W-algebras and Bethe ansatz in 2d CFT

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Fundamental constituents of matter through frontier technologies
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XXX $\mathfrak{su}(2)$ spin chain Bethe equations (Bethe, 1931)

$$1 = q \left(\frac{u_j - \frac{\epsilon}{2}}{u_j + \frac{\epsilon}{2}} \right)^N \prod_{k \neq j} \frac{u_j - u_k + \epsilon}{u_j - u_k - \epsilon}$$

$$\Updownarrow$$

CFT/$\mathcal{W}_{1+\infty}$ Bethe equations
(Litvinov, Nekrasov, Shatashvili, BSTV,... 2013)

$$1 = q \prod_{l=1}^{N} \frac{u_j + a_l - \epsilon_3}{u_j + a_l} \prod_{k \neq j} \frac{(u_j - u_k + \epsilon_1)(u_j - u_k + \epsilon_2)(u_j - u_k + \epsilon_3)}{(u_j - u_k - \epsilon_1)(u_j - u_k - \epsilon_2)(u_j - u_k - \epsilon_3)}$$

Overview

- $\mathcal{W}$-algebras and $\mathcal{W}_\infty$
- affine Yangian
- integrable structure - KdV and BLZ
- instanton R-matrix and ILW Bethe equations

W algebras - motivation

$\mathcal{W}$-algebas: extensions of the Virasoro algebra (2d CFT) by higher spin currents - appear in many different contexts:

- integrable hierarchies of PDE (KdV/KP) $\rightsquigarrow$ $\mathcal{W}$ is quant. KP
- (old) matrix models
- instanton partition functions and AGT
- holographic dual description of 3d higher spin theories
- quantum Hall effect
- topological strings
- higher spin square (Gaberdiel, Gopakumar)
- $4d$ $\mathcal{N} = 4$ SYM at codimension 2 junction of three codimension 1 defects (Gaiotto, Rapčák)
- geometric representation theory (equivariant cohomology of various moduli spaces)

Zamolodchikov $\mathcal{W}_3$ algebra

$\mathcal{W}_3$ algebra constructed by Zamolodchikov (1984) has a stress-energy tensor (Virasoro algebra) with OPE

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + reg.$$

together with spin 3 primary field $W(w)$

$$T(z)W(w) \sim \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} + reg.$$

To close the algebra we need to find the OPE of W with itself consistent with associativity (Jacobi, crossing symmetry...).

The result:

$$\begin{aligned} W(z)W(w) \quad \sim \quad & \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\ & + \frac{1}{(z-w)^2} \left(\frac{32}{5c+22} \Lambda(w) + \frac{3}{10} \partial^2 T(w) \right) \\ & + \frac{1}{z-w} \left(\frac{16}{5c+22} \partial\Lambda(w) + \frac{1}{15} \partial^3 T(w) \right) + reg. \end{aligned}$$

Λ is a quasiprimary 'composite' (spin 4) field,

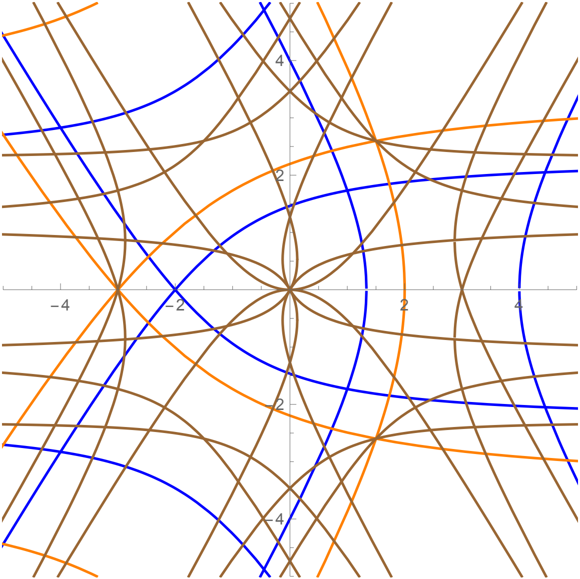
$$\Lambda(z) = (TT)(z) - \frac{3}{10} \partial^2 T(z).$$

