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Stability results for backward time-fractional parabolic equations

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Abstract

Optimal order stability estimates of Hölder type for the backward Caputo time-fractional abstract parabolic equations are obtained. This ill-posed problem is regularized by a non-local boundary value problem method with *a priori* and *a posteriori* parameter choice rules which guarantee error estimates of Hölder type. Numerical implementations are presented to show the validity of the proposed scheme.

Keywords: backward time-fractional parabolic equations, stability estimates, non-local boundary value problem method

(Some figures may appear in colour only in the online journal)

1. Introduction

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, $A : D(A) \subset H \to H$ be a self-adjoint closed operator on H such that $-A$ generates a compact contraction semi-group $\{S(t)\}_{t\geqslant 0}$ on H. Assume that A admits an orthonormal eigenbasis $\{\phi_i\}_{i\geqslant 1}$ in H, associated with the eigenvalues $\{\lambda_i\}_{i\geqslant 1}$ such that

$$0 < \lambda_1 < \lambda_2 < \ldots, \quad \text{and} \quad \lim_{i\to+\infty} \lambda_i = +\infty.$$

For $\gamma \in (0,1)$, consider the backward time-fractional parabolic equation

$$\begin{cases} \frac{\partial^\gamma u}{\partial t^\gamma} + Au = 0, \ 0 < t < T, \\ \|u(T) - f\| \leqslant \varepsilon, \end{cases} \tag{1.1}$$

where

$$\frac{\partial^\gamma u}{\partial t^\gamma} = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} \frac{\partial u(\cdot, s)}{\partial s} \mathrm{d}s, \ \gamma \in (0,1), \tag{1.2}$$

is the Caputo derivative [7, 9] and $\Gamma(\cdot)$ is Euler's Gamma function. Since the first work [8] devoted to the backward time-fractional diffusion equation, several papers on backward time-fractional parabolic equations have been published: the mollification method [12], the non-local boundary value problem method [16–18], Tikhonov regularization [1, 6, 13–15].

In this paper, we first establish stability estimates of Hölder type for problem (1.1) and then we apply a non-local boundary value problem method [2] to it. Namely, we regularize the ill-posed problem (1.1) by the non-local boundary value problem

$$\begin{cases} \frac{\partial^\gamma v_\alpha}{\partial t^\gamma} + A v_\alpha = 0, \ 0 < t < T, \\ \alpha A^k v_\alpha(0) + v_\alpha(T) = f, \end{cases} \tag{1.3}$$

where $0 < \alpha < 1$ and $k = 0, 1, 2, ...$ is fixed. *A priori* and *a posteriori* parameter choice rules are suggested which yield error estimates for $u(t)$ of Hölder type for $t \in [0, T]$.

In [2–5], we used the non-local boundary value problem method with $k = 0$ to regularize backward parabolic equations and we obtained the error estimates of optimal order. However, for the backward time-fractional parabolic equations, we note that if we use $k = 0$, the order of the error estimate does not exceed $1/2$. Therefore, we set $k = 0, 1, 2, ...$ for getting a higher convergence rate of the regularizing method theoretically. The numerical performances of the regularizing schemes for different values of k are also checked.

To compare our results with the previous ones, let us denote

$$D(A^p) = \left\{ \psi \in H : \sum_{n=1}^{\infty} \lambda_n^{2p} \langle \psi, \phi_n \rangle^2 < \infty \right\}$$

and the norm

$$\|\psi\|_p := \|\psi\|_{D(A^p)} = \left(\sum_{n=1}^{\infty} \lambda_n^{2p} \langle \psi, \phi_n \rangle^2 \right)^{\frac{1}{2}}, \ \psi \in D(A^p).$$

In case $H = L^2(\Omega)$, Ω is a bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\Omega$, the expression $\mathcal{A}$ is of the form:

$$\mathcal{A}u = -\sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{d} a_{ij}(x) \frac{\partial}{\partial x_j} u(x) \right) + c(x)u(x), x \in \Omega, \tag{1.4}$$

with $a_{ij} = a_{ji} \in C^1(\overline{\Omega}), i, j = 1, \ldots, d, c \in C(\overline{\Omega}), c \geqslant 0, x \in \overline{\Omega}$ and $\sum_{i,j=1}^{d} a_{ij}\xi_i\xi_j \geqslant \nu \sum_{i=1}^{d} \xi_i^2, x \in \overline{\Omega}$ for some $\nu > 0$. Then the operator A can take the form

$$Au(x) = \mathcal{A}u(x), x \in \Omega, D(A) = H^2(\Omega) \cap H_0^1(\Omega)$$

and we have [10] $D(A^p) \subset H^{2p}(\Omega), p > 0$, $D(A^{\frac{1}{2}}) = H_0^1(\Omega)$.

For similar problems to (1.1), Liu and Yamamoto [8] used the quasi-reversibility method, Yang and Liu [19] used the Fourier method, Yang and Liu [18] applied the non-local boundary value problem method supposing that $\|u(0)\|_1$ is bounded, meanwhile Wang and Liu [12] studied the data regularization method assuming that $\|u(0)\|_p$ is bounded with $p = 1$ or 1/2. Furthermore, these authors applied an *a priori* parameter choice rule to their approaches.

Wang, Zhou and Wei [16], Wei and Wang in [17] applied variations of the non-local boundary value problem method to regularizing (1.1), where in [16] the authors studied *a posteriori* parameter choice rules and in [17] both *a priori* and *a posteriori* parameter choice rules are investigated. In this paper we also study the non-local boundary value problem method for both *a priori* and *a posteriori* parameter choice rules assuming that $\|u(0)\|_p$ is bounded for arbitrary positive constant p. However, the order of our error estimates is better than that of [16] and our strategy for *a posteriori* method is much simpler than that of [16] and [17]. Indeed, the order of our error estimates can be greater than $2/3$ while that of the authors in [12, 16, 18] is not greater than $1/2$ (since these authors only considered $k = 0$). Furthermore, the order of the error estimate in [17] is not greater than $2/3$ for all $p > 0$ for the *a priori* parameter choice rule and is not greater than $1/2$ for the *a posteriori* parameter choice rule (since these authors only considered $k = 1$).

This paper is organized as follow: in the next section we will present our stability estimate for (1.1). In section 3 we describe our regularization method with the error estimates, the proofs of which will be given in section 4. Finally we present some numerical implementations for the proposed regularizing scheme in section 5.

2. Stability estimate

Denote by $E_{\gamma,\beta}(z)$ the Mittag–Leffler function [7, 9]:

$$E_{\gamma,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\gamma+\beta)}, z \in \mathbb{C}, \gamma > 0,\ \beta \in \mathbb{R}. \tag{2.1}$$

For $\gamma \in (0,1)$, $a \in H$, consider the forward time-fractional parabolic equation:

$$\begin{cases} \frac{\partial^\gamma u}{\partial t^\gamma} + Au = 0, 0 < t < T, \\ u(0) = a. \end{cases} \tag{2.2}$$

Definition 1. The Caputo derivative $\frac{\partial^\gamma u}{\partial t^\gamma}$ is defined by

$$\begin{aligned} \frac{\partial^\gamma u}{\partial t^\gamma} &= \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} \frac{\partial u(\cdot,s)}{\partial s} \mathrm{d}s \\ &:= \lim_{\eta \to 0^+} \frac{1}{\Gamma(1-\gamma)} \int_0^{t-\eta} (t-s)^{-\gamma} \frac{\partial u(\cdot,s)}{\partial s} \mathrm{d}s, \gamma \in (0,1). \end{aligned}$$

Definition 2. Let $a \in H$. The function $u(t) : [0,T] \mapsto H$ is called a solution to problem (2.2) if $u(t) \in C^1((0,T),H) \cap C([0,T],H)$, $u(t) \in D(A)$ for all $t \in (0,T)$ and (2.2) holds.

Theorem 1. *Problem (2.2) admits a unique solution, which can be represented in the form:*

$$u(t) = \sum_{n=1}^{\infty} E_{\gamma,1}\left(-\lambda_n t^\gamma\right) \langle a, \phi_n \rangle \phi_n. \tag{2.3}$$

To prove theorem 1, we need following auxiliary results.

Lemma 1 ([16]). *For any λ_n satisfying $\lambda_n \geqslant \lambda_1 > 0$, there exist positive constant $\overline{C}_1, \overline{C}_2$ depending on γ, T, λ_1 such that*

$$\frac{\overline{C}_1}{\lambda_n} \leqslant E_{\gamma,1}(-\lambda_n T^\gamma) \leqslant \frac{\overline{C}_2}{\lambda_n}.$$

Lemma 2. *Let $\gamma \in (0,1)$, $\lambda > 0$ and $t > 0$. We have*

$$(a) \quad \frac{\mathrm{d}}{\mathrm{d}t} E_{\gamma,1}(-\lambda t^\gamma) = -\lambda t^{\gamma-1} E_{\gamma,\gamma}(-\lambda t^\gamma), \tag{2.4}$$

$$(b) \quad \frac{\mathrm{d}^\gamma}{\mathrm{d}t^\gamma} E_{\gamma,1}(-\lambda t^\gamma) = -\lambda E_{\gamma,1}(-\lambda t^\gamma). \tag{2.5}$$

Part (a) can be found in [10]. The proof of part (b) is straightforward and we omit it.

Now we are in a position to prove theorem 1. First, we verify that $u(t)$ defined by (2.3) is a solution to problem (2.2).

We prove that $u(t) \in D(A)$ for all $t \in (0,T)$. For each $t > 0$, using lemma 1, we conclude that there exist positive constants $\overline{C}_3, \overline{C}_4$ depending on γ, t, λ_1 such that $\frac{\overline{C}_3}{\lambda_n} \leqslant E_{\gamma,1}(-\lambda_n t^\gamma) \leqslant \frac{\overline{C}_4}{\lambda_n}$. This implies that $\langle u(t), \phi_n\rangle^2 = (E_{\gamma,1}(-\lambda_n t^\gamma))^2 \langle a, \phi_n\rangle^2 \leqslant \left(\frac{\overline{C}_4}{\lambda_n}\right)^2 \langle a, \phi_n\rangle^2$, $\forall n \in \mathbb{N}^*$ with $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. Consequently, it follows that $\sum_{n=1}^\infty \lambda_n^2 \langle u(t), \phi_n\rangle^2 \leqslant \sum_{n=1}^\infty (\overline{C}_4)^2 \langle a, \phi_n\rangle^2 = (\overline{C}_4)^2 \|a\|^2 < \infty$. Therefore, $u(t) \in D(A)$ for all $t \in (0,T)$.

