Boundary and interface correlators, algebras, and spin chains in 4d N=4

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Subject: local operators at BPS boundaries and interfaces in 4d N=4 SYM

More precisely, boundaries and interfaces preserving 3D N=4 SUSY

Goal: compute boundary and interface correlators (and more)

3D N=4 theories have well-studied protected sectors
[Chester-Lee-Pufu-Yacoby, Beem-Peelaers-Rastelli, MD-Fan-Pufu-Yacoby...]

4D N=4 theories have well-studied SUSY boundary conditions [Gaiotto-Witten]

Combine the two subjects: protected sectors at half-BPS boundaries
[also some progress by Wang and Komatsu]

Protected sector: pick Q (think of "$Q+S$" in flat space SCFT)

$Q^2 = \underline{\text{rotation}} + R\text{-symmetry}$; study cohomology.

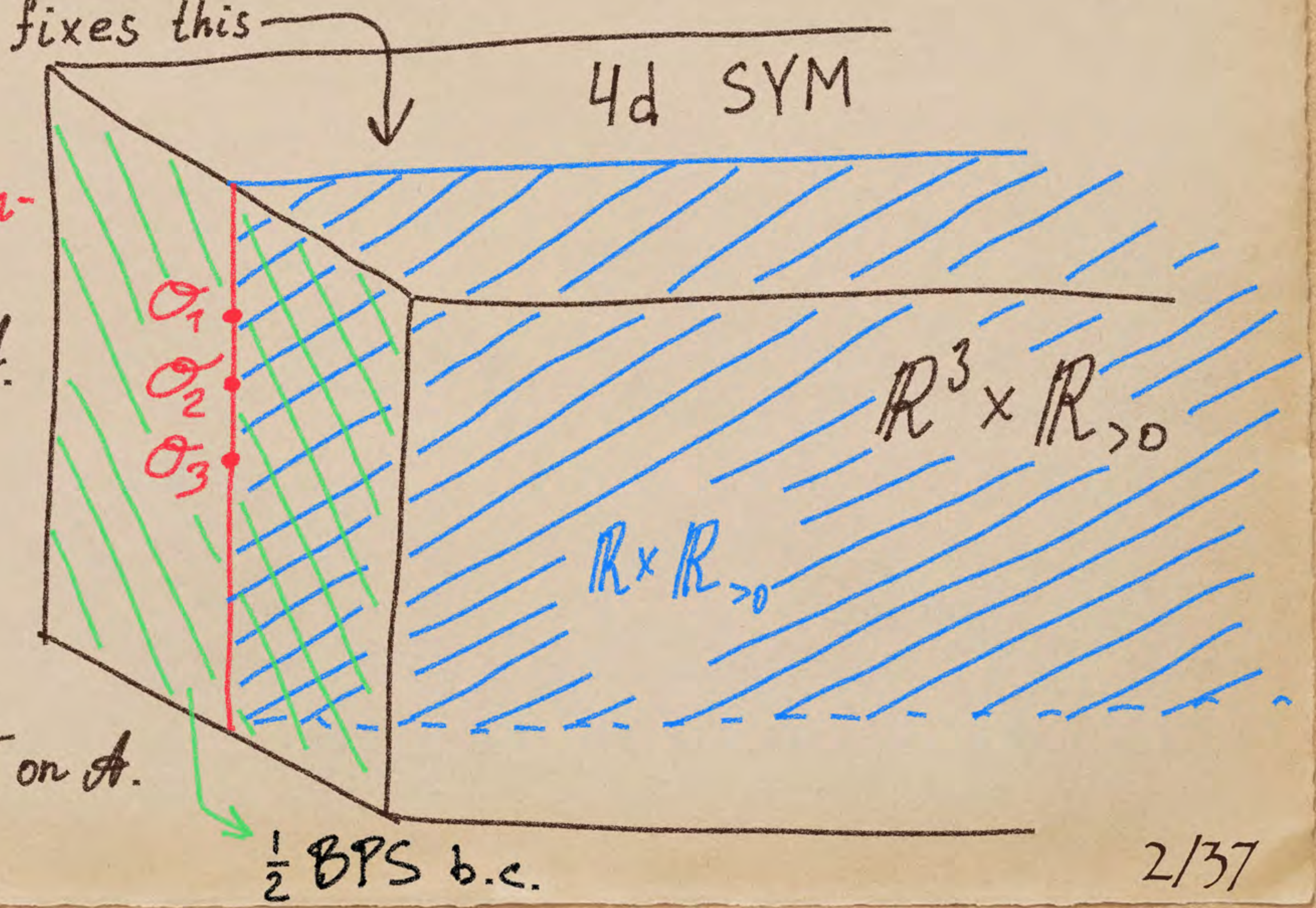
* Bulk operators form a commutative algebra $\mathcal{B}$.

