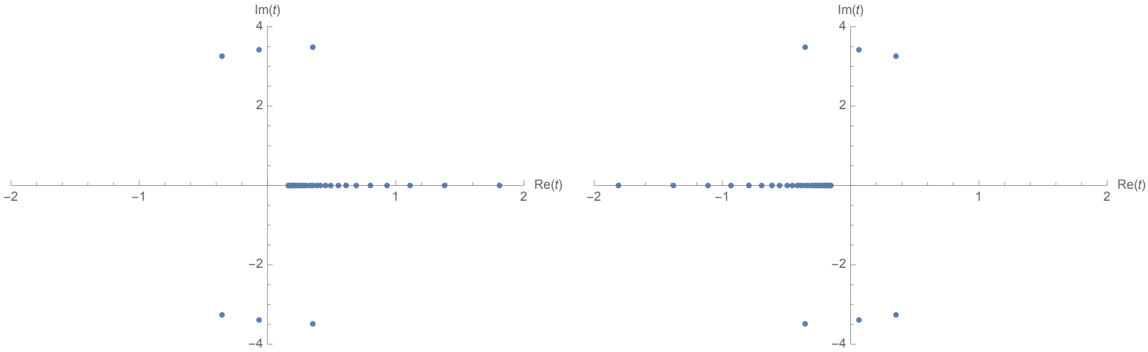
$$\mathfrak{S}^{(k)}_{-}(\tilde{x};\tilde{q}) = \begin{pmatrix} \tilde{x} & \tilde{x} \\ 0 & -1 \end{pmatrix} W_{-1}(\tilde{x};\tilde{q}) W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1})^T \begin{pmatrix} \tilde{x}^{-1} & 0 \\ \tilde{x}^{-1} & -1 \end{pmatrix}, \quad |\tilde{q}| < 1. \tag{3.41}$$

In particular, we notice that the Stokes automorphisms and therefore the Stokes constants do not depend on the level k.

At level $k=1$ numerical evidences for the position of Borel plane singularities as well as (3.36a), (3.36b), (3.36c), (3.36d) were given in detail in [31].

For higher levels, the computation is similar, although the expression becomes more complicated. At level $k=2$, we first choose $x = 6/5$, corresponding to $u = 2\log(6/5)$ and $n = 0$. We truncate the power series to $N = 100$ terms. The poles of the Padé approximant for $\Phi^{(2)}_{\mathbf{4}_1}(x,\tau)$ are shown in figure 3. Notice that the accumulating poles on the real axis hint at a branch cut on the corresponding exact Borel plane.

Clearly, if we consider the vector $\Phi(x,\tau)$, the Borel plane is divided into four regions as promised. It remains to check (3.36a), (3.36b), (3.36c), (3.36d). The comparison is tabulated in table 1.



(a) **(b)**

Figure 3. The distribution of poles of the Padé approximant for (a): $\varphi_{\mathbf{4}_1}^{(k=2,\sigma_1)}(x,\tau)$ and (b): $\varphi_{\mathbf{4}_1}^{(k=2,\sigma_2)}(x,\tau)$ with $x = 6/5$ and $N = 100$ terms.

	$\lvert\frac{s_I(\Phi)(x;\tau)}{F_I(x;\tau)} - 1\rvert$	$\lvert\frac{s_I(\Phi)(x;\tau)}{s'_I(\Phi)(x;\tau)} - 1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert, \lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	8.6×10^{-16}	3.5×10^{-15}	1.9×10^{-8}	0.36
σ_2	3.7×10^{-23}	2.4×10^{-22}		

(a) Region I: $\tau = \frac{1}{8}\mathrm{e}^{\frac{\pi \mathrm{i}}{4}}$

	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{F_{II}(x;\tau)} - 1\rvert$	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{s'_{II}(\Phi)(x;\tau)} - 1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert, \lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.7×10^{-23}	2.4×10^{-22}	1.9×10^{-8}	0.36
σ_2	8.6×10^{-16}	3.5×10^{-15}		

(b) Region II: $\tau = \frac{1}{8}\mathrm{e}^{\frac{3\pi \mathrm{i}}{4}}$

	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{F_{III}(x;\tau)} - 1\rvert$	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{s'_{III}(\Phi)(x;\tau)} - 1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert, \lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.7×10^{-23}	2.4×10^{-22}	1.9×10^{-8}	0.36
σ_2	8.6×10^{-16}	3.5×10^{-15}		

(c) Region III: $\tau = \frac{1}{8}\mathrm{e}^{-\frac{3\pi \mathrm{i}}{4}}$

	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{F_{IV}(x;\tau)} - 1\rvert$	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{s'_{IV}(\Phi)(x;\tau)} - 1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert, \lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	8.6×10^{-16}	3.5×10^{-15}	1.9×10^{-8}	0.36
σ_2	3.7×10^{-23}	2.4×10^{-22}		

(d) Region IV: $\tau = \frac{1}{8}\mathrm{e}^{-\frac{\pi \mathrm{i}}{4}}$

Table 1. We perform the numerical Borel resummation for $\Phi_{\mathbf{4}_1}^{(k=2)}(x;\tau)$ at $x = 6/5$ ($u = 2\log 6/5, n = 0$) with 100 terms, after choosing suitable τ in four different regions. We compare them with equations (3.36a), (3.36b), (3.36c), (3.36d) whose right hand side is denoted as $F_R(x;\tau)$. Meanwhile, we estimate the contribution of higher order terms by resumming $\Phi_{\mathbf{4}_1}^{(k=2)}(x;\tau)$ with 95 terms. The values of $\lvert\tilde{q}(\tilde{q}^{-1})\rvert$ and $\lvert\tilde{x}^{\pm 1}\rvert$ are also provided for comparison.

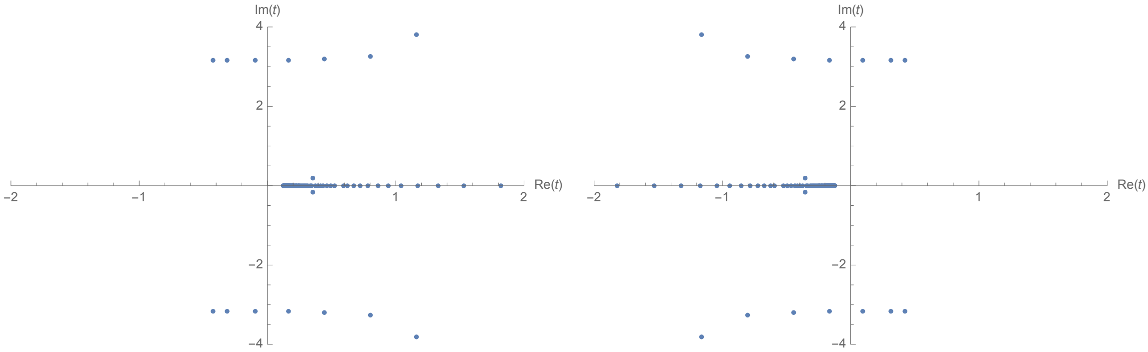


Figure 4. The distribution of poles of the Padé approximant for (a) : $\varphi^{(k=2,\sigma_1)}_{\mathbf{4}_1}(x,\tau)$ and (b) : $\varphi^{(k=2,\sigma_2)}_{\mathbf{4}_1}(x,\tau)$ with $x=-5/4$ and $N=200$ terms.

We consider next $x=-5/4$, corresponding to $u=2\log(5/4)$ and $n=1$, and we truncate the power series up to $N=200$ terms. The positions of poles for the Padé approximant $\Phi^{(k=2)}_{\mathbf{4}_1}(x,\tau)$ are shown in figure 4. Again one observes the emergent branch cut on the real axis. We then present the numerical evidence of (3.36a), (3.36b), (3.36c), (3.36d) for each region in table 2. As we can see, the relative errors between two sides of (3.36a), (3.36b), (3.36c), (3.36d) (the first column) are always within the precision of the Borel resummation (the second column), and are always far smaller than a potential $\tilde{q}$ (or $1/\tilde{q}$ in the lower half plane) or a $\tilde{x}^{\pm 1}$ correction (the third and the fourth columns).

We also perform numerical analysis at level $k=3$. We choose $x=\frac{6}{5}\mathrm{e}^{2\pi n\mathrm{i}/3}$ $(n=0,1,2)$ corresponding to $u=3\log(6/5)$ and $n=0,1,2$ respectively. The power series are truncated up to $N=220$ terms. The poles of the Padé approximant have a similar structure to the cases $k=1,2$, so we omit them here. The numerical evidence for the Borel resummation of the trans-series (3.36a), (3.36b), (3.36c), (3.36d) in each region are given in tables 3, 4 and 5 respectively.

4 Three twists

4.1 Asymptotic series

The state integral of knot $\mathbf{5}_2$ at level k is given as a one-dimensional integral in [13]

$$\begin{aligned}&\chi^{(k)}_{\mathbf{5}_2}(\mu,n;\mathsf{b})\\=&\frac{(\eta_k)^3}{k}\sum_{m\in\mathbb{Z}_k}\int_{\mathbb{R}+\mathrm{i}\,\mathrm{Im}(c_\mathsf{b})-\mathrm{i}0}\mathrm{d}\sigma\,\mathcal{Z}^{(k)}_\mathsf{b}[\Delta](\sigma,m)\mathcal{Z}^{(k)}_\mathsf{b}[\Delta](\sigma+\mu,m+n)\mathcal{Z}^{(k)}_\mathsf{b}[\Delta](\sigma-\mu,m-n)\\&\qquad\times(-1)^n\mathrm{e}^{\frac{\pi\mathrm{i}}{k}(-(\sigma+\mu-c_\mathsf{b})^2-(\sigma-\mu-c_\mathsf{b})^2-\mu^2+(m+n)^2+(m-n)^2+n^2)}.\end{aligned}\tag{4.1}$$

This can also be derived from the generic formula (2.8). We choose the Neumann-Zagier data

$$\boldsymbol{A}=\begin{pmatrix}0&1&0\\-1&0&-1\\0&2&0\end{pmatrix},\quad\boldsymbol{B}=\begin{pmatrix}-1&0&0\\0&1&0\\-1&0&-1\end{pmatrix},\quad\nu=\begin{pmatrix}0\\0\\0\end{pmatrix},\tag{4.2}$$