Since $E_{\gamma,1}(0) = 1$, we have $u(0) = \sum_{n=1}^\infty E_{\gamma,1}(0) \langle a, \phi_n\rangle \phi_n = \sum_{n=1}^\infty \langle a, \phi_n\rangle \phi_n = a$. Let

$$u_N(t) = \sum_{k=1}^N E_{\gamma,1}(-\lambda_n t^\gamma) \langle a, \phi_n\rangle \phi_n, \ \forall t \in [0,T], N \in \mathbb{N}^*. \tag{2.6}$$

We have

$$u_N(t) \to u(t) \ \text{ in } \ H, \text{as } N \to \infty, \ \forall t \in [0,T]. \tag{2.7}$$

Using lemma 2 and (2.6), we get

$$\frac{\partial^\gamma u_N(t)}{\partial t^\gamma} + A u_N(t) = 0, \forall N \in \mathbb{N}^*, \forall t \in (0,T).$$

This implies that

$$A u_N(t) = -\frac{\partial^\gamma u_N(t)}{\partial t^\gamma} \to -\frac{\partial^\gamma u(t)}{\partial t^\gamma} \ \text{ in } \ H \text{ as } N \to \infty, \forall t \in (0,T). \tag{2.8}$$

From (2.7) and (2.8) and the closeness of the operator A, we obtain

$$\frac{\partial^\gamma u(t)}{\partial t^\gamma} + A u(t) = 0, \forall t \in (0,T).$$

Now we prove the continuity of $u(t)$ at $t = 0$, i.e.

$$\lim_{t\to 0} \|u(t) - u(0)\| = 0. \tag{2.9}$$

Indeed, since $a \in H$, there exist positive constants M and n_δ such that $\|a\|^2 = \sum_{n=1}^\infty \langle a, \phi_n\rangle^2 \leqslant M$ and $\sum_{n=n_\delta}^\infty \langle a, \phi_n\rangle^2 \leqslant \delta^2/4$. Furthermore, since $\lim_{t\to 0} E_{\gamma,1}(t) = E_{\gamma,1}(0) = 1$, there exists a constant $\delta_1 = \delta_1(\delta, M)$ such that $|E_{\gamma,1}(-t) - 1| \leqslant \frac{\delta}{2(1+M)}, \forall t \leqslant \delta_1$. If $0 < t \leqslant (\delta_1/\lambda_{n_\delta})^{\frac{1}{\gamma}}$, then $\lambda_n t^\gamma \leqslant \delta_1, \forall n < n_\delta$. Consequently,

$$\begin{aligned}
\|u(t)-u(0)\|^2 &= \left\|\sum_{n=1}^{\infty} E_{\gamma,1}(-\lambda_n t^\gamma)\langle a,\phi_n\rangle - a\right\|^2 \\
&= \sum_{n=1}^{n_\delta-1}\left(E_{\gamma,1}(-\lambda_n t^\gamma)-1\right)^2\langle a,\phi_n\rangle^2 + \sum_{n=n_\delta}^{\infty}\left(E_{\gamma,1}(-\lambda_n t^\gamma)-1\right)^2\langle a,\phi_n\rangle^2 \\
&\leqslant \sum_{n=1}^{n_\delta-1}\left(E_{\gamma,1}(-\lambda_n t^\gamma)-1\right)^2\langle a,\phi_n\rangle^2 + \sum_{n=n_\delta}^{\infty}\langle a,\phi_n\rangle^2 \\
&\leqslant \frac{\delta^2}{4(1+M)^2}\sum_{n=1}^{n_\delta-1}\langle a,\phi_n\rangle^2 + \delta^2/4 \leqslant \frac{\delta^2}{4(1+M)^2}\|a\|^2 + \delta^2/4 \leqslant \delta^2/2.
\end{aligned}$$

This implies $\lim_{t\to 0}\|u(t)-u(0)\| = 0$.

To prove the uniqueness of a solution to problem (2.2) we represent $u(t) = \sum_{n=1}^{\infty}\langle u(t),\phi_n\rangle\phi_n := \sum_{n=1}^{\infty} u_n(t)\phi_n$,. Plugging the last into (2.2), we get $\partial^\gamma u/\partial t^\gamma + Au = 0, 0 < t < T$ and $u_n(0) = \langle a,\phi_n\rangle$. Hence, following theorem 4.3, page 231 in [7], we obtain $u_n(t) = E_{\gamma,1}(-\lambda_n t^\gamma)u_n(0) = E_{\gamma,1}(-\lambda_n t^\gamma)\langle a,\phi_n\rangle$ which follows (2.3).

Remark 1. We note that the case $1 < \gamma < 2$, the equation $\partial^\gamma u/\partial t^\gamma + Au = 0$ requires two initial conditions $u(0)$ and $u_t(0)$ (see,e.g. [10]), is completely different from problem (2.2). Therefore, we do not consider it in this paper.

Theorem 2 (Stability estimate). *Suppose that $u(t)$ is a solution of the equation*

$$\frac{\partial^\gamma u}{\partial t^\gamma} + Au = 0,$$

satisfying $\|u(T)\| \leqslant \varepsilon$ and $\|u(0)\|_p \leqslant E$ for some positive constants E, p. Then there exists a constant C such that

$$\|u(t)\|_q \leqslant C\varepsilon^{\frac{p-q}{p+1}}E^{\frac{q+1}{p+1}}, 0 \leqslant q < p, \forall t \in [0,T]. \tag{2.10}$$

Proof of theorem 2. We have

$$\|u(0)\|_q^2 = \sum_{n=1}^{\infty}\lambda_n^{2q}\langle u(0),\phi_n\rangle^2 = \sum_{n=1}^{\infty}\left(\lambda_n^{\frac{2p(q+1)}{p+1}}|\langle u(0),\phi_n\rangle|^{\frac{2(q+1)}{p+1}}\right)\left(\lambda_n^{\frac{-2(p-q)}{p+1}}|\langle u(0),\phi_n\rangle|^{\frac{2(p-q)}{p+1}}\right).$$

Using the Hölder inequality, we obtain $\|u(0)\|_q^2 \leqslant \left(\sum_{n=1}^{\infty}\lambda_n^{2p}\langle u(0),\phi_n\rangle^2\right)^{\frac{q+1}{p+1}}\left(\sum_{n=1}^{\infty}\lambda_n^{-2}\langle u(0),\phi_n\rangle^2\right)^{\frac{p-q}{p+1}}$. From (2.3), we have $\langle u(T),\phi_n\rangle = E_{\gamma,1}(-\lambda_n T^\gamma)\langle u(0),\phi_n\rangle$. Hence, $\langle u(0),\phi_n\rangle = \frac{\langle u(T),\phi_n\rangle}{E_{\gamma,1}(-\lambda_n T^\gamma)}$. Therefore, $\|u(0)\|_q^2 \leqslant \|u(0)\|_p^{\frac{2(q+1)}{p+1}}\left(\sum_{n=1}^{\infty}\lambda_n^{-2}\frac{\langle u(T),\phi_n\rangle^2}{E_{\gamma,1}^2(-\lambda_n T^\gamma)}\right)^{\frac{p-q}{p+1}}$. Using lemma 1, we have

$$\begin{aligned}
\|u(0)\|_q^2 &\leqslant \|u(0)\|_p^{\frac{2(q+1)}{p+1}}\left(\sum_{n=1}^{\infty}\lambda_n^{-2}\frac{\langle u(T),\phi_n\rangle^2}{\left(\overline{C}_1/\lambda_n\right)^2}\right)^{\frac{p-q}{p+1}} \\
&\leqslant \|u(0)\|_p^{\frac{2(q+1)}{p+1}}\left(\overline{C}_1\right)^{\frac{-2(p-q)}{p+1}}\left(\sum_{n=1}^{\infty}\langle u(T),\phi_n\rangle^2\right)^{\frac{p-q}{p+1}}.
\end{aligned} \tag{2.11}$$

From (2.11), there exists a constant $C > 0$ such that

$$\begin{aligned}\|u(0)\|_q^2 &\leqslant C^2 \|u(0)\|_p^{\frac{2(q+1)}{p+1}} \|u(T)\|^{\frac{2(p-q)}{p+1}} \\ &\leqslant C E^{\frac{2(q+1)}{p+1}} \varepsilon^{\frac{2(p-q)}{p+1}}.\end{aligned}$$

Since $u(t) = \sum_{n=1}^{\infty} E_{\gamma,1}(-\lambda_n T^\gamma) \langle u(0), \phi_n \rangle \phi_n$ and $0 \leqslant E_{\gamma,1}(-\lambda_n T^\gamma) \leqslant 1$, we obtain

$$\|u(t)\|_q \leqslant \|u(0)\|_q \leqslant C \varepsilon^{\frac{p-q}{p+1}} E^{\frac{q+1}{p+1}}, \forall t \in [0, T].$$

The theorem is proved. □

Remark 2. In theorem 2, the estimates at $t = 0$,

$$\|u(0)\|_q \leqslant C \varepsilon^{\frac{p-q}{p+1}} E^{\frac{q+1}{p+1}}, \tag{2.12}$$

is of optimal order. Indeed, set $v = \sum_{n=1}^{\infty} \lambda_n^q \langle u(0), \phi_n \rangle \phi_n$, we have

$$\|u(0)\|_p^2 = \sum_{n=1}^{\infty} \lambda_n^{2p} \langle u(0), \phi_n \rangle^2 = \sum_{n=1}^{\infty} \lambda_n^{2q} \lambda_n^{2(p-q)} \langle u(0), \phi_n \rangle^2 = \sum_{n=1}^{\infty} \lambda_n^{2(p-q)} \langle v, \phi_n \rangle^2.$$

From lemma 1, we have $\frac{\lambda_n}{C_2} \leqslant \frac{1}{E_{\gamma,1}(-\lambda_n T^\gamma)} \leqslant \frac{\lambda_n}{C_1}$. This implies that $\frac{\lambda_n^{1+q}}{C_2} \leqslant \frac{\lambda_n^q}{E_{\gamma,1}(-\lambda_n T^\gamma)} \leqslant \frac{\lambda_n^{1+q}}{C_1}$. Therefore, the condition $\|u(0)\|_p^2 \leqslant E^2$ is equivalent to the condition

$$\sum_{n=1}^{\infty} \left(\lambda_n^{-q} E_{\gamma,1}(-\lambda_n T^\gamma) \right)^{-2(p-q)/(q+1)} \langle v, \phi_n \rangle^2 \leqslant E_1^2. \tag{2.13}$$

Since $u(t)$ is the solution of problem $\frac{\partial^\gamma u}{\partial t^\gamma} + Au = 0$, we have

$$\sum_{n=1}^{\infty} E_{\gamma,1}(-\lambda_n T^\gamma) \langle u(0), \phi_n \rangle \phi_n = u(T).$$

Let us formulate this equation as an operator equation

$$Bv := u(T) = \sum_{n=1}^{\infty} \lambda_n^{-q} E_{\gamma,1}(-\lambda_n T^\gamma) \langle v, \phi_n \rangle \phi_n.$$

Then B is a continuous linear operator. Furthermore, we can easily check that B is a self-conjugated operator which means that $B^* = B$. This implies that

$$\left\langle (B^*B)^{-\frac{p-q}{2(q+1)}} v, (B^*B)^{-\frac{p-q}{2(q+1)}} v \right\rangle = \sum_{n=1}^{\infty} \left(\lambda_n^{-q} E_{\gamma,1}(-\lambda_n T^\gamma) \right)^{-2(p-q)/(q+1)} \langle v, \phi_n \rangle^2.$$

Therefore, condition (2.13) is equivalent to $\|(B^*B)^{-\frac{p-q}{2(q+1)}} v\|^2 \leqslant E_1^2$. With $a \geqslant \|B^*B\|$ we define $\psi : (0, a] \to \mathbb{R}_+$ by $\psi(\lambda) = \lambda^{(p-q)/(q+1)}$ and $\rho(\lambda) = \lambda \psi^{-1}(\lambda)$. Then, $\psi(\lambda)$ is strongly monotonic increasing $(0, a]$ and $\rho(\lambda) = \lambda^{(p+1)/(p-q)}$ is convex in $(0, a]$. Thus, the functions $\psi(\lambda)$ and $\rho(\lambda)$ satisfy assumption 1.1 (p.379) in [11]. Therefore, by theorem 2.1 in [11], with condition (2.13), the estimate of optimal order is

$$\|v\| \leqslant E_1\sqrt{\rho^{-1}\left(\frac{\varepsilon^2}{E_1^2}\right)} = E_1^{\frac{q+1}{p+1}}\varepsilon^{\frac{p-q}{p+1}}.$$

Since $\|v\| = \|u(0)\|_q$, estimate (2.12) of optimal order.

Remark 3. In [15] Wang, Wei and Zhou analyzed the optimal error bound for problem (1.1) with any $p > 0$ but only for the problem in one-dimensional space and $q = 0$. We obtain stability estimate of optimal order for the general problem with any $p > 0$ and $0 \leqslant q < p$. Furthermore, our proof is simpler than that of [15].

Remark 4.

