Counting Calabi-Yau black holes with (mock) modular forms

Boris Pioline

LPTHE LABORATOIRE DE PHYSIQUE THEORIQUE ET HAUTES ENERGIES

cnrs

SORBONNE UNIVERSITÉ

Shing-Tung Yau Center, South East University
Zoom@Nanjing, 30/5/2023

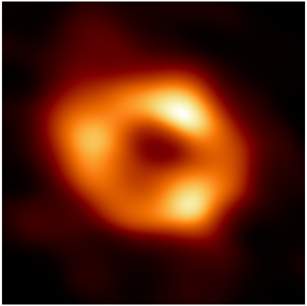
References

- *"Black holes and higher depth mock modular forms"*, with S. Alexandrov, Commun.Math.Phys. 374 (2019) 549 [arXiv:1808.08479]
- *"S-duality and refined BPS indices"*, with S. Alexandrov and J. Manschot, Commun.Math.Phys. 380 (2020) 755 [arXiv:1910.03098]
- *"Modular bootstrap for D4-D2-D0 indices on compact Calabi-Yau threefolds"*, with S. Alexandrov, N. Gaddam, J. Manschot [arXiv:2204.02207]
- *"Quantum geometry, stability and modularity"*, with S. Alexandrov, S. Feyzbakhsh, A. Klemm, T. Schimannek [arXiv:2301.08066]+ work in progress

Introduction

- A driving force in high energy theoretical physics has been the quest for a microscopic explanation of the entropy of black holes. Providing a derivation of the Bekenstein-Hawking formula is a benchmark test of any theory of quantum gravity.

$$S_{BH} = \frac{A}{4G_N}$$

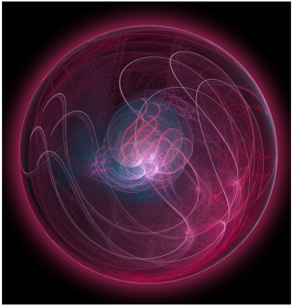


$$S_{BH} \stackrel{?}{=} \log \Omega$$

Sgr A, Event Horizon Telescope 2022*

Black hole microstates as wrapped D-branes

- Back in 1996, Strominger and Vafa argued that String Theory passes this test with flying colors, at least in the context of BPS black holes in vacua with extended SUSY: black hole micro-states can be understood as bound states of D-branes wrapped on the internal manifold, and sometimes can be counted efficiently.



Calabi-Yau black hole, courtesy F. Le Guen

BPS indices and Donaldson-Thomas invariants

- In the context of type IIA strings compactified on a Calabi-Yau three-fold $\mathfrak{X}$, BPS states are described mathematically by stable objects in the derived category of coherent sheaves $\mathcal{C} = D^b\mathrm{Coh}\mathfrak{X}$. The Chern character $\gamma = (\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ is identified as the electromagnetic charge, or D6-D4-D2-D0-brane charge.
- The problem becomes a question in enumerative geometry: for fixed $\gamma \in K(\mathfrak{X})$, compute the Donaldson-Thomas invariant $\Omega_z(\gamma)$ counting (semi)stable objects of class γ for a Bridgeland stability condition $z \in \mathrm{Stab}\,\mathcal{C}$, and determine its growth as $|\gamma| \to \infty$.
- Physical arguments predict that suitable generating series of rank 0 DT invariants (counting D4-D2-D0 bound states) should have specific modular properties. This gives very good control on their asymptotic growth, and allows to check whether $\Omega_z(\gamma) \simeq e^{S_{BH}(\gamma)}$.

Simplest example: Abelian three-fold

- For $\mathfrak{X} = T^6$, $\Omega_z(\gamma)$ depends only on a certain quartic polynomial $I_4(\gamma)$ in the charges, and is moduli independent. It is given by the Fourier coefficient $c(I_4(\gamma)+1)$ of a weak modular form,

$$\frac{\theta_3(2\tau)}{\eta^6(4\tau)} = \sum_{n\geq 0} c(n)\,\mathrm{q}^{n-1} = \frac{1}{\mathrm{q}} + 2 + 8\mathrm{q}^3 + 12\mathrm{q}^4 + 39\mathrm{q}^7 + 56\mathrm{q}^8 + \dots$$

Moore Maldacena Strominger 1999, BP 2005, Shih Strominger Yin 2005

Bryan Oberdieck Pandharipande Yin'15

- Recall that $f(\tau) := \sum_{n\geq 0} c(n)\mathrm{q}^{n-\Delta}$ (with $\mathrm{q} = e^{2\pi\mathrm{i}\tau}$, $\mathrm{Im}\tau > 0$) is a modular form of weight w if $\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset SL(2,\mathbb{Z})$,

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w f(\tau) \quad \Rightarrow \quad c(n) \overset{n\to\infty}{\sim} \exp\left(4\pi\sqrt{\Delta(n-\Delta)}\right)$$

in agreement with $S_{BH}(\gamma) = \frac{1}{4}A(\gamma)$.

Wall-crossing and mock modularity

- For a general CY3, the story is more involved and interesting. First, $\Omega_z(\gamma)$ depends on the Kähler parameters z (more generally, on the stability condition), with a complicated chamber structure.
- Second, the generating series of rank 0 DT invariants in the large volume attractor chamber, denoted by $\Omega_\star(\gamma)$, are generally not modular but rather mock modular of higher depth.
- A (depth one) mock modular form of weight w transforms inhomogeneously under $\Gamma \subset SL(2,\mathbb{Z})$ (or $Mp(2,\mathbb{Z})$ if $w \in \mathbb{Z}+\frac{1}{2}$)

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^w \left[f(\tau) - \int_{-d/c}^{\mathrm{i}\infty} \overline{g(-\bar{\rho})}(\tau+\rho)^{-w} \mathrm{d}\rho \right]$$

where $g(\tau)$ is an ordinary modular form of weight $2-w$, known as the shadow.