	$\lvert\frac{s_I(\Phi)(x;\tau)}{F_I(x;\tau)}-1\rvert$	$\lvert\frac{s_I(\Phi)(x;\tau)}{s'_I(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	8.2×10^{-22}	8.6×10^{-22}	9.6×10^{-9}	0.16
σ_2	3.9×10^{-36}	5.1×10^{-36}		

(a) Region I: $\tau=\frac{1}{10}\mathrm{e}^{\frac{\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{F_{II}(x;\tau)}-1\rvert$	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{s'_{II}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.9×10^{-36}	5.1×10^{-36}	9.6×10^{-9}	0.16
σ_2	8.2×10^{-22}	8.6×10^{-22}		

(b) Region II: $\tau=\frac{1}{10}\mathrm{e}^{\frac{4\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{F_{III}(x;\tau)}-1\rvert$	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{s'_{III}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.9×10^{-36}	5.1×10^{-36}	9.6×10^{-9}	0.16
σ_2	8.2×10^{-22}	8.6×10^{-22}		

(c) Region III: $\tau=\frac{1}{10}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{F_{IV}(x;\tau)}-1\rvert$	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{s'_{IV}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	8.2×10^{-22}	8.6×10^{-22}	9.6×10^{-9}	0.16
σ_2	3.9×10^{-36}	5.1×10^{-36}		

(d) Region IV: $\tau=\frac{1}{10}\mathrm{e}^{-\frac{\pi\mathrm{i}}{5}}$

Table 2. We perform numerical Borel resummation for $\Phi^{(k=2)}_{\mathbf{4}_1}(x;\tau)$ at $x=-5/4$ ($u=2\log 5/4, n=1$) with 200 terms, after choosing suitable τ in four different regions. We compare them with equations (3.36a), (3.36b), (3.36c), (3.36d) whose right hand side is denoted as $F_R(x;\tau)$. Meanwhile, we estimate the contribution of higher order terms by resumming $\Phi^{(k=2)}_{\mathbf{4}_1}(x;\tau)$ with 195 terms. The values of $\lvert\tilde{q}(\tilde{q}^{-1})\rvert$ and $\lvert\tilde{x}^{\pm1}\rvert$ are also provided for comparison.

with which (2.8) yields

$$\begin{aligned}
&\mathcal{Z}^{(k)}_{\gamma}(\mu,n;\mathsf{b})\\
&=\frac{1}{k^3}\sum_{m\in(\mathbb{Z}_k)^3}\int \mathrm{d}^3\sigma\, \mathrm{e}^{\frac{2\pi\mathrm{i}}{k}(\sigma_1(\sigma_2+\mu_1)-m_1(m_2+n_1))}\ \mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma_1,m_1)\\
&\qquad\qquad \mathrm{e}^{\frac{2\pi\mathrm{i}}{k}(\sigma_3(\sigma_2-\mu_1)-m_3(m_2-n_1))}\ \mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma_3,m_3)\ \mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma_2,m_2)\\
&=\frac{\mathrm{e}^{\frac{\pi\mathrm{i}}{6}(k+\frac{8c_{\mathsf{b}}^2}{k})}}{k}\sum_{m\in\mathbb{Z}_k}\int \mathrm{d}\sigma\ \mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma,m)\ \mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma+\mu,m+n)\ \mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma-\mu,m-n)\\
&\qquad\qquad \mathrm{e}^{-\frac{2\pi\mathrm{i}}{k}((\sigma-c_{\mathsf{b}})^2+\mu^2-m^2-n^2)}.
\end{aligned}\tag{4.3}$$

	$\lvert\frac{s_I(\Phi)(x;\tau)}{F_I(x;\tau)}-1\rvert$	$\lvert\frac{s_I(\Phi)(x;\tau)}{s'_I(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.1×10^{-15}	1.6×10^{-15}	1.5×10^{-5}	0.27
σ_2	4.6×10^{-27}	5.2×10^{-26}		

(a) Region I: $\tau=\frac{1}{9}\mathrm{e}^{\frac{\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{F_{II}(x;\tau)}-1\rvert$	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{s'_{II}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	5.8×10^{-27}	1.7×10^{-27}	1.5×10^{-5}	0.27
σ_2	1.1×10^{-15}	8.3×10^{-16}		

(b) Region II: $\tau=\frac{1}{9}\mathrm{e}^{\frac{4\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{F_{III}(x;\tau)}-1\rvert$	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{s'_{III}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	5.8×10^{-27}	1.7×10^{-27}	1.5×10^{-5}	0.27
σ_2	1.1×10^{-15}	8.3×10^{-16}		

(c) Region III: $\tau=\frac{1}{9}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{F_{IV}(x;\tau)}-1\rvert$	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{s'_{IV}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.1×10^{-15}	1.6×10^{-15}	1.5×10^{-5}	0.27
σ_2	4.6×10^{-27}	5.2×10^{-26}		

(d) Region IV: $\tau=\frac{1}{9}\mathrm{e}^{-\frac{\pi\mathrm{i}}{5}}$

Table 3. We perform numerical Borel resummation for $\Phi^{(k=3)}_{\mathbf{4_1}}(x,\tau)$ at $x=6/5$ with 220 terms, after choosing suitable τ in four different regions. We compare them with equations (3.36a), (3.36b), (3.36c), (3.36d) whose right hand side is denoted as $F_R(x,\tau)$. Meanwhile, we estimate the contribution of higher order terms by resumming $\Phi^{(k=3)}_{\mathbf{4_1}}(x,\tau)$ with 215 terms. The values of $\lvert\tilde{q}(\tilde{q}^{-1})\rvert$ and $\lvert\tilde{x}^{\pm1}\rvert$ are also provided for comparison.

where we have used again the formula of Fourier transformation of the tetrahedron partition function (A.13). This is the same as (4.1) up to an overall factor

$$\chi^{(k)}_{\mathbf{5_2}}(\mu,n;\mathsf{b})=\mathrm{e}^{\frac{\pi\mathrm{i}}{3}(k-\frac{c_{\mathsf{b}}^2}{k})}(-1)^n\mathrm{e}^{\frac{\pi\mathrm{i}}{k}(-\mu^2+n^2)}\mathcal{Z}^{(k)}_{\gamma}(\mu,n;\mathsf{b}). \tag{4.4}$$

We now also perform the saddle point expansion of the one-dimensional integral (4.1). We introduce the alternative representation of the tetrahedron partition function (3.5), in terms of which, the state integral (4.1) reads

$$\chi^{(k)}_{\mathbf{5_2}}(\mu,n;\mathsf{b})=\frac{(\eta_k)^3}{k}(-1)^n\mathrm{e}^{\frac{3\pi\mathrm{i}}{k}(n^2-\mu^2)}\sum_{m\in\mathbb{Z}_k}\mathrm{e}^{\frac{2\pi\mathrm{i}m^2}{k}}$$
$$\times\int_{\mathbb{R}+\mathrm{i}0}\mathrm{d}\sigma\,\mathrm{D}^{(k)}_{\mathsf{b}}(\frac{1}{\sqrt{k}}\sigma,m)\,\mathrm{D}^{(k)}_{\mathsf{b}}(\frac{1}{\sqrt{k}}(\sigma-\mu),m+n)\,\mathrm{D}^{(k)}_{\mathsf{b}}(\frac{1}{\sqrt{k}}(\sigma+\mu),m-n)\mathrm{e}^{-\frac{2\pi\mathrm{i}\sigma^2}{k}}. \tag{4.5}$$

	$\lvert\frac{s_I(\Phi)(x;\tau)}{F_I(x;\tau)}-1\rvert$	$\lvert\frac{s_I(\Phi)(x;\tau)}{s'_I(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	4.5×10^{-16}	4.9×10^{-16}	1.5×10^{-5}	0.27
σ_2	6.5×10^{-27}	3.8×10^{-27}		

(a) Region I: $\tau=\frac{1}{9}\mathrm{e}^{\frac{\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{F_{II}(x;\tau)}-1\rvert$	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{s'_{II}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.7×10^{-26}	1.1×10^{-25}	5.3×10^{-5}	0.31
σ_2	6.1×10^{-15}	5.7×10^{-15}		

(b) Region II: $\tau=\frac{1}{8}\mathrm{e}^{\frac{4\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{F_{III}(x;\tau)}-1\rvert$	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{s'_{III}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.2×10^{-26}	1.3×10^{-25}	5.3×10^{-5}	0.31
σ_2	3.0×10^{-15}	9.8×10^{-15}		

(c) Region III: $\tau=\frac{1}{8}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{F_{IV}(x;\tau)}-1\rvert$	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{s'_{IV}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	4.5×10^{-16}	4.9×10^{-16}	1.5×10^{-5}	0.27
σ_2	6.5×10^{-27}	3.8×10^{-27}		

(d) Region IV: $\tau=\frac{1}{9}\mathrm{e}^{-\frac{\pi\mathrm{i}}{5}}$

Table 4. We perform numerical Borel resummation for $\Phi^{(k=3)}_{\mathbf{4_1}}(x,\tau)$ at $x=\frac{6}{5}\mathrm{e}^{4\pi\mathrm{i}/3}$ with 220 terms, after choosing suitable τ in four different regions. We compare them with equations (3.36a), (3.36b), (3.36c), (3.36d) whose right hand side is denoted as $F_R(x,\tau)$. Meanwhile, we estimate the contribution of higher order terms by resumming $\Phi^{(k=3)}_{\mathbf{4_1}}(x,\tau)$ with 215 terms. The values of $\lvert\tilde{q}(\tilde{q}^{-1})\rvert$ and $\lvert\tilde{x}^{\pm1}\rvert$ are also provided for comparison.