1. Liu and Yamamoto in [8], Yang and Liu in [18] considered the case $p = 1$ and $q = 0$. Wang and Liu in [12] considered the cases $p = 1$, $q = 0$ and $p = 1/2$, $q = 0$,
2. Our results is valid for any $p > 0$ and $0 \leqslant q < p$,
3. If $p > 1$ and $q = 0$, then the Hölder exponent in estimate (2.10) $\frac{p}{p+1}$ is strictly greater than $\frac{1}{2}$.

Remark 5. The problem (1.1) is well-posed for $t > 0$ and ill-posed for $t = 0$. In fact, following theorem 2, we have $u(t) = \sum_{n=1}^{\infty} E_{\gamma,1}(-\lambda_n t^\gamma)\langle u(0), \phi_n\rangle \phi_n = \sum_{n=1}^{\infty} \frac{E_{\gamma,1}(-\lambda_n t^\gamma)\langle u(T),\phi_n\rangle\phi_n}{E_{\gamma,1}(-\lambda_n T^\gamma)}$. For $t > 0$, due to lemma 2, we have $0 < \frac{E_{\gamma,1}(-|\xi|^2 t^\gamma)}{E_{\gamma,1}(-|\xi|^2 T^\gamma)} \leqslant \frac{\overline{C}_2}{\overline{C}_1}\left(\frac{T}{t}\right)^\gamma$ which follows that problem (1.1) is well-posed for $t > 0$. At $t = 0$, we have $u(0) = \sum_{n=1}^{\infty} \frac{\langle u(T),\phi_n\rangle\phi_n}{E_{\gamma,1}(-\lambda_n T^\gamma)}$. Due to lemma 2, $\frac{1}{E_{\gamma,1}(-\lambda_n T^\gamma)}$ behaves like λ_n as $n \to \infty$. Therefore, problem (1.1) is ill-posed at $t = 0$.

3. Regularization

In this section, we regularize the *ill-posed* problem (1.1) by the *well-posed* non-local boundary value problem

$$\begin{cases} \frac{\partial^\gamma v_\alpha}{\partial t^\gamma} + Av_\alpha = 0, 0 < t < T, \\ \alpha A^k v_\alpha(0) + v_\alpha(T) = f, \end{cases} \tag{3.1}$$

where $0 < \alpha < 1$ and k is fixed, $k = 0, 1, 2, \ldots$. We shall suggest *a priori* and *a posteriori* methods for choosing the regularization parameter α which yield error estimates of Hölder type.

3.1. A priori parameter choice rule

Theorem 3. *Problem (3.1) is well-posed. For solutions $u(t)$ of problem (1.1) satisfying*

$$\|u(0)\|_p \leqslant E,\ p > 0, E > 0 \tag{3.2}$$

the following statements hold:

(i) If $0 < p < k + 1$, then with $\alpha = \left(\frac{\varepsilon}{E}\right)^{\frac{k+1}{p+1}}$, there exists constant C_1 such that

$$\|u(t) - v_\alpha(t)\| \leqslant C_1\varepsilon^{\frac{p}{p+1}}E^{\frac{1}{p+1}}, \forall t \in [0, T].$$

(ii) If $p \geqslant k+1$, then with $\alpha = \left(\frac{\varepsilon}{E}\right)^{\frac{k+1}{k+2}}$, there exists constant C_2 such that

$$\|u(t) - v_\alpha(t)\| \leqslant C_2 \varepsilon^{\frac{k+1}{k+2}} E^{\frac{1}{k+2}}, \forall t \in [0, T].$$

Remark 6. We note that $\frac{p}{p+1} > \frac{2}{3}$ when $p > 2$ and $\frac{k+1}{k+2} > \frac{2}{3}$ when $k > 1$. Therefore the order of our error estimates is greater than $\frac{2}{3}$ when $2 < p < k+1$ or $p \geqslant k+1 > 2$.

3.2. A posteriori parameter choice rule

Theorem 4. *Let $k \in \mathbb{N}$ and $\beta \in (0,1)$. Suppose that $0 < \varepsilon^\beta < \|f\|$. Choose $\tau > 1$ such that $0 < \tau\varepsilon^\beta \leqslant \|f\|$. Then for solutions $u(t)$ of problem (1.1) satisfying (3.2) the following statements hold:*

(i) If $k > 0$ and ε is sufficiently small, then there exists a unique number $\alpha_\varepsilon > 0$ such that

$$\|v_{\alpha_\varepsilon}(T) - f\| = \tau\varepsilon. \tag{3.3}$$

Further, there exist constants C_3, C_4 such that, for all $t \in [0, T]$,

$$\|u(t) - v_{\alpha_\varepsilon}(t)\| \leqslant \begin{cases} C_3 \varepsilon^{\frac{p}{p+1}} E^{\frac{1}{p+1}} & \text{if} \quad 0 < p < k, \\ C_4 \left(\varepsilon^{\frac{p}{p+1}} E^{\frac{1}{p+1}} + \varepsilon^{\frac{k}{k+1}} E^{\frac{1}{k+1}}\right) & \text{if} \quad p \geqslant k > 0. \end{cases}$$

(ii) If $k = 0$, then there exists a unique number $\alpha_\varepsilon > 0$ such that

$$\|v_{\alpha_\varepsilon}(T) - f\| = \tau\varepsilon^\beta. \tag{3.4}$$

Further, there exists a constant C_5 such that

$$\|u(t) - v_{\alpha_\varepsilon}(t)\| \leqslant C_5 \left(\varepsilon^{\frac{p}{p+1}} E^{\frac{1}{p+1}} + \varepsilon^{\frac{\beta p}{p+1}} E^{\frac{1}{p+1}} + \varepsilon^{1-\beta} E\right), \ t \in [0, T].$$

Remark 7. We note that $\frac{p}{p+1} > \frac{2}{3}$ when $p > 2$ and $\frac{k}{k+1} > \frac{2}{3}$ when $k > 2$. Therefore the order of our error estimates is greater than $\frac{2}{3}$ when $2 < p < k$ or $p \geqslant k > 2$.

Remark 8. Our results in theorems 3 and 4 are better than those of Wang, Zhou and Wei [16] and Wei and Wang [17].

1. Wang, Zhou and Wei [16] proposed an *a posteriori* parameter choice rule by solving the equation

$$\left\| \sum_{n=1}^{\infty} \frac{\alpha}{\alpha + E_{\gamma,1}(-\lambda_n T^\gamma)} \cdot \langle v_\alpha(T) - f, \phi_n \rangle \phi_n \right\| = \tau\varepsilon, \tag{3.5}$$

and got the convergence rate

$$\|u(0) - v_{\alpha_\varepsilon}(0)\| \leqslant \begin{cases} \widetilde{C} \varepsilon^{\frac{p}{p+2}} E^{\frac{1}{p+2}} & \text{if} \quad 0 < p < 2, \\ \widetilde{C} \varepsilon^{\frac{1}{2}} E^{\frac{1}{2}} & \text{if} \quad p \geqslant 2. \end{cases} \tag{3.6}$$

Since $\frac{p}{p+2} < \frac{1}{2}$ for all $p \in (0,2)$, the order of the error estimate in (3.6) is not greater than $1/2$ for all $p > 0$. The order of our error estimates in theorem 3 and part (i) of theorem 4 is better than that of 3.6.

2. Wei and Wang in [17] proposed an *a priori* parameter choice rule and got a convergence rate of form

$$\|u(0) - v_{\alpha_\varepsilon}(0)\| \leqslant \begin{cases} \widetilde{C}\varepsilon^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \text{ if } & 0 < p < 4, \\ \widetilde{C}\varepsilon^{\frac{2}{3}} E^{\frac{1}{3}} \quad \text{if} & p \geqslant 2. \end{cases} \tag{3.7}$$

Since $\frac{p}{p+2} < \frac{2}{3}$ for all $p \in (0,4)$, the order of the error estimate in (3.7) does not exceed $\frac{2}{3}$ for all $p > 0$.

3. Wei and Wang in [17] also proposed an *a posteriori* parameter choice rule by solving the equation (3.5) and got the convergence rate

$$\|u(0) - v_{\alpha_\varepsilon}(0)\| \leqslant \begin{cases} \widetilde{C}\varepsilon^{\frac{p}{p+2}} E^{\frac{1}{p+2}} \text{ if } & 0 < p < 2, \\ \widetilde{C}\varepsilon^{\frac{1}{2}} E^{\frac{1}{2}} \quad \text{if} & p \geqslant 2. \end{cases} \tag{3.8}$$

The error estimates in theorem 3 and part (i) of theorem 4 are better than those of (3.7) and (3.8).

4. Our *a posteriori* methods (3.3) and (3.4) are much simpler than (3.5).

Remark 9. Our results in theorems 3 and 4 are better than those of Al-Jamal [1], Wang, Wei and Zhou in [14, 15]. Indeed, Al-Jamal applied the Tikhonov regularization method but proved no convergence rate. Wang, Wei and Zhou [15] regularized problem (1.1) by the Tikhonov method but only for the problem in one-dimensional space and with the hypothesis $\sum_{n=1}^{\infty}(1+\lambda_n)^p \langle u(0), \phi_n\rangle^2 < E^2$; they proposed an *a priori* parameter choice rule and got a convergence rate of form

$$\|u(0) - v_\varepsilon(0)\| \leqslant \begin{cases} \widetilde{C_1}\varepsilon^{\frac{p}{p+2}} E^{\frac{2}{p+2}} \text{ if } & 0 < p < 4, \\ \widetilde{C_2}\varepsilon^{\frac{2}{3}} E^{\frac{1}{2}} \quad \text{if} & p \geqslant 4. \end{cases} \tag{3.9}$$

Since $\frac{p}{p+2} < \frac{2}{3}$ for all $p \in (0,4)$, the order of the error estimate in (3.9) cannot exceed $\frac{2}{3}$ for all $p > 0$.

Wang, Wei and Zhou in [15] also proposed an *a posteriori* parameter choice rule and got a convergence rate of form

$$\|u(0) - v_\varepsilon(0)\| \leqslant \begin{cases} \widetilde{C_3}\varepsilon^{\frac{p}{p+2}} E^{\frac{1}{p+2}} \text{if} & 0 < p < 2, \\ \widetilde{C_4}\varepsilon^{\frac{1}{2}} E^{\frac{1}{2}} \quad \text{if} & p \geqslant 2. \end{cases} \tag{3.10}$$

The order of the error estimate (3.10) does not exceed $\frac{1}{2}$ for all $p > 0$.

Error estimates in theorem 3 and part (i) of theorem 4 of our are better than (3.9) and (3.10).

In [14] Wang, Wei and Zhou generalized their method to the multi-dimensional problems and obtained the upper bounds $\widetilde{C}\varepsilon^{\frac{p}{p+2}} E^{\frac{2}{p+2}}$ if $0 < p < 4$, and $\widetilde{C}\varepsilon^{\frac{2}{3}} E^{\frac{1}{3}}$ if $p \geqslant 4$ for the *a priori* method (note that $\frac{p}{p+2} < \frac{2}{3}$ if $0 < p < 4$) and $\widetilde{C}\varepsilon^{\frac{p}{p+2}} E^{\frac{2}{p+2}}$ if $0 < p < 2$, $\widetilde{C}\varepsilon^{\frac{1}{2}} E^{\frac{1}{2}}$ if $p \geqslant 2$ for the *a posteriori* method (note that $\frac{p}{p+2} < \frac{1}{2}$ if $0 < p < 2$), which are weaker than that of ours.

We also note that for $k \geqslant p$ theorems 3 and 4 give the convergence rate $E^{\frac{1}{p+1}}\varepsilon^{\frac{p}{p+1}}$ which is of optimal order.