The algebra is non-linear, not a Lie algebra in the usual sense

W_N series and $\mathcal{W}_\infty$ algebra

- W_N: an interesting family of W-algebras associated to $\mathfrak{sl}(N)$ Lie algebras (spins $2, 3, \ldots, N$, Virasoro $\leftrightarrow \mathfrak{sl}(2)$)
- $\mathcal{W}_\infty$: interpolating algebra for W_N series; spins $2, 3, \ldots$
- Gaberdiel-Gopakumar: solving associativity conditions for this field content $\rightsquigarrow$ two-parameter family: central charge c and rank parameter λ
- choosing $\lambda = N \rightarrow$ truncation of $\mathcal{W}_\infty$ to $\mathcal{W}_N = \mathcal{W}[\mathfrak{sl}(N)]$, i.e. $\mathcal{W}_\infty$ is interpolating algebra for the whole $\mathcal{W}_N$ series
- adding spin 1 field, we have $\mathcal{W}_{1+\infty} \rightsquigarrow$ many simplifications
- **triality** symmetry of the algebra (Gaberdiel & Gopakumar) $\mathcal{W}_\infty[c, \lambda_1] \simeq \mathcal{W}_\infty[c, \lambda_2] \simeq \mathcal{W}_\infty[c, \lambda_3]$

$$\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3} = 0, \qquad c = (\lambda_1 - 1)(\lambda_2 - 1)(\lambda_3 - 1)$$

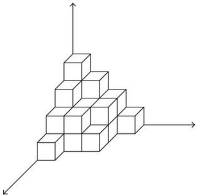


4
2
−4
−2
2
4
−2
−4

- MacMahon function as vacuum character of the algebra (enumerating all the local fields in the algebra)

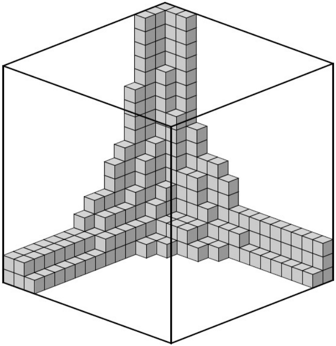
$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + 48q^6 + \cdots$$

- The same generating function is well-known to count the plane partitions (3d Young diagrams)



- triality acts by permuting the coordinate axes
- restriction to $\mathcal{W}_N$ corresponds to max N boxes in one of the directions

- this can be generalized to degenerate primaries (not only vacuum rep) by allowing 2d Young diagram asymptotics
- counting exactly as in topological vertex $\rightsquigarrow$ topological vertex can be interpreted as being a character of degenerate $\mathcal{W}_{1+\infty}$ representations



- box counting generalizes also to minimal models (Ising...) $\rightsquigarrow$ lozenge tilings on cylinder

Yangian of $\widehat{\mathfrak{gl}(1)}$

The Yangian of $\widehat{\mathfrak{gl}(1)}$ (Arbesfeld-Schiffmann-Tsymbaliuk) is an associative algebra with generators $\psi_j, e_j, f_j, j \geq 0$ and relations

$$
\begin{aligned}
0 &= [e_{j+3}, e_k] - 3\,[e_{j+2}, e_{k+1}] + 3\,[e_{j+1}, e_{k+2}] - [e_j, e_{k+3}] \\
&\quad +\sigma_2\,[e_{j+1}, e_k] - \sigma_2\,[e_j, e_{k+1}] - \sigma_3\,\{e_j, e_k\} \\
0 &= [f_{j+3}, f_k] - 3\,[f_{j+2}, f_{k+1}] + 3\,[f_{j+1}, f_{k+2}] - [f_j, f_{k+3}] \\
&\quad +\sigma_2\,[f_{j+1}, f_k] - \sigma_2\,[f_j, f_{k+1}] + \sigma_3\,\{f_j, f_k\} \\
0 &= [\psi_{j+3}, e_k] - 3\,[\psi_{j+2}, e_{k+1}] + 3\,[\psi_{j+1}, e_{k+2}] - [\psi_j, e_{k+3}] \\
&\quad +\sigma_2\,[\psi_{j+1}, e_k] - \sigma_2\,[\psi_j, e_{k+1}] - \sigma_3\,\{\psi_j, e_k\} \\
0 &= [\psi_{j+3}, f_k] - 3\,[\psi_{j+2}, f_{k+1}] + 3\,[\psi_{j+1}, f_{k+2}] - [\psi_j, f_{k+3}] \\
&\quad +\sigma_2\,[\psi_{j+1}, f_k] - \sigma_2\,[\psi_j, f_{k+1}] + \sigma_3\,\{\psi_j, f_k\} \\
0 &= [\psi_j, \psi_k] \\
\psi_{j+k} &= [e_j, f_k]
\end{aligned}
$$