Wall-crossing and mock modularity

- Equivalently, the non-holomorphic completion

$$\widehat{f}(\tau,\bar\tau) := f(\tau) + \int_{-\bar\tau}^{i\infty} \overline{g(-\bar\rho)}(\tau+\rho)^{-w}\mathrm{d}\rho$$

transforms like a modular form of weight w, and satisfies the holomorphic anomaly equation

$$\tau_2^w \partial_{\bar\tau} \widehat{f}(\tau,\bar\tau) \propto \overline{g(\tau)}$$

- Ramanujan's mock θ-functions belong to this class, along with indefinite theta series of Lorentzian signature $(1, n-1)$ *[Zwegers'02]*
- The Fourier coefficients still grow as $c(n) \sim \exp\left(4\pi\sqrt{\Delta(n-\Delta)}\right)$ but subleading corrections are markedly different.

Outline

1. Review some mathematical background on Bridgeland stability conditions on $\mathcal{C} = D^b\mathrm{Coh}\mathfrak{X}$
2. Spell out the modularity properties of rank 0 DT invariants on a general compact CY threefold
3. Test modularity for compact CY threefolds with $b_2(\mathfrak{X}) = 1$, using recent results of S. Feyzbakhsh and R. Thomas
4. Obtain new constraints on higher genus GW/GV invariants from modularity of rank 0 DT invariants

Mathematical preliminaries

- Let $\mathfrak{X}$ a compact CY threefold, and $\mathcal{C} = D^b\mathrm{Coh}\mathfrak{X}$ the bounded derived category of coherent sheaves. Objects $E \in \mathcal{C}$ are bounded complexes of coherent sheaves $\mathcal{E}^k$ on $\mathfrak{X}$,

$$E = (\cdots \xrightarrow{d^{-2}} \mathcal{E}^{-1} \xrightarrow{d^{-1}} \mathcal{E}^0 \xrightarrow{d^0} \mathcal{E}^1 \xrightarrow{d^1} \dots),$$

  with morphisms $d^k : \mathcal{E}^k \to \mathcal{E}^{k+1}$ such that $d^{k+1}d^k = 0$. Physically, $\mathcal{E}^k$ describe D6-branes for k even, or anti D6-branes for k odd, and d^k are open strings .
- $\mathcal{C}$ is graded by the Grothendieck group $K(\mathcal{C})$. Let $\Gamma \subset H^{\mathrm{even}}(\mathfrak{X}, \mathbb{Q})$ be the image of $K(\mathcal{C})$ under $E \mapsto \mathrm{ch}\, E = \sum_k (-1)^k \,\mathrm{ch}\, \mathcal{E}_k$. The lattice of electromagnetic charges Γ is equipped with the skew-symmetric (Dirac-Schwinger-Zwanziger) pairing

$$\langle E, E' \rangle = \chi(E', E) = \int_{\mathfrak{X}} (\mathrm{ch}\, E')^\vee \,\mathrm{ch}(E)\, \mathrm{Td}(T\mathfrak{X}) \in \mathbb{Z}$$

Bridgeland stability conditions

- Let $\mathcal{S} = \mathrm{Stab}(\mathcal{C})$ be the space of Bridgeland stability conditions $\sigma = (Z, \mathcal{A})$, where
  1. $Z : \Gamma \to \mathbb{C}$ is a linear map, known as the central charge. Let $Z(E) := Z(\mathrm{ch}(E))$.
  2. $\mathcal{A} \subset \mathcal{C}$ is an Abelian subcategory (heart of bounded t-structure).
  3. For any non-zero $E \in \mathcal{A}$, $(i)\ \mathrm{Im} Z(E) \geq 0$ and $(ii)\ \mathrm{Im} Z(E) = 0 \Rightarrow \mathrm{Re} Z(E) < 0$. Relax (ii) for weak stability conditions.
  4. Harder-Narasimhan filtration + support property
- If $\mathcal{S}$ is not empty, then it is a complex manifold of dimension $\mathrm{rk}\,\Gamma = b_{\mathrm{even}}(\mathfrak{X})$, locally parametrized by $Z(\gamma_i)$ with γ_i a basis of Γ.
- Stability conditions are known to exist only for a handful of CY threefolds, including the quintic in $\mathbb{P}^4$ *[Li'18]*. Their construction depends on the conjectural Bayer-Macrì-Toda (BMT) inequality. Weak stability conditions are much easier to construct.

Physical stability conditions

- Physics/Mirror symmetry conjecturally selects a subspace $\Pi \subset \mathrm{Stab}\,\mathcal{C}$, known as 'physical slice' or slice of Π-stability conditions, parametrized by complexified Kähler structure of $\mathfrak{X}$, or complex structure of $\widehat{\mathfrak{X}}$. Hence $\dim_{\mathbb{C}} \Pi = b_2(\mathfrak{X}) + 1 = b_3(\widehat{\mathfrak{X}})$.
- Along this slice, the central charge is given by the period

$$Z(\gamma) = \int_{\hat{\gamma}} \Omega_{3,0}$$

of the holomorphic 3-form on $\widehat{\mathfrak{X}}$ on a dual 3-cycle $\hat{\gamma} \in H_3(\hat{\mathfrak{X}}, \mathbb{Z})$.
- Near the large volume point in $\mathcal{M}_K(\mathfrak{X})$, or MUM point in $\mathcal{M}_{cx}(\widehat{\mathfrak{X}})$,

$$Z(E) \sim -\int_{\mathfrak{X}} e^{-z^a H_a} \sqrt{Td(T\mathfrak{X})}\, \mathrm{ch}(E)$$

where H_a is a basis of $H^2(\mathfrak{X}, \mathbb{Z})$, and $z^a = b^a + \mathrm{i} t^a$ are the complexified Kähler moduli.

Generalized Donaldson-Thomas invariants

- Given a (weak) stability condition $\sigma = (Z, \mathcal{A})$, an object $F \in \mathcal{A}$ is called σ-semi-stable if $\arg Z(F') \leq \arg Z(F)$ for every non-zero subobject $F' \subset F$ (where $0 < \arg Z \leq \pi$).
- Let $\mathcal{M}_\sigma(\gamma)$ be the moduli stack of σ-semi-stable objects of class γ in $\mathcal{A}$. Following *[Joyce-Song'08]* one can associate the DT invariant $\bar{\Omega}_\sigma(\gamma) \in \mathbb{Q}$. When γ is primitive and $\mathcal{M}_\sigma(\gamma)$ is a smooth projective variety, then $\bar{\Omega}_\sigma(\gamma) = (-1)^{\dim_{\mathbb{C}} \mathcal{M}_\sigma(\gamma)} \chi(\mathcal{M}_\sigma(\gamma))$.
- Conjecturally, the generalized DT invariant defined by

$$\Omega_\sigma(\gamma) = \sum_{m|\gamma} \frac{\mu(m)}{m^2} \bar{\Omega}_\sigma(\gamma/m)$$

is integer for any γ, and coincides with the physical BPS index along the slice $\Pi \subset \mathrm{Stab}\,\mathcal{C}$.