* Boundary operators form a non-commutative algebra $\mathcal{A}$.

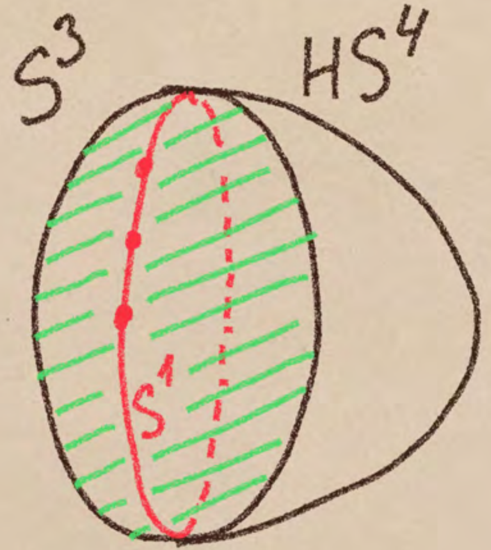
* Bulk-boundary morphism $\rho: \mathcal{B} \to \mathcal{A}$.
  $\rho(\mathcal{B}) \subset Z[\mathcal{A}]$

* Study correlators on $(H)S^4$.

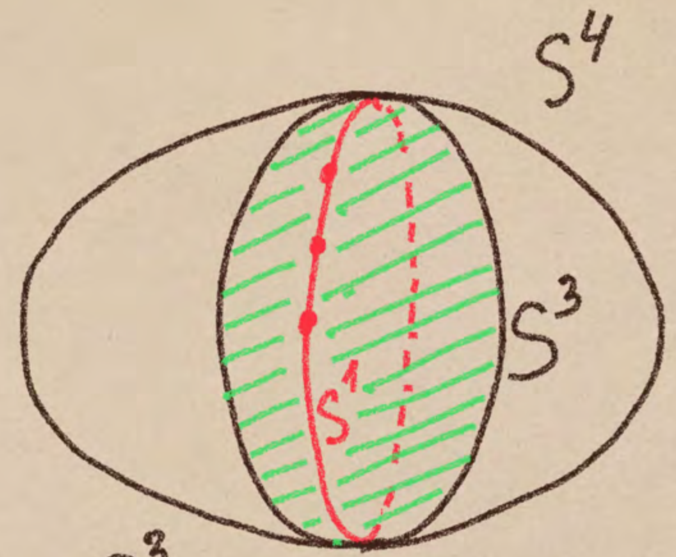
* Boundary correlators are encoded in a twisted trace T on $\mathcal{A}$.



Sphere setup:



or



Operators are inserted on $S^1 \subset S^3 \implies$ appearance of trace.
(twisted by $\mathbb{Z}_2 \subset SU(2)$ R-symm. and boundary flavor symm.)

$$\begin{array}{ccc} S^3 & = & \partial HS^4 \\ \cup & & \cup \\ S^1 & = & \partial HS^2 \end{array}$$

HS^2 supports a 2d theory

What we will find

* Associative algebra $\mathcal{A}$ with a twisted trace T.
* Encodes boundary correlators.
* Correlators : $\langle \mathcal{O}_1 \dots \mathcal{O}_n \rangle = T(\mathcal{O}_1 \dots \mathcal{O}_n)$.
* Examples: $U(\mathfrak{g})$, $W(\mathfrak{g}, e)$, $Y(\mathfrak{g}, N)$.
* Coproduct of $Y(\mathfrak{g}, N)$ plays important role.

$$T[u; N_1 + N_2]^a_c \longmapsto T[u; N_1]^a_b \, T[u; N_2]^b_c$$

$\longrightarrow$ allows to construct traces from simple building blocks

- $Q^2 = \underline{\text{rotation}} + \text{R-symmetry}$
  ↳ fixed locus supports a lower-dimensional QFT.