3.3. Convergence rate of the regularizing solution to the exact one in norm $\|\cdot\|_q$ with $0 < q < p$

Theorem 5. *Let $k \in \mathbb{N}, k \geqslant q > 0$. If $u(t)$ is a solution of problem (1.1) satisfying (3.2) with $p > q > 0$ and $v_\alpha(t)$ is the solution of problem (3.1) then with $\alpha = \left(\frac{\varepsilon}{E}\right)^{\frac{k+1}{p+1}}$ there exist constants C_6, C_7 such that*

$$\|u(t) - v_\alpha(t)\|_q \leqslant \begin{cases} C_6 \varepsilon^{\frac{p-q}{p+1}} E^{\frac{2}{p+1}} & \text{if} \quad q < p \leqslant k+q+1 \\ C_7 \left(\varepsilon^{\frac{k+1}{p+1}} E^{\frac{p-k}{p+1}} + \varepsilon^{\frac{p-q}{p+1}} E^{\frac{q+1}{p+1}}\right) & \text{if} \quad p > k+q+1, \end{cases}$$

$\forall t \in [0, T]$.

Remark 10. Our method is of optimal order in case $k \geqslant p - q - 1$.

Theorem 6. *Suppose that $u(t)$ is a solution of problem (1.1) satisfying (3.2) with $p > q > 0$. If $\alpha_\varepsilon > 0$ satisfying (3.3), then there exist constants C_8, C_9 such that*

$$\|u(t) - v_{\alpha_\varepsilon}(t)\|_q \leqslant \begin{cases} C_8 \varepsilon^{\frac{p-q}{p+1}} E^{\frac{q+1}{p+1}} & \text{if} \quad q < p \leqslant k \\ C_9 \left(\varepsilon^{\frac{p-q}{p+1}} E^{\frac{2}{p+1}} + \varepsilon^{\frac{k-q}{k+1}} E^{\frac{1}{k+1}}\right) & \text{if} \quad p > k > q, \end{cases}$$

$\forall t \in [0, T]$.

Remark 11. Our method is of optimal order in case $k \geqslant p$.

4. Proofs of the main results

4.1. Proof of theorem 3

First, we present some auxiliary results.

Lemma 3 (Young's inequality). *If a, b are nonnegative numbers and m, n are positive numbers such that $\frac{1}{m} + \frac{1}{n} = 1$, then $ab \leqslant \frac{a^m}{m} + \frac{b^n}{n}$.*

Lemma 4. *Problem (3.1) admits a unique solution*

$$v_\alpha(t) = \sum_{n=1}^{\infty} \frac{E_{\gamma,1}\left(-\lambda_n t^\gamma\right) \langle f, \phi_n \rangle \phi_n}{\alpha \lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}, \forall t \in [0, T]. \tag{4.1}$$

The proof if this lemma is straightforward and we omit it.

Lemma 5. *If $v_\alpha(t)$ is the solution of problem (3.1), then there exists a constant $\overline{C}_5$ such that*

$$\|v_\alpha(t)\| \leqslant \overline{C}_5 \alpha^{\frac{-1}{k+1}} \|f\|, \forall t \in [0, T].$$

Proof. By lemma 4, we have $\|v_\alpha(0)\|^2 = \sum_{n=1}^{\infty} \left(\frac{\langle f, \phi_n \rangle}{\alpha \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)}\right)^2$.

For $k > 0$, using lemmas 1 and 3, we obtain

$$\begin{aligned}\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right) &\geqslant \frac{\left(\alpha^{\frac{1}{k+1}}\lambda_n^{\frac{k}{k+1}}\right)^{k+1}}{k+1} + \frac{k}{k+1}\left(\left(E_{\gamma,1}\left(-\lambda_n T^\gamma\right)\right)^{\frac{k}{k+1}}\right)^{\frac{k+1}{k}} \\ &\geqslant \alpha^{\frac{1}{k+1}}\lambda_n^{\frac{k}{k+1}}\left(E_{\gamma,1}\left(-\lambda_n T^\gamma\right)\right)^{\frac{k}{k+1}} \\ &\geqslant \alpha^{\frac{1}{k+1}}\lambda_n^{\frac{k}{k+1}}\left(\overline{C}_1/\lambda_n\right)^{\frac{k}{k+1}} \geqslant \overline{C}_1^{\frac{k}{k+1}}\alpha^{\frac{1}{k+1}}. \end{aligned} \tag{4.2}$$

Therefore, with $k \in \mathbb{N}$ we have

$$\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right) \geqslant \overline{C}_6\alpha^{\frac{1}{k+1}}, \tag{4.3}$$

where $\overline{C}_6 = \min\{1, C_1^{\frac{k}{k+1}}\}$. This implies that $\|v_\alpha(0)\| \leqslant \frac{1}{\overline{C}_6}\alpha^{-\frac{1}{k+1}}\|f\|$.

Since $v_\alpha(t) = \sum_{n=1}^\infty E_{\gamma,1}(-\lambda_n T^\gamma)\langle v_\alpha(0), \phi_n\rangle\phi_n$ and $0 \leqslant E_{\gamma,1}(-\lambda_n T^\gamma) \leqslant 1$, with $\overline{C}_5 = \frac{1}{\overline{C}_6}$ we obtain $\|v_\alpha(t)\| \leqslant \|v_\alpha(0)\| \leqslant \overline{C}_5\alpha^{-\frac{1}{k+1}}\|f\|, \forall t \in [0, T]$. The lemma is proved. □

Lemma 6. *If $\|u(0)\|_p \leqslant E$ holds for some positive constants $p, E > 0$, then there exist constants $\overline{C}_7$ and $\overline{C}_8$ such that*

$$\|u(0) - v_\alpha(0)\|^2 \leqslant \begin{cases} \overline{C}_7\left(\alpha^{\frac{2p}{k+1}}E^2 + \alpha^{\frac{-2}{k+1}}\varepsilon^2\right) & \text{if} \quad p < k+1, \\ \overline{C}_8\left(\alpha^2 E^2 + \alpha^{\frac{-2}{k+1}}\varepsilon^2\right) & \text{if} \quad p \geqslant k+1. \end{cases}$$

Proof. We have

$$\begin{aligned}\|u(0) - v_\alpha(0)\|^2 &= \sum_{n=1}^\infty \langle u(0) - v_\alpha(0), \phi_n\rangle^2 \\ &= \sum_{n=1}^\infty \left(\langle u(0), \phi_n\rangle - \frac{\langle f, \phi_n\rangle}{\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}\right)^2 \\ &= \sum_{n=1}^\infty \left(\langle u(0), \phi_n\rangle - \frac{\langle u(T), \phi_n\rangle}{\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)} + \frac{\langle u(T) - f, \phi_n\rangle}{\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}\right)^2 \\ &\leqslant 2\sum_{n=1}^\infty \left(\langle u(0), \phi_n\rangle - \frac{E_{\gamma,1}(-\lambda_n T^\gamma)\langle u(0), \phi_n\rangle}{\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}\right)^2 + 2\sum_{n=1}^\infty \left(\frac{\langle u(T) - f, \phi_n\rangle}{\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}\right)^2 \\ &= 2\sum_{n=1}^\infty \left(\frac{\alpha\lambda_n^k\langle u(0), \phi_n\rangle}{\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}\right)^2 + 2\sum_{n=1}^\infty \left(\frac{\langle u(T) - f, \phi_n\rangle}{\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}\right)^2. \end{aligned} \tag{4.4}$$

From (4.3) and (4.4), we have

$$\|u(0) - v_\alpha(0)\|^2 \leqslant 2\sum_{n=1}^\infty \left(\frac{\alpha\lambda_n^k\langle u(0), \phi_n\rangle}{\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}\right)^2 + \frac{2\alpha^{\frac{-2}{k+1}}}{\overline{C}_6^2}\sum_{n=1}^\infty \langle u(T) - f, \phi_n\rangle^2. \tag{4.5}$$

If $p < k+1$, using lemmas 1 and 3, we get

$$\begin{aligned}\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right) &\geqslant \frac{k+1-p}{k+1}\left(\left(\alpha\lambda_n^k\right)^{\frac{k+1-p}{k+1}}\right)^{\frac{k+1}{k+1-p}} + \frac{p}{k+1}\left(\left(E_{\gamma,1}\left(-\lambda_n T^\gamma\right)\right)^{\frac{p}{k+1}}\right)^{\frac{k+1}{p}}\\ &\geqslant \alpha^{\frac{k+1-p}{k+1}}\lambda_n^{\frac{k(k+1-p)}{k+1}}\left(E_{\gamma,1}\left(-\lambda_n T^\gamma\right)\right)^{\frac{p}{k+1}}\\ &\geqslant \alpha^{\frac{k+1-p}{k+1}}\lambda_n^{\frac{k(k+1-p)}{k+1}}\left(\overline{C}_1/\lambda_n\right)^{\frac{p}{k+1}} \geqslant \overline{C}_1^{\frac{p}{k+1}}\alpha^{\frac{k+1-p}{k+1}}\lambda_n^{k-p}.\end{aligned}$$

From (4.5) and this inequality, we have

$$\|u(0)-v_\alpha(0)\|^2 \leqslant \overline{C}_1^{\frac{-p}{k+1}}\alpha^{\frac{2p}{k+1}}\sum_{n=1}^{\infty}\lambda_n^{2p}\left\langle u(0),\phi_n\right\rangle^2 + \frac{2\alpha^{\frac{-2}{k+1}}}{\overline{C}_6^2}\|u(T)-f\|^2.$$

Therefore, there exists a constant $\overline{C}_7$ such that

$$\|u(0)-v_\alpha(0)\|^2 \leqslant \overline{C}_7\left(\alpha^{\frac{2p}{k+1}}E^2 + \alpha^{\frac{-2}{k+1}}\varepsilon^2\right).$$

If $p \geqslant k+1$, from (4.5) we obtain

$$\begin{aligned}\|u(0)-v_\alpha(0)\|^2 &\leqslant 2\sum_{n=1}^{\infty}\left(\frac{\alpha\lambda_n^k\left\langle u(0),\phi_n\right\rangle}{E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}\right)^2 + \frac{2\alpha^{\frac{-2}{k+1}}}{\overline{C}_6^2}\|u(T)-f\|^2\\ &\leqslant 2\sum_{n=1}^{\infty}\left(\frac{\alpha\lambda_n^k\left\langle u(0),\phi_n\right\rangle}{\frac{\overline{C}_1}{\lambda_n}}\right)^2 + \frac{2\alpha^{\frac{-2}{k+1}}}{\overline{C}_6^2}\varepsilon^2\\ &\leqslant 2\left(\frac{1}{\overline{C}_1}\right)^2\sum_{n=1}^{\infty}\left(\alpha\lambda_n^{k+1}\left\langle u(0),\phi_n\right\rangle\right)^2 + \frac{2\alpha^{\frac{-2}{k+1}}}{\overline{C}_6^2}\varepsilon^2\\ &\leqslant 2\left(\frac{1}{\overline{C}_1}\right)^2\sum_{n=1}^{\infty}\left(\alpha\lambda_n^{k+1-p}\lambda_n^p\left\langle u(0),\phi_n\right\rangle\right)^2 + \frac{2\alpha^{\frac{-2}{k+1}}}{\overline{C}_6^2}\varepsilon^2\\ &\leqslant 2\left(\frac{1}{\overline{C}_1}\right)^2\alpha^2\sum_{n=1}^{\infty}\left(\lambda_1^{k+1-p}\lambda_n^p\left\langle u(0),\phi_n\right\rangle\right)^2 + \frac{2\alpha^{\frac{-2}{k+1}}}{\overline{C}_6^2}\varepsilon^2\\ &\leqslant 2\left(\frac{\lambda_1^{k+1-p}}{\overline{C}_1}\right)^2\alpha^2\sum_{n=1}^{\infty}\lambda_n^{2p}\left\langle u(0),\phi_n\right\rangle^2 + \frac{2\alpha^{\frac{-2}{k+1}}}{\overline{C}_6^2}\varepsilon^2\end{aligned}$$

and thus arrive at the second estimate of the lemma. □

Now we are in a position to prove theorem 3.