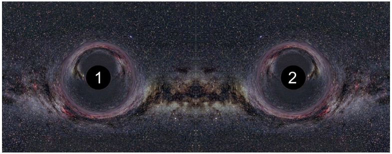
Wall-crossing

- The invariants $\bar\Omega_\sigma(\gamma)$ are locally constant on $\mathcal{S}$, but jump across walls of instability (or marginal stability), where the central charge $Z(\gamma)$ aligns with $Z(\gamma')$ where $\gamma' = \mathrm{ch}\, E'$ for a subobject $E' \subset E$. The jump is governed by a universal wall-crossing formula.

Joyce Song'08; Kontsevich Soibelman'08

- Physically, the jump corresponds to the (dis)appearance of multi-centered black hole bound states. In the simplest case,

$$\Delta\bar\Omega(\gamma_1+\gamma_2) = (-1)^{\langle\gamma_1,\gamma_2\rangle+1} |\langle\gamma_1,\gamma_2\rangle|\, \bar\Omega(\gamma_1)\, \bar\Omega(\gamma_2)$$



S-duality constraints on DT invariants

- Constraints on DT invariants can be derived by studying instanton corrections to the moduli space in IIA/$\mathfrak{X} \times S^1(R)$=M/$\mathfrak{X} \times T^2(\tau)$.
- The moduli space $\mathcal{M}_3$ factorizes into $\mathcal{M}_H \times \widetilde{\mathcal{M}}_V$ where
  1. $\mathcal{M}_H$ parametrizes the complex structure of $\mathfrak{X}$+ dilaton ϕ + Ramond gauge fields in $H^{\rm odd}(\mathfrak{X})$
  2. $\widetilde{\mathcal{M}}_V$ parametrizes the Kähler structure of $\mathfrak{X}$ + radius R + Ramond gauge fields in $H^{\rm even}(\mathfrak{X})$
- Both factors carry a quaternion-Kähler metric. $\mathcal{M}_H$ is largely irrelevant for this talk, but note that $\mathcal{M}_H$ and $\widetilde{\mathcal{M}}_V$ get exchanged under mirror symmetry.

S-duality constraints on DT invariants

- Near $R \to \infty$, $\widetilde{\mathcal{M}}_V$ is a torus bundle over $\mathbb{R}^+ \times \mathcal{M}_K$ with semi-flat QK metric, but the QK metric receives $\mathcal{O}(e^{-R|Z(\gamma)|})$ corrections from Euclidean black holes winding around S^1.
- These corrections are determined from the DT invariants $\Omega_z(\gamma)$ by a twistorial construction à la Gaiotto-Moore-Neitzke *[Alexandrov BP Saueressig Vandoren'08]*
- Since type IIA/$S^1(R)$ is the same as M-theory on $T^2(\tau)$, $\widetilde{\mathcal{M}}_V$ must have an isometric action of $SL(2,\mathbb{Z})$. This strongly constrains the DT invariants in the large volume limit *[Alexandrov, Banerjee, Manschot, BP, Robles-Llana, Persson, Rocek, Saueressig, Theis, Vandoren '06-19]*

S-duality constraints on BPS indices

Requiring that $\widetilde{\mathcal{M}}_V$ admits an isometric action of $SL(2,\mathbb{Z})$ near large volume, one can show that DT invariants $\Omega_z(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2, \mathrm{ch}_3)$ satisfy

- For skyscraper sheaves (or D0-branes), $\Omega_z(0,0,0,n) = -\chi_{\mathfrak{X}}$
- For classes supported on a curve of class $q_a\gamma^a \in \Lambda^* = H_2(\mathfrak{X},\mathbb{Z})$, $\Omega_z(0,0,q_a,n) = \mathrm{GV}^{(0)}_{q_a}$ is given by the genus-zero GV invariant
- For classes supported on an irreducible divisor $\mathcal{D}$ of class $p^a\gamma_a \in \Lambda = H_4(\mathfrak{X},\mathbb{Z})$, the generating series of rank 0 DT invariants

$$h_{p^a,q_a}(\tau) := \sum_n \bar{\Omega}_\star(0,p^a,q_a,n)\,\mathrm{q}^{n-\Delta_{p,q}}$$

  should be a vector-valued, weakly holomorphic modular form of weight $w = -\frac{1}{2}b_2(\mathfrak{X}) - 1$ and prescribed multiplier system.

S-duality constraints on D4-D2-D0 indices

$$h_{p^a,q_a}(\tau) = \sum_n \bar{\Omega}_\star(0, p^a, q_a, n)\, \mathrm{q}^{n+\frac{1}{2}q_a\kappa^{ab}q_b+\frac{1}{2}p^a q_a-\frac{\chi(\mathcal{D})}{24}}$$

- Here, $\bar{\Omega}_\star(0, p^a, q_a, n)$ is the index in the large volume attractor chamber
$$\bar{\Omega}_\star(\gamma) = \lim_{\lambda\to+\infty} \bar{\Omega}_{-\kappa^{ab}q_b+\mathrm{i}\lambda p^a}(\gamma)$$
where κ^{ab} is the inverse of the quadratic form $\kappa_{ab} = \kappa_{abc}p^c$ with Lorentzian signature $(1, b_2(\mathfrak{X}) - 1)$.
- The classical Bogomolov-Gieseker inequality guarantees that n is bounded from below. The BH entropy predicts that $\bar{\Omega}_\star(0, p^a, q_a, n) \sim e^{2\pi\sqrt{\frac{n}{6}\kappa_{abc}p^a p^b p^c}}$ for $n \gg 1$ so the sum should converge for $|\mathrm{q}| < 1$ or $\mathrm{Im}\tau > 0$.