Protected Sectors in 4d $\mathcal{N}=4$ SYM

[Pestun, Giombi Wang]

$\boxed{\text{electric 2d cYM}}$ $\xleftrightarrow{\text{S duality}}$ $\boxed{\text{magnetic 2d cYM}}$

$\boxed{4d\ \mathcal{N}=4}$

$\mathcal{A}_H$ construction @ the boundary

Q_H

$\mathcal{A}_C$ construction @ the boundary

Q_C

$\cup$

$\boxed{\text{chiral algebra [Beem et. al.], } \mathcal{N}=2 \text{ enough}}$

We are looking for: $(\mathcal{A}_H, T_H)$ and $(\mathcal{A}_C, T_C)$

- Higgs and Coulomb branch sectors on the half-space.
- $\mathcal{A}_H$: $\frac{1}{2}\mathbb{Z}$-filtered associative algebra; $T_H: \mathcal{A}_H \to \mathbb{C}$ - twisted trace.

$$T_H(xy) = T_H\left(e^{-m}(-1)^{2R_H}\, y\, x\right)$$

- $(\mathcal{A}_H, T_H)$ encodes boundary correlators for electric 2d cYM
- Same for $(\mathcal{A}_C, T_C)$: boundary correlators for magnetic 2d cYM.

* In 3d, $(\mathcal{A}, T)$ encodes equivariant short star-product on a hyper-Kähler cone (Etingof–Stryker)

* In the 4d/3d system, $(\mathcal{A}, T)$ still encodes quantization, BUT the underlying space is Poisson $\leftarrow$ moduli of vacua on $\mathbb{R}^3 \times \mathbb{R}_+$.

Example # 1

→ Dirichlet boundary conditions

⇒ Dirichlet b.c. in 2d cYM ⇒ pert. 2d BF on $\mathbb{D}^2$
(w/ B @ $\partial\mathbb{D}^2$)

⇒ Poisson sigma-model into $\mathfrak{g}^*_{\mathbb{C}}$

⇒ Kontsevich $\star$-product (Cattaneo-Felder)

$\Longrightarrow$ Quantization of $\mathfrak{g}_{\mathbb{C}}^{*} = U(\mathfrak{g}_{\mathbb{C}})$.

$\mathcal{A}_H = U(\mathfrak{g}_{\mathbb{C}})$ - algebra of boundary operators

Trace T_H is determined by its value on the center $Z[U(\mathfrak{g}_{\mathbb{C}})]$ (follows from Ward id's)

Algebra of bulk operators: (in 2d YM)

$\mathcal{B}_H$ = gauge-inv. poly in $F_{\mu\nu} = F_{12}$

Can be thought of as $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \simeq \mathbb{C}[\mathfrak{t}]^{W}$

Bulk-boundary map:

$\rho_H : \mathcal{B}_H \longrightarrow \mathcal{A}_H$; $\underline{\rho_H(\mathcal{B}_H) \simeq Z[\mathcal{A}_H]}$

For $U(\mathfrak{g}_{\mathbb{C}})$,

$$\rho_H : \mathbb{C}[\mathfrak{t}]^W \xrightarrow{\simeq} Z[U(\mathfrak{g}_{\mathbb{C}})]$$

→ Harish-Chandra isomorphism

→ encodes physics of the map ρ_H

Trace can be expressed through traces on Verma modules $\mathrm{Tr}_{\bar V_\lambda}$ of $U(\mathfrak{g}_{\mathbb{C}})$:

$$T_H(\mathcal{O}) = \# \int_{\mathfrak{t}^\vee} [da]\, e^{-\frac{i\pi}{\tau}\mathrm{Tr}(a^2)}\, \Delta(a)\, \mathrm{Tr}_{\bar V_{-ia-\rho}}\left(e^{-2\pi m\cdot B}\,\mathcal{O}\right)$$

GNO dual (or L)

$$\Delta(a) = \prod_{\alpha\in\Phi_+} \langle \alpha, a\rangle$$

boundary mass

$B \in \mathfrak{g}_{\mathbb{C}}$

- This is compatible with S-duality

- T_H & T_C are finite linear combinations of traces over the Verma modules in 3d. [Gaiotto-Okazaki]

Here: 4d/3d system; continuous linear comb.

Example #2

→ Neumann. b.c. enriched by a 3d theory $\mathcal{T}$.

Equivalent description:

1) Take [Dirichlet] $\otimes$ $\mathcal{T}$

2) Gauge Diag$(G \times G)$ via a 3D vector multiplet

3D gauging corresponds to quantum Hamiltonian reduction of $\mathcal{A}_H$.

$$\mathcal{A}_H = \left(\mathcal{A}_H(T) \otimes \mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \big/ (\mu) \right)^{\mathcal{G}}$$

$$= \left(\mathcal{A}_H(T) \right)^{\mathcal{G}}$$

$\mathcal{A}_C$ is obtained from $\mathcal{A}_C(T)$ as a central extension:

$$0 \to \mathbb{C}[t]^W \xrightarrow{\text{bulk-boundary map}} \mathcal{A}_C \longrightarrow \mathcal{A}_C(T) \longrightarrow 0$$

"$\mathcal{A}_C$ is obtained from $\mathcal{A}_C(T)$ by promoting masses to dynamical fields"

Writing trace is easy.