The well-posedness of problem (3.1) is implied from Lemmas 4 and 5.

Proof of part (i) of theorem 3.

If $p < k+1$, using lemma 6, we have $\|u(0)-v_\alpha(0)\|^2 \leqslant \overline{C}_7\left(\alpha^{\frac{2p}{k+1}}E^2 + \alpha^{\frac{-2}{k+1}}\varepsilon^2\right)$. From $u(t)-v_\alpha(t) = \sum_{n=1}^{\infty}E_{\gamma,1}(-\lambda_n T^\gamma)\left\langle u(0)-v_\alpha(0),\phi_n\right\rangle\phi_n$ and $0 \leqslant E_{\gamma,1}(-\lambda_n T^\gamma) \leqslant 1$, we get $\|u(t)-v_\alpha(t)\|^2 \leqslant \overline{C}_7\left(\alpha^{\frac{2p}{k+1}}E^2 + \alpha^{\frac{-2}{k+1}}\varepsilon^2\right)$. Choosing $\alpha = (\varepsilon/E)^{\frac{k+1}{p+1}}$, we arrive at the conclusion of part (i) of theorem 3.

Proof of part (ii) of theorem 3.

If $p \geqslant k+1$, from $u(t) - v_\alpha(t) = \sum_{n=1}^{\infty} E_{\gamma,1}(-\lambda_n T^\gamma) \langle u(0) - v_\alpha(0), \phi_n \rangle \phi_n$ and $0 \leqslant E_{\gamma,1}(-\lambda_n T^\gamma) \leqslant 1$, and lemma 6, we have $\|u(t) - v_\alpha(t)\|^2 \leqslant \overline{C}_8 \left(\alpha^2 E^2 + \alpha^{\frac{-2}{k+1}} \varepsilon^2\right)$. Choosing $\alpha = \left(\frac{\varepsilon}{E}\right)^{\frac{k+1}{k+2}}$, we arrive at part (ii) of theorem 3.

4.2. *Proof of theorem 4*

We need following auxiliary result.

Lemma 7. *Set $\rho(\alpha) = \|v_\alpha(T) - f\|$ and suppose that $f \neq 0$. Then*

(a) ρ is a continuous function,
(b) $\lim\limits_{\alpha \to 0^+} \rho(\alpha) = 0$,
(c) $\lim\limits_{\alpha \to +\infty} \rho(\alpha) = \|f\|$,
(d) ρ is a strictly increasing function.

Proof.

(a) From

$$\rho^2(\alpha) = \|v_\alpha(T) - f\|^2 = \alpha^2 \|A^k v_\alpha(0)\|^2 = \sum_{n=1}^{\infty} \left(\frac{\alpha \lambda_n^k \langle f, \phi_n \rangle}{\alpha \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} \right)^2 \tag{4.6}$$

we directly verify the continuity of ρ and $\rho(\alpha) > 0$ for all $\alpha > 0$.

(b) Let δ be an arbitrary positive number. Since $\|f\|^2 = \sum_{n=1}^{\infty} \langle f, \phi_n \rangle^2$, there exists a positive integer n_δ such that $\sum_{n=n_\delta+1}^{\infty} \langle f, \phi_n \rangle^2 < \frac{\delta^2}{2}$. For $0 < \alpha < \frac{\overline{C}_1 \delta}{\sqrt{2} \lambda_{n_\delta}^{k+1} \|f\|}$, using lemma 1, we have

$$\begin{aligned}\rho^2(\alpha) &\leqslant \sum_{n=1}^{n_\delta} \frac{\alpha^2 \lambda_n^{2k} \langle f, \phi_n \rangle^2}{(\alpha \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma))^2} + \sum_{n=n_\delta+1}^{\infty} \langle f, \phi_n \rangle^2 \leqslant \sum_{n=1}^{n_\delta} \frac{\alpha^2 \lambda_n^{2k} \langle f, \phi_n \rangle^2}{(E_{\gamma,1}(-\lambda_n T^\gamma))^2} + \frac{\delta^2}{2} \\ &\leqslant \frac{1}{\overline{C}_1^2} \alpha^2 \sum_{n=1}^{n_\delta} \lambda_n^{2k+2} \langle f, \phi_n \rangle^2 + \frac{\delta^2}{2} = \frac{1}{\overline{C}_1^2} \alpha^2 \lambda_{n_\delta}^{2k+2} \|f\|^2 + \frac{\delta^2}{2} \leqslant \delta^2.\end{aligned}$$

This implies that $\lim\limits_{\alpha \to 0^+} \rho(\alpha) = 0$.

(c) From (4.6) we have $\rho(\alpha) \leqslant \|f\|$ and using lemma 1 we get

$$\rho^2(\alpha) = \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle^2}{\left(1 + \frac{E_{\gamma,1}(-\lambda_n T^\gamma)}{\alpha \lambda_n^k}\right)^2} \geqslant \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle^2}{\left(1 + \frac{\overline{C}_2}{\alpha \lambda_n^{k+1}}\right)^2} \geqslant \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle^2}{\left(1 + \frac{\overline{C}_2}{\alpha \lambda_1^{k+1}}\right)^2}.$$

Therefore, $\|f\| \geqslant \rho(\alpha) \geqslant \frac{\|f\|}{1 + \frac{\overline{C}_2}{\alpha \lambda_1^{k+1}}}$. This implies that $\lim\limits_{\alpha \to +\infty} \rho(\alpha) = \|f\|$.

(d) For $0 < \alpha_1 < \alpha_2$, we have $\frac{\alpha_1 \lambda_n^k}{\alpha_1 \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} < \frac{\alpha_2 \lambda_n^k}{\alpha_2 \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)}$. Since $\|f\| > 0$, there exists a positive integer n_0 such that $\langle f, \phi_{n_0} \rangle^2 > 0$. Therefore $\rho(\alpha_1) < \rho(\alpha_2)$. We conclude that ρ is a strictly increasing function.

The lemma is proved. □

Lemma 8. *Let $k \in \mathbb{N}$. If $u(t)$ is a solution of problem (1.1) satisfying (3.2), then there exists a constant $\overline{C}_9$ such that*

$$\|u(0) - v_{\alpha_\varepsilon}(0)\| \leqslant \overline{C}_9 \left(\varepsilon^{\frac{p}{p+1}} E^{\frac{1}{p+1}} + E^{\frac{1}{p+1}} \|v_{\alpha_\varepsilon}(T) - f\|^{\frac{p}{p+1}} + \alpha_\varepsilon^{\frac{-1}{k+1}} \varepsilon \right).$$

Proof. From (4.5) and using the Hölder inequality, we obtain

$$\begin{aligned}\|u(0) - v_{\alpha_\varepsilon}(0)\|^2 &\leqslant 2\sum_{n=1}^{\infty} \left(\frac{\alpha_\varepsilon \lambda_n^k \langle u(0), \phi_n \rangle}{\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} \right)^2 + \frac{2\alpha_\varepsilon^{\frac{-2}{k+1}}}{\overline{C}_6^2} \varepsilon^2 \\ &= 2\sum_{n=1}^{\infty} \left(\lambda_n^{\frac{2p}{p+1}} |\langle u(0), \phi_n \rangle|^{\frac{2}{p+1}} \right) \left(\frac{\alpha_\varepsilon \lambda_n^k \lambda_n^{\frac{-p}{p+1}} |\langle u(0), \phi_n \rangle|^{\frac{p}{p+1}}}{\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} \right)^2 + \frac{2\alpha_\varepsilon^{\frac{-2}{k+1}}}{\overline{C}_6^2} \varepsilon^2.\end{aligned}$$

After some estimations, we get

$$\begin{aligned}&\|u(0) - v_{\alpha_\varepsilon}(0)\|^2 \\ &\leqslant 2\overline{C}_1^{\frac{-2p}{p+1}} E^{\frac{2}{p+1}} \left(2\sum_{n=1}^{\infty} \left(\frac{\alpha_\varepsilon \lambda_n^k \langle u(T) - f, \phi_n \rangle}{\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} \right)^2 + 2\sum_{n=1}^{\infty} \left(\frac{\alpha_\varepsilon \lambda_n^k \langle f, \phi_n \rangle}{\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} \right)^2 \right)^{\frac{p}{p+1}} + \frac{2\alpha_\varepsilon^{\frac{-2}{k+1}}}{\overline{C}_6^2} \varepsilon^2.\end{aligned} \tag{4.7}$$

Noting that $\|v_{\alpha_\varepsilon}(T) - f\|^2 = \sum_{n=1}^{\infty} \left(\frac{\alpha_\varepsilon \lambda_n^k \langle f, \phi_n \rangle}{\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} \right)^2$, we get

$$\begin{aligned}\|u(0) - v_{\alpha_\varepsilon}(0)\|^2 &\leqslant 2\overline{C}_1^{\frac{-2p}{p+1}} E^{\frac{2}{p+1}} \left(2\sum_{n=1}^{\infty} \langle u(T) - f, \phi_n \rangle^2 + 2\|v_{\alpha_\varepsilon}(T) - f\|^2 \right)^{\frac{p}{p+1}} + \frac{2\alpha_\varepsilon^{\frac{-2}{k+1}}}{\overline{C}_6^2} \varepsilon^2 \\ &\leqslant 2\overline{C}_1^{\frac{-2p}{p+1}} E^{\frac{2}{p+1}} \left(2\varepsilon^2 + 2\|v_{\alpha_\varepsilon}(T) - f\|^2 \right)^{\frac{p}{p+1}} + \frac{2\alpha_\varepsilon^{\frac{-2}{k+1}}}{\overline{C}_6^2} \varepsilon^2.\end{aligned} \tag{4.8}$$

The lemma is proved. □

Now we are in a position to prove theorem 4.

Proof of part (i) of theorem 4.

It follows from lemma 7 that there exists a unique number $\alpha_\varepsilon > 0$ satisfying (3.3).

We have

$$\begin{aligned}\tau\varepsilon = \|v_{\alpha_\varepsilon}(T) - f\| &= \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^k \langle f, \phi_n \rangle \phi_n}{\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} \right\| \\ &= \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^k \langle u(T), \phi_n \rangle \phi_n}{\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} - \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^k \langle u(T) - f, \phi_n \rangle \phi_n}{\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} \right\| \\ &\leqslant \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^k E_{\gamma,1}(-\lambda_n T^\gamma) \langle u(0), \phi_n \rangle \phi_n}{\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} \right\| + \left\| \sum_{n=1}^{\infty} \langle u(T) - f, \phi_n \rangle \phi_n \right\| \\ &\leqslant \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^k E_{\gamma,1}(-\lambda_n T^\gamma) \langle u(0), \phi_n \rangle \phi_n}{\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}(-\lambda_n T^\gamma)} \right\| + \varepsilon.\end{aligned} \tag{4.9}$$

It follows from (4.9) that

$$(\tau-1)\varepsilon \leqslant \left\| \sum_{n=1}^{\infty} \frac{\alpha_\varepsilon \lambda_n^k E_{\gamma,1}\left(-\lambda_n T^\gamma\right) \langle u(0), \phi_n\rangle \phi_n}{\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)} \right\|. \tag{4.10}$$