S-duality constraints on D4-D2-D0 indices

- By construction, $\Omega_\star(0, p^a, q_a, n)$ is invariant under tensoring with a line bundle $\mathcal{O}(m^a H_a)$ (aka spectral flow)

$$q_a \to q_a - \kappa_{ab} m^b \ , \quad n \mapsto n - m^a q_a + \tfrac{1}{2}\kappa_{ab} m^a m^b$$

  Thus, the D2-brane charge q_a can be restricted to the finite set Λ^*/Λ, of cardinal $|\det(\kappa_{ab})|$.

- h_{p^a,q_a} transforms under the Weil representation of $\mathrm{Mp}(2,\mathbb{Z})$ associated to the lattice Λ, e.g.

$$h_{p^a,q_a}(-1/\tau) = \sum_{q'_a \in \Lambda^*/\Lambda} \frac{e^{-2\pi\mathrm{i}\kappa^{ab} q_a q'_b + \frac{\mathrm{i}\pi}{4}(b_2(\mathfrak{X}) + 2\chi(\mathcal{O}_\mathcal{D}) - 2)}}{\tau^{1+\frac{1}{2}b_2(\mathfrak{X})}\sqrt{|\det(\kappa_{ab})|}} h_{p^a,q'_a}(\tau)$$

D4-D2-D0 indices from elliptic genus

- Summing over all D2-brane charges and using spectral flow invariance, one gets

$$
\begin{aligned}
Z_p(\tau, v) &:= \sum_{q \in \Lambda, n} \bar{\Omega}_\star(0, p^a, q_a, n)\, \mathrm{q}^{n+\frac{1}{2} q_a \kappa^{ab} q_b} e^{2\pi \mathrm{i} q_a v^a} \\
&= \sum_{q \in \Lambda^*/\Lambda} h_{p,q}(\tau) \Theta_q(\tau, v)
\end{aligned}
$$

  where $\Theta_q(\tau, v)$ is the (non-holomorphic) Siegel theta series for the indefinite lattice (Λ, κ_{ab}). S-duality then requires that Z_p should transform as a (skew-holomorphic) Jacobi form.
- The Jacobi form Z_p can be interpreted as the elliptic genus of the $(0, 4)$ superconformal field theory obtained by wrapping an M5-brane on the divisor $\mathcal{D}$ *[Maldacena Strominger Witten '98]*.

Mock modularity constraints on D4-D2-D0 indices

- For γ supported on a reducible divisor $\mathcal{D} = \sum_{i=1}^{n\geq 2} \mathcal{D}_i$, the generating series h_p (omitting q index for brevity) is no longer expected to be modular. Rather, it should be a vector-valued mock modular form of depth $n-1$ and same weight/multiplier system.

Alexandrov Banerjee Manschot BP '16-19

- There exists explicit non-holomorphic theta series such that

$$\widehat{h}_p(\tau,\bar\tau) = h_p(\tau) + \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \Theta_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} h_{p_i}(\tau)$$

transforms as a modular form of weight $-\frac{1}{2}b_2(\mathfrak{X})-1$. Moreover the completion satisfies an explicit holomorphic anomaly equation,

$$\partial_{\bar\tau}\widehat{h}_p(\tau,\bar\tau) = \sum_{p=\sum_{i=1}^{n\geq 2} p_i} \widehat{\Theta}_n(\{p_i\},\tau,\bar\tau) \prod_{i=1}^{n} \widehat{h}_{p_i}(\tau,\bar\tau)$$

Crash course on indefinite theta series

- Θ_n and $\widehat{\Theta}_n$ belongs to the class of indefinite theta series

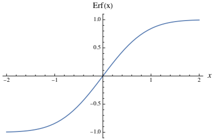
$$\vartheta_{\Phi,q}(\tau,\bar\tau) = \tau_2^{-\lambda} \sum_{k\in\Lambda+q} \Phi\left(\sqrt{2\tau_2}k\right)\, e^{-\mathrm{i}\pi\tau Q(k)}$$

  where (Λ, Q) is an even lattice of signature $(r, d-r)$, $q \in \Lambda^*/\Lambda$, $\lambda \in \mathbb{R}$. The series converges if $f(x) \equiv \Phi(x) e^{\frac{\pi}{2}Q(x)} \in L_1(\Lambda\otimes\mathbb{R})$.
- Theorem (Vignéras, 1978): $\{\vartheta_{\Phi,q}, q\in\Lambda^*/\Lambda\}$ transforms as a vector-valued modular form of weight $(\lambda+\frac{d}{2}, 0)$ provided
  - $R(x)f, R(\partial_x)f \in L_2(\Lambda\otimes\mathbb{R})$ for any polynomial $R(x)$ of degree ≤ 2
  - $\left[\partial_x^2 + 2\pi(x\partial_x - \lambda)\right]\Phi = 0$ [*]
- The relevant lattice for Θ_n and $\widehat{\Theta}_n$ is $\Lambda = H^2(\mathfrak{X},\mathbb{Z})^{\oplus(n-1)}$, with signature $(r, d-r) = (n-1)(1, b_2(\mathfrak{X})-1)$.

Indefinite theta series

- Example 1 (Siegel): $\Phi = e^{\pi Q(x_+)}$, where x_+ is the projection of x on a fixed plane of dimension r, satisfies [*] with $\lambda = -n$. ϑ_Φ is then the usual (non-holomorphic) Siegel-Narain theta series.
- Example 2 (Zwegers): In signature $(1, d-1)$, choose C, C' two vectors such that $Q(C), Q(C'), (C, C') > 0$, then

$$\widehat{\Phi}(x) = \mathrm{Erf}\left(\frac{(C,x)\sqrt{\pi}}{\sqrt{Q(C)}}\right) - \mathrm{Erf}\left(\frac{(C',x)\sqrt{\pi}}{\sqrt{Q(C')}}\right)$$



satisfies [*] with $\lambda = 0$. As $|x| \to \infty$, or if $Q(C) = Q(C') = 0$,

$$\widehat{\Phi}(x) \to \Phi(x) := \mathrm{sgn}(C, x) - \mathrm{sgn}(C', x)$$

- The theta series $\Theta_2(\{p_1, p_2\})$, $\widehat{\Theta}_2(\{p_1, p_2\})$ fall in this class. The generalization to $n \geq 3$ involves generalized error functions $\mathcal{E}_{n-1}(\{C_i\}, x) := e^{\pi Q(x_+)} \star \prod_{i=1}^{n-1} \mathrm{sgn}(C_i, x)$ where $\star$ is the convolution product. *[Alexandrov Banerjee Manschot BP 2016; Nazaroglu 2016]*

