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Towards full instanton trans-series in Hofstadter's butterfly

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Abstract: The trans-series completion of perturbative series of a wide class of quantum mechanical systems can be determined by combining the resurgence program with extra input coming from exact WKB analysis. In this paper, we reexamine the Harper-Hofstadter model and its spectrum, Hofstadter's butterfly in light of recent developments. We demonstrate the connection between the perturbative energy series of the Harper-Hofstadter model and the vev of 1/2-BPS Wilson loop of 5d SYM and clarify the differences between their non-perturbative corrections. Taking insights from the cosine potential model, we construct the full energy trans-series for flux $\phi = 2\pi/Q$ and provide numerical evidence with remarkably high precision. Finally, we revisit the problem of self-similarity of the butterfly and discuss the possibility of a completed version of the Rammal-Wilkinson formula.

Keywords: Nonperturbative Effects, Solitons Monopoles and Instantons, Supersymmetric Gauge Theory, Topological Strings

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1 Introduction

The energy spectrum of the simple system of electrons on a two-dimensional square lattice in a uniform magnetic field has a surprisingly rich structure. After the early studies of Harper [1], in 1976 D. Hofstadter pointed out [2] that this is a very peculiar system where the electron spectrum has different features when the value of the magnetic field is rational or irrational, and he derived a recursion equation which allowed him to plot the energy spectrum of the electron system against the magnetic field when the magnetic field is rational. The resulting beautiful plot is later known as the Hofstadter's butterfly due to its resemblance to a butterfly in shape, and it raises many puzzling questions. Thanks to the periodicity of the square lattice, the energy of the electron system displays a band structure that depends on the Bloch angle. The Hofstadter's butterfly indicates that the energy is a rather intricate function of both the magnetic field and the Bloch angle, but the exact nature of such a function is rather mysterious. In addition, the Hofstadter's butterfly has a fractal structure, which can be described by a strong-weak field duality map $(E, \phi) \to (\tilde{E}, 1/\phi)$, where ϕ is the magnetic flux through a lattice plaquette, but the expression of the mapped energy $\tilde{E}$ is yet unknown. Due to its simple-looking but rather intricate structure, the Hofstadter's butterfly has also attracted attention of many physicists and mathematicians [3–6], and interesting

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connections to quantum integrable systems [7, 8], quantum Hall effect [9, 10] and possibly high-temperature superconductivity [11] were discovered. With the proof of the Ten-Martini problem [12], there are still some unsolved mysteries for the Harper-Hofstadter model.

One way of studying the energy as a function of the magnetic field is to consider the weak field limit, where the energy is treated as a perturbative series in the magnetic field. Such a perturbative series, nevertheless, is oblivious to the Bloch angle and thus cannot explain the band structure. In fact, the rich band structure is known to be caused by non-perturbative effects. For instance, the bandwidth is explained by the instanton effects in the path integral formalism [13]. However, fully understanding the non-perturbative corrections including all-order instanton effects would still be a challenge.

In recent years there have been several new developments that made the solution of this problem a distinct possibility. The first development is the discovery of a surprising connection to an unexpected territory. In 2016, Hatsuda, Katsura and Tachikawa found [14] that the Harper-Hofstadter model is naturally related to the 5d $\mathcal{N} = 1$ $G = \mathrm{SU}(2)$ Super Yang-Mills theory on $S^1 \times \mathbb{R}^4$, or alternatively topological string theory on local $\mathbb{P}^1 \times \mathbb{P}^1$ as its string theory realization. In the IR, the 5d gauge theory is completely characterized by an algebraic curve called the Seiberg-Witten curve, and it was noticed that the curve equation is the same as the Harper Hamiltonian without the magnetic field. Turning on the magnetic field is equivalent to quantizing the Seiberg-Witten curve. This allows us to calculate many quantities efficiently in the Harper-Hofstadter model. For instance, the perturbative energy series of the Harper-Hofstadter model is mapped to the perturbative series of the Wilson loop, while the instanton corrections are controlled by the free energy of the field theory, both of which can be computed efficiently using the holomorphic anomaly equations [15–19]. As a result, the authors of [20] were able to find the complete one-instanton and the partial two-instanton corrections to the energy series of the Harper-Hofstadter model. This connection between electrons in 2d lattices and supersymmetric field theory or string theory was later extended to other models [21–23].

Another development is the powerful resurgence theory [24–27], which claims that a perturbative series and its non-perturbative corrections are intimately related, and that a subset of the non-perturbative corrections can be extracted from the perturbative series itself. Another result of [20] was to use the resurgence technique to confirm the (partial) two-instanton corrections to the energy series in the Harper-Hofstadter model. More importantly, in the 5d SYM, the non-perturbative corrections to both the Wilson loops and the free energies, at least the part accessible by the resurgence techniques have been solved in their entirety [28], which are conjectured to be controlled by the BPS spectrum of the 5d SYM. These results should be reinterpreted in the Harper-Hofstadter model.

Finally, the exact WKB method [29], which is the traditional WKB method enhanced by resurgence techniques, has been very useful in deriving exact quantization conditions for 1d non-relativistic quantum mechanical models. See [30, 31] for earlier analysis of the Harper-Hofstadter model with the WKB method. Recently, the exact WKB method has been revisited [32] so that in many 1d QM models the full energy trans-series including instanton corrections to all orders are written down, and they all share the same universal structure. It implies that even if we do not know the exact quantization conditions of a 1d QM model

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a priori, we can still try to construct the full energy trans-series by looking for a family of well-organized basic building blocks fitting the trans-series coefficients.

In this paper, we will combine the results of all these recent developments to construct the full energy trans-series for the Harper-Hofstadter model in some special cases. The Harper-Hofstadter model is equivalent to a 1d relativistic QM model. We assume the universal structure of the full energy trans-series is still valid. We then borrow elements from the 5d SYM to construct the basic building blocks of this general structure. We use resurgence results from the 5d SYM to find a subset of the trans-series coefficients and use high precision numerical calculation to find the remaining coefficients. With this method, we are able to confirm that the universal structure of the energy trans-series is still valid and find the full energy trans-series when the magnetic flux is $\phi = 2\pi/Q$ for natural number Q. Taking the logic of [32] in reverse, we infer the exact quantization conditions from the full energy trans-series and find that it is in some sense a "double copy" of the exact quantization condition of the Mathieu equation, i.e. the 1d non-relativistic QM model with a cosine potential [33–36].

In this process, we clarify a subtlety in the identification between the non-perturbative corrections to the perturbative Wilson loop in the field theory and the non-perturbative corrections to the perturbative energy series of the Harper-Hofstadter model, which is akin to the transition from the large N expansion to the conventional series discussed recently in [37].

In addition, we also find that the energy trans-series is very sensitive to the nature of the magnetic flux. If the magnetic flux is $\phi = 2\pi P/Q$ with coprime natural numbers P, Q and $P > 1$, the trans-series coefficients change, and they display a peculiar feature related to the strong-weak magnetic field duality, and hence could shed some light on the fractal structure of the spectrum possibly. We also study the expansion of the energy around some rational values of the magnetic flux, extending the Rammal-Wilkinson formula [30, 31].

The remainder of the paper is organized as the following. In section 2, we review the previous results of the Harper-Hofstadter model, including the secular equation that computes the energy spectrum exactly when the magnetic field is rational, and the semi-classical analysis, including the instanton corrections from the path integral formalism. In section 3, we collect results from recent developments, including a short introduction to the resurgence ideas that we will need, the exact WKB method and its implication for the energy trans-series, and the connection between the Harper-Hofstadter model and the 5d SYM. In section 4, using these results, we construct the full energy trans-series for the Harper-Hofstadter model step by step for flux $\phi = 2\pi/Q$. In section 5, we make some attempts to characterize the splitting bands for $P > 1$. We revisit the self-similarity structure of the butterfly and provide evidence for a possible exact version of the Rammal-Wilkinson formula. Finally we conclude and give a list of open problems in section 6.

2 Hofstadter's butterfly and its semiclassical analysis

The Harper-Hofstadter problem concerns the movement of electrons in the square lattice of ions with the presence of uniform magnetic field. According to Bloch's theorem, this can be effectively captured by a single electron wavefunction obeying the almost Mathieu equation, which was first studied by Harper [1]. We will quickly review this story here.

2.1 Harper-Hofstadter equation

Let us consider an electron moving in a two dimensional plane with a doubly periodic electric potential induced by a square lattice of ions. By the tight binding approximation, the Hamiltonian operator of the electron is

$$\mathsf{H} = 2\cos\frac{a}{\hbar}\mathsf{p}_x + 2\cos\frac{a}{\hbar}\mathsf{p}_y. \tag{2.1}$$

Here a is the lattice spacing, $\hbar$ the reduced Planck constant, and $\mathsf{p}_x, \mathsf{p}_y$ are the two independent momentum operators in the x- and y-directions, and they commute with each other. This Hamiltonian allows a single continuous band of energy in the range

$$-4 \leq E \leq 4. \tag{2.2}$$

If we impose in addition a uniform magnetic field of field strength B perpendicular to the plane, the Hamiltonian operator has to be modified where we replace $\mathsf{p}_x, \mathsf{p}_y$ by the operators of canonical momenta Π_x, Π_y

$$\mathsf{H} = \mathrm{e}^{\frac{\mathrm{i}a}{\hbar}\Pi_x} + \mathrm{e}^{-\frac{\mathrm{i}a}{\hbar}\Pi_x} + \mathrm{e}^{\frac{\mathrm{i}a}{\hbar}\Pi_y} + \mathrm{e}^{-\frac{\mathrm{i}a}{\hbar}\Pi_y}. \tag{2.3}$$

Here the canonical momenta are defined by

$$\vec{\Pi} = \vec{\mathsf{p}} + e\vec{A} \tag{2.4}$$

and the two components no longer commute

$$[\Pi_x, \Pi_y] = -\mathrm{i}\hbar e(\partial_x A_y - \partial_y A_x) = -\mathrm{i}\hbar e B. \tag{2.5}$$

We will call this the Harper-Hofstadter model.

We can simplify the notation by defining the scaled operators

$$\mathsf{x} = \frac{a}{\hbar}\Pi_x, \quad \mathsf{y} = -\frac{a}{\hbar}\Pi_y \tag{2.6}$$

with the commutator

$$[\mathsf{x}, \mathsf{y}] = \frac{\mathrm{i}a^2 e B}{\hbar} =: \mathrm{i}\phi, \tag{2.7}$$

so that the Hamiltonian simply reads

$$\mathsf{H} = \mathrm{e}^{\mathrm{i}\mathsf{x}} + \mathrm{e}^{-\mathrm{i}\mathsf{x}} + \mathrm{e}^{\mathrm{i}\mathsf{y}} + \mathrm{e}^{-\mathrm{i}\mathsf{y}}. \tag{2.8}$$

This is equivalent to a relativistic one dimensional quantum mechanical model where x, y play the roles of the position and the momentum operators respectively, and the flux through a lattice plaquette ϕ plays the role of the reduced Planck constant. In the position representation, the time-independent Schrödinger equation reads

$$\psi(x+\phi) + \psi(x-\phi) + 2\cos x\,\psi(x) = E\psi(x). \tag{2.9}$$

Introduce the parametrization

$$x = n\phi + \delta, \quad \psi_n(\delta) = \psi(n\phi + \delta), \tag{2.10}$$

we arrive at the famous Harper's equation

$$\psi_{n+1} + \psi_{n-1} + 2\cos(n\phi + \delta)\psi_n = E\psi_n. \tag{2.11}$$

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2.2 Butterfly at rational fluxes

When the magnetic flux ϕ is rational of the form

$$\phi = 2\pi\alpha = 2\pi\frac{P}{Q}, \quad P, Q \in \mathbb{N}, \ (P, Q) = 1, \tag{2.12}$$

the energy spectrum of the Harper's equation can be derived relatively easily, as first found out by [2]. In this case, the Harper's equation is invariant under the shift $n \to n + Q$, and we can introduce the Bloch wavefunction

$$\psi_n(\delta) = \mathrm{e}^{\mathrm{i}kn} u_n(\delta, k), \tag{2.13}$$

where k is the Bloch wavenumber, and u_n is periodic with

$$u_{n+Q}(\delta, k) = u_n(\delta, k). \tag{2.14}$$

The matrix of the Hamiltonian operator in the Hilbert space then truncates to finite size and we have the eigenvalue equation

$$H_Q \cdot u_Q = E u_Q, \quad u_Q = (u_0, u_1, \ldots, u_{Q-1})^T \tag{2.15}$$

where H_Q is the matrix

$$H_Q(\delta, k) = \begin{pmatrix} 2\cos\delta & \mathrm{e}^{\mathrm{i}k} & & & \mathrm{e}^{-\mathrm{i}k} \\ \mathrm{e}^{-\mathrm{i}k} & 2\cos\left(\delta + 2\pi\frac{P}{Q}\right) & \mathrm{e}^{\mathrm{i}k} & & \\ & \mathrm{e}^{-\mathrm{i}k} & 2\cos\left(\delta + 4\pi\frac{P}{Q}\right) & \mathrm{e}^{\mathrm{i}k} & \\ & & \ddots & \ddots & \\ \mathrm{e}^{\mathrm{i}k} & & & \mathrm{e}^{\mathrm{i}k} & 2\cos\left(\delta + 2\pi(Q-1)\frac{P}{Q}\right) \end{pmatrix}. \tag{2.16}$$

The energy spectrum is solved from the secular equation

$$F_{P/Q}(E, \delta, k) := \det(H_Q - E\mathbf{1}_Q) = 0. \tag{2.17}$$

The left hand side defines a degree Q polynomial in E, which we denote by $F_{P/Q}(E, \delta, k)$, and it indicates that for fixed δ, k, there are Q eigen-energies.

It can be shown (see e.g. [38]) that the secular equation can be equivalently written as

$$F_{P/Q}(E, 0, 0) = 2(\cos Qk + \cos Q\delta) =: 2(\cos\theta_x + \cos\theta_y). \tag{2.18}$$

Here we have denoted $Qk, Q\delta$ respectively by θ_x, θ_y. They can be treated on equal footing: both of them are periodic with $\theta_{x,y} \to \theta_{x,y} + 2\pi$, and the secular equation is not changed by exchanging θ_x, θ_y. In fact, it was pointed out in [20] that the Harper-Hofstadter model is special in the sense that when the flux ϕ is rational there can be Bloch angles in both the x- and y-directions, and θ_x, θ_y defined here are precisely these two Bloch angles.

The secular equation (2.18) also indicates that by varying θ_x, θ_y in their respective domain, the Q eigen-energies are broadened to Q continuous energy bands, where the top and the bottom edges correspond respectively to $\theta_x = \theta_y = 0$ and $\theta_x = \theta_y = \pi$. The spectrum

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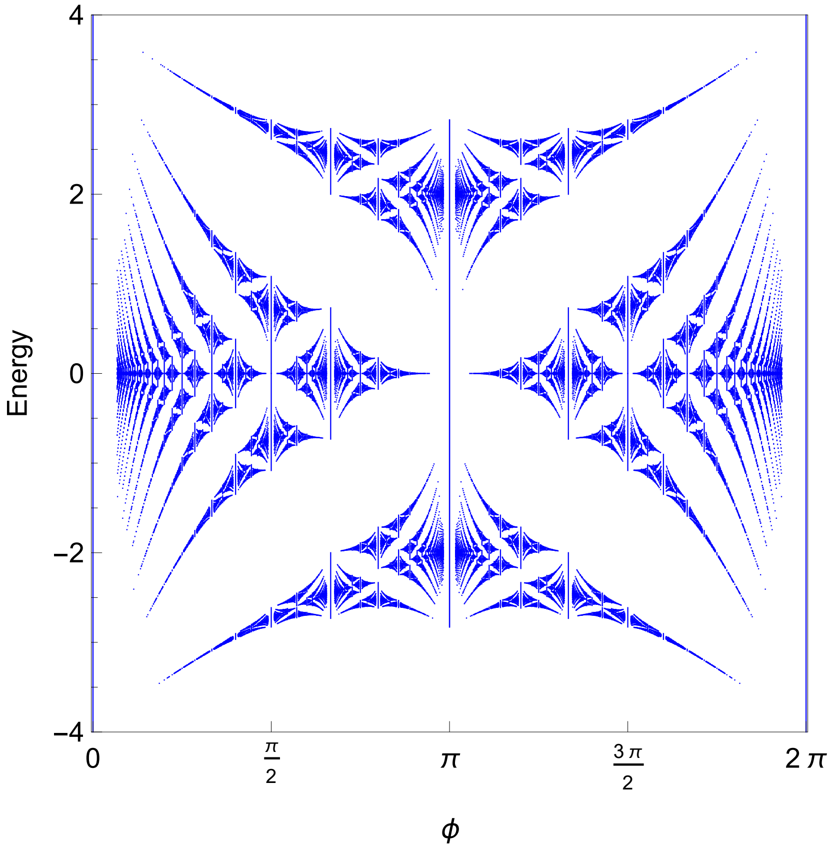

Figure 1. Hofstadter. We plot the band structure for P/Q with $(P,Q)=1$ and Q up to 60.

of energy as a function of the flux is plotted in figure 1.[1] As the Harper's equation (2.11), and therefore the energy E, is invariant under the shift $\phi \to \phi + 2\pi$, the plot is restricted to the domain of $\phi \in [0, 2\pi]$.