We scale the variables

$$Z=2\pi\mathsf{b}\sigma,\quad u=2\pi\mathsf{b}\mu \tag{4.6}$$

and then perform asymptotic expansion of the tetrahedron partition functions in the small b limit using (A.15).

$$\chi^{(k)}_{\mathbf{5_2}}(\mu,n;\mathsf{b})\sim\frac{(\eta_k)^3}{2\pi\mathsf{b}k}(-1)^n\mathrm{e}^{\frac{3\pi\mathrm{i}}{k}(n^2-\mu^2)}\sum_{m\in\mathbb{Z}_k}\mathrm{e}^{\frac{2\pi\mathrm{i}}{k}m^2}\int\mathrm{d}Z\exp\sum_{\ell\geq0}(2\pi\mathrm{i}\mathsf{b}^2)^{\ell-1}U_\ell(u,Z) \tag{4.7}$$

where the potential functions are

$$U_0(u,Z)=\frac{1}{k}\left(Z^2+\mathrm{Li}_2(-\mathrm{e}^Z)+\mathrm{Li}_2(-\mathrm{e}^{Z-u})+\mathrm{Li}_2(-\mathrm{e}^{Z+u})\right), \tag{4.8a}$$

$$U_{\ell\geq1}(u,Z)=\frac{1}{\ell!}\sum_{j\in\mathbb{Z}_k}B_\ell(1-1/(2k)-j/k)\Big(\mathrm{Li}_{2-\ell}(\zeta^{m-j-1/2}\mathrm{e}^{\frac{Z}{k}})+\mathrm{Li}_{2-\ell}(\zeta^{m+n-j-1/2}\mathrm{e}^{\frac{Z-u}{k}})+\mathrm{Li}_{2-\ell}(\zeta^{m-n-j-1/2}\mathrm{e}^{\frac{Z+u}{k}})\Big). \tag{4.8b}$$

	$\lvert\frac{s_I(\Phi)(x;\tau)}{F_I(x;\tau)}-1\rvert$	$\lvert\frac{s_I(\Phi)(x;\tau)}{s'_I(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	2.4×10^{-16}	7.5×10^{-16}	1.5×10^{-5}	0.27
σ_2	4.4×10^{-27}	7.9×10^{-27}		

(a) Region I: $\tau=\frac{1}{9}\mathrm{e}^{\frac{\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{F_{II}(x;\tau)}-1\rvert$	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{s'_{II}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.2×10^{-26}	1.3×10^{-25}	5.3×10^{-5}	0.31
σ_2	3.0×10^{-15}	9.8×10^{-15}		

(b) Region II: $\tau=\frac{1}{8}\mathrm{e}^{\frac{4\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{F_{III}(x;\tau)}-1\rvert$	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{s'_{III}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.7×10^{-26}	1.1×10^{-25}	5.3×10^{-5}	0.31
σ_2	6.1×10^{-15}	5.7×10^{-15}		

(c) Region III: $\tau=\frac{1}{8}\mathrm{e}^{-\frac{4\pi\mathrm{i}}{5}}$

	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{F_{IV}(x;\tau)}-1\rvert$	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{s'_{IV}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	4.5×10^{-16}	7.5×10^{-16}	1.5×10^{-5}	0.27
σ_2	4.4×10^{-27}	7.9×10^{-27}		

(d) Region IV: $\tau=\frac{1}{9}\mathrm{e}^{-\frac{\pi\mathrm{i}}{5}}$

Table 5. We perform numerical Borel resummation for $\Phi^{(k=3)}_{\mathbf{4}_1}(x,\tau)$ at $x=\frac{6}{5}\mathrm{e}^{2\pi\mathrm{i}/3}$ with 220 terms, after choosing suitable τ in four different regions. We compare them with equations (3.36a), (3.36b), (3.36c), (3.36d) whose right hand side is denoted as $F_R(x,\tau)$. Meanwhile, we estimate the contribution of higher order terms by resumming $\Phi^{(k=3)}_{\mathbf{4}_1}(x,\tau)$ with 215 terms. The values of $\lvert\tilde{q}(\tilde{q}^{-1})\rvert$ and $\lvert\tilde{x}^{\pm1}\rvert$ are also provided for comparison.

The critical equation of the leading order potential function reads

$$0=\frac{\partial}{\partial Z}U_0(u,Z)=\frac{1}{k}\left(2Z-\log(1+\mathrm{e}^{Z})-\log(1+\mathrm{e}^{-u+Z})-\log(1+\mathrm{e}^{u+Z})\right), \tag{4.9}$$

which can be simplified to the algebraic equation

$$Y^2=(1-Y)(1-XY)(1-X^{-1}Y), \tag{4.10}$$

with

$$X=\mathrm{e}^u=x^k,\quad Y=-\mathrm{e}^Z. \tag{4.11}$$

We also introduce the variable θ such that

$$Y=-\theta^k. \tag{4.12}$$

The critical point equation (4.10) has three solutions in terms of Y, corresponding to the geometric, conjugate, and the real flat connections $\sigma_1,\sigma_2,\sigma_3$ on the knot complement

$S^3\backslash\mathbf{5}_2$. Numerically they are the ones which in the limit $X \to 1$ reduce respectively to the solutions $0.78492 + 1.30714\ldots\mathsf{i}$, $0.78492 - 1.30714\ldots\mathsf{i}$ and $0.43016\ldots$ to (4.10). Near each critical point σ_j, the integrand is approximated by the exponential

$$\exp\left(\frac{V^{(k)}(\sigma_j)}{2\pi\mathsf{i}\tau}\right), \tag{4.13}$$

where

$$V^{(k)}(\sigma_j) = U_0(\log(X), -\log(Y_j)) + \frac{3u^2}{2k} + \frac{\pi^2}{2k}. \tag{4.14}$$

The last two terms come from the prefactors outside the summation over $m \in \mathbb{Z}_k$. We again notice the curious fact that $V^{(k)}(\sigma_j) = V^{(1)}(\sigma_j)/k$.

By expanding near the critical point and performing the Gaussian integration order by order, for each critical point σ_j we can find an asymptotic power series $\varphi^{(k,\sigma_j)}(x;\tau)$, which, together with the exponential factor (4.13), forms the trans-series

$$\Phi^{(k,\sigma_j)}_{\mathbf{5}_2}(x;\tau) = \mathrm{e}^{V^{(k)}(\sigma_j)/(2\pi\mathsf{i}\tau)}\varphi^{(k,\sigma_j)}_{\mathbf{5}_2}(x;\tau), \quad j = 1,2,3. \tag{4.15}$$

The one-loop contribution $\varphi^{(k,\sigma_j)}_{\mathbf{5}_2}(x;0) = \omega^{(k,\sigma_j)}_{\mathbf{5}_2}(x)$ has a universal expression

$$\begin{aligned}
&\omega^{(k,\sigma_j)}_{\mathbf{5}_2}(x)\\
&= \frac{(-1)^{n-1}\mathrm{e}^{\frac{\pi\mathsf{i}}{4}+\frac{\pi\mathsf{i}k}{2}+\frac{\pi\mathsf{i}}{2k}(6n^2-1)}}{\sqrt{-k(3+2s(x^k)\theta^k+(s(x^k)-1)\theta^{2k})}}\theta D^*_k(\zeta^{-1/2}\theta)^{1/k}D^*_k(\zeta^{-1/2}\theta x)^{1/k}D^*_k(\zeta^{-1/2}\theta/x)^{1/k}\\
&\quad\times\sum_{m\in\mathbb{Z}_k}\zeta^{m^2}\theta^{2m}(\zeta^{1/2}\theta;\zeta)^{-1}_m(\zeta^{1/2}\theta x;\zeta)^{-1}_m(\zeta^{1/2}\theta/x;\zeta)^{-1}_m,
\end{aligned} \tag{4.16}$$

where

$$s(x) = 1 + x + 1/x. \tag{4.17}$$

The power series $\varphi^{(k,\sigma_j)}_{\mathbf{5}_2}(x;\tau)/\omega^{(k,\sigma_j)}_{\mathbf{5}_2}(x)$ on the other hand depends in a complicated way on the level k.

The case of $k = 1$ is particularly simple. The dependence on $n \in \mathbb{Z}_1$ drops out, and the state integral reduces to [31]

$$\chi_{\mathbf{5}_2}(\mu;\tau) = \chi^{(1)}_{\mathbf{5}_2}(\mu,1;\tau) = \Phi_{\mathsf{b}}(0)^{-6}\int_{\mathbb{R}+\mathsf{i}0}\mathrm{d}v\Phi_{\mathsf{b}}(v)\Phi_{\mathsf{b}}(v+\mu)\Phi_{\mathsf{b}}(v-\mu)\mathrm{e}^{-\pi\mathsf{i}(2v^2+3\mu^2)} \tag{4.18}$$

where we have used $\eta_1 = \Phi_{\mathsf{b}}(0)^{-2}$. The power series $\varphi_1^{(\sigma_i)}(x;\tau)$ are given by [31]

$$\begin{aligned}
&(\omega_{\mathbf{5}_2}^{(\sigma_j)})^{-1}\varphi_{\mathbf{5}_2}^{(\sigma_1)}\left(x;\frac{\tau}{2\pi\mathsf{i}}\right)\\
&=1+\frac{Y_j^4}{12\gamma_j^3}\Big(-12+18s-4s^2-5s^3+s^4+(27-16s-s^2+6s^3-s^4)Y_j+s(-19+2s)Y_j^2\Big)\tau\\
&\quad+\frac{Y_j^{10}}{288\gamma_j^6}\Big(24201-34032s+7438s^2+8872s^3-7337s^4+5128s^5-3479s^6+1135s^7-93s^8\\
&\quad-13s^9+s^{10}+(4680+2562s-1132s^2-8688s^3+6400s^4-4338s^5+3353s^6-1145s^7\\
&\quad+94s^8+13s^9-s^{10})Y_j+(-5832+6972s+4921s^2-6026s^3+3034s^4-2454s^5+1030s^6\\
&\quad-105s^7-12s^8+s^9)Y_j^2\Big)\tau^2+\mathcal{O}(\tau^3),
\end{aligned}\tag{4.19}$$

where

$$s = s(x) = 1 + x + x^{-1}, \tag{4.20}$$

and

$$\gamma_j(x) = 3 - 2s(x)Y_j(x) + (s(x)-1)Y_j(x)^2 \tag{4.21}$$

At level $k = 2$, the first few terms of the power series are given in appendix B.2. It is clear that the coefficients of the power series depend only on the solution Y_j to the NZ equation, but not on a k-th root of Y_j. Furthermore, they depend on the deformation parameters u, n only through the holonomy parameter x. Finally, the power series are increasingly more complicated at higher levels k.