If $0 < p < k$, using lemmas 1 and 3, we get

$$\begin{aligned}
\alpha_\varepsilon \lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right) &\geqslant \frac{k-p}{k+1}\left(\left(\alpha_\varepsilon \lambda_n^k\right)^{\frac{k-p}{k+1}}\right)^{\frac{k+1}{k-p}} + \frac{p+1}{k+1}\left(\left(E_{\gamma,1}\left(-\lambda_n T^\gamma\right)\right)^{\frac{p+1}{k+1}}\right)^{\frac{k+1}{p+1}} \\
&\geqslant \alpha_\varepsilon^{\frac{k-p}{k+1}} \lambda_n^{\frac{k(k-p)}{k+1}} \left(E_{\gamma,1}\left(-\lambda_n T^\gamma\right)\right)^{\frac{p+1}{k+1}} \\
&\geqslant \alpha_\varepsilon^{\frac{k-p}{k+1}} \lambda_n^{\frac{k(k-p)}{k+1}} \left(\frac{\overline{C}_1}{\lambda_n}\right)^{\frac{p+1}{k+1}} \geqslant \overline{C}_1^{\frac{p+1}{k+1}} \alpha_\varepsilon^{\frac{k-p}{k+1}} \lambda_n^{k-p-1}.
\end{aligned} \tag{4.11}$$

From (4.10) and (4.11), we have

$$\begin{aligned}
(\tau-1)\varepsilon &\leqslant \overline{C}_1^{-\frac{p+1}{k+1}} \alpha_\varepsilon^{\frac{p+1}{k+1}} \left\| \sum_{n=1}^{\infty} \lambda_n^{p+1} E_{\gamma,1}\left(-\lambda_n T^\gamma\right) \langle u(0), \phi_n\rangle \phi_n \right\| \\
&\leqslant \overline{C}_1^{-\frac{p+1}{k+1}} \alpha_\varepsilon^{\frac{p+1}{k+1}} \left\| \sum_{n=1}^{\infty} \lambda_n^{p+1} \frac{\overline{C}_2}{\lambda_n} \langle u(0), \phi_n\rangle \phi_n \right\| \\
&\leqslant \overline{C}_2 \overline{C}_1^{-\frac{p+1}{k+1}} \alpha_\varepsilon^{\frac{p+1}{k+1}} \left\| \sum_{n=1}^{\infty} \lambda_n^{p} \langle u(0), \phi_n\rangle \phi_n \right\| \leqslant \overline{C}_2 \overline{C}_1^{-\frac{p+1}{k+1}} \alpha_\varepsilon^{\frac{p+1}{k+1}} E.
\end{aligned} \tag{4.12}$$

Further, from lemma 8, (3.3) and (4.12), we get

$$\|u(0) - v_{\alpha_\varepsilon}(0)\| \leqslant C_3 \varepsilon^{\frac{p}{p+1}} E^{\frac{1}{p+1}}. \tag{4.13}$$

If $p \geqslant k > 0$, from (4.10), we have

$$\begin{aligned}
(\tau-1)\varepsilon &\leqslant \alpha_\varepsilon \left\| \sum_{n=1}^{\infty} \lambda_n^k \langle u(0), \phi_n\rangle \phi_n \right\| \leqslant \alpha_\varepsilon \left\| \sum_{n=1}^{\infty} \lambda_n^{k-p} \lambda_n^p \langle u(0), \phi_n\rangle \phi_n \right\| \\
&\leqslant \lambda_1^{k-p} \alpha_\varepsilon \left\| \sum_{n=1}^{\infty} \lambda_n^{p} \langle u(0), \phi_n\rangle \phi_n \right\| \leqslant \lambda_1^{k-p} \alpha_\varepsilon E.
\end{aligned} \tag{4.14}$$

Hence, from lemma 8, (3.3) and (4.14), we get

$$\|u(0) - v_{\alpha_\varepsilon}(0)\| \leqslant C_4 \left(\varepsilon^{\frac{p}{p+1}} E^{\frac{1}{p+1}} + \varepsilon^{\frac{k}{p+1}} E^{\frac{1}{k+1}}\right). \tag{4.15}$$

From $u(t) - v_\alpha(t) = \sum_{n=1}^{\infty} E_{\gamma,1}(-\lambda_n T^\gamma) \langle u(0) - v_\alpha(0), \phi_n\rangle \phi_n$ and $0 \leqslant E_{\gamma,1}(-\lambda_n T^\gamma) \leqslant 1$, and (4.13) and (4.15), part (i) of theorem 4 is proved.

Proof of part (ii) of theorem 4

It follows from lemma 7 that there exists a unique number $\alpha_\varepsilon > 0$ satisfying (3.4).

We have

$$\begin{aligned}
\tau\varepsilon^{\beta} &= \|v_{\alpha_\varepsilon}(T)-f\| = \left\|\sum_{n=1}^{\infty}\frac{\alpha_\varepsilon\langle f,\phi_n\rangle\phi_n}{\alpha_\varepsilon+E_{\gamma,1}(-\lambda_n T^\gamma)}\right\| \\
&\leqslant \left\|\sum_{n=1}^{\infty}\frac{\alpha_\varepsilon E_{\gamma,1}(-\lambda_n T^\gamma)\langle u(0),\phi_n\rangle\phi_n}{\alpha_\varepsilon+E_{\gamma,1}(-\lambda_n T^\gamma)}\right\| + \left\|\sum_{n=1}^{\infty}\frac{\alpha_\varepsilon\langle u(T)-f,\phi_n\rangle\phi_n}{\alpha_\varepsilon+E_{\gamma,1}(-\lambda_n T^\gamma)}\right\| \\
&\leqslant \alpha_\varepsilon\left\|\sum_{n=1}^{\infty}\langle u(0),\phi_n\rangle\phi_n\right\| + \|u(T)-f\| \leqslant \alpha_\varepsilon\left\|\sum_{n=1}^{\infty}\frac{\lambda_n^p}{\lambda_n^p}\langle u(0),\phi_n\rangle\phi_n\right\| + \varepsilon \\
&\leqslant \lambda_1^{-p}\alpha_\varepsilon\left\|\sum_{n=1}^{\infty}\lambda_n^p\langle u(0),\phi_n\rangle\phi_n\right\| + \varepsilon \leqslant \lambda_1^{-p}\alpha_\varepsilon E + \varepsilon.
\end{aligned} \tag{4.16}$$

If ε is sufficiently small, then

$$(\tau-1)\varepsilon^{\beta} \leqslant \lambda_1^{-p}\alpha_\varepsilon E. \tag{4.17}$$

If $k=0$, from lemma 8, (3.4) and (4.17), there exists a constant $\overline{C}_{10}$ such that

$$\|u(0)-v_{\alpha_\varepsilon}(0)\| \leqslant \overline{C}_{10}\left(\varepsilon^{\frac{p}{p+1}}E^{\frac{1}{p+1}} + \varepsilon^{\frac{\beta p}{p+1}}E^{1-\frac{\beta}{p+1}} + \varepsilon^{1-\beta}E\right). \tag{4.18}$$

From $u(t)-v_\alpha(t)=\sum_{n=1}^{\infty}E_{\gamma,1}(-\lambda_n T^\gamma)\langle u(0)-v_\alpha(0),\phi_n\rangle\phi_n$ and $0\leqslant E_{\gamma,1}(-\lambda_n T^\gamma)\leqslant 1$, and (4.18), then there exists a constant C_5 such that

$$\|u(t)-v_{\alpha_\varepsilon}(t)\| \leqslant C_5\left(\varepsilon^{\frac{p}{p+1}}E^{\frac{1}{p+1}} + \varepsilon^{\frac{\beta p}{p+1}}E^{1-\frac{\beta}{p+1}} + \varepsilon^{1-\beta}E\right).$$

Part (ii) of theorem 4 is proved.

4.3. Proof of theorem 5

Since $u(t)=\sum_{n=1}^{\infty}E_{\gamma,1}(-\lambda_n t^\gamma)\langle u(0),\phi_n\rangle\phi_n$, $v_\alpha(t)=\sum_{n=1}^{\infty}\frac{E_{\gamma,1}(-\lambda_n t^\gamma)\langle f,\phi_n\rangle\phi_n}{\alpha\lambda_n^k+E_{\gamma,1}(-\lambda_n T^\gamma)}$ and $0\leqslant E_{\gamma,1}(-\lambda_n t^\gamma)\leqslant 1$, we have

$$\begin{aligned}
\|u(t)-v_\alpha(t)\|_q^2 &= \sum_{n=1}^{\infty}\left(\lambda_n^q E_{\gamma,1}(-\lambda_n t^\gamma)\langle u(0),\phi_n\rangle - \frac{\lambda_n^q E_{\gamma,1}(-\lambda_n t^\gamma)\langle f,\phi_n\rangle}{\alpha\lambda_n^k+E_{\gamma,1}(-\lambda_n T^\gamma)}\right)^2 \\
&\leqslant \sum_{n=1}^{\infty}\left(\lambda_n^q\langle u(0),\phi_n\rangle - \frac{\lambda_n^q\langle f,\phi_n\rangle}{\alpha\lambda_n^k+E_{\gamma,1}(-\lambda_n T^\gamma)}\right)^2 \\
&= \sum_{n=1}^{\infty}\left(\lambda_n^q\langle u(0),\phi_n\rangle - \frac{\lambda_n^q\langle u(T),\phi_n\rangle}{\alpha\lambda_n^k+E_{\gamma,1}(-\lambda_n T^\gamma)} + \frac{\lambda_n^q\langle u(T)-f,\phi_n\rangle}{\alpha\lambda_n^k+E_{\gamma,1}(-\lambda_n T^\gamma)}\right)^2 \\
&= \sum_{n=1}^{\infty}\left(\lambda_n^q\langle u(0),\phi_n\rangle - \frac{\lambda_n^q E_{\gamma,1}(-\lambda_n T^\gamma)\langle u(0),\phi_n\rangle}{\alpha\lambda_n^k+E_{\gamma,1}(-\lambda_n T^\gamma)} + \frac{\lambda_n^q\langle u(T)-f,\phi_n\rangle}{\alpha\lambda_n^k+E_{\gamma,1}(-\lambda_n T^\gamma)}\right)^2 \\
&= \sum_{n=1}^{\infty}\left(\frac{\alpha\lambda_n^{k+q}\langle u(0),\phi_n\rangle}{\alpha\lambda_n^k+E_{\gamma,1}(-\lambda_n T^\gamma)} + \frac{\lambda_n^q\langle u(T)-f,\phi_n\rangle}{\alpha\lambda_n^k+E_{\gamma,1}(-\lambda_n T^\gamma)}\right)^2 \\
&\leqslant 2\sum_{n=1}^{\infty}\left(\frac{\alpha\lambda_n^{k+q}\langle u(0),\phi_n\rangle}{\alpha\lambda_n^k+E_{\gamma,1}(-\lambda_n T^\gamma)}\right)^2 + 2\sum_{n=1}^{\infty}\left(\frac{\lambda_n^q\langle u(T)-f,\phi_n\rangle}{\alpha\lambda_n^k+E_{\gamma,1}(-\lambda_n T^\gamma)}\right)^2.
\end{aligned} \tag{4.19}$$

With $k > q$, using lemmas 1 and 3, we have

$$\begin{aligned}\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right) &\geqslant \frac{q+1}{k+1}\left(\left(\alpha\lambda_n^k\right)^{\frac{q+1}{k+1}}\right)^{\frac{k+1}{q+1}} + \frac{k-q}{k+1}\left(\left(E_{\gamma,1}\left(-\lambda_n T^\gamma\right)\right)^{\frac{k-q}{k+1}}\right)^{\frac{k+1}{k-q}}\\ &\geqslant \alpha^{\frac{q+1}{k+1}}\lambda_n^{\frac{k(q+1)}{k+1}}\left(E_{\gamma,1}\left(-\lambda_n T^\gamma\right)\right)^{\frac{k-q}{k+1}}\\ &\geqslant \alpha^{\frac{q+1}{k+1}}\lambda_n^{\frac{k(q+1)}{k+1}}\left(\frac{\overline{C}_1}{\lambda_n}\right)^{\frac{k-q}{k+1}} = \overline{C}_1^{\frac{k-q}{k+1}}\alpha^{\frac{q+1}{k+1}}\lambda_n^q.\end{aligned}$$

Therefore, with $k \geqslant q$ there exists a constant $\overline{C}_{11}$ such that

$$\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right) \geqslant \overline{C}_{11}\alpha^{\frac{q+1}{k+1}}\lambda_n^q. \tag{4.20}$$