This plot of spectrum in figure 1 is the famous Hofstadter's butterfly. It has a striking fractal structure, which implies that the energy spectrum as a function of the flux has very rich non-perturbative structures, which we try to understand.

2.3 Semi-classical analysis

As mentioned in section 2.1, the Harper's equation can be viewed as the Schrödinger equation of a relativistic one dimensional quantum mechanical model with ϕ plays the role of the reduced Planck constant. It is natural then to treat the spectrum problem semiclassically, and consider the energy E first as a perturbative series in ϕ.

In one-dimensional non-relativistic quantum mechanical problems, in principle the perturbative energy series can be calculated by the Rayleigh-Schrödinger perturbation theory, but in practise, one cannot go very far. Instead, it is more efficient to use the method of Bender and Wu [40, 41], which makes the ansatz that the wavefunction is a deformation of that of the harmonic oscillator, and which allows very fast calculation of the perturbative

[1] There's also a useful open-source package [39] that can be used to compute the band structure of a generalized Hofstadter model on any regular Euclidean lattice, as well as its key properties.

energy around any local minimum of a polynomial potential where the second derivative of the potential does not vanish. This algorithm was made into a `Mathematica` package called `BenderWu` in [42], which was expanded in [43] to allow relativistic systems whose Hamiltonians are polynomials of $e^{\pm x}, e^{\pm y}, x, y$. As pointed out in [20], after a Wick rotation $x, y \to ix, iy$, our Hamiltonian (2.8) falls into this category. With the help of the `BenderWu` package, one can easily calculate the perturbative energy series for the Harper-Hofstadter model up to close to 100 terms, and the first few terms are

$$E^{\text{pert}}(\nu,\phi) = 4 - 2\nu\phi + \left(\frac{1}{16} + \frac{\nu^2}{4}\right)\phi^2 + \left(-\frac{\nu}{128} - \frac{\nu^3}{96}\right)\phi^3 + \dots \tag{2.19}$$

with the Landau level $\nu = N + 1/2,\ N = 0, 1, 2, \dots$.

As discussed at the end of section 2.2, the spectrum of the Harper-Hofstadter model has significant non-perturbative corrections, which presumably come from instanton effects. The leading instanton corrections can be computed by the path integral formalism [20]. For this purpose, we recall that in the more standard one dimensional periodic quantum mechanical model with a single Bloch angle θ, we can define the Bloch wavefunction

$$\psi_\theta(x) = \sum_{N\in\mathbb{Z}} e^{-iN\theta}\psi_0(x + Na) \tag{2.20}$$

where $\psi_0(x)$ is the approximate eigenstate wavefunction centered around the origin, as well as the twisted thermal partition function

$$\langle\psi_{\theta'}|\, e^{-HT/\phi}\, |\psi_\theta\rangle = 2\pi\delta(\theta-\theta')Z_\theta(T) = 2\pi\delta(\theta-\theta') \sum_{\nu\in\mathbb{N}+1/2} e^{-E_\theta(\nu)T/\phi}. \tag{2.21}$$

Assuming the energy spectrum is not degenerate, the ground state energy can then be computed using the twisted thermal partition function

$$E_\theta(1/2) = -\lim_{T\to\infty}\frac{\phi}{T}\log Z_\theta(T). \tag{2.22}$$

The twisted partition function can be computed in path integral, and in the semiclassical limit with $\phi \to 0$, it decomposes by

$$Z_\theta = Z_\theta^{(0)} + Z_\theta^{(+1)} + Z_\theta^{(-1)} + Z_\theta^{(+2)} + Z_\theta^{(-2)} + \dots \tag{2.23}$$

where

$$Z_\theta^{(n)} = \int [\mathcal{D}x]_n e^{-S_E/\phi + in\theta}, \tag{2.24}$$

with the boundary conditions

$$[\mathcal{D}x]_n:\ x(-T/2) = 0,\ x(+T/2) = na, \tag{2.25}$$

which describe precisely an $|n|$-instanton configuration.

In the case of the Harper-Hofstadter model, this path integral analysis was performed in [20] for the cases of $\phi = 2\pi/Q$, i.e. $P = 1$. It was observed that one can find 1-instanton

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configurations in both the x- and y-directions, with the corresponding Bloch angles θ_x, θ_y, and the instanton action is identically

$$S_c = 8C \tag{2.26}$$

C being the Catalan's number. With instantons in both x- and y-directions, and in both positive and negative directions, the one-instanton correction of the ground state energy is[2]

$$E^{(1)}_{\theta_x,\theta_y}(1/2,\phi) = 16(\cos\theta_x + \cos\theta_y)\left(\frac{\phi}{2\pi}\right)^{1/2} \mathrm{e}^{-S_c/\phi}(1+\dots). \tag{2.27}$$

This can be checked by comparing with the bandwidth of each energy band

$$\mathrm{bw}_N(\phi) \approx |E^{(1)}_{0,0}(\nu,\phi) - E^{(1)}_{\pi,\pi}(\nu,\phi)|, \tag{2.28}$$

which at the leading order is controlled by the 1-instanton correction. In fact, with this method, one finds numerically that at any Landau level [20][3]

$$E^{(1)}_{\theta_x,\theta_y}(\nu,\phi) = (\cos\theta_x + \cos\theta_y)(-1)^N \frac{16\cdot 8^N}{\pi^N N!}\left(\frac{\phi}{2\pi}\right)^{1/2-N} \mathrm{e}^{-S_c/\phi}(1+\dots). \tag{2.29}$$

If we take into account instanton corrections of all orders, we expect the non-perturbative energy series to be of the form

$$E(\nu,\phi) = E^{(0)}(\nu,\phi) + \sum_{n\geq 1} E^{(n)}_{\theta_{x,y}}(\nu,\phi), \tag{2.30}$$

where the n-instanton correction is of the order

$$E^{(n)} \sim \mathrm{e}^{-nS_c/\phi}. \tag{2.31}$$

We will make this expression more concrete in section 4.

3 Resurgence, exact WKB, and 5d SYM

3.1 Resurgence program

We give here a quick overview of the resurgence theory. See [24–27] for more detailed discussion. Given a perturbative series $\varphi(z)$, which is of 1-Gevrey type, meaning that its coefficients grow factorially fast

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n \sim n! \tag{3.1}$$

such that it has zero radius of convergence, there is a well-studied procedure called Borel resummation to evaluate, or to resum such a divergent series.

For this purpose, we first construct the Borel transform of the 1-Gevrey series

$$\widehat{\varphi}(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!}\zeta^n, \tag{3.2}$$

[2]There is a typo in [20], where 1-instanton amplitude should be increased by a factor of two.
[3]See footnote 2.

which is regular in the neighborhood of the origin. It can be analytically continued to the entire complex plane, and let us make the mild assumption that it has a discrete set $\Omega \subset \mathbb{C}$ of singular points, known as the Borel singularities.

Let us define Stokes lines in the complex z-plane, which are rays from the origin and whose inclinations are the arguments of the Borel singularities. These Stokes lines divide the complex z-plane into disjoint cones. For any value of z inside a cone, we can define the Borel resummation

$$\mathscr{S}\varphi(z) = \frac{1}{z}\int_0^{\mathrm{e}^{\mathrm{i}\arg z}\infty} \mathrm{e}^{-\zeta/z}\widehat{\varphi}(\zeta)\mathrm{d}\zeta. \tag{3.3}$$

If z is on a Stokes line, naive definition above of Borel resummation would fail as the integration contour will be obstructed by a Borel singularity. In this case, we have to define not one but a pair of lateral Borel resummations by slightly raising or lowering the inclination of the integration contour to bypass the Borel singularity

$$\mathscr{S}^{(\pm)}\varphi(z) = \frac{1}{z}\int_0^{\mathrm{e}^{\mathrm{i}(\arg z\pm\epsilon)}\infty} \mathrm{e}^{-\zeta/z}\widehat{\varphi}(\zeta)\mathrm{d}\zeta, \tag{3.4}$$

and the two resummations differ by an exponentially suppressed discrepancy known as the Stokes discontinuity

$$\mathrm{disc}_\theta\varphi(z) = \mathscr{S}^{(+)}\varphi(z) - \mathscr{S}^{(-)}\varphi(z) \sim \mathrm{e}^{-1/z}, \tag{3.5}$$

where θ is the inclination of the Stokes line.

Suppose there is a sequence of Borel singularities $kA = A, 2A, 3A, \ldots$ which share the same argument as z and which obstruct the naive integration contour for the Borel resummation. According to the resurgence theory, the Stokes discontinuity can be attributed in a precise manner to these Borel singularities. In fact, each such Borel singularity kA represents a non-trivial saddle point in the theory, and to each is associated a new 1-Gevrey series $\varphi^{(k)}$, such that

$$\mathrm{disc}_\theta\varphi(z) = \sum_{k=1}^{\infty} \mathsf{S}_k \mathrm{e}^{-kA/z}\mathscr{S}^{(-)}\varphi^{(k)}(z). \tag{3.6}$$

The proportionality constants S_k are known as the Borel residues and they depend on the normalization of the series $\varphi^{(k)}$.

It is sometimes more useful to encode in a different manner contributions of individual singular points to the Stokes discontinuity. For instance, we can introduce a map of power series known as Stokes automorphism

$$\mathfrak{S}_\theta\varphi(z) := \varphi(z) + \sum_{k=1}^{\infty} \mathsf{S}_k \mathrm{e}^{-kA/z}\varphi^{(k)}(z). \tag{3.7}$$

so that

$$\mathscr{S}^{(+)}\varphi(z) = \mathscr{S}^{(-)}\mathfrak{S}_\theta\varphi(z). \tag{3.8}$$

which has the property that it is an automorphism in the ring of power series. Alternatively, we can introduce pointed alien derivatives associated to each of the Borel singularities [24][4]

$$\begin{aligned}\mathfrak{S}_\theta\varphi(z) &= \exp\left(\sum_{k=1}^{\infty} \dot{\Delta}_{kA}\right)\varphi(z) \\ &= \varphi(z) + \dot{\Delta}_A\varphi(z) + \left(\dot{\Delta}_{2A} + \frac{1}{2}(\dot{\Delta}_A)^2\right)\varphi(z) + \dots\end{aligned} \tag{3.9}$$

Each alien derivative is a map of 1-Gevrey power series, and in particular, we have

$$\dot{\Delta}_{kA}\varphi(z) = \mathcal{S}_k \mathrm{e}^{-kA/z}\varphi^{(k)}(z) \tag{3.10}$$

The coefficients $\mathcal{S}_k$ here are called the Stokes constants, and they are related to the Borel residues by simple combinatoric formulas. The alien derivatives have very nice properties: they follow the Leibniz rule and chain rule, just like ordinary derivatives, and furthermore commute with ordinary derivations.

As the new series $\varphi^{(k)}$ uncovered from the original perturbative series φ are also 1-Gevrey, the same resurgence analysis of Borel singularities and Stokes discontinuities can be repeated, revealing even more Borel singularities and the associated additional 1-Gevrey power series. Together all these 1-Gevrey series are said to form a *minimal resurgent structure* starting from φ [44].

From the discussion of resurgent structure, a paradigm to study generic perturbative series called resurgence program can be formulated. One distinguishes between the weak resurgence program and the strong resurgence program [45]. The *weak resurgence program* conjectures that any physical quantity that allows a perturbative expansion φ can be expressed in terms of the Borel resummation of a trans-series, whose leading contribution is the perturbative series φ. Trans-series is a rather broad concept, see [46] for a good exposition. The most common form of trans-series, which is enough for us, is

$$\Phi(z) = \varphi(z) + \sum_k c_k \mathrm{e}^{-A_k/z}\varphi^{(A_k)}(z) \tag{3.11}$$

where $\varphi^{(A_k)}(z)$ are usually power series just like $\varphi(z)$, but may also contain terms with $\log(z)$. The *strong resurgence program* in addition requires that all $\varphi^{(A_k)}$ belong to the minimal resurgence structure starting from φ. In many scenarios, the strong resurgence program is too strong, and only a subset of $\varphi^{(A_k)}$, which is sometimes called the *minimal trans-series* [32], belong to the minimal resurgent structure. Regardless, the trans-series coefficients c_k associated to this subset of power series will jump as we cross a Stokes line in order to compensate for the Stokes discontinuity so that the exact physical quantity can be a continuous function of z and is ambiguity free. In general, as z moves in the complex plane, crossing various Stokes lines, all ingredients of the minimal resurgent structure will appear in the full trans-series.

[4]Alien derivatives can also be introduced even if the Borel singularities are spaced unevenly.

3.2 Structure of trans-series from exact WKB

Following the weak resurgence program, the exact energy eigenvalue should be the Borel resummation of an energy trans-series. In 1d QM models, a particularly powerful method to derive such an energy trans-series is to solve exact quantization conditions (EQCs) obtained via the exact WKB method [29], which is based on the resurgence theory. It was implied in [32] that full energy trans-series seems to have a universal structure, which we explain. In later sections, we will demonstrate that the full energy trans-series of the Harper-Hofstadter model shares this universal structure.

Suppose we have the Schrödinger equation for a 1d non-relativistic QM model

$$H(\mathsf{x},\mathsf{y})\psi(x) = E\psi(x), \tag{3.12}$$

which is a second order ODE. We can write down the WKB ansatz for the wavefunction

$$\psi(x) = \exp\left(\frac{\mathsf{i}}{\hbar}\int_*^x P(x',\hbar)\mathrm{d}x'\right). \tag{3.13}$$

Here $P(x,\hbar)$ is a formal power series

$$P(x,\hbar) = \sum_{n=0}^{\infty} P_n(x)\hbar^n. \tag{3.14}$$

The coefficients $P_n(x)$ can be solved by plugging in the WKB ansatz into the Schrödinger equation. The leading coefficient $P_0(x) = \pm y(x)$ is the momentum satisfying the classical equation

$$H(x,y) = E. \tag{3.15}$$

Higher order coefficients $P_{n\geq 1}(x)$ can be solved recursively.

If we promote x,y to complex variables, the classical equation (3.15) defines a complex curve known as the WKB curve Σ. We will assume that the WKB curve is of genus one, so that it has two independent 1-cycles, called the A-cycle γ_A and the B-cycle γ_B with intersection number $\langle\gamma_A,\gamma_B\rangle = 1$. We choose the A-cycle γ_A and B-cycle γ_B so that when projected to the complex x-plane, they are mapped to respectively the classical allowed and classically forbidden regions, cf. figure 2. We can then define the quantum A- and B-periods

$$t(E,\hbar) = \frac{1}{2\pi}\sum_{n=0}^{\infty}\hbar^{2n}\oint_{\gamma_A} P_{2n}(x)\mathrm{d}x, \tag{3.16a}$$

$$t_D(E,\hbar) = -\,\mathsf{i}\sum_{n=0}^{\infty}\hbar^{2n}\oint_{\gamma_B} P_{2n}(x)\mathrm{d}x. \tag{3.16b}$$

They are also known as the perturbative and the non-perturbative quantum periods, as they are responsible for respectively the perturbative and non-perturbative contributions to the quantization conditions that we will see momentarily. And the normalization in (3.16a) and (3.16b) are chosen so that they are positive in the leading order. Both quantum periods are power series in $\hbar$, and the leading terms are classical periods of the 1-form $\lambda = y(x)\mathrm{d}x$. In addition, both quantum periods are 1-Gevrey, as

$$\oint_\gamma P_{2n}(x)\mathrm{d}x \sim (2n)! \tag{3.17}$$

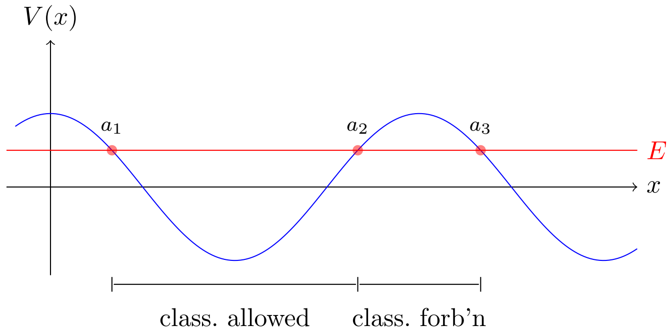

Figure 2. Classically allowed and forbidden regions.