Modularity for one-modulus compact CY

- Let $\mathfrak{X}$ be a compact CY3 with $H^2(\mathfrak{X}, \mathbb{Z}) = \mathbb{Z}H$. Can we compute rank 0 DT invariants $\bar{\Omega}_\star(0, N, q, n)$ and test (mock) modularity ?
- We focus on smooth complete intersections in weighted projective space (CICY), $\mathfrak{X} = X_{\{d_i\}}(\{w_j\})$ with $\sum d_i = \sum w_j$. There are 13 such models, with Kähler moduli space $\mathcal{M}_K = \mathbb{P}^1 \backslash \{0, 1, \infty\}$, with a large volume point at $z = 0$ and a conifold singularity at $z = 1$.
- The central charge $Z_z(\gamma)$ is expressed in terms of hypergeometric functions. GV invariants $\mathrm{GV}_Q^{(g)}$ are known up to high genus *[Huang Klemm Quackenbush'06]*.
- I will concentrate on $N = 1$, and discuss $N = 2$ if time permits.

Gaiotto Strominger Yin '06-07, Collinucci Wyder '08, ...

Alexandrov Gaddam Manschot BP'22, Alexandrob Feyzbakhsh Klemm BP Schimannek'23

Modularity for one-modulus compact CY

$\mathfrak{X}$	$\chi_{\mathfrak{X}}$	κ	$c_2(T\mathfrak{X})$	$\chi(\mathcal{O}_{\mathcal{D}})$	n_1	C_1
$X_5(1^5)$	-200	5	50	5	7	0
$X_6(1^4,2)$	-204	3	42	4	4	0
$X_8(1^4,4)$	-296	2	44	4	4	0
$X_{10}(1^3,2,5)$	-288	1	34	3	2	0
$X_{4,3}(1^5,2)$	-156	6	48	5	9	0
$X_{4,4}(1^4,2^2)$	-144	4	40	4	6	1
$X_{6,2}(1^5,3)$	-256	4	52	5	7	0
$X_{6,4}(1^3,2^2,3)$	-156	2	32	3	3	0
$X_{6,6}(1^2,2^2,3^2)$	-120	1	22	2	1	0
$X_{3,3}(1^6)$	-144	9	54	6	14	1
$X_{4,2}(1^6)$	-176	8	56	6	15	1
$X_{3,2,2}(1^7)$	-144	12	60	7	21	1
$X_{2,2,2,2}(1^8)$	-128	16	64	8	33	3

Abelian D4-D2-D0 invariants

- For $N = 1$, the generating series

$$h_{1,q} = \sum_{n\in\mathbb{Z}} \Omega_\star(0,1,q,n)\, \mathrm{q}^{n+\frac{q^2}{2\kappa}+\frac{q}{2}-\frac{\chi(\mathcal{D})}{24}}\ , \quad q \in \mathbb{Z}/\kappa\mathbb{Z}$$

should transform as a vector-valued modular form of weight $-\frac{3}{2}$ in the Weil representation of $\mathbb{Z}[\kappa]$ with $\kappa = H^3$.

- An overcomplete basis is given for κ even by

$$\frac{E_4^a E_6^b}{\eta^{4\kappa+c_2}} D^\ell(\vartheta_q^{(\kappa)}) \quad \text{with} \quad \vartheta_q^{(\kappa)} = \sum_{k\in\mathbb{Z}+\frac{q}{\kappa}} \mathrm{q}^{\frac{1}{2}\kappa k^2}$$

where $D = \mathrm{q}\partial_{\mathrm{q}} - \frac{w}{12}E_2$, is the Serre derivative and $4a + 6b + 2\ell - 2\kappa - \frac{c_2}{2} + \frac{1}{2} = -\frac{3}{2}$.

- For κ odd, the same works with $\vartheta_q^{(\kappa)} = \sum_{k\in\mathbb{Z}+\frac{q}{\kappa}+\frac{1}{2}} (-1)^{\kappa k}\, k\mathrm{q}^{\frac{1}{2}\kappa k^2}$.

A naive Ansatz for the polar terms

- $h_{1,q}$ is uniquely determined by the polar terms $n < \frac{\chi(\mathcal{D})}{24} - \frac{q^2}{2\kappa} - \frac{q}{2}$, but the dimension $d_1 = n_1 - C_1$ of the space of modular forms may be smaller than the number n_1 of polar terms !
- Physically, we expect that polar coefficients arise as bound states of D6-brane and anti D6-branes *[Denef Moore'07]*
- Earlier studies *[Gaiotto Strominger Yin'06, Collinucci Wyder'08]* suggest that only bound states of the form $(D6 + qD2 + nD0, \overline{D6(-1)})$ contribute to polar coeffs:

$$\Omega(0,1,q,n) = (-1)^{\chi(\mathcal{O}_\mathcal{D})-q-n+1}\left(\chi(\mathcal{O}_\mathcal{D}) - q - n\right) DT(q,n)$$

where $DT(q,n)$ counts ideal sheaves with $\mathrm{ch}_2 = q$ and $\mathrm{ch}_3 = n$ *[Alexandrov Gaddam Manschot BP'22]*

GV/DT/PT relation

- For a single D6-brane, the DT-invariant $DT(q,n)=\Omega(1,0,q,n)$ at large volume can be computed via the GV/DT relation

$$\sum_{Q,n} DT(Q,n)\, \mathrm{q}^n v^Q = M(-\mathrm{q})^{\chi_{\mathfrak{X}}} \prod_{Q,g,\ell} \left(1-(-\mathrm{q})^{g-\ell-1} v^Q\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell}\mathrm{GV}_Q^{(g)}}$$

where $M(\mathrm{q})=\prod_{n\geq 1}(1-\mathrm{q}^n)^{-n}$ is the Mac-Mahon function.

Maulik Nekrasov Okounkov Pandharipande'06

- The topological string partition function is given by

$$\Psi_{\mathrm{top}}(z,\lambda)=M(-\mathrm{q})^{-\chi_{\mathfrak{X}}/2} Z_{DT}\ , \quad \mathrm{q}=e^{\mathrm{i}\lambda}, v=e^{2\pi \mathrm{i} z/\lambda}$$

can be computed by the direct integration method, assuming conifold gap conditions and Castelnuovo-type bounds $g\leq g_{\max}(Q)$ *[BCOV 93, Huang Klemm Quackenbush'06]*.