4.2 Holomorphic blocks

As in the example of knot $\mathbf{4}_1$, here we introduce descendants of the state integral and demonstrate their property of factorisation into holomorphic blocks.

The descendant state integral for the knot $\mathbf{5}_2$ takes the form,

$$\begin{aligned}
\chi^{(k)}_{\mathbf{5}_2;r,s}(\mu,n;\mathsf{b}) = \frac{(\eta_k)^3}{k}\sum_{m\in\mathbb{Z}_k}\int_{\mathcal{C}}\mathrm{d}\sigma\,\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma,m)\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma+\mu,m+n)\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\sigma-\mu,m-n)\\
\times(-1)^n\mathrm{e}^{\frac{\pi\mathsf{i}}{k}(-(\sigma+\mu-c_{\mathsf{b}})^2-(\sigma-\mu-c_{\mathsf{b}})^2-\mu^2+(m+n)^2+(m-n)^2+n^2)}\\
\times\mathrm{e}^{\frac{2\pi\mathsf{i}}{k}(-\mathsf{i}(r\mathsf{b}+s\mathsf{b}^{-1})(c_{\mathsf{b}}-\sigma)+(r-s)m)},
\end{aligned}\tag{4.22}$$

where the contour of integral $\mathcal{C}$ is such that it asymptotes at both $\pm\infty$ to the horizontal line $\operatorname{Im}\sigma = \sigma_0$ with $\sigma_0 < \operatorname{Im}(c_{\mathsf{b}}) - |\operatorname{Re}(r\mathsf{b}+s\mathsf{b}^{-1})|$ but is deformed near the origin so that all the poles of the integrand are below the contour.

The integrand only has poles coming from the tetrahedron partition functions and they are located at

$$v = -\mathsf{i}\mathsf{b}\alpha - \mathsf{i}\mathsf{b}^{-1}\beta, \qquad \alpha-\beta\equiv m \ (\mathrm{mod}\ k), \tag{4.23}$$

$$v = \mu-\mathsf{i}\mathsf{b}\alpha - \mathsf{i}\mathsf{b}^{-1}\beta, \qquad \alpha-\beta\equiv m-n \ (\mathrm{mod}\ k), \tag{4.24}$$

$$v = -\mu-\mathsf{i}\mathsf{b}\alpha - \mathsf{i}\mathsf{b}^{-1}\beta, \qquad \alpha-\beta\equiv m+n \ (\mathrm{mod}\ k). \tag{4.25}$$

We can evaluate the descendant state integral by completing the contour from below and summing up residues from all these poles. We find in this way that

$$\begin{aligned}\chi^{(k)}_{\mathbf{5}_2;r,s}(\mu,n;\mathsf{b}) =&(-1)^n q^{r/2}\tilde{q}^{-s/2}\Big(\mathrm{e}^{-\frac{\pi\mathrm{i}}{12}+\frac{5\pi\mathrm{i}}{6k}+\frac{5\pi\mathrm{i}}{12k}(\tau+\tau^{-1})}A_r(x;q)A_s(\tilde{x},\tilde{q}^{-1})\\ &+\mathrm{e}^{-\frac{5}{12}\pi\mathrm{i}k+\frac{\pi\mathrm{i}}{6k}+\frac{\pi\mathrm{i}}{12k}(\tau+\tau^{-1})}B_r(x;q)B_s(\tilde{x},\tilde{q}^{-1})\\ &+\mathrm{e}^{-\frac{5}{12}\pi\mathrm{i}k+\frac{\pi\mathrm{i}}{6k}+\frac{\pi\mathrm{i}}{12k}(\tau+\tau^{-1})}C_r(x;q)C_s(\tilde{x},\tilde{q}^{-1})\Big).\end{aligned} \tag{4.26}$$

Here $A_r(x,q), B_r(x,q), C_r(x,q)$ are descendant holomorphic blocks defined by

$$A_r(x;q) = \mathcal{H}(x, x^{-1}, q^r; q), \tag{4.27a}$$

$$B_r(x;q) = \theta(-q^{1/2}x;q)^{-2}x^r\mathcal{H}(x, x^2, q^r x^2; q), \tag{4.27b}$$

$$C_r(x;q) = \theta(-q^{-1/2}x;q)^{-2}x^{-r}\mathcal{H}(x^{-1}, x^{-2}, q^r x^{-2}; q), \tag{4.27c}$$

where $H(x,y,z;q^\varepsilon)$ for $|q|<1$ and $\epsilon=\pm1$ and

$$H^+(x,y,z;q) = (qx;q)_\infty(qy;q)_\infty\sum_{n=0}^{\infty}\frac{q^{n(n+1)}z^n}{(q;q)_n(qx;q)_n(qy;q)_n} \tag{4.28a}$$

$$H^-(x,y,z;q) = \frac{1}{(x;q)_\infty(y;q)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{q^{\frac{1}{2}n(n+1)}x^{-n}y^{-n}z^n}{(q;q)_n(qx^{-1};q)_n(qy^{-1};q)_n}. \tag{4.28b}$$

Note again that the definition of descendant holomorphic blocks as functions of x,q are independent of k. They thus enjoy the same properties as discussed in [31]. If we define the descendant Wronskian matrix

$$W_r(x;q) = \begin{pmatrix} A_r(x;q) & B_r(x;q) & C_r(x;q)\\ A_{r+1}(x;q) & B_{r+1}(x;q) & C_{r+1}(x;q)\\ A_{r+2}(x;q) & B_{r+2}(x;q) & C_{r+2}(x;q)\end{pmatrix}, \tag{4.29}$$

it enjoys the identity

$$W_m(x^{-1};q) = W_m(x;q)\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}, \tag{4.30}$$

as well as the inversion property

$$W_m(x;q)W_\ell(x;q^{-1})^T \in \mathrm{PSL}(3,\mathbb{Z}[x^{\pm1},q^{\pm1}]). \tag{4.31}$$

In particular

$$W_{-1}(x;q)W_{-1}(x;q^{-1})^T = \begin{pmatrix}1&0&0\\0&0&1\\0&1&x+x^{-1}\end{pmatrix}. \tag{4.32}$$

4.3 Resurgent structure

We collect the trans-series associated to the geometric and conjugate flat connections in a vector

$$\Phi^{(k)}_{\mathbf{5}_2}(x;\tau) = \begin{pmatrix}\Phi^{(k,\sigma_1)}_{\mathbf{5}_2}\\ \Phi^{(k,\sigma_2)}_{\mathbf{5}_2}\\ \Phi^{(k,\sigma_2)}_{\mathbf{5}_3}\end{pmatrix}(x;\tau). \tag{4.33}$$

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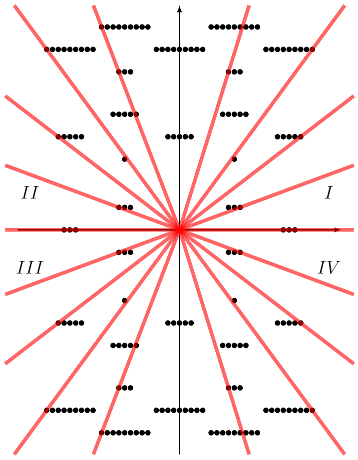

Figure 5. Singular points of the Borel transform of the vector of trans-series of the $sl(2,\mathbb{C})$ Chern-Simons theory at level $k=1$ on the complement of the knot $\mathbf{5}_2$.

The singularities of the Borel transform of each component trans-series are located in $\Lambda_j^{(k)}$ given in (2.63). In the case of level $k=1$, we superimpose the Borel singularities of the components of $\Phi^{(1)}_{\mathbf{5}_2}(x;\tau)$ and plot them in figure 5. At a generic level $k\geq 1$, the distrubtion of the Borel plane singularities is the same and their actual locations are reduced by a factor of $1/k$ according to (2.63).

As explained in section 2, Stokes constants of individual Borel plane singularities can be extracted from global Stokes automorphisms. The entire Borel plane is divided by Stokes rays of components of $\Phi^{(k)}_{\mathbf{5}_2}(x;\tau)$ into infinitely many cones. We label the four cones above and below the positive and negative real axes by I, II, III, IV, ordered in counter-clockwise direction, as illustrated in figure 5. We choose to calculate the global Stokes automorphism $\mathfrak{S}^{(k)}_+(\tilde{x};\tilde{q}) = \mathfrak{S}^{(k)}_{IV\to II}(\tilde{x};\tilde{q})$ and $\mathfrak{S}^{(k)}_-(\tilde{x};\tilde{q}) = \mathfrak{S}^{(k)}_{II\to IV}(\tilde{x};\tilde{q})$, both in the anti-clockwise direction, which encode all the Stokes constants.

As in the previous section, in order to derive the Stokes automorphisms, we first calculate the Borel resummation of the trans-series. Following [31] and with numerical evidence presented shortly after, We claim that the Borel resummation of trans-series can be written as bilinear products of holomorphic blocks in q and $\tilde{q}$. Concretely,

$$s_I(\Phi)(x;\tau) = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(\tilde{x};\tilde{q}^{-1})\Delta^{(k)}(\tau)B(x;q), \quad |\tilde{q}|<1, \tag{4.34a}$$

$$s_{II}(\Phi)(x;\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1})\Delta^{(k)}(\tau)B(x;q), \quad |\tilde{q}|<1, \tag{4.34b}$$

$$s_{III}(\Phi)(x;\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -s(\tilde{x}) & 1 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1})\Delta^{(k)}(\tau)B(x;q), \quad |\tilde{q}|>1, \tag{4.34c}$$

$$s_{IV}(\Phi)(x;\tau) = \begin{pmatrix} 0 & -s(\tilde{x}) & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(\tilde{x};\tilde{q}^{-1})\Delta^{(k)}(\tau)B(x;q), \quad |\tilde{q}|>1. \tag{4.34d}$$

Here $W_{-1}(x,q)$ is the Wronskian of holomorphic blocks defined in (3.32), and

$$B(x;q) = \begin{pmatrix} A_0(x;q) \\ B_0(x;q) \\ C_0(x;q) \end{pmatrix} \tag{4.35}$$