The difference is that when $\hbar > 0$, the perturbative quantum period is not Borel summable, so that a prescription of lateral resummation is needed, while the non-perturbative quantum period is Borel summable, and a vanilla version of Borel resummation is applicable.

In general, the EQCs for the eigen-energy E take the form

$$1 + \mathcal{V}_A = f(\mathcal{V}_B^{1/2}, \mathcal{V}_A^{1/2}), \tag{3.18}$$

with the Voros symbols

$$\mathcal{V}_A = \mathrm{e}^{2\pi \mathrm{i} t(E,\hbar)/\hbar}, \quad \mathcal{V}_B = \mathrm{e}^{-t_D(E,\hbar)/\hbar}. \tag{3.19}$$

Here $f(u, v)$ is certain single-valued function of u, v, and it vanishes in the $u \to 0$ limit, corresponding to the semi-classical limit $\hbar \to 0$. For instance this is true for the cubic mode, the double-well model (see e.g. [32]), and in particular for the cosine model, whose Schrödinger equation is the famous Mathieu equation. This is the one dimensional quantum mechanical model with the Hamiltonian

$$H = \frac{y^2}{2} + 1 - \cos(x), \tag{3.20}$$

and it can be regarded as the non-relativistic limit of the Hamiltonian (2.8) of the Harper-Hofstadter model. The EQC for this model is well-known and it reads [33–36]

$$D_\theta^\pm = 1 + \mathcal{V}_A^{\mp 1}(1 + \mathcal{V}_B) - 2\sqrt{\mathcal{V}_A^{\mp 1}\mathcal{V}_B}\cos\theta = 0. \tag{3.21}$$

Depending on the choice of the lateral resummation $\mathscr{S}^\pm$ of the perturbative quantum period, one of the two quantization conditions $D_\theta^\pm$ is used. Here θ is the Bloch angle.

To solve the energy trans-series, we first consider the semi-classical limit $\hbar \to 0$, with the non-perturbative contributions due to $\mathrm{e}^{-t_D/\hbar}$ turned off. The EQC is reduced to

$$1 + \mathcal{V}_A = 0 \tag{3.22}$$

which is equivalent to the all-orders Bohr-Sommerfeld quantization conditions

$$t(E, \hbar) = \hbar\nu, \tag{3.23}$$

where $\nu = N + 1/2$, $N = 0, 1, 2, \ldots$. Let $\mathcal{E}(t, \hbar)$ be the inverse of $t(E, \hbar)$ as a function of E, the perturbative energy series is

$$E^{(0)}(\nu, \hbar) = \mathcal{E}(t = \hbar\nu, \hbar). \tag{3.24}$$

To solve the EQC (3.18) with the non-perturbative corrections turned on, one can assume that t is a small deviation from $\hbar\nu$

$$t = \hbar(\nu + \Delta\nu), \tag{3.25}$$

and solve Δt from the equation

$$1 - \mathrm{e}^{2\pi \mathrm{i} \Delta\nu} = f(\mathrm{e}^{-\frac{1}{2\hbar} t_D(\nu+\Delta\nu,\hbar)}, \pm\mathrm{i}\, \mathrm{e}^{\pi\mathrm{i}\Delta\nu}), \tag{3.26}$$

where the exponent $t_D(\nu, \hbar)$ is

$$t_D(\nu, \hbar) := t_D(E = E^{(0)}(\nu, \hbar), \hbar), \tag{3.27}$$

while the full enery trans-series is then obtained by substituting the deformed $\nu + \Delta\nu$ for ν in the perturbative series

$$E(\nu, \hbar) = e^{\Delta\nu\partial\nu} E^{(0)}(\nu, \hbar) = E^{(0)}(\nu + \Delta\nu, \hbar). \tag{3.28}$$

This is the strategy pursued in [32], where it is proposed to recast the eq. (3.26) in the form of[5]

$$\Delta\nu = R(\lambda(\nu + \Delta\nu)) = \sum_{k=1}^{\infty} r_k \lambda(\nu + \Delta\nu)^k, \quad \lambda(\nu) = \mathrm{e}^{-\frac{1}{2\hbar} t_D(\nu,\hbar)}, \tag{3.29}$$

whose solution as a trans-series can be explicitly written down via the Lagrange inversion theorem (see e.g. [47, 48]). One then finds that the full energy trans-series obtained via (3.28) has the general structure [32]

$$E(\nu, \hbar) = E^{(0)}(\nu, \hbar) + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} u_{n,m}(\underline{r}) E^{(n,m)}(\nu, \hbar). \tag{3.30}$$

The basic building blocks are the basic trans-series

$$E^{(n,m)}(\nu, \hbar) = \left(\frac{\partial}{\partial\nu}\right)^m \left(\frac{\partial E^{(0)}(\nu, \hbar)}{\partial\nu} \mathrm{e}^{-n t_D(\nu,\hbar)/\hbar}\right). \tag{3.31}$$

All of $E^{(n,m)}$ with $m = 0, 1, 2, \ldots, n-1$ account for the n-instanton corrections. Note that $E^{(n,m)}$ with $n \geq 2, m \geq 1$ may contain $\log(\hbar)$ terms and they arise due to instanton / anti-instanton interactions. The trans-series coefficients read

$$u_{n,m}(\underline{r}) = \frac{1}{n!} B_{n,m+1}(1!r_1, 2!r_2, \ldots, (n-m)!r_{n-m}), \tag{3.32}$$

[5] Note that the power series on the right hand side must start with $k = 1$ as it should vanish in the semi-classical limit $\hbar \to 0$.

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where $B_{n,m+1}$ are the incomplete Bell's polynomials. Finally the weak resurgence program requires that the Borel resummation of the full energy trans-series gives the exact value of energy in the regime $\hbar\nu \ll 1$.

Compared to the general full trans-series (2.30), the structure (3.30) is much simpler. The basic building blocks $E^{(n,m)}$ only depend on two ingredients, the perturbative energy series $E^{(0)}(\nu,\hbar)$ and the non-perturbative quantum period $t_D(\nu,\hbar)$, and once they are identified, the remaining job is to fix relatively simpler trans-series coefficients $u_{n,m}$.

For non-relativistic one dimensional quantum mechanical models whose Schrödinger equations are second order difference equations, the exact WKB method is still applicable, although it is difficult to write down EQCs in this way as the connection formulas are yet not competely clear (see [49] though for recent progress). The EQCs have been written down in some examples by other methods [50–53], for instance via the TS/ST corresondence [50, 54], but these EQCs have a more complicated form. Nevertheless, we will see in later sections that the universal structure (3.30) for energy trans-series also holds for the Harper-Hofstadter model, as least when $\phi = 2\pi/Q$. Furthermore, the basic building blocks can be readily written down. For instance, it has already been shown [20] via examples of 1-instanton corrections that the non-perturbative quantum period can be easily computed as it has an interesting interpretation in supersymmetric field theories, which we quickly review.

3.3 5d SYM and its resurgent structure

We will be interested in 5d $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with gauge group $G = \mathrm{SU}(2)$ on $S^1 \times \mathbb{R}^4$. The IR effective theory is described by the Seiberg-Witten curve given by the equation [55, 56][6]

$$\Sigma: \quad \mathrm{e}^x + \mathrm{e}^{-x} + \mathrm{e}^y + \mathrm{e}^{-y} - u = 0 \tag{3.33}$$

The Seiberg-Witten curve is equipped with the meromorphic 1-form

$$\lambda = y\mathrm{d}x, \tag{3.34}$$

and its integration along closed 1-cycles on Σ are known as classical periods.

A 5d $\mathcal{N} = 1$ supersymmetric theory usually has degenerate vacuum states, and they form a moduli space $\mathcal{M}$. Due to $\mathcal{N} = 1$ supersymmetry, the moduli space has the structure of a special Kähler manifold, which means that in any patch of the moduli space, one can choose a basis of flat coordinates to locally parametrise the moduli space, and these flat coordinates are paired with their conjugates (see e.g. [57])

$$\left(t^a, \frac{\partial F_0}{\partial t^a}\right), \quad a = 1, \dots, \frac{1}{2}\dim\mathcal{M}, \tag{3.35}$$

so that they are related to each other via a single function called the prepotential F_0. Such a choice of flat coordinates is called choosing a frame. Here both t^a and $\partial_{t^a} F_0$ are integral periods of the meromorphic form λ over the Seiberg-Witten curve, and togethe with $4\pi^2\mathsf{i}$ they span the period lattice.

[6]We have chosen the special so-called diagonal slice in the moduli space where the radius of S^1 is one.

In the case of SYM, the moduli space is $\mathbb{P}^1$, parametrised by $z = 1/u^2$, and when quantum corrections are taken into account, it has three singular points, located at $z = 0, 1/16, \infty$, known as the large radius point, the conifold point, and the orbifold point. The neighborhood of the conifold point will be of particular interest for us. Here, the suitable flat coordinate and its conjugate are (see e.g. [58])

$$t_c = \frac{1}{\pi}\left(\frac{1}{\pi} G^{3,2}_{3,3}\left(\begin{smallmatrix}\frac{1}{2},\frac{1}{2};1\\0,0,0\end{smallmatrix}; 16z\right) - \pi^2\right), \tag{3.36a}$$

$$\frac{\partial F_0(t_c)}{\partial t_c} = -\pi\left(\log(z) + 4z\,{}_4F_3\left(1,1,\frac{3}{2},\frac{3}{2};2,2,2;16z\right)\right) + \pi \mathrm{i} t_c, \tag{3.36b}$$

which have the property that $t_c(z = 1/16) = 0$. We use a slightly different convention of $\partial_{t_c} F_0(t_c)$ than in the literature, as we will be interested in the regime $z > 1/16$, where our convention has the property that $\partial_{t_c} F_0(t_c) \in \mathbb{R}_+$.

We couple the gauge theory to a background gravity by turning on the Omega background [59] and restrict ourselves to the so-called Nekrasov-Shatashvili (NS) limit [60]. The Seiberg-Witten curve is promoted to a quantum operator known as the quantum Seiberg-Witten curve [61]. It is a relativistic Schrödinger equation

$$\mathsf{H}^{\mathrm{SYM}}\psi = u\psi \tag{3.37}$$

with the Hamiltonian operator

$$\mathsf{H}^{\mathrm{SYM}} = \mathrm{e}^{\mathsf{x}} + \mathrm{e}^{-\mathsf{x}} + \mathrm{e}^{\mathsf{y}} + \mathrm{e}^{-\mathsf{y}}, \tag{3.38}$$

and x, y satisfy the canonical quantization condition

$$[\mathsf{x}, \mathsf{y}] = \mathrm{i}\hbar. \tag{3.39}$$

As shown in [52], the quantum Seiberg-Witten curve can be identified with the Hamiltonian of the two particle closed relativisitc Toda lattice [62]. On the other hand, if we make the Wick rotation [14]

$$(x, y) \to (\mathrm{i}x, \mathrm{i}y), \tag{3.40}$$

as well as the map

$$\hbar \to -\phi, \tag{3.41}$$

the quantum Seiberg-Witten curve (3.37) can be identified with the Hamiltonian operator (2.8) of the Harper-Hofstadter model. While the EQCs of the relativistic Toda lattice have been written down, with no distinction between the rational and irrational $\hbar$ [50, 51], those for the Harper-Hofstadter model are much more complicated. This is akin to the difference between the Mathieu equation and the modified Mathieu equation.

We will be interested in finding in the Harper-Hofstadter model the full energy trans-series and the implied EQCs in the form of (3.18). We fix our convention and define the perturbative quantum period and non-perturbative quantum periods (3.16a), (3.16b) so that their leading terms are respectively t_c and $\partial_{t_c} F_0(t_c)$ in (3.36a), (3.36b).

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Two interesting observables can be defined in the 5d SYM. The first is the vev $W_{\mathbf{r}}(t,\hbar)$ of the half-BPS Wilson loop operator in the fundamental representation $\mathbf{r}=\square$ where the Wilson loop wraps the S^1 and is located at the center of $\mathbb{R}^4$. In the NS limit, the perturbative Wilson loop vev is a power series in $\hbar$ [18, 19]

$$W_{\square}(t,\hbar)=\sum_{n=0}^{\infty}W_n(t)\hbar^{2n}. \tag{3.42}$$

where the coefficients $W_n(t)$ are functions over the moduli space. More importantly it is identified with the perturbative eigenvalue of the Hamiltonian operator H^{SYM} [60, 63, 64]

$$u=W_{\square}(t,\hbar), \tag{3.43}$$

and therefore also with the perturbative energy series of the Harper-Hofstadter model with the dictionary (3.41). In the semi-classical limit with $\hbar=-\phi\to 0$, the energy of the Harper-Hofstadter model is 4, corresponding to the conifold point singularity at $z=1/u^2=1/16$. We should thus evaluate the Wilson loop vev in the conifold frame. The first few coefficients are

$$W_0(t_c)=4+2t_c+\frac{t_c^2}{4}+\ldots, \tag{3.44a}$$

$$W_1(t_c)=\frac{1}{16}+\frac{t_c}{128}+\frac{3t_c^2}{1024}+\ldots, \tag{3.44b}$$

$$W_2(t_c)=\frac{13}{24576}-\frac{151t_c}{393216}+\frac{159t_c^2}{524288}+\ldots. \tag{3.44c}$$

It is easy to see that indeed (3.42) reproduces (2.19) through

$$E^{(0)}(\nu,\phi)=W_{\square}(t_c,\hbar)\Big|_{t_c=-\phi\nu,\hbar=-\phi}. \tag{3.45}$$

Another interesting physical observable is the NS free energy, which is also a perturbative power series in $\hbar$ [60]

$$F_{\text{NS}}(t_c,\hbar)=\sum_{n=0}^{\infty}F_n(t_c)\hbar^{2n}. \tag{3.46}$$

In the conifold frame, the perturbative free energy can be decomposed in terms

$$F_n(t_c)=F^{\text{sing}}(t_c)+F^{\text{reg}}(t_c) \tag{3.47}$$

where the singular parts are

$$F_0^{\text{sing}}(t_c)=\frac{t_c^2}{2}\left(\log\left(-\frac{t_c}{16}\right)-\frac{3}{2}\right), \tag{3.48a}$$

$$F_1^{\text{sing}}(t_c)=-\frac{1}{24}\log\left(-\frac{t_c}{16^2}\right), \tag{3.48b}$$

$$F_n^{\text{sing}}(t_c)=\frac{(1-2^{1-2n})B_{2n}}{(2n)(2n-1)(2n-2)t_c^{2n-2}},\quad n\geq 2. \tag{3.48c}$$

while the first few terms of the regular parts are

$$F_0^{\rm reg}(t_c) = -8Ct_c - \frac{t_c^3}{48} + \frac{5t_c^4}{4608} - \frac{7t_c^5}{61440} + \dots \tag{3.49a}$$

$$F_1^{\rm reg}(t_c) = -\frac{11t_c}{192} + \frac{49t_c^2}{9216} - \frac{77t_c^3}{73728} + \dots \tag{3.49b}$$

$$F_2^{\rm reg}(t_c) = -\frac{101}{221184} - \frac{889t_c}{2949120} + \frac{181981t_c^2}{707788800} + \dots. \tag{3.49c}$$

It was found out by calculations in the 1-instanton sector that the non-perturbative quantum period t_D can be identified with the free energy through [20]

$$t_D(\nu,\phi) = \frac{\partial}{\partial t_c}F(t_c,\hbar)\Big|_{t_c=-\phi\nu,\hbar=-\phi}. \tag{3.50}$$

These identifications between quantities in the Harper-Hofstadter model and observables in 5d SYM is very useful, as both the perturbative Wilson loop vev and the perturbative free energy can be computed very efficiently through the holomorphic anomaly equations [15, 17–19, 65]. More importantly, both of them turn out to be 1-Gevrey divergent power series, and their resurgent structures have been recently completely understood [28].