Skip to save some time.

Example # 3

$\longrightarrow$ Nahm pole b. c.

$\vec{X} \sim \frac{\vec{t}}{y} \longleftarrow$ su(2) triple corresponding to

Triplet of $SU(2)_H$

$\rho : su(2) \longrightarrow \mathfrak{g}$

*A modification of Dirichlet b.c. ; $\mathcal{A}_C \simeq \mathbb{C}$; $\mathcal{A}_H$ modified.

We find:

moduli space of Nahm's eqn's on $\mathbb{R}^3 \times \mathbb{R}_+$

$\rightarrow$ $\boxed{\text{Space of boundary operators}}$

$\cong$

$\boxed{\text{Regular functions on Slodowy slice } S_{t_+}}$

Fact: S_{t_+} has a natural Poisson structure, [Gan-Ginzburg; I. Losev]

Quantization $\longrightarrow$ finite $\mathcal{W}$ algebra

New challenges:

- $SU(2)_H$ R-symmetry mixes with gauge symmetry at the boundary
  $\Rightarrow$ boundary R-charges are shifted.
- Singularity restricts boundary values of fields.
- Identification of boundary operators is interesting.

Conjecture

$\mathcal{A}_H\{\text{Nahm pole } \rho\} \simeq$ finite W-algebra $\mathcal{W}(\mathfrak{g}_{\mathbb{C}}, t_+)$

→ Reminder: ρ determines grading on $\mathfrak{g}_{\mathbb{C}}$,
$\mathfrak{n} \subset \mathfrak{g}_{\mathbb{C}}$ – nilpotent subalgebra of deg < 0.

$\mathcal{W}(\mathfrak{g}_{\mathbb{C}}, t_+)$ is roughly a quantum Hamiltonian reduction of $U(\mathfrak{g}_{\mathbb{C}})$ over $\mathfrak{n}$.

The most convincing check:

S-dual: Neumann b.c. + $T_\rho[G^\vee]$

e.g. for $SU(N)$: $V_{k-1} - V_{k-2} - \cdots - V_1 - \boxed{N}$

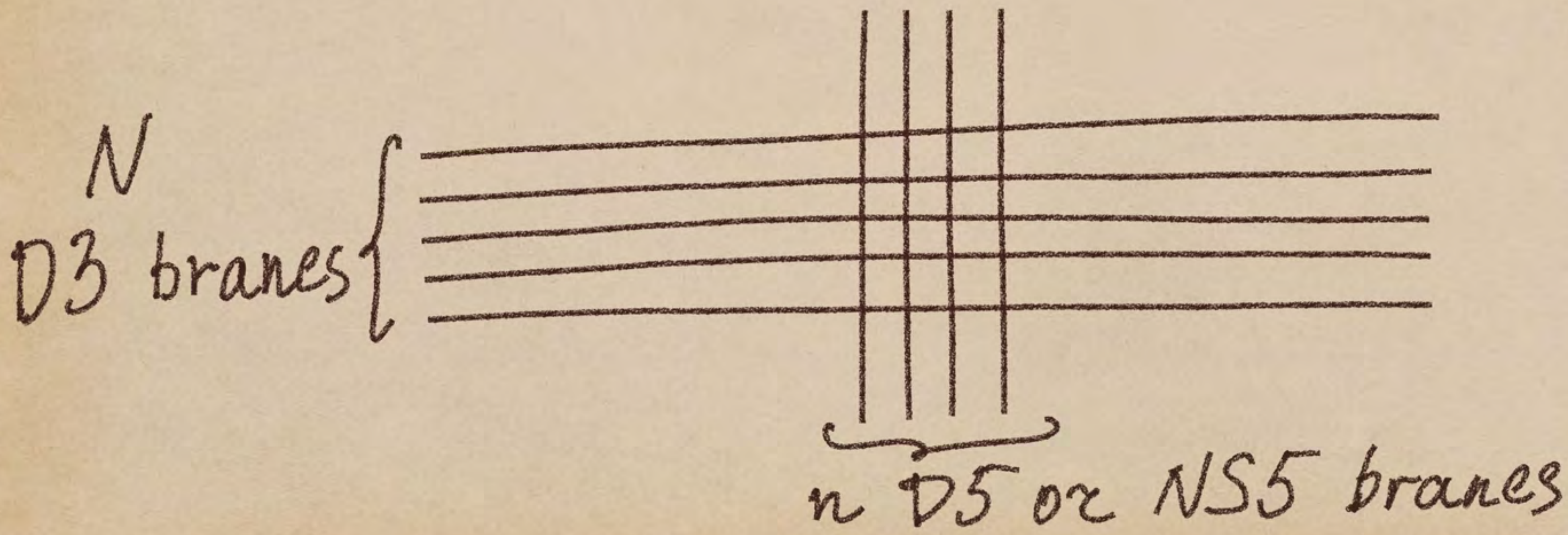
This theory is known to have (central quotient) of $W(\mathfrak{g}_{\mathbb{C}}, t_+)$ for its $\mathcal{A}_{\mathbb{C}}$.