'initial/boundary conditions'

$$
\begin{aligned}
&[\psi_0, e_j] = 0, \qquad [\psi_1, e_j] = 0, \qquad [\psi_2, e_j] = 2e_j, \\
&[\psi_0, f_j] = 0, \qquad [\psi_1, f_j] = 0, \qquad [\psi_2, f_j] = -2f_j
\end{aligned}
$$

and finally the Serre relations

$$
0 = \mathrm{Sym}_{(j_1,j_2,j_3)} [e_{j_1}, [e_{j_2}, e_{j_3+1}]], \quad 0 = \mathrm{Sym}_{(j_1,j_2,j_3)} [f_{j_1}, [f_{j_2}, f_{j_3+1}]].
$$

Parameters $\epsilon_1, \epsilon_2, \epsilon_3 \in \mathbb{C}$ constrained by $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ and

$$
\begin{aligned}
\sigma_2 &= \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3 \\
\sigma_3 &= \epsilon_1\epsilon_2\epsilon_3.
\end{aligned}
$$

We have both commutators and anticommutators in defining quadratic relations (but no $\mathbb{Z}_2$ grading) - for $\sigma_3 \neq 0$ not a Lie (super)-algebra.

Introducing generating functions (Drinfel'd currents)

$$e(u) = \sum_{j=0}^{\infty} \frac{e_j}{u^{j+1}}, \quad f(u) = \sum_{j=0}^{\infty} \frac{f_j}{u^{j+1}}, \quad \psi(u) = 1 + \sigma_3 \sum_{j=0}^{\infty} \frac{\psi_j}{u^{j+1}}$$

the first set of formulas above (almost!) simplify to

$$\begin{aligned} e(u)e(v) &\sim \varphi(u-v)e(v)e(u), \quad f(u)f(v) \sim \varphi(v-u)f(v)f(u), \\ \psi(u)e(v) &\sim \varphi(u-v)e(v)\psi(u), \quad \psi(u)f(v) \sim \varphi(v-u)f(v)\psi(u) \end{aligned}$$

with rational structure function (scattering phase in BAE)

$$\boxed{\varphi(u) = \frac{(u+\epsilon_1)(u+\epsilon_2)(u+\epsilon_3)}{(u-\epsilon_1)(u-\epsilon_2)(u-\epsilon_3)}}$$

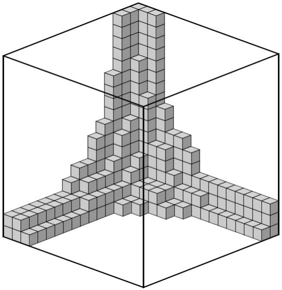
The representation theory of the algebra is much simpler in this Yangian formulation and is controlled by this function

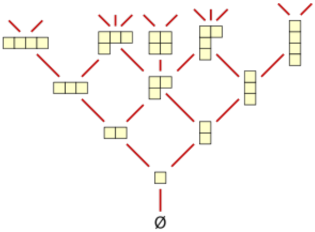
$\psi(u), e(u)$ and $f(u)$ in representations act like

$$\psi(u)\,|\Lambda\rangle = \psi_0(u) \prod_{\square\in\Lambda} \varphi(u-\epsilon_\square)\,|\Lambda\rangle$$

$$e(u)\,|\Lambda\rangle = \sum_{\square\in\Lambda^+} \frac{E(\Lambda\to\Lambda+\square)}{u-\epsilon_\square}\,|\Lambda+\square\rangle$$

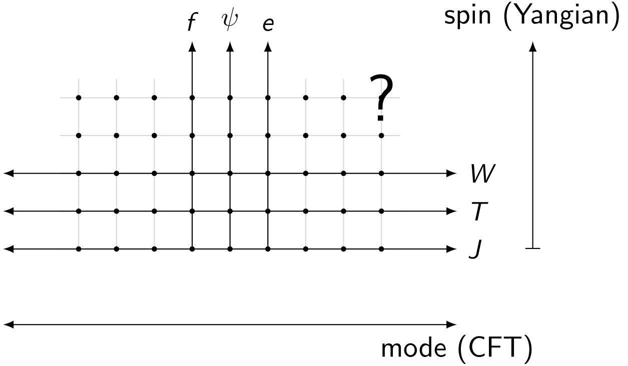
where the states $|\Lambda\rangle$ are associated to geometric configurations of boxes (plane partitions,...) and where $\epsilon_\square = \sum_j \epsilon_j x_j(\square)$ is the weighted geometric position of the box.