$\Delta^{(k)}(\tau)$ is the diagonal matrix defined by

$$\Delta^{(k)}(\tau) = \mathrm{diag}(\mathrm{e}^{-\frac{\pi\mathrm{i}}{12}+\frac{5\pi\mathrm{i}}{6k}+\frac{5\pi\mathrm{i}}{12k}(\tau+\tau^{-1})}, \mathrm{e}^{-\frac{5}{12}\pi\mathrm{i}k+\frac{\pi\mathrm{i}}{6k}+\frac{\pi\mathrm{i}}{12k}(\tau+\tau^{-1})}, \mathrm{e}^{-\frac{5}{12}\pi\mathrm{i}k+\frac{\pi\mathrm{i}}{6k}+\frac{\pi\mathrm{i}}{12k}(\tau+\tau^{-1})}). \tag{4.36}$$

As in the example of knot $\mathbf{4}_1$, we use (4.34a) and (4.34b) to derive Stokes automorphism in the upper half plane, and use (4.34a) and (4.34d) to calculate Stokes constants for Borel plane singularities in the positive axis, which involves analytic continuation of the holomorphic blocks with the help of (4.32). Combining these results, we can write down the global Stokes automorphism $\mathfrak{S}_+^{(k)}$ from sector IV to sector II in anti-clockwise diretion. Similar calculations can be done to write down $\mathfrak{S}_-^{(k)}$ from sector II to sector IV in anti-clockwise direction.

$$\mathfrak{S}_+^{(k)}(\tilde{x};\tilde{q}) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1})W_{-1}(\tilde{x};\tilde{q})^T \begin{pmatrix} 0 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad |\tilde{q}|<1 \tag{4.37a}$$

$$\mathfrak{S}_-^{(k)}(\tilde{x};\tilde{q}) = \begin{pmatrix} 0 & -s(\tilde{x}) & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} W_{-1}(\tilde{x};\tilde{q})W_{-1}(\tilde{x}^{-1};\tilde{q}^{-1})^T \begin{pmatrix} 0 & 0 & -1 \\ -s(\tilde{x}) & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad |\tilde{q}|<1. \tag{4.37b}$$

we notice that the Stokes automorphisms and therefore the Stokes constants do not depend on the level k.

At level $k=1$, the numerical evidences for the Borel plane singularities as well the Borel resummation formulas (4.34a), (4.34b), (4.34c), (4.34d) were provided in detail in [31].

We verify these results numerically at level $k=2$. We choose $x=(-1)^n\frac{6}{5}$ $(n=0,1)$ corresponding to $u=2\log 6/5, n=0,1$ with power series truncated to $N=200$ terms. The Stokes plane is divided into four regions similar to figure 5. For example, the positions of poles for the Padé approximant $\Phi_{\mathbf{5}_2}^{(k=2)}(x,\tau)$ at $x=\frac{6}{5}$ are shown in figure 6. Finally we present the numerical results in tables 6 and 7 respectively. As we can see, the relative errors between two sides of (4.34a), (4.34b), (4.34c), (4.34d) (the first column) are always within the precision of the Borel resummation (the second column), and in regions I, II are always far smaller than a potential $\tilde{q}$ (or $1/\tilde{q}$ in the lower half plane) or a $\tilde{x}^{\pm 1}$ correction (the third and the fourth columns). In region III, the relative error between two sides of (4.34b) is relatively high, due to discrete singular points that break off from the branch cut of the Borel transform of the truncated series $\Phi^{(k=2,\sigma_2)}_{\mathbf{5}_2}(x;\tau)$ and stray into region III, as seen in figure 6b. The relative error is still much smaller than a $\tilde{x}^{\pm 1}$ correction, although it does not preclude a potential $\tilde{q}^{-1}$ correction. The latter, however, is not possible. Figure 6b indicates that $\Phi^{(k=2,\sigma_2)}_{\mathbf{5}_2}(x;\tau)$ in regions II and III must be related by a Stokes automorphism that accounts for the three Borel plane singularities in the negative real axis. Indeed from (4.34b), (4.34c) we find that

$$s_{III}(\Phi)(x;\tau) = \mathfrak{S}_{II\to III}(\tilde{x}) s_{II}(\Phi)(x;\tau), \quad \mathfrak{S}_{II\to III}(\tilde{x}) = \begin{pmatrix} 1 & 0 & 0 \\ -1 - s(\tilde{x}) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.38}$$

The $(2,1)$ entry of $\mathfrak{S}_{II\to III}(\tilde{x})$

$$\mathfrak{S}_{II\to III}(\tilde{x})_{2,1} = -\tilde{x} - 2 - \tilde{x}^{-1} \tag{4.39}$$

should encode the Stokes constants of these three Borel plane singularities. Any correction to the right hand side of (4.34c) should involve a correction of the coefficients of $\mathfrak{S}_{II\to III}(\tilde{x})_{2,1}$ and thus must be of relative order at least $\tilde{x}^{\pm 1}$. Likewise, from (4.34a), (4.34d) we find that

$$s_{I}(\Phi)(x;\tau) = \mathfrak{S}_{IV\to I}(\tilde{x}) s_{IV}(\Phi)(x;\tau), \quad \mathfrak{S}_{IV\to I}(\tilde{x}) = \begin{pmatrix} 1 & 1 + s(\tilde{x}) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.40}$$

The $(1,2)$ entry of $\mathfrak{S}_{IV\to I}(\tilde{x})$

$$\mathfrak{S}_{IV\to I}(\tilde{x})_{1,2} = \tilde{x} + 2 + \tilde{x}^{-1} \tag{4.41}$$

accounts for the three singularities of the Borel transform of $\Phi^{(k=2,\sigma_1)}_{\mathbf{5}_2}(x;\tau)$ in the positive real axis shown in figure 6a. Any correction to the right hand side of (4.34d) should involve a correction of the coefficients of $\mathfrak{S}_{IV\to I}(\tilde{x})_{1,2}$ and thus must be of relative order $\tilde{x}^{\pm 1}$, which is also excluded in tables 6, 7.

5 Conclusion and discussion

In this short note we study the resurgent properties of the $sl(2,\mathbb{C})$ Chern-Simons state integral model on knot complemnets $S^3\backslash\mathbf{4}_1$, $S^3\backslash\mathbf{5}_2$ at generic level $k \geq 1$. When the level k increases, the saddle point expansion of the $sl(2,\mathbb{C})$ Chern-Simons state integral becomes increasingly more complicated. But there are several interesting universal featrues. First of all, the coefficients of the power series always live in the trace field of the knot extended by the holonomy paramter x. Second, these power series enjoy universal resurgent properies. When the holonomy deformed is weak so that $|\log x^k| \ll 1$, the distribution of the singularities of the Borel transform is the same and they are all related to the level $k = 1$ singularites by a factor of $1/k$. In addition, the Stokes constants associated to these singularities are independent of the level k. We speculate that this is related to the fact

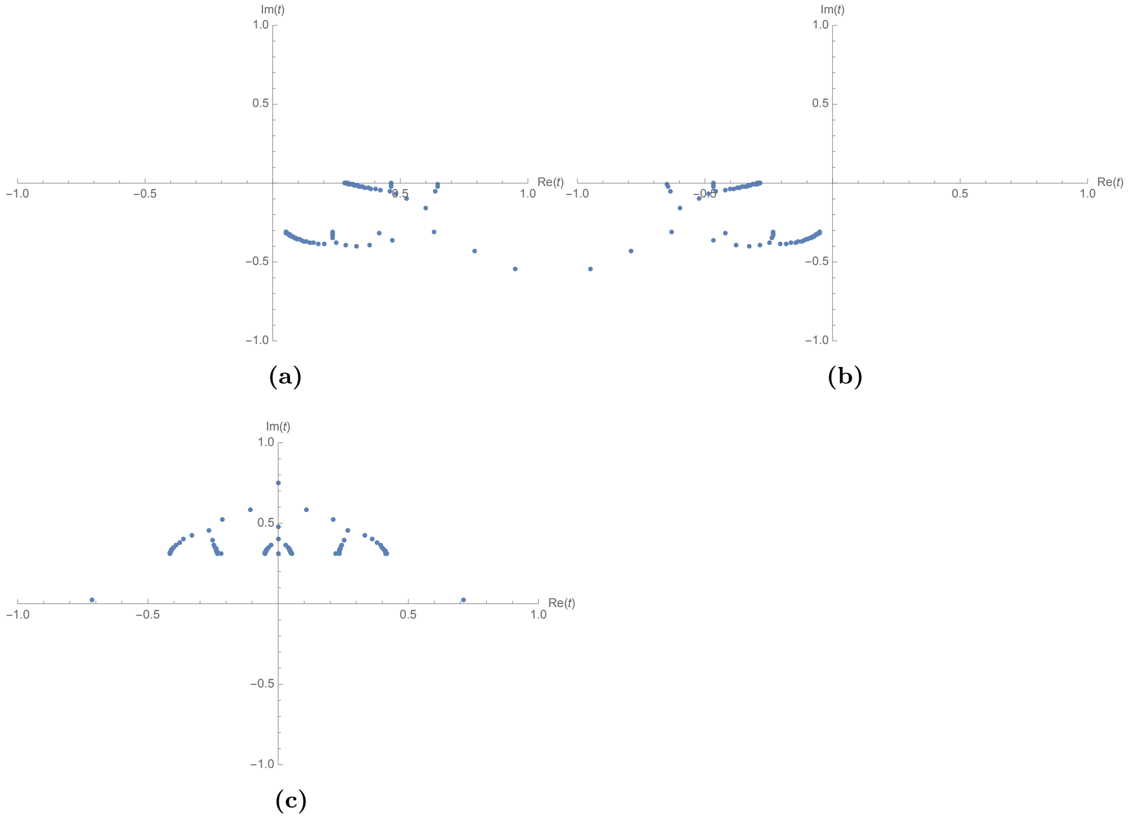

Figure 6. The distribution of poles of the Padé approximant for (a) : $\Phi_{\mathbf{5_2}}^{(k=2,\sigma_1)}(x,\tau)$, (b) : $\Phi_{\mathbf{5_2}}^{(k=2,\sigma_2)}(x,\tau)$ and (c) : $\Phi_{\mathbf{5_2}}^{(k=2,\sigma_3)}(x,\tau)$ with $x = 6/5$ and $N = 200$ terms.

the Hilbert space of the quantum Chern-Simons theory at level k is simply a k copy of the level one theory, and that they all coincide with the same BPS invariants of the dual 3d SCFT.