First of all, each Borel singularity corresponds conjecturally to a BPS state of the 5d SYM. The position of the Borel singularity is a classical period[7]

$$\mathcal{A}_\gamma = p\partial_{t_c}F_0(t_c) + 2\pi \mathsf{i} q t_c + 4\pi^2\mathsf{i} r, \quad \gamma = (p,q,r), \tag{3.51}$$

which is the central charge of the BPS state, and the lattice charge γ is the electromagnetic charge of the BPS state. For free energies, all the BPS states are conjectured to appear, while for Wilson loop vevs, only those whose charges have non-zero Dirac pairing with the charge vector of the flat coordinate appear. In the conifold frame where the flat coordinate is t_c, this means those BPS states with $p \neq 0$. We give examples of plots of Borel singularities for Wilson loop vevs for z on the real axis smaller than and greater than the conifold point $1/16$ respectively in figures 3. These two plots indicate that in the case of $z < 1/16$, the BPS states with small central charges are[8]

$$\gamma = \pm(2,-1,0),\ \pm(2,0,0),\ \pm(2,-1,1). \tag{3.52}$$

while in the case of $z > 1/16$, the BPS states with small central charges are

$$\gamma = \pm(2,0,0),\ \pm(2,1,1). \tag{3.53}$$

The difference of the BPS spectrum in different chambers of the moduli space is known as the wall-crossing phenomenon [66–68], and here it is clearly demonstrated via the change of Borel singularities of Wilson loop vevs. See [37] for additional demonstrations via the change of Borel singularities of free energy.[9]

[7]We use a slightly different convention from [28].

[8]The charge vectors differ from the usual convention in the literatuer by $(0,-1,0)$, as we shifted the definition of $\partial_{t_c}F_0(t_c)$.

[9]To be precise, Borel singularities of self-dual free energy are considered in [37], but they should be one-to-one correspondent with Borel singularitie of NS free energy [69].

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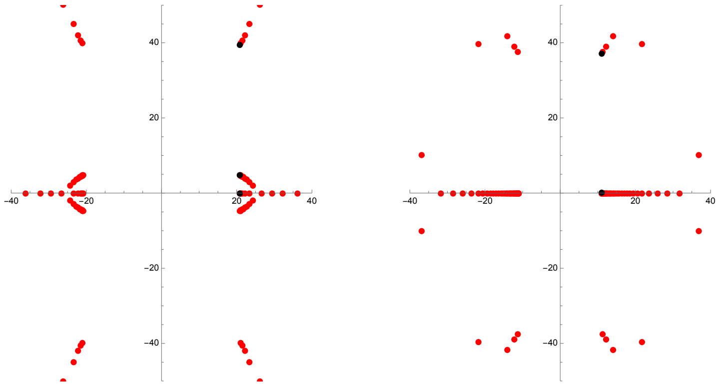

(a) $z = 1/32$ (b) $z = 3/32$

Figure 3. Borel singularities for Wilson loop vev in the 5d SYM theory with (a) $z < 1/16$ and (b) $z > 1/16$ respectively. In the left figure with $z < 1/16$, the Borel singularities marked by black dots have charge vectors $\gamma = (2, -1, 0)$ (on the real axis), $(2, 0, 0)$ (slightly away), $(2, -1, 1)$ (far off in the first quadrant). In the right figure with $z > 1/16$, the Borel singularities marked by black dots have charge vectors $\gamma = (2, 0, 0)$ (on the positive real axis), $(2, 1, 1)$ (far off in the first quadrant).

Secondly, the Stokes discontinuities across Stokes rays are best illustrated through the alien derivatives. If these are a sequence of Borel singularities $A, 2A, \ldots$, the alien derivatives associated to these singular points are

$$\dot{\Delta}_{\ell A_\gamma} W(t_c, \hbar) = \frac{S_\gamma^{\mathrm{BPS}}}{2\pi\mathsf{i}} \hbar \frac{(-1)^\ell}{\ell} p \partial_{t_c} W(t_c, \hbar) \mathrm{e}^{-\ell p \partial_t F_{\mathrm{NS}}^\sharp(t,\hbar)/\hbar}, \tag{3.54a}$$

$$\dot{\Delta}_{\ell A_\gamma} F_{\mathrm{NS}}(t_c, \hbar) = \frac{S_\gamma^{\mathrm{BPS}}}{2\pi\mathsf{i}} \hbar^2 \frac{(-1)^{\ell-1}}{\ell^2} \mathrm{e}^{-\ell p \partial_{t_c} F_{\mathrm{NS}}^\sharp(t_c,\hbar)/\hbar}, \tag{3.54b}$$

where the superscript $\sharp$ means the leading term of free energy is shifted

$$F_0(t_c) \to F_0^\sharp(t_c) = F_0(t_c) + \frac{\pi\mathsf{i}q}{p} t_c^2 + \frac{4\pi^2\mathsf{i}r}{p} t_c, \tag{3.55}$$

so that

$$A_\gamma = p \partial_{t_c} F_0^\sharp(t_c). \tag{3.56}$$

Most importantly, the Stokes constant S_γ^{BPS} is conjectured to coincide with the multiplicity Ω_γ of the BPS state with charge vector γ. For Borel singularities of the perturbative Wilson loop vev at $z < 1/16$ in figure 3(a), which corresponds to the weak coupling regime of the 5d SYM, the Stokes constant of the singularity on the real axis with $\gamma = (2, -1, 0)$ and that of the singularity slightly away with $\gamma = (2, 0, 0)$ and $\gamma = (2, -2, 0)$ are respectively [37]

$$S_{(2,-1,0)}^{\mathrm{BPS,[weak]}} = -4, \quad S_{(2,0,0)}^{\mathrm{BPS,[weak]}} = S_{(2,-2,0)}^{\mathrm{BPS,[weak]}} = 2, \tag{3.57}$$

while for the Borel singularities at $z > 1/16$ in figure 3(b), which corresponds to the strong coupling regime of the 5d SYM, the Stokes constant of the singularity on the positive real axis with $\gamma = (2, 0, 0)$ is [37]

$$S_{(2,0,0)}^{\mathrm{BPS,[strong]}} = 2. \tag{3.58}$$

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4 Full trans-series of Hofstadter's butterfly

In this section, following the weak resurgence program, we will demonstrate that for the Harper-Hofstadter model, at least in the case of $\phi = 2\pi/Q$, the exact energy spectrum is the Borel resummation of the full energy trans-series with different Landau levels. We assume that the universal structure of the full trans-series (3.30) inspired from the analysis of the exact WKB method, which we copy below,

$$E(\nu,\phi) = E^{(0)}(\nu,\phi) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} u_{n,m} E^{(n,m)}(\nu,\phi), \tag{4.1}$$

where we have replaced $\hbar$ by ϕ, and in addition to the perturbative series $E^{(0)}(\nu,\phi)$, the basic building blocks are the basic trans-series

$$E^{(n,m)}(\nu,\phi) = \left(\frac{\partial}{\partial\nu}\right)^m \left(\frac{\partial E^{(0)}(\nu,\phi)}{\partial\nu} \mathrm{e}^{-n t_D(\nu,\phi)/\phi}\right). \tag{4.2}$$

We have also discussed in section 3.3 that $E^{(0)}(\nu,\phi)$ and $t_D(\nu,\phi)$ can be identified with perturbative Wilson loop vev and perturbative free energy from 5d SU(2) SYM via (3.45) and (3.50). We will justify the assumption (4.1) by calculating the trans-series coefficients $u_{n,m}$ and then by making precision comparison with the exact spectrum from the secular equation (2.18).

4.1 Borel summability of perturbative energy series

We first discuss the Borel summability of the perturbative energy series, i.e. whether we can perform the vanilla version of the Borel resummation or lateral resummations are needed.

We collect about 200 terms of the perturbative energy series $E^{(0)}(\nu,\phi)$ computed via either the `BenderWu` package or via Wilson loop vev, the position of Borel singularities for the energy series at different Landau levels are given in figure 4. In all these plots, the dominant Borel singularities are

$$16C, \quad 16C \pm 4\pi^2\mathsf{i}, \tag{4.3}$$

where C is the Catalan number. There are several consequences of this pattern of Borel singularities.

The first consequence is that since the singular point $16C$ is on the positive real axis, the naive version of Borel resummation fails. We have to adopt either choice of lateral Borel resummations, and the ambiguity thus entailed should be compensated by an appropriate jump of the trans-series coefficients. We should update the structure of full trans-series (4.1) to

$$E_{\theta_{x,y},\epsilon}(\nu,\phi) = E^{(0)}(\nu,\phi) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} u_{n,m}(\theta_{x,y},\epsilon) E^{(n,m)}(\nu,\phi), \tag{4.4}$$

such that the exact energy spectrum is

$$E^{\text{ex}}_{\theta_{x,y}}(\nu,\phi) = \mathscr{S}^{\pm}(E_{\theta_{x,y}},\pm 1)(\nu,\phi). \tag{4.5}$$

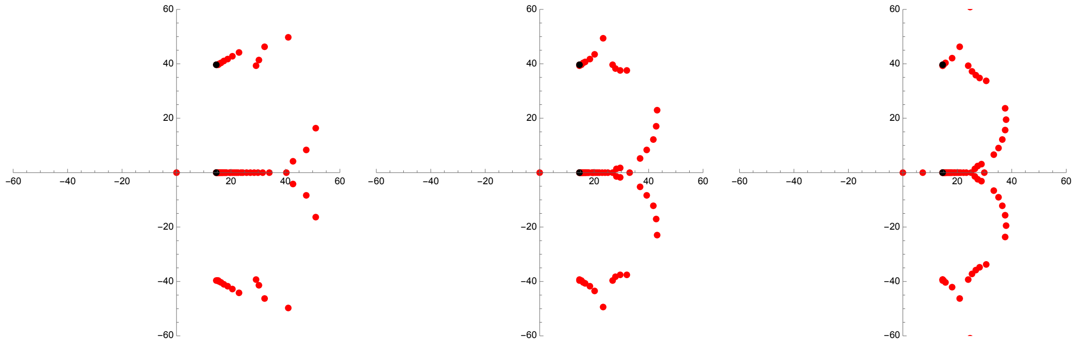

Figure 4. Borel singularities of perturbative energy series at Landau levels $N = 0, 1, 2$. The singularities marked by black dots on the positive real axis and off in the first quadrant in all three plots are $16C$ and $16C + 4\pi^2\mathrm{i}$, where C is the Catalan number. The arcs of singular points on the right periphery of each plot are due to numerical instability and thus are spurious.

Note that only the trans-series coefficients $u_{n,m}$ depend on the contour of lateral resummation as well as the Bloch angles, while the trans-series building blocks $E^{(n,m)}$ do not.

Furthermore, the position of the most dominant Borel singularity coincides with twice the 1-instantion $S_c = 8C$ discussed in section 2.3. This means that the 1-instanton contribution $E^{(1,0)}$ is invisible from the Borel transform of the perturbative series, and cannot be extracted from the latter by the resurgence technique. In other words, the strong resurgence program fails for the Harper-Hofstadter energy spectrum, which is the second consequence of the pattern of Borel singularities. This failure of the strong resurgence program is due to the presence of Bloch angles, and similar phenomena are already well-known, for instance in the non-relativistic cosine model [35].[10]

Finally, we comment that even though the perturbative energy series of the Harper-Hofstadter model and the perturbative Wilson loop vev of the 5d SYM are identified via (3.45), the distribution of their Borel singularities look rather different, cf. figures 4 and 3. The most conspicuous discrepancy is that the singular points in figures 3 for the Wilson loop are left-right symmetric while those in figures 4 for the Harper-Hofstadter energy are one-sided. This can be explained by the observation that the perturbative Wilson loop vev (3.42) has the symmetry

$$W(t_c, -\hbar) = W(t_c, \hbar), \tag{4.6}$$

which is broken in the perturbative Harper-Hofstadter energy. More detailed explanation is the following.

In the identification (3.45) we use the dictionary

$$t_c = -\phi\nu, \quad \hbar = -\phi, \tag{4.7}$$

which means in the limit $\hbar = -\phi \to 0$, both t_c and $\hbar$ are sent to zero simultaneously. The relation between the series $E(\nu, \phi)$ and the series $W(t_c, \hbar)$ is akin to the relation between a

[10] Or even in simpler models like the double-well model, where the parity is a discrete analogue of the Bloch angle.

$1/N$ expansion and its 't Hooft limit, and the relation between their respective resurgent structures, including the Borel singularities, the non-perturbative series, and the Stokes constants, is recently discussed in detail in [37]. We will follow their discussion and explain the relation between the resurgent structure of $E(\nu,\phi)$ and $W(t_c,\hbar)$.

When we map from the resurgent structure of $W(t_c,\hbar)$ to that of $E(\nu,\phi)$, several changes will happen. The first change is that half of the non-perturbative series will vanish and thus the associated Borel singularities will disappear. Let us examines how this happens by deriving carefully the non-perturbative series for $E(\nu,\phi)$ from those for $W(t_c,\hbar)$. Recall that if there is a sequence of Borel singularities $\mathcal{A}_\gamma, 2\mathcal{A}_\gamma,\ldots$ for $W(t_c,\hbar)$, the alien derivatives for these singular points are given in (3.54a), from which we read off the lowest non-perturbative series, the contribution at $\ell=1$ excluding the Stokes constant

$$W^{(1)}(t_c,\hbar) = \hbar\, p \frac{\partial W^{(0)}(t_c,\hbar)}{\partial t_c} \exp\left(-\frac{p}{\hbar}\frac{\partial F^{\sharp}_{\mathrm{NS}}(t_c,\hbar)}{\partial t_c}\right). \tag{4.8}$$

Here p is the magnetic charge of the BPS state associated to the Borel singularity $\mathcal{A}_\gamma$, and we will assume that it is positive.

As seen in (3.46), (3.47), the free energy consists of both the singular part and the regular part, and we discuss their behavior after the dictionary (4.7) is applied. The coefficients of the regular part (3.49) have the form

$$F^{\mathrm{reg}}_n(t_c) = \sum_{m\geq 1} f_{n,m} t_c^m, \tag{4.9}$$

where we ignore the constant term. After applying the dictionary (4.7), its derivative becomes[11]

$$\frac{\partial F^{\mathrm{reg}}(t_c,\hbar)}{\partial t_c} \;\to\; f_{0,1} + \sum_{n\geq 2}(-\phi)^n \sum_{m=0}^{[n/2]}(n+1-2m) f_{m,n+1-2m}\nu^{n-2m}, \tag{4.10}$$

where the leading constant is, cf. (3.49)

$$f_{0,1} = \left.\frac{\partial F^{\mathrm{reg}}_0(t_c)}{\partial t_c}\right|_{t_c\to 0} = -8C. \tag{4.11}$$

Next, we consider the singular part. With (3.48), one finds, after applying the dictionary (4.7),

$$-\frac{1}{\hbar}\frac{\partial F^{\mathrm{sing}}(t_c,\hbar)}{\partial t_c} = \nu - \nu\log\left(\frac{\nu}{16}\right) + \frac{1}{24\nu} + \sum_{n\geq 2}\frac{1-2^{1-2n}}{(2n)(2n-1)}\frac{B_{2n}}{\nu^{2n-1}} - \nu\log\phi. \tag{4.12}$$

It is proposed in [70] that this can be regularised as

$$-\frac{1}{\hbar}\frac{\partial F^{\mathrm{sing}}(t_c,\hbar)}{\partial t_c} \to \log\left(\frac{\sqrt{2\pi}\,16^\nu}{\Gamma(\nu+1/2)}\right) - \nu\log\phi, \tag{4.13}$$

the reason being that the large ν expansion of the right hand side reproduces the power series in $\hbar$ in the left hand side. The similar idea is used in [37].

[11]The $n=1$ terms vanishes because it only involves $f_{0,2}$, which according to (3.49) is zero.