Modular predictions for D4-D2-D0 indices (naive)

- Remarkably, there exists a vv modular form with integer Fourier coefficients matching these polar terms for almost all CICY (except $X_{4,2}, X_{3,2,2}, X_{2,2,2,2}$), reproducing earlier results *[Gaiotto Strominger Yin]* for X_5, X_6, X_8, X_{10} and $X_{3,3}$.
- $X_5 = \mathbb{P}^4[5]$:

$$h_{1,0} = \mathrm{q}^{-\frac{55}{24}} \left(\underline{5 - 800\mathrm{q} + 58500\mathrm{q}^2} + 5817125\mathrm{q}^3 + \dots \right)$$

$$h_{1,\pm 1} = \mathrm{q}^{-\frac{55}{24}+\frac{3}{5}} \left(\underline{0 + 8625\mathrm{q}} - 1138500\mathrm{q}^2 + 3777474000\mathrm{q}^3 + \dots \right)$$

$$h_{1,\pm 2} = \mathrm{q}^{-\frac{55}{24}+\frac{2}{5}} \left(\underline{0 + 0\mathrm{q}} - 1218500\mathrm{q}^2 + 441969250\mathrm{q}^3 + \dots \right)$$

- $X_{10} = \mathbb{P}^4_{5,2,1,1,1}[10]$:

$$h_{1,0} \stackrel{?}{=} \frac{541 E_4^4 + 1187 E_4 E_6^2}{576\, \eta^{35}} = \mathrm{q}^{-\frac{35}{24}} \left(\underline{3 - 576\mathrm{q}} + 271704\mathrm{q}^2 + \cdots \right)$$

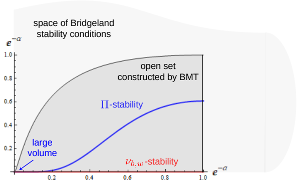
- Our Ansatz for polar terms was an educated guess. Fortunately, recent progress in Donaldson-Thomas theory allows to compute D4-D2-D0 indices rigorously, and compare with modular predictions.

Bayer Macri Toda'11; Toda'11; Feyzbakhsh Thomas'20-22

- The key idea is to consider a family of weak stability conditions on the boundary of Stab $\mathcal{C}$, called tilt stability, with central charge

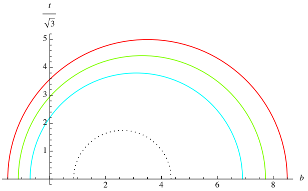
$$Z_{b,t} = \frac{\mathrm{i}}{6} t^3 \,\mathrm{ch}_0 - \frac{1}{2} t^2 \,\mathrm{ch}_1^b - \mathrm{i} t \,\mathrm{ch}_2^b + 0 \,\mathrm{ch}_3^b$$

$$\mathrm{ch}_k^b = \int_{\mathfrak{X}} H^{3-k} e^{-bH} \,\mathrm{ch}(E)$$



Rank 0 DT invariants from GV invariants

- Tilt stability agrees with physical stability at large volume, but the chamber structure is much simpler: walls are nested half-circles in the Poincaré upper half-plane spanned by $z = b + \mathrm{i}\frac{t}{\sqrt{3}}$.



- Importantly, there is a conjectural inequality on Chern classes $C_k := \mathrm{ch}_k^0$ required for existence of tilt-semistable objects,

$$(C_1^2 - 2C_0C_2)(\tfrac{1}{2}b^2 + \tfrac{1}{6}t^2) + (3C_0C_3 - C_1C_2)b + (2C_2^2 - 3C_1C_3) \geq 0$$

Bayer Macri Toda'11; Bayer Macri Stellari'16

Rank 0 DT invariants from GV invariants

- The BMT inequality is known to hold for $X_5, X_6, X_8, X_{4,2}$ *[Li'19,Koseki'20]*, and plays a key role in the construction of Bridgeland stability conditions.
- The BMT inequality provides an empty chamber whenever the discriminant at $t = 0$ is positive. This happens exactly when single centered black hole solutions are ruled out !

$$8C_0C_2^3 + 6C_1^3C_3 + 9C_0^2C_3^2 - 3C_1^2C_2^2 - 18C_0C_1C_2C_3 \geq 0$$
$$\Updownarrow$$
$$\frac{8}{9\kappa}p_0q_1^3 - \frac{2}{3}\kappa q_0(p^1)^3 - (p^0q_0)^2 + \frac{1}{3}(p^1q_1)^2 - 2p^0p^1q_0q_1 \leq 0$$

- By studying wall-crossing between the empty chamber provided by BMT bound and large volume, *[Feyzbakhsh Thomas]* show that D4-D2-D0 indices can be computed from rank 1 DT, which are in turn related to GV invariants.

Rank 0 DT invariants from GV invariants

- More precisely, for a D4-D2-D0 charge $(0, r, q, n)$ close enough to the (usual) Bogomolov-Gieseker bound, *[Toda'13, Feyzbakhsh'22]*

$$\bar{\Omega}_{r,q}(n) = \sum_{r_i,Q_i,n_i} (-1)^{\langle\gamma_1,\gamma_2\rangle}\,\mathrm{DT}(Q_1,n_1)\,\mathrm{PT}(Q_2,n_2)$$

  where $\mathrm{DT}(Q_1,n_1), \mathrm{PT}(Q_2,n_2)$ counts BPS states with $\gamma_1 = (1,0,-Q_1,-n_1), \gamma_2 = (-1,0,Q_2,-n_2)$, respectively
- Similar to DT invariants, the PT/GV correspondence gives

$$\sum_{Q,n} PT(Q,n)\,\mathrm{q}^n v^Q = \prod_{Q,g,\ell}\left(1-(-\mathrm{q})^{g-\ell-1}v^Q\right)^{(-1)^{g+\ell}\binom{2g-2}{\ell}\mathrm{GV}_Q^{(g)}}$$

- The contribution from $Q_2 = n_2 = 0$ reproduces our naive Ansatz ☺. Unfortunately the formula only holds for the most polar term ☹.