From (4.19) and (4.20) there exists a constant $\overline{C}_{12}$ such that

$$\begin{aligned}\|u(t) - v_\alpha(t)\|_q^2 &\leqslant 2\sum_{n=1}^{\infty}\left(\frac{\alpha\lambda_n^{k+q}\langle u(0),\phi_n\rangle}{\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}\right)^2 + \overline{C}_{12}\alpha^{\frac{-2(q+1)}{k+1}}\|u(T) - f\|^2\\ &\leqslant 2\sum_{n=1}^{\infty}\left(\frac{\alpha\lambda_n^{k+q}\langle u(0),\phi_n\rangle}{\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}\right)^2 + \overline{C}_{12}\alpha^{\frac{-2(q+1)}{k+1}}\varepsilon^2.\end{aligned} \tag{4.21}$$

If $q < p < k+q+1$, using lemmas 1 and 3, we have

$$\begin{aligned}\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right) &\geqslant \frac{k+q+1-p}{k+1}\left(\left(\alpha\lambda_n^k\right)^{\frac{k+q+1-p}{k+1}}\right)^{\frac{k+1}{k+q+1-p}} + \frac{p-q}{k+1}\left(\left(E_{\gamma,1}\left(-\lambda_n T^\gamma\right)\right)^{\frac{p-q}{k+1}}\right)^{\frac{k+1}{p-q}}\\ &\geqslant \alpha^{\frac{k+q+1-p}{k+1}}\lambda_n^{\frac{k(k+q+1-p)}{k+1}}\left(E_{\gamma,1}\left(-\lambda_n T^\gamma\right)\right)^{\frac{p-q}{k+1}}\\ &\geqslant \alpha^{\frac{k+q+1-p}{k+1}}\lambda_n^{\frac{k(k+q+1-p)}{k+1}}\left(\frac{\overline{C}_1}{\lambda_n}\right)^{\frac{p-q}{k+1}} = \overline{C}_1^{\frac{p-q}{k+1}}\alpha^{\frac{k+q+1-p}{k+1}}\lambda_n^{k+q-p}.\end{aligned}$$

Therefore, with $q < p \leqslant k+q+1$, there exists a constant $\overline{C}_{13}$ such that

$$\alpha\lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right) \geqslant \overline{C}_{13}\alpha^{\frac{2(k+q+1-p)}{k+1}}\lambda_n^{k+q-p}. \tag{4.22}$$

From (4.21) and (4.22), we conclude that there exists a constant $\overline{C}_{14}$ such that

$$\begin{aligned}\|u(t) - v_\alpha(t)\|_q^2 &\leqslant \overline{C}_{14}\left(\alpha^{\frac{2(p-q)}{k+1}}\sum_{n=1}^{\infty}\lambda_n^{2p}\langle u(0),\phi_n\rangle^2 + \alpha^{\frac{-2(q+1)}{k+1}}\varepsilon^2\right)\\ &\leqslant \overline{C}_{14}\left(\alpha^{\frac{2(p-q)}{k+1}}E^2 + \alpha^{\frac{-2(q+1)}{k+1}}\varepsilon^2\right).\end{aligned} \tag{4.23}$$

If $p > k+q+1$, from (4.21) and lemma 1, after some estimations, we have

$$\begin{aligned}\|u(t) - v_\alpha(t)\|_q^2 &\leqslant 2\sum_{n=1}^{\infty}\left(\frac{\alpha\lambda_n^{k+q}\langle u(0),\phi_n\rangle}{E_{\gamma,1}\left(-\lambda_n T^\gamma\right)}\right)^2 + \overline{C}_{12}\alpha^{\frac{-2(q+1)}{k+1}}\varepsilon^2\\ &\leqslant \frac{2\alpha^2}{\overline{C}_1^2}\lambda_1^{2(k+q+1-p)}E^2 + \overline{C}_{12}\alpha^{\frac{-2(q+1)}{k+1}}\varepsilon^2.\end{aligned} \tag{4.24}$$

From (4.23) and (4.24), we conclude that there exists a constant $\overline{C}_{15}$ such that

$$\|u(t)-v_\alpha(t)\|_q^2 \leqslant \begin{cases} \overline{C}_{14}\left(\alpha^{\frac{2(p-q)}{k+1}}E^2+\alpha^{\frac{-2(q+1)}{k+1}}\varepsilon^2\right) & \text{if} \quad q<p\leqslant k+q+1 \\ \overline{C}_{15}\left(\alpha^2E^2+\alpha^{\frac{-2(q+1)}{k+1}}\varepsilon^2\right) & \text{if} \quad p>k+q+1. \end{cases}$$

Choosing $\alpha=(\varepsilon/E)^{\frac{k+1}{p+1}}$, we arrive at the conclusion of theorem 5.

4.4. Proof of theorem 6

From (4.21), using the Hölder inequality, we obtain

$$\begin{aligned}
\|u(t)-v_{\alpha_\varepsilon}(t)\|_q^2 &\leqslant 2\sum_{n=1}^{\infty}\left(\frac{\alpha_\varepsilon\lambda_n^{k+q}\langle u(0),\phi_n\rangle}{\alpha_\varepsilon\lambda_n^k+E_{\gamma,1}(-\lambda_nT^\gamma)}\right)^2+\overline{C}_{12}\alpha_\varepsilon^{\frac{-2(q+1)}{k+1}}\varepsilon^2 \\
&= 2\sum_{n=1}^{\infty}\left(\lambda_n^{\frac{2(q+1)p}{p+1}}|\langle u(0),\phi_n\rangle|^{\frac{2(q+1)}{p+1}}\right)\left(\frac{\alpha_\varepsilon\lambda_n^k\lambda_n^{\frac{q-p}{p+1}}|\langle u(0),\phi_n\rangle|^{\frac{p-q}{p+1}}}{\alpha_\varepsilon\lambda_n^k+E_{\gamma,1}(-\lambda_nT^\gamma)}\right)^2+\overline{C}_{12}\alpha_\varepsilon^{\frac{-2(q+1)}{k+1}}\varepsilon^2 \\
&\leqslant 2\left(\sum_{n=1}^{\infty}\lambda_n^{2p}\langle u(0),\phi_n\rangle^2\right)^{\frac{q+1}{p+1}}\left(\sum_{n=1}^{\infty}\left(\frac{\alpha_\varepsilon\lambda_n^k\lambda_n^{\frac{q-p}{p+1}}|\langle u(0),\phi_n\rangle|^{\frac{p-q}{p+1}}}{\alpha_\varepsilon\lambda_n^k+E_{\gamma,1}(-\lambda_nT^\gamma)}\right)^{\frac{2(p+1)}{p-q}}\right)^{\frac{p-q}{p+1}} \\
&\quad+\overline{C}_{12}\alpha_\varepsilon^{\frac{-2(q+1)}{k+1}}\varepsilon^2 \\
&\leqslant 2E^{\frac{2(q+1)}{p+1}}\left(\sum_{n=1}^{\infty}\left(\frac{\alpha_\varepsilon\lambda_n^k}{\alpha_\varepsilon\lambda_n^k+E_{\gamma,1}(-\lambda_nT^\gamma)}\right)^{\frac{2(p+1)}{p-q}}\lambda_n^{-2}\langle u(0),\phi_n\rangle^2\right)^{\frac{p-q}{p+1}}+\overline{C}_{12}\alpha_\varepsilon^{\frac{-2(q+1)}{k+1}}\varepsilon^2.
\end{aligned}$$

Since $\frac{\alpha_\varepsilon\lambda_n^k}{\alpha_\varepsilon\lambda_n^k+E_{\gamma,1}(-\lambda_nT^\gamma)}\leqslant 1$, it follows that

$$\left(\frac{\alpha_\varepsilon\lambda_n^k}{\alpha_\varepsilon\lambda_n^k+E_{\gamma,1}(-\lambda_nT^\gamma)}\right)^{\frac{2(p+1)}{p-q}}\leqslant\left(\frac{\alpha_\varepsilon\lambda_n^k}{\alpha_\varepsilon\lambda_n^k+E_{\gamma,1}(-\lambda_nT^\gamma)}\right)^2.$$

From that, after some calculations, we can conclude that

$$\begin{aligned}
\|u(t)-v_{\alpha_\varepsilon}(t)\|_q^2 &\leqslant 2E^{\frac{2(q+1)}{p+1}}(\overline{C}_1)^{\frac{-2(p-q)}{p+1}}\left\{2\sum_{n=1}^{\infty}\left(\frac{\alpha_\varepsilon\lambda_n^k\langle u(T)-f,\phi_n\rangle}{\alpha_\varepsilon\lambda_n^k+E_{\gamma,1}(-\lambda_nT^\gamma)}\right)^2\right. \\
&\quad\left.+2\sum_{n=1}^{\infty}\left(\frac{\alpha_\varepsilon\lambda_n^k\langle f,\phi_n\rangle}{\alpha_\varepsilon\lambda_n^k+E_{\gamma,1}(-\lambda_nT^\gamma)}\right)^2\right\}^{\frac{p-q}{p+1}}+\overline{C}_{12}\alpha_\varepsilon^{\frac{-2(q+1)}{k+1}}\varepsilon^2.
\end{aligned} \tag{4.25}$$

Note that, with $k\geqslant 1$ then $\tau\varepsilon=\|v_{\alpha_\varepsilon}(T)-f\|^2=\sum_{n=1}^{\infty}\left(\frac{\alpha_\varepsilon\lambda_n^k\langle f,\phi_n\rangle}{\alpha_\varepsilon\lambda_n^k+E_{\gamma,1}(-\lambda_nT^\gamma)}\right)^2$. From (4.25), we conclude that there exist constants $\overline{C}_{16},\overline{C}_{17}$ such that

$$\begin{aligned}
\|u(t)-v_{\alpha_\varepsilon}(t)\|_q^2 &\leqslant \overline{C}_{16}E^{\frac{2(q+1)}{p+1}}\left(\|u(T)-f\|^2+2\tau^2\varepsilon^2\right)^{\frac{p-q}{p+1}}+\overline{C}_{12}\alpha_\varepsilon^{\frac{-2(q+1)}{k+1}}\varepsilon^2 \\
&\leqslant \overline{C}_{17}E^{\frac{2(q+1)}{p+1}}\varepsilon^{\frac{2(p-q)}{p+1}}+\overline{C}_{12}\alpha_\varepsilon^{\frac{-2(q+1)}{k+1}}\varepsilon^2.
\end{aligned} \tag{4.26}$$

From (4.12) and (4.14), with $k > q, p > q$, we have

$$(\tau - 1)\varepsilon \leqslant \begin{cases} \overline{C}_2 \overline{C}_1^{-\frac{p+1}{k+1}} \alpha_\varepsilon^{\frac{p+1}{k+1}} E \text{ if } & q < p \leqslant k \\ \lambda_1^{k-p} \alpha_\varepsilon E \quad \text{if} & p > k > q \end{cases}$$

and we arrive at the conclusion of theorem 6.

5. Numerical implementation

In this section, we give some numerical implementations for our inversion scheme with different k by regularized scheme (3.1) in 1-dimension space domain $\Omega = [0, l]$. Using the eigenvalue expansions, we get the representation of regularized solution for $k = 0, 1, 2, \cdots$

$$v^k_{\alpha,\gamma}(x,t) = \sum_{n=1}^{\infty} \frac{\langle f, \phi_n \rangle}{\alpha(-1)^k \lambda_n^k + E_{\gamma,1}\left(-\lambda_n T^\gamma\right)} E_{\gamma,1}\left(-\lambda_n t^\gamma\right) \phi_n(x), \tag{5.1}$$

where, the eigenvalue system $(\lambda_n, \phi_n) = \left(\frac{n^2\pi^2}{l^2}, \sqrt{\frac{2}{l}} \sin(\sqrt{\lambda_n} x)\right)$.