Some interesting questions remain unanswered following this work. Both this work and its predecessor [31] focus on the case when the holonomy is weak.[6] It will be interesting to understand the resurgent structure of the Chern-Simons theory in the entire holonomy space. Due to the identification of the Stokes constants with the BPS invariants in the dual 3d SCFT [31, 32], the evolution of the resurgent structure as the holonomy changes may enjoy features similar to the Wall Crossing Formulas of Kontsevich and Soibelman [55].

Second, it was shown in [24] that the resurgent structures presented in [31, 32] are not complete: they involve only asymptotic series $\varphi^{(k=1,\sigma_j)}(x;\tau)$ $(j \geq 1)$ associated to non-Abelian flat connections, while there still remains the asymptotic series $\varphi^{(k=1,\sigma_0)}(x;\tau)$ associated to the Abelian flat connection. It was revealed in [24] with examples of $\mathbf{4}_1$ and $\mathbf{5}_2$ that there are non-trivial Stokes automorphisms that relate $\varphi^{(k=1,\sigma_0)}(x;\tau)$ to $\varphi^{(k=1,\sigma_j)}(x;\tau)$ $(j \geq 1)$ but not the other way around. This is similar to the discovery in the case of compact three manifolds in [27]. It would be interesting to also generalise this connection to generic levels.

[6] There are some limited but unsystematic discussion for large holonomy in [31].

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	$\lvert\frac{s_I(\Phi)(x;\tau)}{F_I(x;\tau)}-1\rvert$	$\lvert\frac{s_I(\Phi)(x;\tau)}{s_I'(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	1.4×10^{-17}	1.7×10^{-17}		
σ_2	1.1×10^{-27}	1.0×10^{-26}	1.7×10^{-5}	0.33
σ_3	1.9×10^{-21}	1.9×10^{-21}		

(a) Region I: $\tau=\frac{1}{7}\mathrm{e}^{\frac{\pi\mathrm{i}}{6}}$

	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{F_{II}(x;\tau)}-1\rvert$	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{s_{II}'(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	1.1×10^{-27}	1.0×10^{-26}		
σ_2	1.4×10^{-17}	1.7×10^{-17}	1.7×10^{-5}	0.33
σ_3	1.9×10^{-21}	1.9×10^{-21}		

(b) Region II: $\tau=\frac{1}{7}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}$

	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{F_{III}(x;\tau)}-1\rvert$	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{s_{III}'(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.8×10^{-27}	7.2×10^{-26}		
σ_2	1.3×10^{-2}	1.5×10^{-3}	1.7×10^{-5}	0.36
σ_3	1.1×10^{-23}	1.9×10^{-24}		

(c) Region III: $\tau=\frac{1}{7}\mathrm{e}^{-\frac{5\pi\mathrm{i}}{6}}$

	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{F_{IV}(x;\tau)}-1\rvert$	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{s_{IV}'(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert,\lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	1.3×10^{-2}	1.5×10^{-3}		
σ_2	3.8×10^{-27}	7.2×10^{-26}	1.7×10^{-5}	0.36
σ_3	1.1×10^{-23}	1.9×10^{-24}		

(d) Region IV: $\tau=\frac{1}{7}\mathrm{e}^{-\frac{\pi\mathrm{i}}{6}}$

Table 6. We perform the numerical Borel resummation for $\Phi^{(k=2)}_{\mathbf{5_2}}(x,\tau)$ at $x=6/5$ ($u=2\log 6/5$, $n=0$) with 200 terms, after choosing suitable τ in regions I and II. We compare them with equations (4.34a)(4.34b)(4.34c)(4.34d) whose right hand side is denoted as $F_R(x,\tau)$. Meanwhile, we estimate the contribution of higher order terms by resumming $\Phi^{(k=2)}_{\mathbf{5_2}}(x,\tau)$ with 196 terms. The values of $\lvert\tilde{q}(\tilde{q}^{-1})\rvert$ and $\lvert\tilde{x}^{\pm1}\rvert$ are also provided for comparison.

In fact, the power series $\varphi^{(k=1,\sigma_0)}(x;\tau)$ were computed by expanding colored Jones polynomial near $q\to1$. Both $\varphi^{(k=1,\sigma_0)}(1;\tau)$ and $\varphi^{(k=1,\sigma_j)}(1;\tau)$ with holonomy turned off appear in the Refined Quantum Modularity Conjecture [29, 56] in the case of S-transformation (see also [34, 35]). For a generic $sl(2,\mathbb{Z})$ transformation $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$, the power series should be replaced by $\varphi^{(k=c,\sigma_0)}(1;\tau)$ and $\varphi^{(k=c,\sigma_j)}(1;\tau)$ or their Galois transformation in the trace field. The power series $\varphi^{(k=c,\sigma_0)}(x;\tau)$ can be computed by expanding colored Jones polynomial near $q\to\exp 2\pi\mathrm{i}/c$. We conjecture then that $\varphi^{(k,\sigma_0)}(x;\tau)$ constructed this way is also related to $\varphi^{(k,\sigma_j)}(x;\tau)$ ($j\geq1$) computed in this work by non-trivial Stokes automorphisms, where the Stokes constants coincide with those in [24].

	$\lvert\frac{s_I(\Phi)(x;\tau)}{F_I(x;\tau)}-1\rvert$	$\lvert\frac{s_I(\Phi)(x;\tau)}{s'_I(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert, \lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	3.9×10^{-18}	4.0×10^{-18}		
σ_2	2.9×10^{-28}	2.5×10^{-28}	1.7×10^{-5}	0.33
σ_3	2.2×10^{-21}	7.3×10^{-21}		

(a) Region I: $\tau=\frac{1}{7}\mathrm{e}^{\frac{\pi\mathrm{i}}{6}}$

	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{F_{II}(x;\tau)}-1\rvert$	$\lvert\frac{s_{II}(\Phi)(x;\tau)}{s'_{II}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert, \lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	2.9×10^{-28}	2.5×10^{-28}		
σ_2	3.9×10^{-18}	4.0×10^{-18}	1.7×10^{-5}	0.33
σ_3	2.2×10^{-21}	7.3×10^{-21}		

(b) Region II: $\tau=\frac{1}{7}\mathrm{e}^{\frac{5\pi\mathrm{i}}{6}}$

	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{F_{III}(x;\tau)}-1\rvert$	$\lvert\frac{s_{III}(\Phi)(x;\tau)}{s'_{III}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert, \lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	2.4×10^{-29}	1.7×10^{-29}		
σ_2	8.1×10^{-3}	2.0×10^{-4}	3.5×10^{-6}	0.36
σ_3	2.7×10^{-25}	1.1×10^{-24}		

(c) Region III: $\tau=\frac{1}{8}\mathrm{e}^{-\frac{5\pi\mathrm{i}}{6}}$

	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{F_{IV}(x;\tau)}-1\rvert$	$\lvert\frac{s_{IV}(\Phi)(x;\tau)}{s'_{IV}(\Phi)(x;\tau)}-1\rvert$	$\lvert\tilde{q}(\tau)^{-1}\rvert$	$\mathrm{Min}(\lvert\tilde{x}(x,\tau)\rvert, \lvert\tilde{x}(x,\tau)^{-1}\rvert)$
σ_1	8.1×10^{-3}	2.0×10^{-4}		
σ_2	2.4×10^{-29}	1.7×10^{-29}	3.5×10^{-6}	0.36
σ_3	2.7×10^{-25}	1.1×10^{-24}		

(d) Region IV: $\tau=\frac{1}{8}\mathrm{e}^{-\frac{\pi\mathrm{i}}{6}}$

Table 7. We perform the numerical Borel resummation for $\Phi^{(k=2)}_{\mathbf{5_2}}(x,\tau)$ at $x=-6/5$ ($u=2\log 6/5$, $n=1$) with 200 terms, after choosing suitable τ in regions I and II. We compare them with equations (4.34a), (4.34b), (4.34c), (4.34d) whose right hand side is denoted as $F_R(x,\tau)$. Meanwhile, we estimate the contribution of higher order terms by resumming $\Phi^{(k=2)}_{\mathbf{5_2}}(x,\tau)$ with 196 terms. The values of $|\tilde{q}(\tilde{q}^{-1})|$ and $|\tilde{x}^{\pm1}|$ are also provided for comparison.

We have been focusing on the generalisation of [31, 32] to $sl(2,\mathbb{C})$ Chern-Simons theory with higher levels $k\geq1$. Other directions of generalisation are also possible. For instance one can consider $sl(N,\mathbb{C})$ Chern-Simons theory with $N\geq2$. State integral models have also been written down in [5]. Alternatively, one can also study the resurgent structures of the Chern-Simons theory at level $k=0$. This is a special case, and by the 3d-3d correspondence, it is dual to the 3d SCFT on $S^2\times S^1$ [14], whose partition function is nothing else but the supersymmetric indices of the 3d SCFT. In other words, one could study the saddle point expansion of the indices and their resurgent structures. See related work in [57]. Furthermore, three manifolds obtained from Dehn filling of hyperbolic knot complements are also interesting objects to study [58, 59].

Finally, following the arguments in [60] (see a similar application of the same idea [61]), the Stokes automorphism together with the asymptotic behavior define for us a Riemann-Hilbert problem for the Borel resummed asymptotic series. The latter may be solved by a TBA-like equation. It is interesting to explore if such a TBA-like equation can be written down for Borel resummed asymptotic series in complex Chern-Simons theory. As showcased in [61], such an equation could help answer the question of the evolution of the resurgent structure as we change the moduli of the system, which are the holonomy parameter x here.