Modular predictions for D4-D2-D0 (rigorous)

- Alternatively, one can study wall crossing for $\gamma = (-1, 0, q, n)$. For (q, n) large enough, there is an empty chamber and a single wall corresponding to $\overline{D6} \to \overline{D6} + D4$ contributes to $PT(q, n)$:

$$PT(q,n) = (-1)^{\langle \overline{D6(1)}, \gamma_{D4}\rangle + 1} \langle \overline{D6(1)}, \gamma_{D4}\rangle \, \bar{\Omega}(\gamma_{D4})$$

  with $\overline{D6(1)} := \mathcal{O}_{\mathfrak{X}}(H)[1]$ and $\gamma_{D4} = (0, 1, q, n)$ *[Feyzbakhsh'22]*.

- Conversely, using spectral flow invariance, one finds

$$\boxed{\Omega(\gamma) = \frac{(-1)^{\langle \overline{D6(1-m)}, \gamma\rangle + 1}}{\langle \overline{D6(1-m)}, \gamma\rangle} PT(q', n')} \qquad \begin{cases} q' = q + \kappa m \\ n' = n - mq - \frac{\kappa}{2} m(m+1) \end{cases}$$

  for sufficiently large $m \geq m_0(q, n)$.

- As a spin off, we obtain rigorous Castelnuovo-type bounds $g \leq \frac{Q^2}{2\kappa} + \frac{Q}{2} + 1$ on GV invariants ! (see also Liu Ruan'22)

Modular predictions for D4-D2-D0 (rigorous)

- Using an extension of this idea, we have computed most of the polar terms, and many non-polar ones, for all models except $X_{3,2,2}, X_{2,2,2,2}$. In all cases, modularity holds with flying colors !

Alexandrov, Feyzbakhsh, Klemm, BP, Schimannek'23

- E.g. for X_5:

$$h_{1,0} = \mathrm{q}^{-\frac{55}{24}} \Big(\underline{5 - 800\mathrm{q} + 58500\mathrm{q}^2} + 5817125\mathrm{q}^3 + 75474060100\mathrm{q}^4 + 28096675153255\mathrm{q}^5 + 3756542229485475\mathrm{q}^6 + 277591744202815875\mathrm{q}^7 + 13610985014709888750\mathrm{q}^8 + \dots \Big),$$

$$h_{1,\pm1} = \mathrm{q}^{-\frac{55}{24}+\frac{3}{5}} \Big(\underline{0 + 8625\mathrm{q}} - 1138500\mathrm{q}^2 + 3777474000\mathrm{q}^3 + 3102750380125\mathrm{q}^4 + 577727215123000\mathrm{q}^5 + \dots \Big)$$

$$h_{1,\pm2} = \mathrm{q}^{-\frac{55}{24}+\frac{2}{5}} \Big(\underline{0 + 0\mathrm{q}} - 1218500\mathrm{q}^2 + 441969250\mathrm{q}^3 + 953712511250\mathrm{q}^4 + 217571250023750\mathrm{q}^5 + 22258695264509625\mathrm{q}^6 + \dots \Big)$$

Modular predictions for D4-D2-D0 (rigorous)

- We find that our educated guess is correct for $X_5, X_6, X_8, X_{3,3}, X_{4,4}, X_{6,6}$ ☺ , but fails for $X_{10}, X_{6,2}, X_{6,4}, X_{4,3}$ ☹
- E.g. for X_{10},

$$h_{1,0} = \frac{203 E_4^4 + 445 E_4 E_6^2}{216\,\eta^{35}} = \mathrm{q}^{-\frac{35}{24}} \left(\underline{3 - 575\mathrm{q}} + 271955\mathrm{q}^2 + \cdots \right)$$

rather than $\underline{3 - 576\mathrm{q}} + \ldots$, as anticipated by *[van Herck Wyder'09]*.

- The nature of the bound state responsible for this discrepancy is still mysterious...

Mock modularity for non-Abelian D4-D2-D0 indices

- Let us consider D4-D2-D0 indices with $N = 2$ units of D4-brane charge. In that case, $\{h_{2,q}, q \in \mathbb{Z}/(2\kappa\mathbb{Z})\}$ should transform as a vv mock modular form with modular completion

$$\widehat{h}_{2,q}(\tau,\bar{\tau}) = h_{2,q}(\tau) + \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, \Theta^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

where

$$\Theta^{(\kappa)}_q = \frac{(-1)^q}{8\pi} \sum_{k\in 2\kappa\mathbb{Z}+q} |k|\, \beta\left(\frac{\tau_2 k^2}{\kappa}\right) e^{-\frac{\pi i\tau}{2\kappa} k^2},$$

and $\beta(x^2) = 2|x|^{-1} e^{-\pi x^2} - 2\pi \mathrm{Erfc}(\sqrt{\pi}|x|)$.

- The series $\Theta^{(\kappa)}_q$ is convergent but not modular invariant.

Mock modularity for non-Abelian D4-D2-D0 indices

- Suppose there exists a holomorphic function $g_q^{(\kappa)}$ such that $\Theta_q^{(\kappa)} + g_q^{(\kappa)}$ transforms as a vv modular form. Then

$$\widetilde{h}_{2,q}(\tau,\bar\tau) = h_{2,q}(\tau) - \sum_{q_1,q_2=0}^{\kappa-1} \delta^{(\kappa)}_{q_1+q_2-q}\, g^{(\kappa)}_{q_2-q_1+\kappa}\, h_{1,q_1}\, h_{1,q_2}$$

  will be an ordinary weak holomorphic vv modular form, hence uniquely determined by its polar part.
- For $\kappa = 1$, the series $\Theta_q^{(1)}$ is the one appearing in the modular completion of rank 2 Vafa-Witten invariants on $\mathbb{P}^2$! Thus we can choose $g_q^{(1)} = H_q(\tau)$, the generating series of Hurwitz class numbers *[Hirzebruch Zagier 1973]*

$$H_0(\tau) = -\tfrac{1}{12} + \tfrac{1}{2}\mathrm{q} + \mathrm{q}^2 + \tfrac{4}{3}\mathrm{q}^3 + \tfrac{3}{2}\mathrm{q}^4 + \dots$$
$$H_1(\tau) = \mathrm{q}^{\frac{3}{4}} \left(\tfrac{1}{3} + \mathrm{q} + \mathrm{q}^2 + 2\mathrm{q}^3 + \mathrm{q}^4 + \dots \right)$$