Two different descriptions of the algebra:

- usual CFT point of view with local fields $J(z), T(z), W(z), \ldots$ with increasingly complicated OPE as we go to higher spins
- Yangian point of view (Arbesfeld-Schiffmann-Tsymbaliuk) where all the spins are included in the generating functions $\psi(u), e(u)$ and $f(u)$ but accessing higher mode numbers is difficult



Integrable structures

- there are two natural distinct infinite familites of commuting quantities (Hamiltonians):
  1. the Yangian (Bejamin-Ono) family of generators ψ_j
  2. the family of *local* conserved charges (BLZ, quantum KdV)
- the Yangian charges are very easy to diagonalize and their spectrum is determined by combinatorics of plane partitions
- the diagonalization of the local commuting quantities on the other hand is quite non-trivial and has a long history
- we will relate these two families by constructing a family of quantum ILW Hamiltonians (Litvinov) that interpolate between these two families

Classical KdV/KP

- in the classical limit, the theory reduces to the theory of integrable hierarchies of PDEs (KdV, KP)
- the classical object associated to Virasoro algebra is the one-dimensional Schrödinger operator

$$L^2 = \partial_x^2 + u(x)$$

- there exists an infinite dimensional family of continuous deformations of $u(x)$ which preserve the spectrum of L^2 and are organized into commuting flows
- the first such deformation is the trivial rigid translation of the potential

$$\partial_{t_1} u = \partial_x u$$

- the next one is already rather non-trivial and is captured by the Korteweg-de-Vries equation (Boussinesq 1877)

$$4\partial_{t_3} u = 6u\partial_x u + \partial_x^3 u.$$

- the space of Schödinger potentials is a Hamiltonian system if we equip it with Poisson bracket

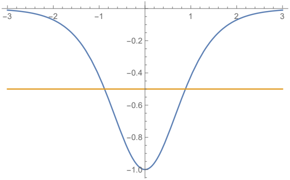
$$\{u(x), u(y)\} = -\delta'''(x-y) - 4u(x)\delta'(x-y) - 2u'(x)\delta(x-y)$$

(whose Fourier transform is just the classical Virasoro algebra)

- the deformations are generated by Hamiltonians which are at the same time conserved quantities capturing the spectral data of the family of Schrödinger operators

$$I_1 = \int u(x)dx, \qquad I_3 = \int u^2(x)dx, \quad \ldots$$

- e.g. KdV soliton (Pöschl-Teller potential) with a single bound state



$$\exp\left(\sum_j \frac{I_j}{j\lambda^j}\right) = \frac{\lambda+1}{\lambda-1}$$

- the KdV conserved charges survive quantization in the form

$$I_1 = \int T(x)dx = L_0 - \frac{c}{24}$$

$$I_3 = \int (TT)(x)dx = L_0^2 + 2\sum_{m=1}^{\infty} L_{-m}L_m - \frac{c+2}{12}L_0 + \frac{c(5c+22)}{2880}$$

so it makes sense to ask what their spectrum is

- since L_0 is part of the family, the problem is to diagonalize finite dimensional matrices level by level
- a surprising description of their spectrum was found by Bazhanov-Lukyanov-Zamolodchikov (in the context ODE/IM correspondence initiated by Dorey and Tateo)

- consider a Schrödinger operator

$$-\partial_z^2 + \frac{\ell(\ell+1)}{z^2} + \frac{\#}{z} + \lambda z^{h^2-2}$$

associated to a CFT primary state (central charge c and conformal dimension Δ are encoded in h and ℓ) and dress it by allowing for additional collection of regular singular points

$$\sum_{j=1}^{M} \left(\frac{2}{(z-z_j)^2} + \frac{\gamma_j}{z-z_j} \right)$$

(M is the Virasoro level)

- the requirement of trivial monodromy around these singularities leads to a system of BLZ Bethe equations