Combining (4.10) and (4.13), non-perturbative correction $W^{(1)}(t_c,\hbar)$ becomes

$$W^{(1)}(t_c,\hbar) \;\to p\frac{\partial E^{(0)}(\nu,\phi)}{\partial \nu}\left(\frac{\sqrt{2\pi}}{\Gamma(\nu+1/2)}\right)^p\left(\frac{16}{\phi}\right)^{p\nu}\mathrm{e}^{-\frac{p\mathcal{A}_{(p,r)}}{\phi}-2\pi\mathsf{i}q\nu}$$
$$\exp\left(p\sum_{n\geq 2}(-\phi)^{n-1}\sum_{m=0}^{[n/2]}(n+1-2m)f_{m,n+1-2m}\nu^{n-2m}\right), \tag{4.14}$$

where

$$\mathcal{A}_{(p,r)} = -pf_{0,1} - 4\pi^2\mathsf{i}r. \tag{4.15}$$

The Wilson loop vevs have Borel singularities not only at $A_\gamma, 2A_\gamma, \ldots$, but also at the opposite sites $-A_\gamma, -2A_\gamma, \ldots$, and the associated non-perturbative corrections are denoted by $W^{(0|\ell)}$. The non-perturbative corrections to the Harper-Hofstadter energy series inherited from $W^{(0|\ell)}$, according to [37], are obtained from (4.14) (in the $\ell = 1$ case) via

$$(\phi,\nu) \;\to (-\phi,-\nu), \tag{4.16}$$

which can be justified by the following symmetry of the perturbative energy series

$$E^{(0)}(-\nu,-\phi) = E^{(0)}(\nu,\phi). \tag{4.17}$$

Therefore, for instance, the leading one of these non-perturbative corrections is

$$W^{(0|1)}(t_c,\hbar) \;\to\; -\frac{p}{2\pi\mathsf{i}}\frac{\partial E^{(0)}(-\nu,-\phi)}{\partial \nu}\left(\frac{\sqrt{2\pi}}{\Gamma(-\nu+1/2)}\right)^p\left(-\frac{16}{\phi}\right)^{p\nu}\mathrm{e}^{+\frac{p\mathcal{A}_{p,r}}{\phi}-2\pi\mathsf{i}q\nu}$$
$$\exp\left(p\sum_{n\geq 2}(+\phi)^{n-1}\sum_{m=0}^{[n/2]}(n+1-2m)f_{m,n+1-2m}(-\nu)^{n-2m}\right), \tag{4.18}$$

which vanishes for $\nu = 1/2, 3/2, \ldots$ due to the pole of the Gamma function in the denominator. This explains why we do not see these Borel singularities at opposite sites for the perturbative energy series of the Harper-Hofstadter model.

A corollary of this analysis and (4.14) is that the set of Borel singularities $A_{(p,q,r)}$ of the perturbative Wilson loop vev with fixed p,r but different q all collapse to a single Borel singularity $\mathcal{A}_{(p,r)}$ of the perturbative Harper-Hofstadter energy series. In addition, the alien derivative at this Borel singularity is

$$\dot{\Delta}_{\mathcal{A}_{(p,r)}}E^{(0)}(\nu,\phi) = \frac{S_{(p,r)}}{2\pi\mathsf{i}}\mathcal{E}^{(p,r)}(\nu,\phi), \tag{4.19}$$

where

$$\mathcal{E}^{(p,r)}(\nu,\phi) = \frac{\partial E^{(0)}(\nu,\phi)}{\partial \nu}\left(\frac{\sqrt{2\pi}}{\Gamma(\nu+1/2)}\right)^p\left(\frac{16}{\phi}\right)^{p\nu}\mathrm{e}^{-\frac{p\mathcal{A}_{(p,r)}}{\phi}}$$
$$\exp\left(p\sum_{n\geq 2}(-\phi)^{n-1}\sum_{m=0}^{[n/2]}(n+1-2m)f_{m,n+1-2m}\nu^{n-2m}\right), \tag{4.20}$$

and

$$S_{(p,r)} = p \sum_q S^{\rm BPS}_{(p,q,r)} \mathrm{e}^{-2\pi \mathrm{i} q\nu}. \tag{4.21}$$

Since $\nu = N+1/2$ and $q \in \mathbb{Z}$, the right hand side of (4.21) does not depend on the Landau level.

One important subtlety is that after using the dictionary (4.7), in the limit $\phi \to 0$, the flat coordinate t_c is sent to zero, and we are approaching the wall of marginal stability across which the Borel singularities as well as the Stokes constants change, a phenomenon related to the wall-crossing phenomenon in 5d SYM as we mentioned at the end of section 3.3. It is then ambiguous which Stokes constants of the perturbative Wilson loop vev should be used to compute the Stokes constants of the perturbative Harper-Hofstadter energy series, a question already posed in a similar contextd in [37]. The answer from [37], which was found emprically, is that we should use Stokes constants from the strong coupling regime, which corresponds to $z > 1/16$ in this example. We verify that this is also the case here.

Let us consider the dominant Borel singularity

$$\mathcal{A}_{(2,0)} = 16C \tag{4.22}$$

for the perturbative energy series on the positive real axis. We recognise that $\mathcal{E}^{(2,0)}(\nu,\phi) = E^{(2,0)}(\nu,\phi)$, and in particular, this implies that

$$E^{(1,0)}(\nu,\phi) = \frac{\partial E^{(0)}(\nu,\phi)}{\partial \nu} \frac{\sqrt{2\pi}}{\Gamma(\nu+1/2)} \left(\frac{16}{\phi}\right)^{\nu} \mathrm{e}^{-8C/\phi} \left(1 + \ldots\right), \tag{4.23}$$

which agrees with (2.29) up to a trans-series coefficient, i.e.

$$E^{(1)}_{\theta_x,\theta_y}(\nu,\phi) = w_{1,0}(\theta_{x,y}) E^{(1,0)}(\nu,\phi) \tag{4.24}$$

and the trans-series coefficient $w_{1,0}(\theta_{x,y})$ is given by

$$w_{1,0}(\theta_{x,y}) = (-1)^{N+1} \frac{\cos\theta_x + \cos\theta_y}{\pi} \tag{4.25}$$

as we will confirm in section 4.3. Then (4.19) becomes

$$\dot{\Delta}_{\mathcal{A}_{(2,0)}} E^{(0)}(\nu,\phi) = \frac{S_{(2,0)}}{2\pi\mathrm{i}} E^{(2,0)}(\nu,\phi). \tag{4.26}$$

This Borel singularity can descend via (4.21) either from the three Borel singularities with $\gamma = (2,-1,0), (2,-1\pm 1,0)$ and respective Stokes constants in (3.57) of the perturbative Wilson loop in the weak coupling regime, in which case, the predicted Stokes constant associated to $\mathcal{A}_{(2,0)}$ is

$$S^{[\rm weak]}_{(2,0)} = 16, \tag{4.27}$$

or from the single Borel singularity with $\gamma = (2,0,0)$ and Stokes constant in (3.58) of the Wilson loop in the strong coupling regime, in which case, the predicted Stokes constant associated to $\mathcal{A}_{(2,0)}$ is

$$S^{[\rm strong]}_{(2,0)} = 4. \tag{4.28}$$

The actual numerical calculation of the Stokes discontinuity of $E^{(0)}$ across the positive real axis compared with the right hand side of (4.26) shows the latter is the case, i.e.

$$S_{(2,0)} = S_{(2,0)}^{[\text{strong}]} = 4. \tag{4.29}$$

Another way to see that this has to be the case is notice that the energy of the Harper-Hofstadter model has the property that $-4 < E < 4$, and this is translated to the modulus in the 5d SYM as $z = 1/u^2 = 1/E^2 > 1/16$, which corresponds to the strong coupling regime.

4.2 Minimal trans-series

We would like to study the general resurgent structure of the perturbative energy series, not only the dominant Borel singularity. Since we will be interested in the Borel resummation with real and positive ϕ, we focus on the Borel singularities on the positive real axis. We conjecture the only Borel singularities of this type are $\mathcal{A}_{(2,0)} = 16C$ and its multiples, and we will denote them simply by

$$\ell\mathcal{A}, \quad \mathcal{A} := \mathcal{A}_{(2,0)}, \ \ell = 1, 2, \ldots. \tag{4.30}$$

The action of the alien derivatives of these Borel singularities $\dot{\Delta}_{k\mathcal{A}}$ on the perturbative energy series should follow from the alien derivatives of the Wilson loop vev (3.54a). By a similar calculation as in the previous section, or by simply comparing the right hand side of (3.54a) with the definition of $E^{(n,m)}$ given in (3.31) together with the dictionary (3.50), we can conclude that

$$\dot{\Delta}_{\ell\mathcal{A}} E^{(0)}(\nu,\phi) = \frac{S_{\mathcal{A}}}{2\pi\mathsf{i}} \frac{(-1)^{\ell-1}}{\ell} E^{(2\ell,0)}(\nu,\phi), \tag{4.31}$$

where, as we discussed in the previous section,

$$S_{\mathcal{A}} := S_{(2,0)} = 4. \tag{4.32}$$

It is also useful to consider the resurgent structure of the trans-series building blocks $E^{(n,m)}(\nu,\phi)$. Starting from (3.54b), and using the chain rule of alien derivatives as well as that it commutes with ordinary derivatives, one finds

$$\dot{\Delta}_{\ell A_{(2,0,0)}} \mathrm{e}^{-n\partial_{t_c} F_{\text{NS}}(t_c,\hbar)/\hbar} = \frac{S^{\text{BPS}}_{(2,0,0)}}{2\pi\mathsf{i}} \frac{\hbar(-1)^{(\ell-1)}}{\ell} (2n\partial_{t_c}^2 F_{\text{NS}}(t_c,\hbar)) \mathrm{e}^{-(n+2\ell)\partial_{t_c} F_{\text{NS}}(t_c,\hbar)/\hbar}. \tag{4.33}$$

Using this result, the Leibniz rule of alien derivatives, and following the derivation as in the previous section, together with (4.21), (4.29), one finds that

$$\dot{\Delta}_{\ell\mathcal{A}} E^{(n,m)}(\nu,\phi) = \frac{S_{\mathcal{A}}}{2\pi\mathsf{i}} \frac{(-1)^{\ell-1}}{\ell} E^{(n+2\ell,m+1)}(\nu,\phi). \tag{4.34}$$

It also implies that $E^{(n,m)}(\nu,\phi)$ has the same Borel singularities as $E^{(0)}(\nu,\phi)$.

It was then argued in [32] that we can include all corrections to the energy perturbative series from all of its Borel singularities on the positive real axis via the minimal trans-series

$$E^{(0)}_{\text{min}}(\nu,\phi;\sigma) = E^{(0)}(\nu,\phi) + \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'-1} \sigma^{m'+1} v_{n',m'} E^{(2n',m')}(\nu,\phi), \tag{4.35}$$

m	0	1	2	3
$v_{1,m}$	$-\frac{\mathsf{i}}{2\pi}$			
$v_{2,m}$	$\frac{\mathsf{i}}{4\pi}$	$-\frac{1}{8\pi^2}$		
$v_{3,m}$	$-\frac{\mathsf{i}}{6\pi}$	$\frac{1}{8\pi^2}$	$-\frac{\mathsf{i}}{48\pi^3}$	
$v_{4,m}$	$\frac{\mathsf{i}}{8\pi}$	$-\frac{11}{96\pi^2}$	$-\frac{\mathsf{i}}{32\pi^3}$	$\frac{1}{384\pi^4}$

Table 1. Trans-series coefficients $v_{n,m}$ in minimal trans-series.

where the trans-series coefficients are

$$v_{n,m} = \frac{1}{n!} B_{n,m+1}(1!s_1, 2!s_2, \ldots, (n-m)!s_{n-m}), \tag{4.36}$$

with $B_{n,m+1}$ being the incomplete Bell's polynomials and

$$s_j = \frac{(-1)^{j-1}}{j} \frac{1}{2\pi\mathsf{i}}, \quad j = 1, 2, \ldots. \tag{4.37}$$

The first few trans-series coefficients $v_{n,m}$ are given in table 1. The minimal trans-series $E^{(0)}_{\text{min}}$ has the nice property that its Stokes automorphism across the positive real axis is given by

$$\mathfrak{S}_0 E^{(0)}_{\text{min}}(\nu, \phi; \sigma) = E^{(0)}_{\text{min}}(\nu, \phi; \sigma + S_{\mathcal{A}}). \tag{4.38}$$

To see this, we first notice that

$$\begin{aligned}
\mathfrak{S}_0 E^{(0)}(\nu, \hbar) &= \left(\exp \sum_{\ell=1}^{\infty} \dot{\Delta}_{\ell\mathcal{A}} \right) E^{(0)}(\nu, \hbar) \\
&= E^{(0)} + \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'-1} \frac{1}{n'!} B_{n',m'+1}(j! \dot{\Delta}_{j\mathcal{A}}) E^{(0)} \\
&= E^{(0)} + \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'-1} S_{\mathcal{A}}^{m'+1} \frac{1}{n'!} B_{n',m'+1}(j! s_j) E^{(2n',m')} \\
&= E^{(0)} + \sum_{n'=1}^{\infty} \sum_{m'=0}^{n'-1} S_{\mathcal{A}}^{m'+1} v_{n',m'} E^{(2n',m')},
\end{aligned} \tag{4.39}$$

where from the first line to the second line we used the Faà di Bruno formula, and from the second line to the third line we have used the resurgent properties (4.31), (4.34) as well as the homogeneity property of incomplete Bell's polynomials

$$B_{n,m+1}(\alpha\beta x_1, \alpha^2\beta x_2, \ldots, \alpha^{n-m}\beta x_{n-m}) = \alpha^n \beta^{m+1} B_{n,m+1}(x_1, x_2, \ldots, x_{n-m}) \tag{4.40}$$

Comparing (4.39) with (4.35), we conclude that

$$\mathfrak{S}_0 E^{(0)}(\nu, \hbar) = \mathfrak{S}_0 E^{(0)}_{\text{min}}(\nu, \hbar; 0) = E^{(0)}_{\text{min}}(\nu, \hbar; S_{\mathcal{A}}). \tag{4.41}$$

Furthermore, let us define

$$\dot{\Delta}^{(s)}_{\ell\mathcal{A}} E^{(n,m)} = \frac{(-1)^{\ell-1}}{\ell} E^{(n+2\ell,m+1)} \tag{4.42}$$

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n	$\mathscr{S}^{(+)}E^{(0)}_{\text{min}}(0,\phi,-2)$	$\mathscr{S}^{(-)}E^{(0)}_{\text{min}}(0,\phi,+2)$
0	$3.545 + 3.794 \times 10^{-12}\mathrm{i}$	$3.545 - 3.794 \times 10^{-12}\mathrm{i}$
2	$3.545 - 2.485 \times 10^{-23}\mathrm{i}$	$3.545 + 2.485 \times 10^{-23}\mathrm{i}$
4	$3.545 - 6.074 \times 10^{-33}\mathrm{i}$	$3.545 + 6.074 \times 10^{-33}\mathrm{i}$
6	$3.545 - 2.074 \times 10^{-38}\mathrm{i}$	$3.545 + 2.074 \times 10^{-38}\mathrm{i}$

Table 2. Borel resummation of minimal trans-series $E^{(0)}_{\text{min}}(N,\phi;\mp 2)$ at Landau level $N=0$ with $\phi = 2\pi/13$. n is the level of instanton corrections included. As higher level instanton corrections are included, the imaginary part of the resummation becomes smaller.

n	$\mathscr{S}^{(+)}E^{(0)}_{\text{min}}(0,\phi,-2)$	$\mathscr{S}^{(-)}E^{(0)}_{\text{min}}(0,\phi,+2)$
0	$3.736 + 2.985 \times 10^{-22}\mathrm{i}$	$3.736 - 2.985 \times 10^{-22}\mathrm{i}$
2	$3.736 - 2.651 \times 10^{-43}\mathrm{i}$	$3.736 + 2.651 \times 10^{-43}\mathrm{i}$
4	$3.736 + 2.247 \times 10^{-60}\mathrm{i}$	$3.736 - 2.247 \times 10^{-60}\mathrm{i}$

Table 3. Borel resummation of minimal trans-series $E^{(0)}_{\text{min}}(N,\phi;\mp 2)$ at Landau level $N=0$ with $\phi = 2\pi/23$. n is the level of instanton corrections included. As higher level instanton corrections are included, the imaginary part of the resummation becomes smaller.

so that $\dot{\Delta}_{\ell\mathcal{A}} = S_{\mathcal{A}} \dot{\Delta}^{(s)}_{\ell\mathcal{A}}$ and

$$\mathfrak{S}_0[S_{\mathcal{A}}] = \exp\left(S_{\mathcal{A}} \sum_{\ell=1}^{\infty} \dot{\Delta}^{(s)}_{\ell\mathcal{A}} \right). \tag{4.43}$$

Since

$$\mathfrak{S}_0[S_1]\mathfrak{S}_0[S_2] = \mathfrak{S}_0[S_1+S_2], \tag{4.44}$$

we finally arrive at

$$\begin{aligned}\mathfrak{S}_0 E^{(0)}_{\text{min}}(\nu,\phi;\sigma) &= \mathfrak{S}_0[S_{\mathcal{A}}]\mathfrak{S}_0[\sigma]E^{(0)}_{\text{min}}(\nu,\phi;0) = \mathfrak{S}_0[S_{\mathcal{A}}+\sigma]E^{(0)}_{\text{min}}(\nu,\phi;0) \\ &= E^{(0)}_{\text{min}}(\nu,\phi;\sigma+S_{\mathcal{A}}).\end{aligned} \tag{4.45}$$

In the example of the Harper-Hofstadter model, with $S_{\mathcal{A}} = 4$ for the Borel singularities $\mathcal{A} = 16C$, this implies that there is an ambiguity-free prescription of performing Borel resummation of the minimal trans-series

$$\mathscr{S}^{(+)}E^{(0)}_{\text{min}}(\nu,\phi;-2) = \mathscr{S}^{(-)}E^{(0)}_{\text{min}}(\nu,\phi;+2), \tag{4.46}$$

which has the additional nice property that it is a real value, in contrast to lateral resummations of $E^{(0)}(\nu,\phi)$ which are always complex. Some numerical evidences are provided in tables 2, 3, 4, 5.