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A Tetrahedron partition function at level k

The partition function of $sl(2,\mathbb{C})$ Chern-Simons theory at level k on an ideal tetrahedron is given by

$$\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\mu,n) = \prod_{(r,s)\in\Gamma(k;n)} \Phi_{\mathsf{b}}\Big(c_{\mathsf{b}} - \frac{1}{k}(\mu + \mathsf{i}\mathsf{b}r + \mathsf{i}\mathsf{b}^{-1}s)\Big) \tag{A.1}$$

where

$$\Gamma(k;n) = \{(r,s)\in\mathbb{Z}^2 \mid 0\le r,s<k, r-s\equiv n (\mathrm{mod}\ k)\} \tag{A.2}$$

and $c_{\mathsf{b}} = \frac{\mathsf{i}}{2}(\mathsf{b}+\mathsf{b}^{-1})$. $\Phi_{\mathsf{b}}(x)$ is Faddeev's quantum dilogarithm. This defines a meromorphic function of $\mu\in\mathbb{C}$ for each $n\in\mathbb{Z}_k$, and it is defined for all values of b with b^2 in the cut plane $\mathbb{C}' = \mathbb{C}\backslash\mathbb{R}_{\le 0}$.

When $\mathrm{Im}\,\mathsf{b}>0$ or $\mathrm{Im}\,\mathsf{b}<0$, it has the factorisation form

$$\mathcal{Z}^{(k)}_{\mathsf{b}}[\Delta](\mu,n) = (qx^{-1};q)_\infty(\tilde{q}^{-1}\tilde{x}^{-1};\tilde{q}^{-1})_\infty \tag{A.3}$$

where we use the notation

$$\begin{aligned} q &= \exp\frac{2\pi\mathsf{i}}{k}(\mathsf{b}^2+1), & \tilde{q} &= \exp -\frac{2\pi\mathsf{i}}{k}(\mathsf{b}^{-2}+1),\\ x &= \exp\left(\frac{2\pi\mathsf{b}\mu}{k} - \frac{2\pi\mathsf{i}n}{k}\right), & \tilde{x} &= \exp\left(\frac{2\pi\mathsf{b}^{-1}\mu}{k} + \frac{2\pi\mathsf{i}n}{k}\right), \end{aligned} \tag{A.4}$$

and $(a,q)_\infty$ is the q-Pochhammer symbol defined to be $\prod_{j=1}^\infty(1-aq^j)$ if $|q|<1$ or $1/(q^{-1}a;q^{-1})_\infty$ if $|q|>1$.

A.1 Fundamental properties

- Inversion relation

$$\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](c_{\mathsf{b}}+x,n)\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](c_{\mathsf{b}}-x,-n)=(-1)^n\mathrm{e}^{\frac{\pi\mathrm{i}}{k}(x^2-n^2)}\eta_k^{-1}, \tag{A.5}$$

  where

$$\eta_k=\mathrm{e}^{\pi\mathrm{i}(k+2c_{\mathsf{b}}^2k^{-1})/6}. \tag{A.6}$$

  Note that this implies the special value

$$\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](c_{\mathsf{b}},0)^2=\eta_k^{-1}. \tag{A.7}$$

- Quasi-periodicity (Faddeev's difference equations). For $x,\mathsf{b}\in\mathbb{C}$, $\mathrm{Im}(\mathsf{b})\neq 0$, $n\in\mathbb{Z}$

$$\begin{aligned}\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta]\left(x-\mathrm{i}\mathsf{b}^{\pm1},n\pm1\right)&=\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](x,n)\left(1-q_{\pm}x_{\pm}^{-1}\right)^{-1}\\ \mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta]\left(x+\mathrm{i}\mathsf{b}^{\pm1},n\mp1\right)&=\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](x,n)\left(1-x_{\pm}^{-1}\right).\end{aligned} \tag{A.8}$$

  where $q_+=q, q_-=\tilde{q}$, $x_+=x$, $x_-=\tilde{x}$.

- Asymptotic behavior

$$\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](\mu,n)\sim\begin{cases}\mathrm{e}^{\frac{\pi\mathrm{i}}{k}(c_{\mathsf{b}}-\mu)^2} & \mathrm{Re}(\mu)\ll 0\\ 1 & \mathrm{Re}(\mu)\gg 0.\end{cases} \tag{A.9}$$

A.2 Zeros and poles

For $\mathrm{Im}(\mathsf{b})>0$, the tetrahedron partition function $\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](x,n)$ has zeros

$$\begin{cases}x=2c_{\mathsf{b}}+\mathrm{i}\mathsf{b}^{-1}l+\mathrm{i}\mathsf{b}\,m\\ n=l-m \pmod k\end{cases} \tag{A.10}$$

and poles

$$\begin{cases}x=-\mathrm{i}\mathsf{b}^{-1}l-\mathrm{i}\mathsf{b}\,m\\ n=m-l \pmod k\end{cases} \tag{A.11}$$

for $l,m\in\mathbb{Z}_{\geq 0}$, and the residue is

$$\frac{k}{2\pi\mathsf{b}^{-1}}\frac{(q;q)_\infty}{(\tilde{q};\tilde{q})_\infty}\frac{1}{(q;q)_m}\frac{1}{(\tilde{q}^{-1};\tilde{q}^{-1})_l}. \tag{A.12}$$

A.3 Integral identities

The tetrahedron partition function has the following property of Fourier transformation,

$$\frac{1}{k}\sum_{n\in\mathbb{Z}_k}\int_{\mathbb{R}+c_{\mathsf{b}}}\mathrm{d}u\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](u,n)\mathrm{e}^{\frac{2\pi\mathrm{i}}{k}(uw-nm)}=\mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](w,m)(-1)^m\mathrm{e}^{-\frac{\pi\mathrm{i}}{k}((w-c_{\mathsf{b}})^2-m^2)+\frac{\pi\mathrm{i}}{12}(k+8c_{\mathsf{b}}^2k^{-1})}. \tag{A.13}$$

A.4 Semi-classical limit

In the double scaling limit

$$\mathsf{b} \to 0, \quad \mu \to \infty, \quad \mathsf{b}\mu \text{ fixed} \tag{A.14}$$

the tetrahedron partition function has the asymptotic expansion

$$\log \mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](\mu, m) = \sum_{\ell=0}^{\infty} \frac{(2\pi \mathsf{i} \mathsf{b}^2)^{\ell-1}}{\ell!} \sum_{j \in \mathbb{Z}_k} B_\ell(j/k) \operatorname{Li}_{2-\ell}\left(\mathrm{e}^{\frac{2\pi\mathsf{i}}{k}(j+m)} \mathrm{e}^{-\frac{2\pi \mathsf{b}\mu}{k}}\right). \tag{A.15}$$

Alternatively, define

$$\mathrm{D}_{\mathsf{b}}^{(k)}(u, m) = \mathcal{Z}_{\mathsf{b}}^{(k)}[\Delta](c_{\mathsf{b}} - \sqrt{k}u, m). \tag{A.16}$$

In the double scaling limit

$$\mathsf{b} \to 0, \quad u \to \infty, \quad \mathsf{b}u \text{ fixed}, \tag{A.17}$$

it has the asymptotic expansion

$$\log \mathrm{D}_{\mathsf{b}}^{(k)}(u, m) = \sum_{\ell=0}^{\infty} \frac{(2\pi \mathsf{i} \mathsf{b}^2)^{\ell-1}}{\ell!} \sum_{j \in \mathbb{Z}_k} B_\ell\left(1 - \frac{1}{2k} - \frac{j}{k}\right) \operatorname{Li}_{2-\ell}(\mathrm{e}^{-\frac{\pi\mathsf{i}}{k}(1+2j-2m)} \mathrm{e}^{\frac{2\pi\mathsf{b}u}{\sqrt{k}}}). \tag{A.18}$$

B Power series of state integrals at different levels

B.1 Knot 4_1

At level $k = 2$, the power series are

$$(\omega_{\mathbf{4}_1}^{(k=2,\sigma_j)}(x))^{-1} \varphi_{\mathbf{4}_1}^{(k=2,\sigma_j)}\left(x; \frac{\tau}{2\pi\mathsf{i}}\right) = 1 + a_1^{(\sigma_j)}(x)\tau + a_2^{(\sigma_j)}(x)\tau^2 + a_3^{(\sigma_j)}(x)\tau + \mathcal{O}(\tau^4), \tag{B.1}$$

where $a_1^{(\sigma_j)}(x)$ is given in (3.21) and some additional coefficients are

$$\begin{aligned} a_2^{(\sigma_j)}(x) =& (1+12x+46x^2-576x^3+1413x^4+1608x^5+202x^6+7788x^7+3258x^8-34572x^9 \\ &+1094x^{10}+27816x^{11}-3555x^{12}+27816x^{13}+1094x^{14}-34572x^{15}+3258x^{16} \\ &+7788x^{17}+202x^{18}+1608x^{19}+1413x^{20}-576x^{21}+46x^{22}+12x^{23}+x^{24})/ \\ &(4608(-1-x+x^2)^3(1-x+x^2)^3(-1+x+x^2)^3(1+x+x^2)^3), \end{aligned} \tag{B.2a}$$

$$\begin{aligned} a_3^{(\sigma_j)}(x) =& -((5-8550x-10329x^2-84870x^3+534081x^4+156600x^5+907732x^6+7978680x^7 \\ &+4743126x^8-47249100x^9+10187454x^{10}+68274900x^{11}-8491270x^{12}+156024000x^{13} \\ &+16794912x^{14}-565084800x^{15}+12734439x^{16}+387780030x^{17}-42351651x^{18} \\ &+387780030x^{19}+12734439x^{20}-565084800x^{21}+16794912x^{22}+156024000x^{23} \\ &-8491270x^{24}+68274900x^{25}+10187454x^{26}-47249100x^{27}+4743126x^{28}+7978680x^{29} \\ &+907732x^{30}+156600x^{31}+534081x^{32}-84870x^{33}-10329x^{34}-8550x^{35}+5x^{36}) \\ &(-1-x^2+x^4+2x^2Y_j))/(3317760(-1-x+x^2)^5(1-x+x^2)^5(-1+x+x^2)^5(1+x+x^2)^5). \end{aligned} \tag{B.2b}$$