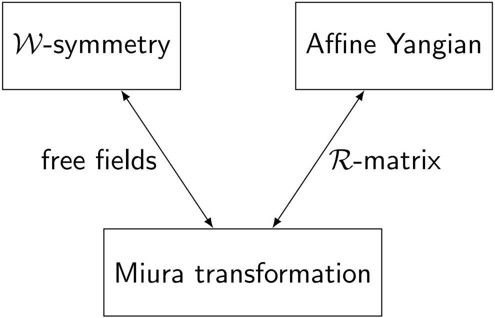
$$\sum_{k\neq j} \frac{z_j(h^4 z_j^2 - (h^2-2)(2h^2+1)z_j z_k + (h^2-1)(h^2-2)z_k^2)}{(z_j-z_k)^3} = (1-h^2)z_j - h^4\Delta.$$

- given any solution of BLZ Bethe equations, the eigenvalues of I_j are determined, for instance

$$I_3 = (\Delta+M)^2 - \frac{c+2}{12}(\Delta+M) + \frac{c(5c+22)}{2880} + 4(h^{-4}-h^{-2}) \sum_{j=1}^{M} z_j.$$

- how does this generalize to higher ranks?
- how are these local Hamiltonians related to Yangian conserved quantities?

Miura transformation and $\mathcal{R}$-matrix



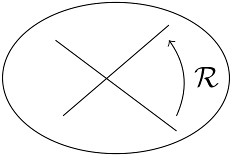
- consider the following factorization of N-th order differential operator

$$(\partial + \partial\phi_1(z)) \cdots (\partial + \partial\phi_N(z)) = \sum_{j=0}^{N} U_j(z)\partial^{N-j}$$

with N commuting free fields $\partial\phi_j(z)\partial\phi_k(w) \sim \delta_{jk}(z-w)^{-2}$

- OPEs of U_j generate $\mathcal{W}_N$ and furthermore are quadratic
- $\mathcal{W}_N \leftrightarrow$ quantization of N-th order differential operators
- the embedding of $\mathcal{W}_N$ in the bosonic Fock space depends on the way we order the fields on the LHS
- Maulik-Okounkov: $\mathcal{R}$-matrix as intertwiner between two embeddings, $\mathcal{R} : \mathcal{F}^{\otimes 2} \to \mathcal{F}^{\otimes 2}$

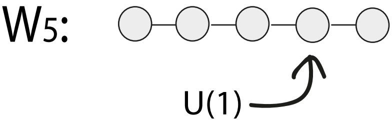
$$(\partial + \partial\phi_1)(\partial + \partial\phi_2) = \mathcal{R}^{-1}(\partial + \partial\phi_2)(\partial + \partial\phi_1)\mathcal{R}$$



- $\mathcal{R}$ defined in this way satisfies the Yang-Baxter equation (two ways of reordering $321 \to 123$)

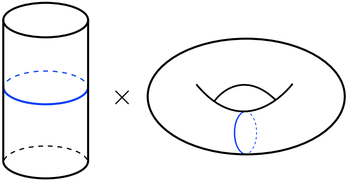
$$\mathcal{R}_{12}(u_1 - u_2)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{23}(u_2 - u_3) = \\ = \mathcal{R}_{23}(u_2 - u_3)\mathcal{R}_{13}(u_1 - u_3)\mathcal{R}_{12}(u_1 - u_2)$$

- the spectral parameter u - the global $U(1)$ charge
- $\mathcal{R}$-matrix satisfying YBE $\rightsquigarrow$ apply the algebraic Bethe ansatz



- spin chain of length $N \rightsquigarrow \widehat{\mathfrak{gl}}(1) \times \mathcal{W}_N$ algebra

- once we have an $\mathcal{R}$-matrix, we can couple a CFT to a probe, in our case this is another CFT
- consider an *auxiliary* Fock space $\mathcal{F}_A$ and a *quantum* space $\mathcal{F}_Q \equiv \mathcal{F}_1 \otimes \ldots \otimes \mathcal{F}_N$
- we associate to this the monodromy matrix $\mathcal{T}_{AQ} : \mathcal{F}_A \otimes \mathcal{F}_Q \to \mathcal{F}_A \otimes \mathcal{F}_Q$ defined as $\mathcal{T}_{AQ} = \mathcal{R}_{A1} \cdots \mathcal{R}_{AN}$
- in the usual algebraic Bethe ansatz the next step is to take the trace over the auxiliary space
- since our auxiliary spaces are infinite dimensional Fock spaces, we have to regularize the trace, $\mathcal{H}_q(u) = \mathrm{Tr}_A\, q^{L_{A,0}} \mathcal{T}_{AQ}(u)$