For later purpose, we will also introduce the minimal trans-series for the building blocks $E^{(n,m)}$ and they read

$$E^{(n,m)}_{\text{min}}(\nu,\phi;\sigma) := E^{(n,m)}(\nu,\phi) + \sum_{n'=1}^{\infty}\sum_{m'=0}^{n'-1} \sigma^{m'+1} v_{n',m'} E^{(n+2n',m+m'+1)}(\nu,\phi). \tag{4.47}$$

n	$\mathscr{S}^{(+)}E^{(0)}_{\text{min}}(1,\phi,-2)$	$\mathscr{S}^{(-)}E^{(0)}_{\text{min}}(1,\phi,+2)$
0	$2.691+3.190\times 10^{-9}\mathrm{i}$	$2.691-3.190\times 10^{-9}\mathrm{i}$
2	$2.691-2.001\times 10^{-17}\mathrm{i}$	$2.691+2.001\times 10^{-17}\mathrm{i}$
4	$2.691-2.454\times 10^{-24}\mathrm{i}$	$2.691+2.454\times 10^{-24}\mathrm{i}$
6	$2.691+7.118\times 10^{-32}\mathrm{i}$	$2.691-7.118\times 10^{-32}\mathrm{i}$

Table 4. Borel resummation of minimal trans-series $E^{(0)}_{\text{min}}(N,\phi;\mp 2)$ at Landau level $N=1$ with $\phi=2\pi/13$. n is the level of instanton corrections included. As higher level instanton corrections are included, the imaginary part of the resummation becomes smaller.

n	$\mathscr{S}^{(+)}E^{(0)}_{\text{min}}(1,\phi,-2)$	$\mathscr{S}^{(-)}E^{(0)}_{\text{min}}(1,\phi,+2)$
0	$3.226+8.873\times 10^{-19}\mathrm{i}$	$3.226-8.873\times 10^{-19}\mathrm{i}$
2	$3.226-2.514\times 10^{-36}\mathrm{i}$	$3.226+2.514\times 10^{-36}\mathrm{i}$
4	$3.226-2.138\times 10^{-52}\mathrm{i}$	$3.226+2.138\times 10^{-52}\mathrm{i}$

Table 5. Borel resummation of minimal trans-series $E^{(0)}_{\text{min}}(N,\phi;\mp 2)$ at Landau level $N=1$ with $\phi=2\pi/23$. n is the level of instanton corrections included. As higher level instanton corrections are included, the imaginary part of the resummation becomes smaller.

Using a similar argument with (4.34), one can show that across the positive real axis

$$\mathfrak{S}_0 E^{(n,m)}_{\text{min}}(\nu,\phi;\sigma) = E^{(n,m)}_{\text{min}}(\nu,\phi;\sigma+S_{\mathcal{A}}). \tag{4.48}$$

4.3 Full trans-series and the exact quantization condition

The minimal trans-series $E^{(0)}_{\text{min}}(\nu,\phi;\sigma)$ encodes the minimal resurgent structure starting from $E^{(0)}(\nu,\phi)$ accessible via Borel singularities on the positive real axis. If the strong resurgence program were to hold here, it would be the entire story, and the Borel resummation (4.46) would be the exact energy spectrum. But as we have discussed in section 4.1, it misses at least the 1-instanton sector, and the full energy trans-series would be a superset of the minimal energy trans-series.

The way to construct a larger and full trans-series which includes the minimal trans-series as a consistent component is via the procedure of "tensor product" of trans-series introduced in [32]. We assume that the full energy trans-series still has the form of (3.30). Suppose the right hand side of (3.29) can split into the sum of two functions

$$R(\lambda) = R^A(\lambda) + R^B(\lambda) = \sum_{k=1}^{\infty}(r_k^A + r_k^B)\lambda^k \tag{4.49}$$

where we have replaced $\hbar$ by ϕ, we can define two implicit equations[12]

$$\Delta\nu_A = R^A(\lambda(\nu+\Delta\nu_A)), \tag{4.50}$$

$$\Delta\nu_B = R^B(\lambda(\nu+\Delta\nu_B)), \tag{4.51}$$

[12] Note that $\Delta\nu$ as solution to (3.29) is not equal to the sum of $\Delta\nu_A$,$\Delta\nu_B$ as solutions to (4.50), (4.51).

with $\lambda(\nu) = \mathrm{e}^{-\frac{1}{2\phi}t_D(\nu,\phi)}$, and the solutions define respectively two sets of trans-series: the trans-series of A type

$$\begin{aligned} E_A^{(0)}(\nu,\phi) &= \mathrm{e}^{\Delta\nu_A\partial_\nu}E^{(0)}(\nu,\phi) \\ &= E^{(0)}(\nu,\phi) + \sum_{n'=1}^{\infty}\sum_{m'=0}^{n'-1} u_{n',m'}(\underline{r}^A)E^{(n',m')}(\nu,\phi), \qquad (4.52a) \\ E_A^{(n,m)}(\nu,\phi) &= \mathrm{e}^{\Delta\nu_A\partial_\nu}E^{(n,m)}(\nu,\phi) \\ &= E^{(n,m)}(\nu,\phi) + \sum_{n'=1}^{\infty}\sum_{m'=0}^{n'-1} u_{n',m'}(\underline{r}^A)E^{(n+n',m+m'+1)}(\nu,\phi), \qquad (4.52b) \end{aligned}$$

and the trans-series of B type

$$\begin{aligned} E_B^{(0)}(\nu,\phi) &= \mathrm{e}^{\Delta\nu_B\partial_\nu}E^{(0)}(\nu,\phi) \\ &= E^{(0)}(\nu,\phi) + \sum_{n'=1}^{\infty}\sum_{m'=0}^{n'-1} u_{n',m'}(\underline{r}^B)E^{(n',m')}(\nu,\phi), \qquad (4.53a) \\ E_B^{(n,m)}(\nu,\phi) &= \mathrm{e}^{\Delta\nu_B\partial_\nu}E^{(n,m)}(\nu,\phi) \\ &= E^{(n,m)}(\nu,\phi) + \sum_{n'=1}^{\infty}\sum_{m'=0}^{n'-1} u_{n',m'}(\underline{r}^B)E^{(n+n',m+m'+1)}(\nu,\phi), \qquad (4.53b) \end{aligned}$$

and the full trans-series

$$E(\nu,\phi) = E^{(0)}(\nu,\phi) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} u_{n,m}(\underline{r})E^{(n,m)}(\nu,\phi), \qquad (4.54)$$

can be formulated as the "tensor product",

$$\text{full trans-series} \simeq \text{trans-series } A \otimes \text{trans-series } B, \qquad (4.55)$$

in the sense that it can be equally written as

$$E(\nu,\phi) = E_A^{(0)}(\nu,\phi) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} u_{n,m}(\underline{r}^B)E_A^{(n,m)}(\nu,\phi). \qquad (4.56)$$

This can be understood as arising from a two step application of (3.28), (3.30),

$$\begin{aligned} E(\nu,\phi) = e^{\Delta\nu_A\partial\nu}e^{\Delta\nu_B\partial\nu}E^{(0)}(\nu) &= E^{(0)}(\nu+\Delta\nu^A) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} u_{n,m}(\underline{r}^B)E^{(n,m)}(\nu+\Delta\nu^A,\phi) \\ &= E_A^{(0)}(\nu) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} u_{n,m}(\underline{r}^B)E_A^{(n,m)}(\nu,\phi), \qquad (4.57) \end{aligned}$$

and it can be verified by checking the identities of trans-series coefficients

$$\begin{aligned} u_{n,m}(\underline{r}) = u_{n,m}(\underline{r}^A) + u_{n,m}(\underline{r}^B) &+ \sum_{n'=1}^{n-1}\sum_{m'=\max(m-n+n',0)}^{\min(m-1,n'-1)} u_{n-n',m-m'-1}(\underline{r}^B)u_{n',m'}(\underline{r}^A), \\ n = 1,2,\ldots,\infty,\ m &= 0,1,\ldots,n-1, \qquad (4.58) \end{aligned}$$

obtained by comparing coefficients of (4.54) and (4.56).

In the case of the Harper-Hofstadter model, we first notice that the minimal trans-series (4.35) can be put in the form of (3.30) with

$$u_{n,m}(\underline{r}^{\min}) = \begin{cases} v_{n/2,m}, & \text{even } n, \\ 0, & \text{odd } n, \end{cases} \qquad r_j^{\min} = \begin{cases} s_{j/2}, & \text{even } j, \\ 0, & \text{odd } j. \end{cases} \tag{4.59}$$

Therefore, the minimal trans-series can be written as

$$E^{(0)}_{\min}(\nu,\phi;\sigma) = E^{(0)}(\nu+\sigma\Delta\nu^{\min},\phi) \tag{4.60}$$

where $\Delta\nu^{\min}$ is solution to

$$\Delta\nu^{\min} = \sum_{j=1}^{\infty} s_j \lambda^{2j} =: R^{\min}(\lambda), \qquad \lambda = \mathrm{e}^{-\frac{1}{2\phi}t_D(\nu+\Delta\nu^{\min},\phi)} \tag{4.61}$$

Now without loss of generality, we can assume that for the Harper-Hofstadter model, the right hand side of (3.29) can indeed be split as

$$\Delta\nu = \sigma R^{\min}(\lambda) + R^{\mathrm{med}}(\lambda) = \sum_{j\geq 1}(\sigma r_j^{\min} + r_j^{\mathrm{med}})\lambda^j \tag{4.62}$$

where r_j^{med} are yet unknown. Then the full trans-series can be written as

$$E_{\theta_{x,y},\sigma}(\nu,\phi) = E^{(0)}_{\min}(\nu,\phi;\sigma) + \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} w_{n,m}(\theta_{x,y}) E^{(n,m)}_{\min}(\nu,\phi;\sigma), \tag{4.63}$$

where we have denoted the trans-series coefficients

$$w_{n,m} := u_{n,m}(\underline{r}^{\mathrm{med}}(\theta_{x,y})) \tag{4.64}$$

which depend on the Bloch angles θ_x, θ_y.

The weak resurgence program dictates that in the regime $\phi\nu \ll 1$, the exact energy spectrum is given by

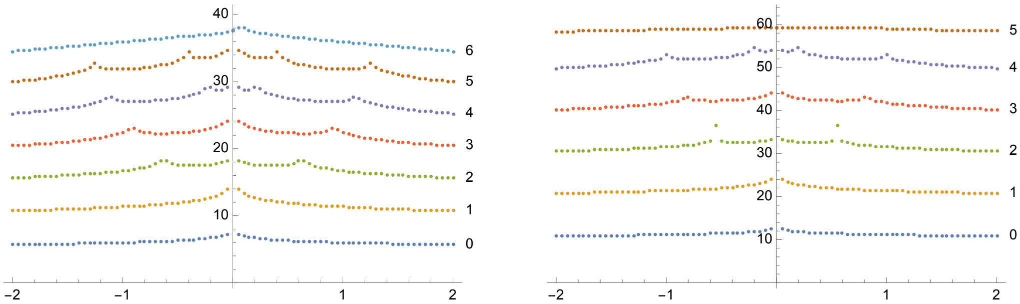
$$E^{\mathrm{ext}}_{\theta_x,\theta_y}(\nu,\phi) = \mathscr{S}^{(+)} E_{\theta_x,\theta_y,-2}(\nu,\phi) = \mathscr{S}^{(-)} E_{\theta_x,\theta_y,+2}(\nu,\phi). \tag{4.65}$$

The resurgent properties (4.38) and (4.48) make sure that the two prescriptions of lateral Borel resummation yield the same result. This gives us a method to fix the unknown trans-series coefficients $w_{n,m}$. By comparing with the exact energy spectrum solved from the secular equation (2.18) at $\phi = 2\pi/Q$, with high precision numerical calculations, we find the first few trans-series coefficients $w_{n,m}$ up to $n = 6$, i.e. up to 6-instanton order, as tabulated in table 6. Here we have introduced notation

$$\Theta := (-1)^{N+1}(\cos\theta_x + \cos\theta_y). \tag{4.66}$$

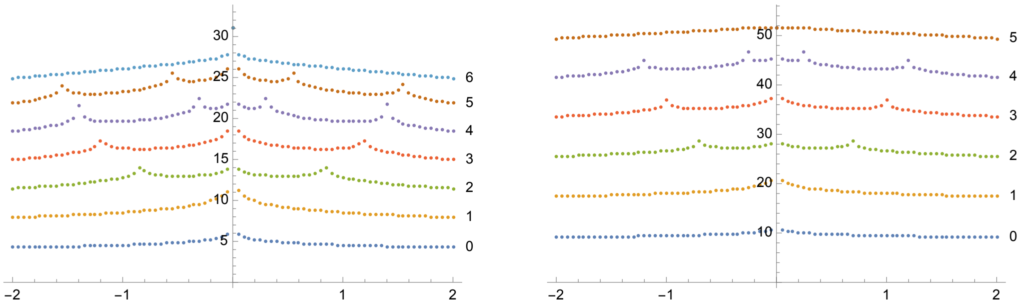
Some numerical evidences are provided in figure 5, 6. It turns out that these trans-series coefficients can indeed be written in the form of (3.32). In fact, we find the trans-series coefficients $w_{n,m}$ can be expressed as

$$w_{n,m} = \frac{1}{n!} B_{n,m+1}(1!t_1, 2!t_2, \ldots, (n-m)!t_{n-m}) \tag{4.67}$$



(a) $\phi = 2\pi/13$ (b) $\phi = 2\pi/23$

Figure 5. The order of magnitude ($-\log_{10}(|*|)$, vertical axis) of the difference between the exact spectrum and the Borel resummation of full energy trans-series in the form of (4.63) at Landau level 0 with varying Θ (horizontal axis). We include progressively contributions of increasing instanton orders $n = 0, 1, 2, \ldots$ from lower data points to higher data points.



(a) $\phi = 2\pi/13$ (b) $\phi = 2\pi/23$

Figure 6. The order of magnitude ($-\log_{10}(|*|)$, vertical axis) of the difference between the exact spectrum and the Borel resummation of full energy trans-series in the form of (4.63) at Landau level 1 with varying Θ (horizontal axis). We include progressively contributions of increasing instanton orders $n = 0, 1, 2, \ldots$ from lower data points to higher data points.

where the parameters t_j are such that the generating function for t_j is

$$\sum_{j\geq 1} t_j \lambda^j = \frac{1}{\pi} \arcsin \frac{\Theta}{\lambda + \lambda^{-1}}. \tag{4.68}$$

This in turn validates our conjecture that the full energy trans-series can be written as a tensor product of the minimal trans-series and a secondary trans-series, which is called the medium in the sense of (4.56).