→ Can also write trace as a continuous linear combination of Verma traces.

→ A fact to appreciate: a Higgs/Coulomb branch of vacua on $\mathbb{R}^3 \times \mathbb{R}_{>0}$ is parametrized by the vevs of the appropriate boundary operators.

Interfaces

(Example # 4)



$$\mathcal{A}_H(n\ \mathrm{D5's}) \cong \mathcal{A}_C(n\ \mathrm{NS5's})$$

$\mathcal{A}_H(n\ D5\text{'s}) \equiv A_{N,n}$ admits description as quantum Hamiltonian reduction of:

$$U(\mathfrak{gl}_n) \otimes W^{Nn} \otimes U(\mathfrak{gl}_N)$$

Left D3 branes
generators $(B_-)_\alpha{}^\beta$

n fund. hypers
at the interface
Generators
$X^a{}_\alpha\ ,\ Y_b{}^\beta \leftarrow$ Weyl algebra

Right D3 branes
generators $(B_+)_\alpha{}^\beta$

Yangian Truncation.

Recall: $Y(\mathfrak{gl}_n)$ is generated by $T[z]_a{}^b = \delta_a{}^b + \sum_{n=1}^{\infty} \frac{t^{(n)}{}_a{}^b}{z^n}$, $a, b = 1 \ldots n$, s.t.

$$(z-w)\left[T[z]_a{}^b, T[w]_c{}^d\right] = T[z]_c{}^b T[w]_a{}^d - T[w]_c{}^b T[z]_a{}^d$$

Define: • $T[z] = \mathbb{1} - X \frac{1}{z - B_+} Y$ or • $T[z] = \mathbb{1} + X \frac{1}{z + B_-} Y$

These obey Yangian relations, give surjective morphisms: $Y(\mathfrak{gl}_n) \twoheadrightarrow A_{N,n}$

- "Isomorphism" in the large-N limit.
- T_H on $A_{N,n}$ induces traces on $Y(\mathfrak{gl}_n)$.

On the S-dual side look at $\mathcal{A}_C$ (n NS5's).

Engineered by a 3d $\mathcal{N}=4$ quiver: $\boxed{N} - \underbrace{(N) - (N) - \cdots - (N) - (N)}_{n-1 \text{ gauge nodes}} - \boxed{N}$

- 3d algebra $\mathcal{A}_C$ is a truncation of $Y(\mathfrak{sl}_n)$ [Bullimore-Dimofte-Gaiotto, Braverman-Finkelberg-Nakajima]
- Couple to the bulk by promoting masses to fields $\Rightarrow$ truncation of $Y(\mathfrak{gl}_n) = \mathcal{A}_C \equiv A_{N,n}$
- Apply [MD-Fan-Pufu-Yacoby] to find the trace. $\Rightarrow$ Representation of $Y(\mathfrak{sl}_n)$ in terms of shift operators as in [Gerasimov-Kharchev-Lebedev-Oblezin].
- We will compute $\langle T[z_1]_{a_1}^{b_1} T[z_2]_{a_2}^{b_2} \dots T[z_L]_{a_L}^{b_L} \rangle$ from the Coulomb branch.

Let me first sketch the answer:

$a_i, b_i = 1 \dots n$

$$\left\langle T[z_1]_{a_1}{}^{b_1}\, T[z_2]_{a_2}{}^{b_2} \dots T[z_L]_{a_L}{}^{b_L} \right\rangle = \langle a_1, \dots, a_L | \underbrace{M_L}_{n^L \times n^L \text{ matrix}} | b_1, \dots, b_L \rangle,$$

Interpret M_L as acting on $\mathbb{C}^{n^L}$, a Hilbert space of the sl_n spin chain of length L.

M_L = linear combination of transfer matrices.

We will write M_L for $n=2$ explicitly

$\rightarrow$ inhomogeneous XXX spin chain.

[c.f. Bethe/Gauge correspondence of Nekrasov-Shatashvili]

Remark:

- Why do integrable spin chains appear?

  (1) Finite-dimensional representations of the Yangian $\leftrightarrow$ spin chains.