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At level $k=3$, the power series are

$$(\omega_{\mathbf{4}_1}^{(k=3,\sigma_j)}(x))^{-1}\varphi_{\mathbf{4}_1}^{(k=3,\sigma_j)}\left(x;\frac{\tau}{2\pi\mathsf{i}}\right)=1+c_1^{(\sigma_j)}(x)\tau+c_2^{(\sigma_j)}(x)\tau^2+\mathcal{O}(\tau^3), \tag{B.3}$$

where

$$\begin{aligned}c_1^{(\sigma_j)}(x)=&(1+8x+17x^2+8(2+3\zeta)x^3-24(1+\zeta)x^4-24(1+\zeta)x^5-4(1+6\zeta)x^6\\&+8(-1+3\zeta)x^7+36(-1+2\zeta)x^8+2(47-48\zeta)x^9+24(1+2\zeta)x^{10}+142x^{11}\\&+(95-24\zeta)x^{12}+8(4+15\zeta)x^{13}++3(53-8\zeta)x^{14}-194x^{15}+8(5+\zeta)x^{16}\\&-2(89+48\zeta)x^{17}+2(25+36\zeta)x^{18}-8(1-3\zeta)x^{19}-2(7+12\zeta)x^{20}+2(17-12\zeta)x^{21}\\&+8(2-3\zeta)x^{22}+2(5+12\zeta)x^{23}-17x^{24}-8x^{25}-x^{26}\\&+2x^3(1+8x+17x^2+3x^3-20x^4-29x^5+6x^6+8x^7-26x^8+139x^9-28x^{10}\\&+139x^{11}-26x^{12}+8x^{13}+6x^{14}-29x^{15}-20x^{16}+3x^{17}+17x^{18}+8x^{19}+x^{20})Y_j)/\\&(72(1+x^2)(1-3x^3+x^6)^2(1+x^3+x^6)^2),\end{aligned} \tag{B.4}$$

$$\begin{aligned}c_2^{(\sigma_j)}(x)=&(1+16(5+8\zeta)x+(33-128\zeta)x^2+2(207+440\zeta)x^3-16(129+103\zeta)x^4\\&-2(-167+1144\zeta)x^5+(3101+6256\zeta)x^6-112(-64+49\zeta)x^7+(-5923+2928\zeta)x^8\\&-2(-14241+3544\zeta)x^9-16(484+459\zeta)x^{10}+2(40457+6680\zeta)x^{11}\\&+2(-6751+13744\zeta)x^{12}-32(1406+615\zeta)x^{13}-2(63311+27248\zeta)x^{14}\\&+2(6179+48568\zeta)x^{15}-16(635+1871\zeta)x^{16}-2(54965+10456\zeta)x^{17}\\&+(228909-59120\zeta)x^{18}-16(1402+1109\zeta)x^{19}+(329133+141328\zeta)x^{20}\\&-2(72397+45320\zeta)x^{21}-16(-128+345\zeta)x^{22}-10(4033+824\zeta)x^{23}\\&+2(-32943+33488\zeta)x^{24}+32(-964+269\zeta)x^{25}-2(26287+25328\zeta)x^{26}\\&-2(-36257+1720\zeta)x^{27}+16(-249+11\zeta)x^{28}+2(20473+8920\zeta)x^{29}\\&+(-11011-7248\zeta)x^{30}+16(680+121\zeta)x^{31}+(-2275-4496\zeta)x^{32}\\&+2(1247+1016\zeta)x^{33}+16(-18+119\zeta)x^{34}-2(233+440\zeta)x^{35}+(161+128\zeta)x^{36}\\&-16(3+8\zeta)x^{37}+x^{38}-16(1+2\zeta)x^4(8-8x+55x^2-111x^3-135x^4+336x^5\\&-224x^6+310x^7-724x^8-346x^9+390x^{10}+2778x^{11}-1108x^{12}-3486x^{13}+2569x^{14}\\&-1109x^{15}+2569x^{16}-3486x^{17}-1108x^{18}+2778x^{19}+390x^{20}-346x^{21}-724x^{22}\\&+310x^{23}-224x^{24}+336x^{25}-135x^{26}-111x^{27}+55x^{28}-8x^{29}+8x^{30})Y)/\\&(10368(1+x^2)(1-3x^3+x^6)^3(1+x^3+x^6)^3);\end{aligned} \tag{B.5}$$

B.2 Knot $\mathbf{5_2}$

At level $k=2$, the power series are

$$(\omega_{\mathbf{5}_2}^{(k=2,\sigma_j)}(x))^{-1}\varphi_{\mathbf{5}_2}^{(k=2,\sigma_j)}\left(x;\frac{\tau}{2\pi\mathsf{i}}\right)=1+a_1^{(\sigma_j)}\tau+a_2^{(\sigma_j)}\tau^2+\mathcal{O}(\tau^3), \tag{B.6}$$

where

$$\begin{aligned}a_1^{(\sigma_j)}=&(1+6x+14x^2-30x^3-245x^4-216x^5+944x^6+1428x^7-1231x^8-2346x^9+1326x^{10}\\&+3042x^{11}-1873x^{12}-6168x^{13}-3400x^{14}-2556x^{15}-3935x^{16}-4950x^{17}-5474x^{18}\\&-4950x^{19}-3935x^{20}-2556x^{21}-3400x^{22}-6168x^{23}-1873x^{24}+3042x^{25}+1326x^{26}\\&-2346x^{27}-1231x^{28}+1428x^{29}+944x^{30}-216x^{31}-245x^{32}-30x^{33}+14x^{34}+6x^{35}\end{aligned}$$

$$
\begin{aligned}
&+x^{36}+(-1-3x-10x^2+15x^3+262x^4+381x^5-964x^6-2118x^7+697x^8+2886x^9\\
&+469x^{10}-873x^{11}+1656x^{12}+4797x^{13}+2911x^{14}+903x^{15}+899x^{16}+984x^{17}+1185x^{18}\\
&+984x^{19}+899x^{20}+903x^{21}+2911x^{22}+4797x^{23}+1656x^{24}-873x^{25}+469x^{26}+2886x^{27}\\
&+697x^{28}-2118x^{29}-964x^{30}+381x^{31}+262x^{32}+15x^{33}-10x^{34}-3x^{35}-x^{36})Y_j\\
&+(x^2+3x^3+9x^4-27x^5-273x^6-318x^7+973x^8+1737x^9-916x^{10}-2070x^{11}\\
&+193x^{12}+288x^{13}-2922x^{14}-6345x^{15}-4154x^{16}-1221x^{17}-825x^{18}-1221x^{19}\\
&-4154x^{20}-6345x^{21}-2922x^{22}+288x^{23}+193x^{24}-2070x^{25}-916x^{26}+1737x^{27}\\
&+973x^{28}-318x^{29}-273x^{30}-27x^{31}+9x^{32}+3x^{33}+x^{34})Y_j^2)/\\
&(24(1+3x+3x^2+3x^3+x^4)(1-2x-x^2+8x^3-11x^4+8x^5-x^6-2x^7+x^8)^2\\
&(1+2x-x^2-8x^3-11x^4-8x^5-x^6+2x^7+x^8)^2), \qquad \text{(B.7a)}
\end{aligned}
$$

$$
\begin{aligned}
a_2^{(\sigma_j)} =&(1+9x+183x^2+27x^3-4394x^4-6915x^5+24656x^6+55122x^7-7429x^8-73344x^9\\
&-227211x^{10}-497652x^{11}+35060x^{12}+948795x^{13}+659308x^{14}+607485x^{15}+3506489x^{16}\\
&+4523433x^{17}-7706054x^{18}-16876941x^{19}+1410725x^{20}+16099737x^{21}-1232206x^{22}\\
&-18943725x^{23}-8872703x^{24}+6776679x^{25}+10490726x^{26}+6776679x^{27}-8872703x^{28}\\
&-18943725x^{29}-1232206x^{30}+16099737x^{31}+1410725x^{32}-16876941x^{33}-7706054x^{34}\\
&+4523433x^{35}+3506489x^{36}+607485x^{37}+659308x^{38}+948795x^{3}9+35060x^{40}-497652x^{41}\\
&-227211x^{42}-73344x^{43}-7429x^{44}+55122x^{45}+24656x^{46}-6915x^{47}-4394x^{48}+27x^{49}\\
&+183x^{50}+9x^{51}+x^{52}+(-1-9x-318x^2+546x^3+7812x^4+5967x^5-42735x^6\\
&-59742x^7+74478x^8+179832x^9+140847x^{10}+36363x^{11}-270391x^{12}-424173x^{13}\\
&+219178x^{14}+505557x^{15}-3042062x^{16}-4952124x^{17}+5787235x^{18}+12365862x^{19}\\
&-1529844x^{20}-11109657x^{21}+1982588x^{22}+14755839x^{23}+7961410x^{24}-3333816x^{25}\\
&-7012037x^{26}-3333816x^{27}+7961410x^{28}+14755839x^{29}+1982588x^{30}-11109657x^{31}\\
&-1529844x^{32}+12365862x^{33}+5787235x^{34}-4952124x^{35}-3042062x^{36}+505557x^{37}\\
&+219178x^{38}-424173x^{39}-270391x^{40}+36363x^{41}+140847x^{42}+179832x^{43}+74478x^{44}\\
&-59742x^{45}-42735x^{46}+5967x^{47}+7812x^{48}+546x^{49}-318x^{50}-9x^{51}-x^{52})Y\\
&+x^2(1+9x+319x^2-531x^3-7770x^4-6363x^5+43157x^6+66417x^7-72273x^8\\
&-213864x^9-169562x^{10}+14532x^{11}+375824x^{12}+532452x^{13}-180770x^{14}-559596x^{15}\\
&+2833078x^{16}+4801236x^{17}-5971218x^{18}-14640504x^{19}-223658x^{20}+13591086x^{21}\\
&+3954688x^{22}-7274442x^{23}-8512366x^{24}-7274442x^{25}+3954688x^{26}+13591086x^{27}\\
&-223658x^{28}-14640504x^{29}-5971218x^{30}+4801236x^{31}+2833078x^{32}-559596x^{33}\\
&-180770x^{34}+532452x^{35}+375824x^{36}+14532x^{37}-169562x^{38}-213864x^{39}-72273x^{40}\\
&+66417x^{41}+43157x^{42}-6363x^{43}-7770x^{44}-531x^{45}+319x^{46}+9x^{47}+x^{48})Y^2)/\\
&(1152(1+3x+3x^2+3x^3+x^4)(1-2x-x^2+8x^3-11x^4+8x^5-x^6-2x^7+x^8)^3\\
&(1+2x-x^2-8x^3-11x^4-8x^5-x^6+2x^7+x^8)^3) \qquad \text{(B.7b)}
\end{aligned}
$$

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