Note that the medium trans-series coefficients all have the property that they vanish in the van Hove singularity with $\Theta = 0$. In particular this implies that

$$E^{\text{ext}}_{0,0}(\nu, \phi) = \mathscr{S}^{(+)} E^{(0)}_{\text{min}}(\nu, \phi; -2) = \mathscr{S}^{(-)} E^{(0)}_{\text{min}}(\nu, \phi; +2). \tag{4.69}$$

Moreover, taking the difference between full trans-series evaluated at $\Theta = 2$ and $\Theta = -2$, one finds the exact formula for the energy bandwidths at $P = 1$ to be

$$\text{bw}^{\text{ext}}_N(\phi) = 2 \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} w_{2n-1,m}(\Theta = 2) \mathscr{S}^{(\pm)} E^{(2n-1,m)}_{\text{min}} \left(N + \frac{1}{2}, \phi; \mp 2 \right). \tag{4.70}$$

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Once we have calculated the coefficients $w_{n,m}$ for the medium trans-series, we can use (4.58) to build the coefficients $u_{n,m}$ for the full trans-series from $v_{n,m}$ and $w_{n,m}$. The first few examples are in table 7. They reduce to $v_{n,m}$ if we set $\Theta = 0$ and reduce to $w_{n,m}$ if we set $\epsilon = 0$. Alternatively, we can add up the generating series for s_j and t_j and find the generating series for the parameter r_j of the full trans-series,

$$\sum_{j\geq 1} r_j \lambda^j = \frac{\mathrm{i}}{\epsilon\pi} \log\left(\sqrt{1+(2-\Theta^2)\lambda^2+\lambda^4} - \mathrm{i}\epsilon\Theta\lambda\right). \tag{4.71}$$

Taking the logic in section 3.2 backwards, this implies the EQCs

$$D^{\pm}_{\theta_x,\theta_y}: \quad 1+\mathcal{V}_A^{\pm 1}(1+\mathcal{V}_B)^2 - 2\sqrt{\mathcal{V}_A^{\pm 1}\mathcal{V}_B}\,\Theta = 0. \tag{4.72}$$

The two conditions $D^{\pm}_{\theta_x,\theta_y}$ are suitable for the two choices of the lateral Borel resummations $\mathscr{S}^{(\pm)}$ respectively. These two quantization conditions lead to the same energy spectrum as they are correctly related by the Stokes transformation of the Voros symbols. As explained in [28, 69], the Stokes transforms of Voros symbols are controled by the BPS invariants of the corresponding supersymmetric field theory

$$\mathfrak{S}_\theta: \quad \mathcal{V}_A \to \mathcal{V}_A(1+\mathcal{V}_B)^{\langle\gamma_A,\gamma_B\rangle\Omega(\gamma_B)}, \tag{4.73}$$

where we take the convention for the Dirac pairing of EM charges,

$$\langle\gamma_A,\gamma_B\rangle = p_B q_A - p_A q_B, \quad \gamma_A = (p_A,q_A,r_A),\ \gamma_B = (p_B,q_B,r_B). \tag{4.74}$$

As we discussed in the previous sections, the supersymmetric field theory corresponding to the Harper-Hofstader model is the 5d SYM on $S^1\times\mathbb{R}^4$ in the strong coupling regime. The charge vectors associated to $\mathcal{V}_A,\mathcal{V}_B$ are respectively

$$\gamma_A = (0,1,0), \quad \gamma_B = (2,0,0), \tag{4.75}$$

with

$$\langle\gamma_A,\gamma_B\rangle = 2, \quad \Omega(\gamma_B) = 2 \tag{4.76}$$

as discussed in section 3.3, so that

$$\mathfrak{S}_0: \quad \mathcal{V}_A \to \mathcal{V}_A(1+\mathcal{V}_B)^4, \tag{4.77}$$

which makes sure that the two conditions in (4.72) are equivalent to each other. Note that this is different from Mathieu equation, where the corresponding supersymmetric field theory is 4d SYM, and the BPS invariant $\Omega(\gamma_B) = 1$. This implies a slightly different form of Stokes transformation of Voros symbols

$$\mathfrak{S}_0: \quad \mathcal{V}_A \to \mathcal{V}_A(1+\mathcal{V}_B)^2, \tag{4.78}$$

which also makes sure that the two forms of EQC in (3.21) are equivalent to each other.

If we introduce the medium resummation

$$\mathscr{S}^{(\mathrm{med})} = \frac{1}{2}\left(\mathscr{S}^{(+)}+\mathscr{S}^{(-)}\right), \tag{4.79}$$

m	0	1	2	3	4	5
$w_{1,m}$	$\frac{\Theta}{\pi}$					
$w_{2,m}$	0	$\frac{\Theta^2}{2\pi^2}$				
$w_{3,m}$	$-\frac{\Theta}{\pi}+\frac{\Theta^3}{6\pi}$	0	$\frac{\Theta^3}{6\pi^3}$			
$w_{4,m}$	0	$-\frac{\Theta^2}{\pi^2}+\frac{\Theta^4}{6\pi^2}$	0	$\frac{\Theta^4}{24\pi^4}$		
$w_{5,m}$	$\frac{\Theta}{\pi}-\frac{\Theta^3}{2\pi}+\frac{3\Theta^5}{40\pi}$	0	$-\frac{\Theta^3}{2\pi^3}+\frac{\Theta^5}{12\pi^3}$	0	$\frac{\Theta^5}{120\pi^5}$	
$w_{6,m}$	0	$\frac{3\Theta^2}{2\pi^2}-\frac{2\Theta^4}{3\pi^2}+\frac{4\Theta^6}{45\pi^2}$	0	$-\frac{\Theta^4}{6\pi^4}+\frac{\Theta^6}{36\pi^4}$	0	$\frac{\Theta^6}{720\pi^6}$

Table 6. Trans-series coefficients $w_{n,m}$ in medium trans-series.

m	0	1	2	3	4	5
$u_{1,m}$	$\frac{\Theta}{\pi}$					
$u_{2,m}$	$\frac{\mathrm{i}\epsilon}{\pi}$	$\frac{\Theta^2}{2\pi^2}$				
$u_{3,m}$	$-\frac{\Theta}{\pi}+\frac{\Theta^3}{6\pi}$	$\frac{\mathrm{i}\epsilon\Theta}{\pi^2}$	$\frac{\Theta^3}{6\pi^3}$			
$u_{4,m}$	$-\frac{\mathrm{i}\epsilon}{2\pi}$	$-\frac{\epsilon^2}{2\pi^2}-\frac{\Theta^2}{\pi^2}+\frac{\Theta^4}{6\pi^2}$	$\frac{\mathrm{i}\epsilon\Theta}{2\pi^3}$	$\frac{\Theta^4}{24\pi^4}$		
$u_{5,m}$	$\frac{\Theta}{\pi}-\frac{\Theta^3}{2\pi}+\frac{3\Theta^5}{40\pi}$	$-\frac{3\mathrm{i}\epsilon\Theta}{2\pi^2}+\frac{\mathrm{i}\epsilon\Theta^3}{6\pi^2}$	$-\frac{\Theta^2}{2\pi^3}-\frac{\Theta^3}{2\pi^3}+\frac{\Theta^5}{12\pi^3}$	$\frac{\mathrm{i}\epsilon\Theta^3}{6\pi^4}$	$\frac{\Theta^5}{120\pi^5}$	
$u_{6,m}$	$\frac{\mathrm{i}\epsilon}{3\pi}$	$\frac{1}{2\pi^2}+\frac{3\Theta^2}{2\pi^2}-\frac{2\Theta^4}{3\pi^2}+\frac{4\Theta^6}{45\pi^2}$	$-\frac{\mathrm{i}\epsilon}{6\pi^3}-\frac{5\mathrm{i}\epsilon\Theta^2}{4\pi^3}+\frac{\mathrm{i}\epsilon\Theta^4}{6\pi^3}$	$-\frac{\Theta^2}{4\pi^4}-\frac{\Theta^4}{6\pi^4}+\frac{\Theta^6}{36\pi^4}$	$\frac{\mathrm{i}\epsilon\Theta^4}{24\pi^5}$	$\frac{\Theta^6}{720\pi^6}$

Table 7. Trans-series coefficients $u_{n,m}$ for the full energy trans-series.

the EQC can be written as

$$D^{\mathrm{med}}_{\theta_x,\theta_y}: \quad (1+\mathcal{V}_A)(1+\mathcal{V}_B) - 2\sqrt{\mathcal{V}_A\mathcal{V}_B}\,\Theta = 0. \tag{4.80}$$

which is more symmetric between the perturbative and the non-perturbative Voros symbols. Note that the medium trans-series with coefficients (4.67) can be solved from this quantization condition, which explains its name.

5 Characterization of splitting bands

The result of the last section provides an alternative quantization method for the Harper-Hofstadter model with flux $\phi = 2\pi/Q$. From the left graph of figure 7, we can easily tell that this approach is valid pretty well into the non-perturbative regime. In fact, we have checked that the alternative quantization method is valid for $Q \geq 2N+3$. For $P > 1$, an important difference from the $P = 1$ case is that a single energy band at $\phi = 2\pi P/Q$, which we call the primary Landau level, splits to P smaller secondary energy bands, which is also visible on the right graph of figure 7. How to characterize this phenomenon would be the main goal of this section.

5.1 Self-similarity of the butterfly revisited

The discussion of the resurgent properties of the energy trans-series as well as the construction of minimal trans-series in section 4.2 is universal and it holds true for any rational value of ϕ. The construction of the medium trans-series and the consequent matching with the exact

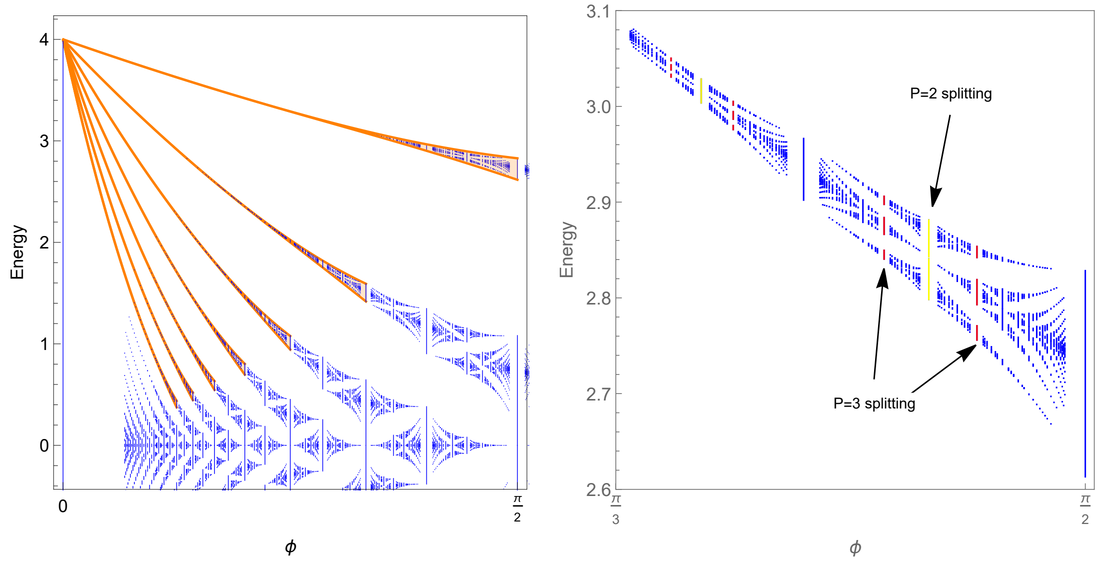


Figure 7. Left: Hofstadter's butterfly with energy trans-series for $P = 1$ up to the sixth Landau level. Right: zooming in on the lowest Landau level. We depict $P = 2$ band splitting in yellow and $P = 3$ band splitting in red.

energy spectrum, however, depends surprisingly on the numerator of $\phi = 2\pi P/Q$. If the flux ϕ is such that $P > 1$, the generating series (4.67) with (4.68) are no longer valid. In fact, the coefficients $w_{n,m}$ for the medium trans-series become vastly more complicated.

For instance, we find the 1-instanton coefficient $w_{1,0}$ is one of the P solutions to

$$\Theta = \frac{1}{2} F_{Q/P}(2\pi u_{1,0}, 0, 0) =: \frac{1}{2} F_{Q/P}(2\pi u_{1,0}) \tag{5.1}$$

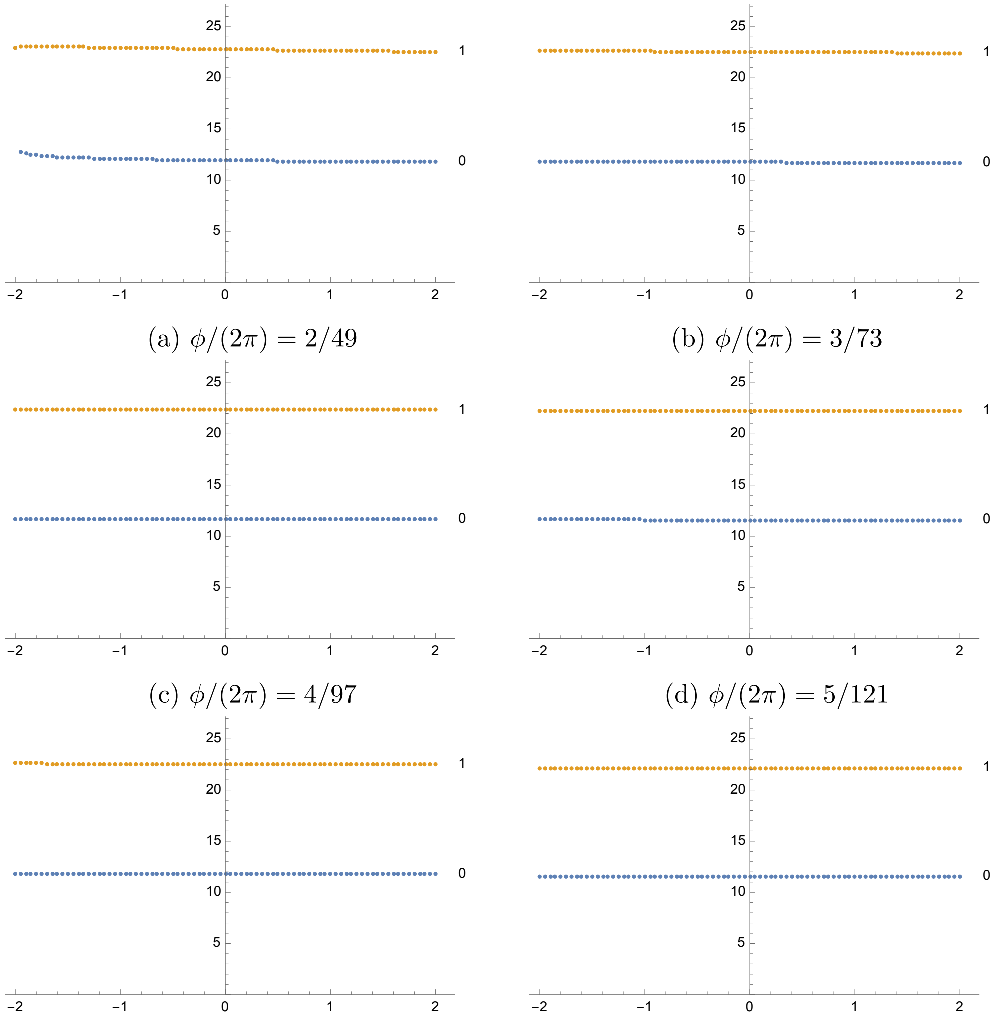
where $F_{Q/P}(x)$ is the secular polynomial defined in section 2.2. Note that here the subscript is inverted from P/Q to Q/P, which may be related to the fractal structure of the energy spectrum. The secular polynomial has the property that

$$F_{Q/P}(x) = F_{1-Q/P}(x). \tag{5.2}$$

Some examples are

$$F_1(x) = x, \tag{5.3a}$$
$$F_{1/2}(x) = -4 + x^2, \tag{5.3b}$$
$$F_{1/3}(x) = -6x + x^3, \tag{5.3c}$$
$$F_{1/4}(x) = 4 - 8x^2 + x^4, \tag{5.3d}$$
$$F_{1/5}(x) = \frac{5}{2}(7 - \sqrt{5})x - 10x^3 + x^5, \tag{5.3e}$$
$$F_{2/5}(x) = \frac{5}{2}(7 + \sqrt{5})x - 10x^3 + x^5, \tag{5.3f}$$
$$F_{1/6}(x) = -4 + 24x^2 - 12x^4 + x^6. \tag{5.3g}$$

Some numerical evidence of (5.1) are provided in figure 8 and figure 9.