  (2) Traces in finite-dim. repr. generate the algebra of traces*.

- Such traces are related to transfer matrices.

- We will also use special properties of truncated Yangians (otherwise — too hard)

* Coproduct defines a commutative product on traces.

• One can build general traces starting from the traces over evaluation modules.

• Evalation module of $Y(\mathfrak{g})$ is constructed from a given $\mathfrak{g}$-module:

L-operator: $\mathbb{L}(z) = z \cdot \overset{n\times n}{\mathbb{1}} + \sum E_{ij} \otimes J_{ij}$

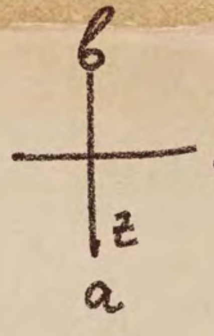
generators of $\mathfrak{sl}_n$

elementary $\mathfrak{sl}_n$ matrices

Satisfies: $R(x-y)(\mathbb{L}(x)\otimes \mathbb{1})(\mathbb{1}\otimes \mathbb{L}(y)) = (\mathbb{1}\otimes \mathbb{L}(y))(\mathbb{L}(x)\otimes \mathbb{1})R(x-y)$

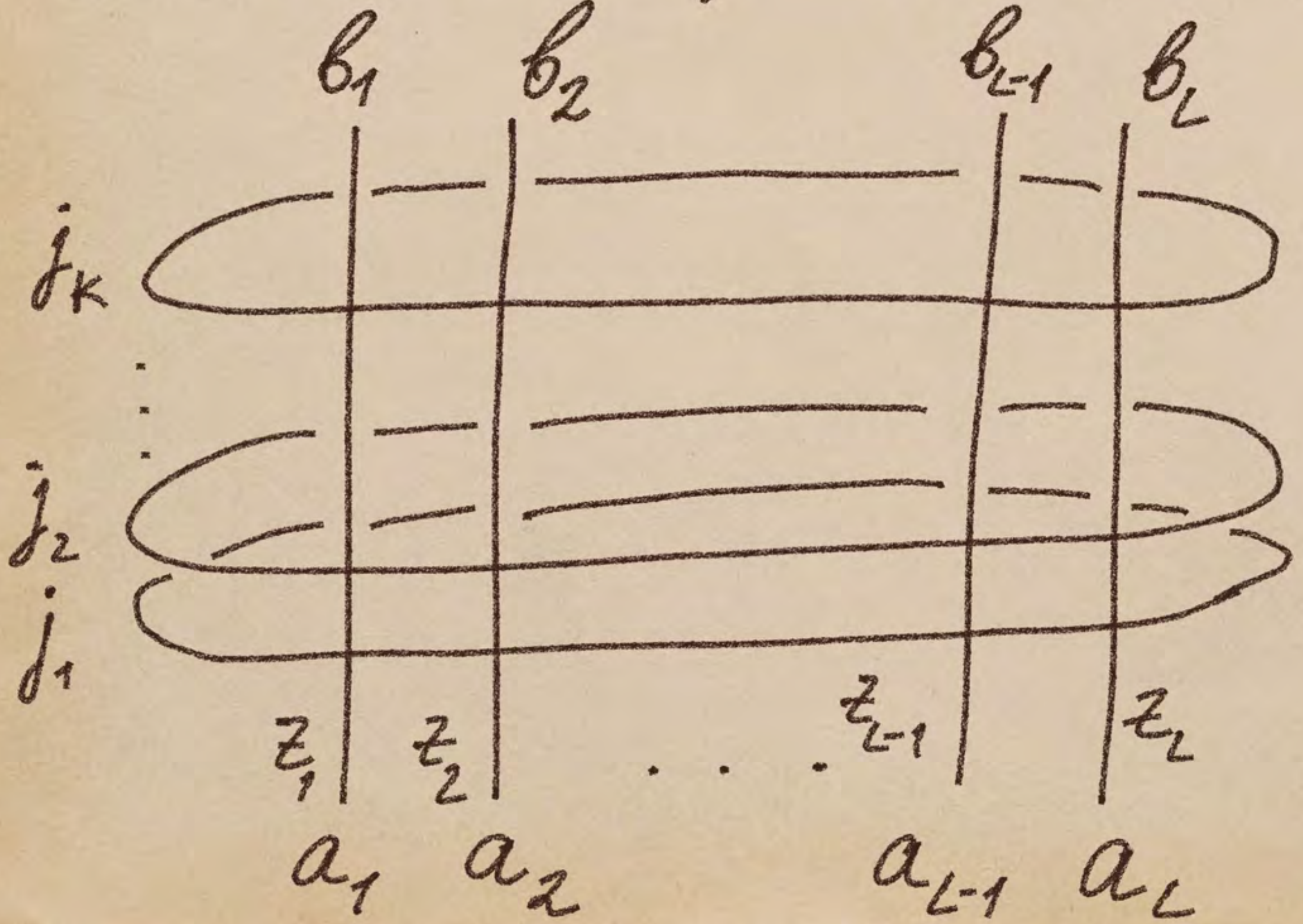
$R(z) = z + P : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n$;