- this leads for every q to a different infinite family of commuting Hamiltonians, Hamiltonians of *intermediate long wave equation*, the first non-trivial being

$$H_3 = (\Phi_3)_0 + \sum_{m>0} m \frac{1+q^m}{1-q^m} J_{-m} J_m$$

- interpolates between Yangian/BO Hamiltonians at $q \to 0$, local quantum KP/BLZ Hamiltonians at $q \to 1$ limit and to charge conjugate Yangian/BO Hamiltonians as $q \to \infty$

$$m \frac{1+q^m}{1-q^m} \to |m|, q \to 0, \qquad m \frac{1+q^m}{1-q^m} \to \frac{2}{1-q} - 1 + ..., q \to 1$$

- these Hamiltonians can be diagonalized by Bethe ansatz equations (Litvinov, Nekrasov, Shatashvili, Bonelli, Sciarappa, Tanzini, Vasko)

$$1 = q \prod_{l=1}^{N} \frac{u_j + a_l - \epsilon_3}{u_j + a_l} \prod_{k \neq j} \frac{(u_j - u_k + \epsilon_1)(u_j - u_k + \epsilon_2)(u_j - u_k + \epsilon_3)}{(u_j - u_k - \epsilon_1)(u_j - u_k - \epsilon_2)(u_j - u_k - \epsilon_3)}$$

- these equations are the same as in the simplest Heisenberg XXX $SU(2)$ spin chain, except for the fact that the interaction between Bethe roots is now a degree 3 rational function instead of degree 1!
- very rich structure of solutions: capture all the representation theory of Virasoro of $\mathcal{W}_N$ algebras (singular vectors / null states / minimal models, ...)

- the parameter q is very natural from various points of view:
  1. the twist parameter from spin chain point of view
  2. encodes the shape (complex structure) of the auxiliary torus
  3. controls the non-locality of the Hamiltonians
  4. serves as a natural homotopy parameter for numerical solution of the equations
- once we solve Bethe ansatz equations, the spectrum of $\mathcal{H}_q(u)$ can be written as

$$\frac{\mathcal{H}_q(u)}{\mathcal{H}_{q=0}(u)} \to \frac{1}{\sum_\lambda q^{|\lambda|}} \sum_\lambda q^{|\lambda|} \prod_{\square\in\lambda} \psi_\Lambda(u - \epsilon_\square + \epsilon_3)$$

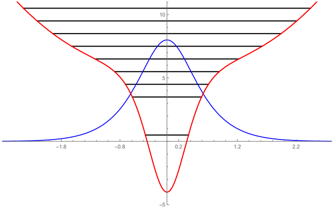
  where

$$\psi_\Lambda(u) = A(u) \prod_j \varphi(u - x_j)$$

  which is a conjectural Yangian version of a formula by Feigin-Jimbo-Miwa-Mukhin (TP & Akimi Watanabe)
- somewhat similar to functions related to qq characters?

- the local limit $q \to 1$ is rather singular (actually any q a root of unity!), but in this limit the Heisenberg subalgebra and $\mathcal{W}_\infty$ decouple
- in particular, the Bethe roots associated to $\mathcal{W}_\infty$ remain finite in the $q \to 1$ limit while those associated to Heisenberg subalgebra diverge
- Bethe equations for free CFT: the singular behaviour of Heisenberg roots encodes in rather subtle way the shape of the Young diagram - connection to equilibrium positions of rational Calogero model, to rational deformations of harmonic oscillator or to Airault–McKean–Moser locus of KdV potentials (work in progress with Matěj Kudrna)

$$-\partial_x^2 + x^2 + \frac{8(2x^2-1)}{(2x^2+1)^2} + 4$$



Questions

Many questions

- how are the ILW and BLZ Bethe ansatz equations related? understanding this could shed light on mysterious fiber-base duality / Miki automorphism in Yangian setting (spectral duality)
- another set of Bethe ansatz equations based on affine Gaudin model (nested BA structure)
- how can the ILW generating function be regularized to extract interesting information in $q \to 1$ limit? qq-characters?
- refined characters & modularity (Dijkgraaf, Maloney-Ng-Ross-Tsiares)
- quantum periods, TBA, mirror symmetry in topological string
- elliptic Calogero model (TBA equations of Nekrasov-Shatashvili)

Thank you!