(a) $\phi/(2\pi) = 2/49$ (b) $\phi/(2\pi) = 3/73$

(c) $\phi/(2\pi) = 4/97$ (d) $\phi/(2\pi) = 5/121$

(e) $\phi/(2\pi) = 5/122$ (f) $\phi/(2\pi) = 6/145$

Figure 8. The orders of magnitude ($-\log_{10}(|*|)$, vertical axis) of the difference between the exact spectrum and the Borel resummation of full energy trans-seriesin the form of (4.63) at Landau level 0 with varying Θ (horizontal axis). We include progressively contributions of increasing instanton orders $n = 0, 1$ from lower data points to higher data points. The six plots are examples of (a) $P = 2, Q = 1 \bmod P$ (a) $P = 3, Q = 1 \bmod P$ (a) $P = 4, Q = 1 \bmod P$ (a) $P = 5, Q = 1 \bmod P$ (a) $P = 5, Q = 2 \bmod P$ (a) $P = 6, Q = 1 \bmod P$ for $\phi = 2\pi P/Q$.

It's quite clear from the right figure of figure 7 that the entire structure of Hofstadter's butterfly reemerges within each primary Landau level. From our preliminary analysis on higher instanton corrections for splitting bands, the higher trans-series coefficients are more complicated to be determined numerically and they are not naive generalizations of trans-series coefficients for $P = 1$ case.

5.2 Evidence for exact Rammal-Wilkinson formula

Another approach to characterize these splitting bands is identifying the proper expansion base point of these bands such that the secondary Landau levels become primary Landau

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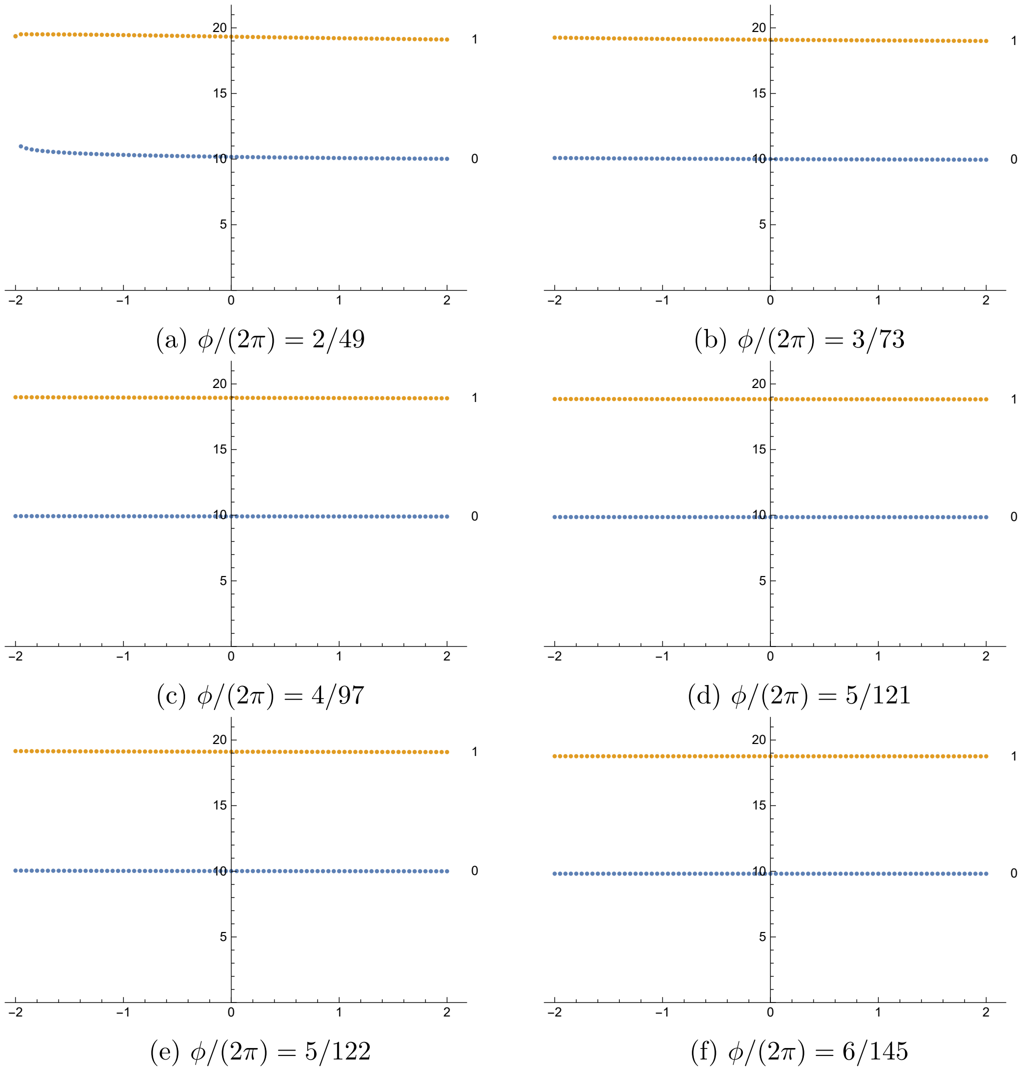

Figure 9. The orders of magnitude ($-\log_{10}(|*|)$, vertical axis) of the difference between the exact spectrum and the Borel resummation of full energy trans-seriesin the form of (4.63) at Landau level 1 with varying Θ (horizontal axis). We include progressively contributions of increasing instanton orders $n = 0, 1$ from lower data points to higher data points. The six plots are examples of (a) $P = 2, Q = 1 \bmod P$ (a) $P = 3, Q = 1 \bmod P$ (a) $P = 4, Q = 1 \bmod P$ (a) $P = 5, Q = 1 \bmod P$ (a) $P = 5, Q = 2 \bmod P$ (a) $P = 6, Q = 1 \bmod P$ for $\phi = 2\pi P/Q$.

levels in this new expansion scenario. In [31, 71], they discovered a formula for perturbative energy expansion near arbitrary rational values. However, due to limitations of technology at that age, their computation of the perturbative energy from the quantization condition were quite primitive and incomplete, i.e, the perturbative expansion of Rammal-Wilkinson formula was calculated only to the first order and their consideration of non-perturbative instanton corrections are hand-waving without any determination of instanton action or prefactor.

In order to perform similar numerical analysis for bands near certain rational values as in [20] and [22], we need to introduce the concept of almost canonical continued fraction. For a given non-negative rational number α, it can always be expressed as

$$\alpha = n_0 + \cfrac{1}{n_1 + \frac{1}{n_2 + \cdots + \frac{1}{n_l}}} \tag{5.4}$$

and be denoted as $[n_0, n_1, n_2, \cdots, n_l]$. This is the canonical continued fraction. Due to the symmetry property of the butterfly, we can restrict our attention to real numbers that satisfy $0 \le r \le 1/2$ and n_0 can be set to 0. The almost canonical continued fraction representation of a real number can be achieved by allowing n_i to be negative and requiring

$$|n_i| \ge 2. \tag{5.5}$$

The representation of a rational number written in almost canonical representation should be unique since we can always start with a canonical continued fraction and rewrite

$$\begin{aligned} &[0, n_1, n_2, \cdots, n_{i-1}, 1, n_{i+1}, n_{i+2}, \cdots, n_l] \\ \to &[0, n_1, n_2, \cdots, n_{i-1} + 1, -(n_{i+1} + 1), -n_{i+2}, \cdots, -n_l] \end{aligned} \tag{5.6}$$

whenever we find $n_i = 1$ in the sequence.

After representing the rational magnetic flux $\alpha = \phi/(2\pi)$ in the almost canonical fashion, n_1 is nothing but the number of principal Landau levels, and n_k is in general the number of sub-levels at that nested layer assuming there's no merging of subbands. For a rational magnetic flux $\alpha = [0, n_1, n_2, \cdots, n_{l-1}, n_l]$, we can regard it as a small deviation from $\alpha_0 = [0, n_1, n_2, \cdots, n_{l-1}]$ and consider the perturbative expansion of the energy from certain exact energy value at α_0, usually at an edge of energy bands. In the examples shown below, we will focus on bands at $[0, n_1, n_2]$ expanded around $[0, n_1] = 1/n_1$.

The simplest possible example to illustrate the expansion around rational points other than zero would be considering the base point $\alpha_0 = 1/2$, and taking the energy value at the top edge of the first energy band $E_0 = 2\sqrt{2}$, solved from

$$F_{1/2}(E, 0, 0) - 4 = 0. \tag{5.7}$$

If we use $\phi' = \phi - \phi_0$ with $\phi_0 = 2\pi\alpha_0 = \pi$ as our expansion parameter, the perturbative series can be approximated numerically from the exact spectra

$$\begin{aligned} E(\phi'; N) = 2\sqrt{2} - \frac{(2N+1)\phi'}{\sqrt{2}} &+ \frac{(4N^2 + 4N + 3)\,\phi'^2}{8\sqrt{2}} \\ &+ \frac{(8N^3 + 12N^2 + 30N + 13)\,\phi'^3}{96\sqrt{2}} + \mathcal{O}(\phi'^4). \end{aligned} \tag{5.8}$$

The leading instanton contribution to the bandwidths near $[0, 2]$ can also be approximated in reasonable precision,

$$\mathrm{bw}_N(\phi') \simeq \frac{2^{2N+4}}{\sqrt{\pi}\, N!} \mathrm{e}^{-2C/\phi'} \left(1 - \frac{3N^2 + 9N + 5}{12}\phi' - \frac{9N^4 + 34N^3 + 9N^2 - 46N - 56}{288}\phi'^2 + \mathcal{O}(\phi'^3)\right). \tag{5.9}$$

We next consider the bands at $\alpha = [0, 3, n_2]$ away from the base point $\alpha_0 = 1/3$ expanded around the top band edge $E_0 = \sqrt{3} + 1$ solved from

$$F_{1/3}(E, 0, 0) - 4 = 0, \tag{5.10}$$

as another case study. If we use $\phi' = \phi - \phi_0$ with $\phi_0 = 2\pi\alpha_0 = 2\pi/3$ as our expansion parameter, then

$$\begin{aligned} E(\phi'; N) = \sqrt{3} + 1 - \frac{3}{4}(\sqrt{3} - 1)(2N + 1)|\phi'| + \left(\frac{\sqrt{3}}{2} - 1\right)\phi' \\ + \frac{1}{32}\left(18(7\sqrt{3} - 11)N(N + 1) + 65\sqrt{3} - 95\right)\phi'^2 \\ - \frac{3}{8}(5\sqrt{3} - 8)(2N + 1)\mathrm{sgn}(\phi')\phi'^2 + \mathcal{O}(\phi^3), \end{aligned} \tag{5.11}$$

and the bandwidth is approximated by

$$\mathrm{bw}_N(\phi') \simeq \mathrm{const.}\frac{2^{4N+\frac{9}{2}}}{\sqrt{\pi}\, N!\, 3^{2N}}\phi'^{\frac{1}{2}-N}\mathrm{e}^{-8C/9\phi'}\mathcal{P}_1^{\mathrm{inst}}(\phi'; N), \tag{5.12}$$

where $\mathcal{P}_1^{\mathrm{inst}}(\phi'; N)$ is a power series starting from 1 that represents the instanton fluctuation. If we use instead $\tilde{\phi}$ as our expansion parameter, which is given by

$$\phi = \frac{2\pi}{3 - \frac{\tilde{\phi}}{2\pi}}, \tag{5.13}$$

the perturbative energy series is

$$E(\tilde{\phi}; N) = \sqrt{3} + 1 - \frac{1}{12}(\sqrt{3} - 1)(2N + 1)|\tilde{\phi}| + \left(\frac{\sqrt{3} - 2}{18}\right)\tilde{\phi} + \mathcal{O}(\tilde{\phi}^2), \tag{5.14}$$

and the energy bandwidth is approximated by

$$\mathrm{bw}_N(\tilde{\phi}) \simeq \frac{2^{4N+\frac{9}{2}}}{\sqrt{\pi}\, N!}\tilde{\phi}^{\frac{1}{2}-N}\mathrm{e}^{-8C/\tilde{\phi}}\mathcal{P}_2^{\mathrm{inst}}(\tilde{\phi}; N), \tag{5.15}$$

where $\mathcal{P}_2^{\mathrm{inst}}$ is again a power series starting from 1. The prefactor and the instanton action of (5.15) is identical to the one appearing for bandwidths formula for $[0, n_1]$, which suggests that (5.13) perhaps is a more natural way of performing the expansion. Extracting information of the instanton fluctuation can help us learn about the quantum periods expanded near the corresponding quantum conifold points. We wish to come back to this problem in future works.

6 Conclusion and discussion

In this paper, we begin to study the full energy trans-series for the Harper-Hofstader model. Using inspiration from the structure of energy trans-series of 1d non-relativistic QM models obtained by the exact WKB method, and the connection between the Harper-Hofstadter model and the 5d SYM theory, we are able to write down a conjectural full energy trans-series including instanton corrections at all levels when the magnetic flux is $\phi = 2\pi/Q$,

$Q \in \mathbb{N}$, and we checked our conjectural formula with very high numerical precisions, up to six instanton levels.

One prominent feature of the full energy trans-series is that the perturbative series only determines even instanton sectors via resurgence but not the odd instanton sectors, which are in different topological sectors, so that the strong resurgence program does not hold.

When the magnetic flux is $\phi = 2\pi P/Q$ with $P > 1$, although we argue the resurgent structure of the perturbative series remains the same, the coefficients of the full energy trans-series could be quite different. For instance, the coefficient of the 1-instanton sector is given by roots of the secular equation with the inverted flux. In addition, we also made progress in the expansion of energy around a rational value of magnetic flux instead of at the zero flux, including both the perturbative energy series and the leading contribution to energy bandwidth, extending the Rammal-Wilkinson formula.

There are many open problems following this work. The energy spectrum of the Harper-Hofstadter model is mesmerizing for the distinction between rational and irrational values of the magnetic flux, and for the self-similarity structure of the energy spectrum. To understand the self-similarity structure of the energy spectrum, it will be worthwhile to push further the calculation of the trans-series coefficients for higher instanton levels when the magnetic flux is $\phi = 2\pi P/Q$ with $P > 1$. One should also explore further the expansion of energy around a non-zero rational value of magnetic flux. One important line of attack is to use the supersymmetric localization results of the Wilson loop vev of 5d SYM [64, 72], which we argued to coincide with the energy of the Harper-Hofstadter model, as it is more suitable for expansion around the rational value of magnetic flux.[13] To understand the distinction between rational and irrational values of the magnetic flux, it would be very beneficial to exploit the relation between the Harper-Hofstadter model and the quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ [4, 73] and quantum integrable models [7, 8]. It would also be interesting to consider other lattices which are related supersymmetric gauge theories or topological string [21–23]. It would also be interesting to perform a systematic exact WKB analysis on the Harper-Hofstadter model as previous studies on 4d SYM [36, 74] as another line of attack. We would like to return to these problems in the near future.

In an orthogonal direction, as we discussed in section 4.1, we can calculate the Stokes constants of the Harper-Hofstadter perturbative energy series from those of the perturbative Wilson loop vevs in 5d SYM. However, in this process, we face the problem of choosing between using the Stokes constants in the strong coupling regime or in the weak coupling regime from the 5d SYM. A similar problem was already encountered in [37] where one wished to reconstruct the Stokes constants of topological string free energy in the conifold limit from the Stokes constants of conventional topological string free energy. The authors of [37] proposed to use the Stokes constants in the strong coupling regime, but could not provide an explanation. Here we find the same prescription is true, and we argue that the reason is because the range of energy of the Harper-Hofstadter model is mapped to the Coulomb modulus of the 5d SYM in the strong coupling regime. We hope this argument can shed some light on the mystery in [37].

[13] The Wilson loop vev, or equivalently the inverse quantum mirror map should be expanded around the rational value flux together with the identification of the perturbative quantum period as the perturbative quantization condition near the rational value flux in consideration.

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