• $\mathbb{L}(z)$ determines an <u>eval. module</u>: $T[z] \mapsto \frac{1}{z}\mathbb{L}(z)$

Pictorially: $\mathbb{L}(z)_a{}^b =$ 

j , j labels sl_n rep.

Twisted trace of $T[z_1]_{a_1}{}^{b_1} \dots T[z_L]_{a_L}{}^{b_L}$ in $[j_1]\otimes\dots\otimes[j_K]$:



Matrix element of a product of K transfer matrices of a length-L periodic inhomogeneous sl_n spin chain.

* Explain co-product.

can skip

! In general, the space of traces on $Y(\mathfrak{sl}_n)$ is huge, and we'd rather not work with it.

We have $Y(\mathfrak{sl}_n) \to \mathcal{A}_C$[quiver theory], and we only need traces on this algebra. It has finitely many Verma modules. We need traces over those!

Additional input: $\mathcal{A}_C$[quiver theory] = "truncated Yangian". It also admits the coproduct [thesis of A. Weekes]

Consider $n=2$. [$Y(\mathfrak{gl}_2)$, $Y(\mathfrak{sl}_2)$, two fivebranes]

Using techniques of [MD-Fan-Pufu-Yacoby], can directly compute $\langle T[z_1]_{a_1}{}^{b_1} \cdots T[z_L]_{a_L}{}^{b_L} \rangle^*$ ...

It turns out that...

The answer involves N $\mathfrak{sl}_2$ Verma modules.

I.e., $[j_1], \ldots, [j_N]$ – eval. modules corresp. to $\mathfrak{sl}_2$ Verma modules of h.w. j_m.

* Only compute some correlators; enough to fix the answer.

Let me give more details. The quiver: $\boxed{N} - (N) - \boxed{N}$ [U(N) SQCD]

Call its A_C algebra $\mathcal{A}_{N;2}[m]$.

- First solve the $N=1$ case using [MD-Fan-Pufu-Yacoby]

→ $\mathcal{A}_{1;2}[m]$ has two Verma modules ⟹ two traces: [and two vacua]

$$Q_+(\vec{z}-\mu_1\vec{e})Q_-(\vec{z}-\mu_2\vec{e}) \quad \text{and} \quad Q_+(\vec{z}-\mu_2\vec{e})Q_-(\vec{z}-\mu_1\vec{e})$$

$$\overset{+\ \ \ \parallel}{2\sinh(\pi\zeta)\,\mathbb{T}_{-\frac{1}{2}+i\frac{\mu_1-\mu_2}{2}}\left(\vec{z}-\frac{\mu_1+\mu_2}{2}\vec{e}\right)} \qquad \overset{+\ \ \ \parallel}{2\sinh(\pi\zeta)\,\mathbb{T}_{-\frac{1}{2}+i\frac{\mu_2-\mu_1}{2}}\left(\vec{z}-\frac{\mu_1+\mu_2}{2}\vec{e}\right)}$$

→ Here $\mathbb{T}^+_\alpha(\vec{z})$ is a transfer matrix for the Verma module of h.w. α

→ $\vec{z}=(z_1,\dots,z_L)$; $\vec{e}=(1,\dots,1)$; μ_1,μ_2 – masses; $Q_\pm$ – Baxter's Q-operators.

- Use the coproduct $\mathcal{A}_{N;2}[m] \to \mathcal{A}_{1;2}[m]\otimes\dots\otimes\mathcal{A}_{1;2}[m]$ to argue:

$$\prod_{a=1}^{N}\underbrace{Q_+(\vec{z}-\mu_{\sigma(a)}\vec{e})Q_-(\vec{z}-\mu_{\sigma(a+N)})}_{\text{a trace on }\mathcal{A}_{1;2}[m]},\quad \sigma\in\frac{S_{2N}}{S_N\times S_N}$$

are the $\binom{2N}{N}$ Verma traces

Gaiotto–Okazaki '19

[$\binom{2N}{N}$ choices: which N out of $2N$ masses appear inside Q_+'s]

Using localization results allows to uniquely determine the linear comb.