Mock modularity for non-Abelian D4-D2-D0 indices

- For X_{10}, we computed the 7 polar terms + 1 non-polar and found a unique mock modular form reproducing this data:

$$
\begin{aligned}
h_{2,\mu} =& \frac{5397523 E_4^{12} + 70149738 E_4^9 E_6^2 - 12112656 E_4^6 E_6^4 - 61127530 E_4^3 E_6^6 - 2307075 E_6^8}{46438023168\, \eta^{100}} \vartheta_\mu^{(1,2)} \\
& + \frac{-10826123 E_4^{10} E_6 - 14574207 E_4^7 E_6^3 + 20196255 E_4^4 E_6^5 + 5204075 E_4 E_6^7}{1934917632\, \eta^{100}} D\vartheta_\mu^{(1,2)} \\
& + (-1)^{\mu+1} H_{\mu+1}(\tau)\, h_1(\tau)^2
\end{aligned}
$$

leading to integer DT invariants

$$
\begin{aligned}
h_{2,0}^{(\mathrm{int})} =& \mathrm{q}^{-\frac{19}{6}} \left(\underline{7 - 1728\mathrm{q} + 203778\mathrm{q}^2 - 13717632\mathrm{q}^3} - 23922034036\mathrm{q}^4 + \dots \right) \\
h_{2,1}^{(\mathrm{int})} =& \mathrm{q}^{-\frac{35}{12}} \left(\underline{-6 + 1430\mathrm{q} - 1086092\mathrm{q}^2} + 208065204\mathrm{q}^3 + \dots \right)
\end{aligned}
$$

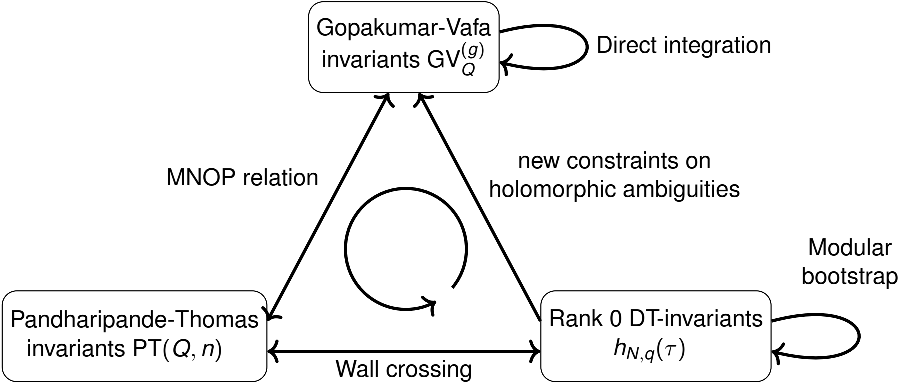
- The extension to other one-parameter models is in progress.

Mock modularity for non-Abelian D4-D2-D0 indices

$\mathfrak{X}$	$\chi_{\mathfrak{X}}$	κ	c_2	$\chi(\mathcal{O}_{2\mathcal{D}})$	n_2	C_2
$X_5(1^5)$	-200	5	50	15	36	1
$X_6(1^4,2)$	-204	3	42	11	19	1
$X_8(1^4,4)$	-296	2	44	10	14	1
$X_{10}(1^3,2,5)$	-288	1	34	7	7	0
$X_{4,3}(1^5,2)$	-156	6	48	16	42	0
$X_{4,4}(1^4,2^2)$	-144	4	40	12	25	1
$X_{6,2}(1^5,3)$	-256	4	52	14	30	1
$X_{6,4}(1^3,2^2,3)$	-156	2	32	8	11	1
$X_{6,6}(1^2,2^2,3^2)$	-120	1	5	2	5	0
$X_{3,3}(1^6)$	-144	9	54	21	78	3
$X_{4,2}(1^6)$	-176	8	56	20	69	3
$X_{3,2,2}(1^7)$	-144	12	60	26	117	0
$X_{2,2,2,2}(1^8)$	-128	16	64	32	185	4

Quantum geometry from stability and modularity

Conversely, we can use our knowledge of Abelian D4-D2-D0 invariants to compute GV invariants and push the direct integration method to higher genus !



Alexandrov Feyzbakhsh Klemm BP Schimannek'23

$\mathfrak{X}$	$\chi_{\mathfrak{X}}$	κ	type	$g_{\rm integ}$	$g_{\rm mod}$	$g_{\rm avail}$
$X_5(1^5)$	-200	5	F	53	69	64
$X_6(1^4,2)$	-204	3	F	48	63	48
$X_8(1^4,4)$	-296	2	F	60	80	60
$X_{10}(1^3,2,5)$	-288	1	F	50	65	65
$X_{4,3}(1^5,2)$	-156	6	F	20	24	24
$X_{6,4}(1^3,2^2,3)$	-156	2	F	14	17	17
$X_{6,6}(1^2,2^2,3^2)$	-120	1	K	18	21	21
$X_{4,4}(1^4,2^2)$	-144	4	K	26	34	34
$X_{3,3}(1^6)$	-144	9	K	29	33	33
$X_{4,2}(1^6)$	-176	8	C	50	64	50
$X_{6,2}(1^5,3)$	-256	4	C	63	78	42

Conclusion

- The existence of an isometric action of S-duality on the vector-multiplet moduli space in $D = 3$, leads to strong modularity constraints on rank 0 DT invariants in the large volume limit.
- For $p = \sum_{i=1}^{n} p_i$ the sum of n irreducible divisors, the generating function h_p is a mock modular form of depth $n - 1$ with an explicit shadow, thus it is uniquely determined by its polar coefficients.
- While modularity is clear physically, its mathematical origin is mysterious. For vertical D4 branes in K3-fibered CY3, it follows from Noether-Lefschetz theory *[Bouchard Creutzig Diaconescu Doran Quigley Sheshmani'16]*.
- Using modularity and GV/DT/PT relations, we can not only compute D4D2-D0 indices, but also push $\Psi_{\rm top}$ to higher genus !
- Mock modularity affects the growth of Fourier coefficients, hence the microscopic entropy of supersymmetric black holes. It should have an imprint on the macroscopic side as well...

Thanks for your attention !