$$M_L = \sum_{\sigma \in \frac{S_{2N}}{S_N \times S_N}} \frac{i^{-N^2} e^{i\pi\zeta \sum_{j=1}^{2N} \mu_j}}{(2\sinh \pi\zeta)^N \prod_{a=1}^{N} \prod_{k=N+1}^{2N} 2\sinh \pi(\mu_{\sigma(a)} - \mu_{\sigma(k)})} \prod_{a=1}^{N} Q_+(\vec{z} - \mu_{\sigma(a)}\vec{e})\, Q_-(\vec{z} - \mu_{\sigma(a+N)}\vec{e})$$

$\Longrightarrow$ Twisted trace on $Y[\mathfrak{sl}_2]$ [answer for a 3d theory]

Coupling this to the bulk, and after some combinatorics...

$$M_L = \frac{1}{N!} \int_{\mathbb{R}^N \times \mathbb{R}^N} [d\mu^L][d\mu^R] e^{-\frac{i\pi}{\tau} tr(\mu^L)^2 - \frac{i\pi}{\tau} tr(\mu^R)^2 + 2\pi i \zeta \sum_{a=1}^{N} \bar{\mu}_a} \Delta(\mu^L)\,\Delta(\mu^R) \prod_{j=1}^{N} \frac{\Pi^+_{-\frac{1}{2} + i(\mu^L_j - \bar{\mu}_j)}(\vec{z} - \bar{\mu}_j \vec{e}) - \Pi^+_{\frac{1}{2} + i(\mu^R_j - \bar{\mu}_j)}(\vec{z} - \bar{\mu}_j \vec{e})}{2i \sinh \pi(\mu^L_j - \mu^R_j)}$$

Here $\bar{\mu}_a = \frac{1}{2}(\mu^L_a + \mu^R_a)$.

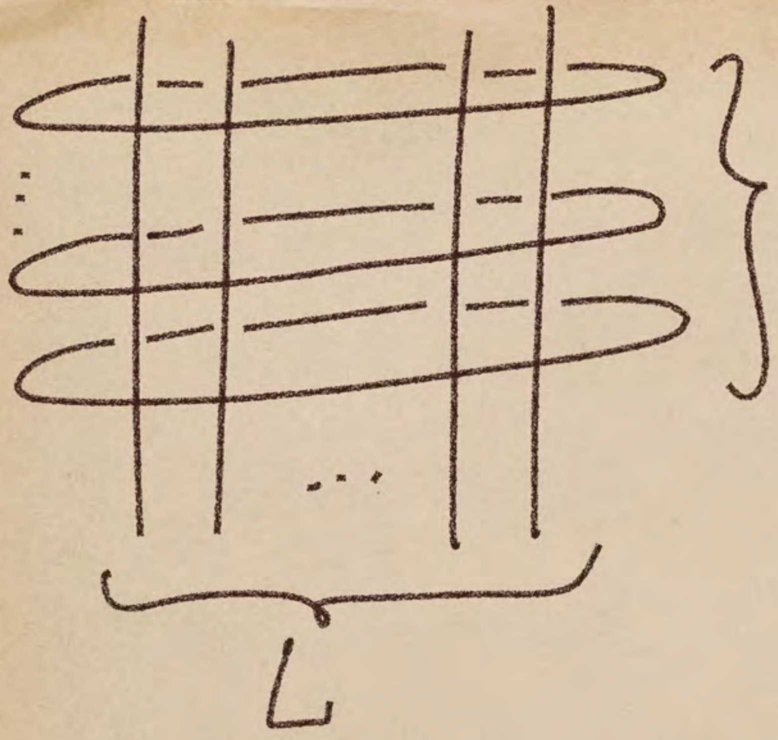
$\Longrightarrow$ Twisted trace on $Y[\mathfrak{gl}_2]$ [answer for an interface theory]

Remarkably explicit answer!

$Q_\pm$ can be computed algorithmically using the results of [Bazhanov-Lukowski-Meneghelli-Staudacher] (as traces over an auxiliary oscillator Fock space)

Compute $Q_\pm$ for each $L \Rightarrow$ answers for $\forall N$

Can do large-N etc... [especially for the interface, complicated matrix integral...]



$N \leftarrow$ the same as # of D3 branes!

Ask me: relation to 4d CS.

<u>To do</u>: generalize to $n > 2$

<u>To do</u>: Analyze the large-N limit of the interface correlators. Study AdS dual.

To do: this is a good setting for another example of twisted holography (both 2d YM and 1d TQM at the interface)
/c.f. Ishtiaque - Faroogh Moosavian - Zhou '18 /
Faroogh Moosavian - Zhou '21

Thank You!