For numerical tests, we consider the special cases where the solutions can be expressed by the eigenvalue system explicitly, so that we can check the numerical errors of our scheme by comparing the numerics with exact solutions. We take the time fractional derivative order $\gamma = 1/2$ where the function $E_{\gamma,1}(x)$ has an integral expression and thus can be computed accurately in (2.3). Of course, we can also consider other values $\gamma \in (0, 1)$ for which $E_{\gamma,1}(x)$ should be computed efficiently by some package, for example, see www.mathworks.com/matlabcentral/fileexchange/48154-the-mittag-leffler-function.

We test our regularizing scheme for two 1-dimensional examples with smooth and non-smooth initial status, respectively. We implement the regularizing schemes with $k = 0, 1, 2$ and compare their performances on the regularizing solutions.

Example 1. Consider the following problem

$$\begin{cases} \frac{\partial^\gamma u}{\partial t^\gamma} = \frac{\partial^2 u}{\partial x^2}, & (x,t) \in (0,\pi) \times (0,T], \\ u(0,t) = u(\pi,t) = 0, & t \in [0,T], \\ u(x,0) = u_0(x) = \sin x + \sin 2x, & x \in [0,\pi]. \end{cases} \tag{5.2}$$

The forward problem for $\gamma = 1/2$ has the solution

$$u_{1/2}(x,t) = E_{1/2,1}(-t^{1/2}) \sin x + E_{1/2,1}(-4t^{1/2}) \sin 2x, \quad x \in [0,\pi], t > 0, \tag{5.3}$$

where

$$E_{1/2,1}(x) = \mathrm{e}^{x^2} \mathrm{erfc}(-x) = \frac{2}{\sqrt{\pi}} \mathrm{e}^{x^2} \int_{-x}^{+\infty} \mathrm{e}^{-s^2} \mathrm{d}s. \tag{5.4}$$

For exact input data $u_{1/2}(\cdot, T)$, the regularizing solutions for $k = 0, 1, \cdots$ are

$$\begin{aligned} v^k_{\alpha,1/2}[u_{\frac{1}{2}}(\cdot,T)](x,t) = {} & E_{1/2,1}(-T^{1/2}) \frac{E_{1/2,1}(-t^{1/2})}{\alpha(-1)^k + E_{1/2,1}(-T^{1/2})} \sin x \\ & + E_{1/2,1}(-4T^{1/2}) \frac{E_{1/2,1}(-4t^{1/2})}{\alpha(-1)^k 2^{2k} + E_{1/2,1}(-4T^{1/2})} \sin 2x. \end{aligned} \tag{5.5}$$

Now we generate the final measurement data at $T = 1$ with noise in the form

$$u^{\delta}_{1/2}(x,T) = u_{1/2}(x,T) + \sqrt{\frac{2}{\pi}}\delta \times rand(x), \tag{5.6}$$

at discrete points $x_i \in [0,\pi]$, where $rand(x_i) \in [-1,1]$ with $i = 1,\cdots 101$ are the standard random numbers, δ is the error level. Using the above noisy data, we compute the regularizing solution

$$\begin{aligned} v^{k}_{\alpha,\delta,1/2}[u^{\delta}_{1/2}(\cdot,T)](x,t) &= \sum_{n=1}^{\infty} f^{k}_{n,\delta} E_{1/2,1}\left(-n^2 t^{1/2}\right)\sin(nx) \\ &\approx \sum_{n=1}^{5} f^{k}_{n,\delta} E_{1/2,1}\left(-n^2 t^{1/2}\right)\sin(nx), \end{aligned} \tag{5.7}$$

with the coefficients

$$\begin{aligned} f^{k}_{n,\delta} &= \frac{1}{\alpha(-1)^k n^{2k} + E_{1/2,1}\left(-n^2 T^{1/2}\right)} \frac{2}{\pi}\int_0^{\pi} u^{\delta}_{1/2}(x,T)\sin nx \mathrm{d}x \\ &\approx \frac{1}{\alpha(-1)^k n^{2k} + E_{1/2,1}\left(-n^2 T^{1/2}\right)} \frac{2}{100}\sum_{i=1}^{100} u^{\delta}_{1/2}(x_i,T)\sin nx_i \end{aligned} \tag{5.8}$$

with $0 = x_1 < \cdots < x_{101} = \pi$ the grids dividing the interval $[0,\pi]$.

We take the first five-term in (5.7) for our computations, which will lead to an optimal numerical behavior by our numerical tests. In fact, the number of finite summation terms for truncations is also a regularizing parameter which should be chosen suitably in terms of δ. The authors in [17, 18] considered the case $k = 0$ and $k = 1$ respectively for regularized schemes (3.1). The order of the error estimate is found to be not greater than $1/2$ in [18], and not greater than $2/3$ in [17] for all $p > 0$ for the *a-priori* parameter choice rule and is not greater than $1/2$ for the *a-posteriori* parameter choice rule.

Now we take $k = 2, p = 3$ for the numerical tests. In this case, the convergence order is $3/4$ theoretically in terms of the *a-priori* parameter choice rule (ii), which is greater than $1/2$ in [18] and greater than $2/3$ in [17] for the *a-priori* parameter choice rule.

To show the accuracy of numerical results, we compute the approximate L^2 error denoted by

$$e_a(u_0,\delta) = \|u(\cdot,0) - v^{k}_{\alpha,\delta,1/2}(\cdot,0)\|$$

and the approximate relative error in L^2 norm denoted by

$$e_r(u_0,\delta) = \frac{\|u(\cdot,0) - v^{k}_{\alpha,\delta,1/2}(\cdot,0)\|}{\|u(\cdot,0)\|}.$$

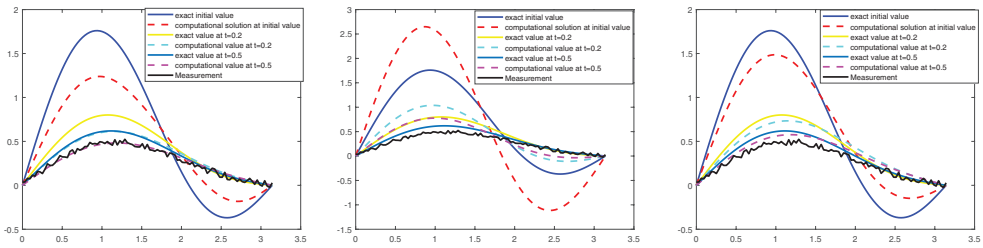
To verify the convergence rate, we use the index

$$\text{Convergence Order} := \log_2 \frac{e_a(u_0,2\delta)}{e_a(u_0,\delta)}.$$

The numerical errors and convergence orders for example 1 with different δ are shown in table 1, where the choice strategy of regularized parameter α by using the *a-priori* parameter choice rule (ii), from which we can see that the numerical error is decreasing as the noise level becomes smaller and the convergence order is close to 0.75, which support our convergence estimate in the theoretical analysis.

Table 1. Reconstruction results with $k = 2, p = 3$.

δ	0.0005	0.001	0.002	0.004	0.008	0.016	0.032
$e_a(u_0, \delta)$	0.0861	0.1437	0.2383	0.3917	0.6346	1.0056	1.5415
$e_r(u_0, \delta)$	0.0086	0.0144	0.0238	0.0392	0.0635	0.1006	0.1542
Order		0.7407	0.7297	0.7170	0.6961	0.6641	0.6163



(a) (b) (c)

Figure 1. Reconstruction results for $k = 0$(a), $k = 1$(b) and $k = 2$(c), respectively.

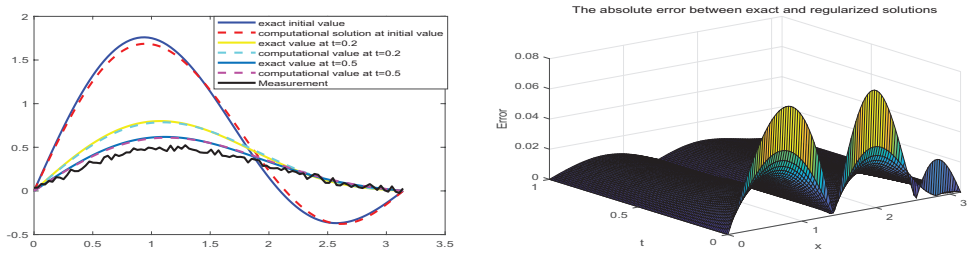
In figure 1, we show the reconstructions applying the same noisy data of level $\delta = 0.05$ with fixed regularized parameter choice strategy for $k = 0, 1, 2$, respectively. From these spatial distributed results for different time t, we can see, except those near $t = 0$, the numerical results are relatively good at the other time points. Especially, the reconstructions presented in (c) corresponding to $k = 2$ are much better than those for $k = 0, 1$ shown in (a) and (b), which support our theoretical result.

In figure 2(a), the computational results for $t = 0, 0.2, 0.5$ from the noise data given at $T = 1$ with $\delta = 0.05$ are shown using the explicit expressions, where the regularizing parameter is fixed as $\alpha = 0.0005$. It should be noticed that reconstructions for such a value α are better than those shown in figure 1(c) where α is chosen for the optimal value specified in theorem 3(ii). The reason is that the optimal value should be $C^* \left(\frac{\epsilon}{E}\right)^{\frac{k+1}{p+1}}$ for some constant $C^* > 0$ which may not be 1, see the proof of lemma 6. In figure 2(b), we show the absolute error between exact distributions and regularized ones.

To show the performances of our scheme for recovering the initial status, we present our reconstructions at $t = 0$ from final measurement time $T = 1$ in figure 3 for fixed noisy level $\delta = 0.02$, with different variable regularizing $\alpha = 0.02, 0.002, 0.0002, 0.000\,02$ (left) and variable noisy level $\delta = 0.005, 0.01, 0.05, 0.1$ with fixed regularizing parameter chosen as $\alpha = (\frac{\sqrt{2/\pi}\delta}{E})^{3/4}$ (right), which comes from the *a priori* rule (ii). By comparing the results in (a) and (b), we can see that the reconstructions using Tikhonov regularizing parameters from our theoretical strategy are indeed good.

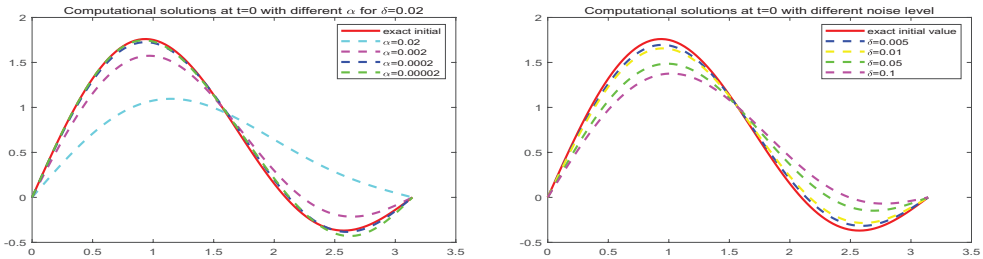
Example 2. Consider a backward problem for non-smooth initial distribution containing all frequencies.

$$\begin{cases} \frac{\partial^{1/2} u}{\partial t^{1/2}} = \frac{\partial^2 u}{\partial x^2}, & (x,t) \in (0,\pi) \times (0,T], \\ u(0,t) = u(\pi,t) = 0, & t \in [0,T], \\ u(x,0) = u_0(x) = \begin{cases} x, & x \in [0,\pi/2] \\ \pi - x, & x \in [\pi/2,\pi]. \end{cases} \end{cases} \tag{5.9}$$



(a) (b)

Figure 2. Numerical performances at different times for fixed noise level and regularizing parameter for example 1. (a): reconstructions at $t = 0, 0.2, 0.5$ from noisy data. (b): the absolute error between exact and regularized solutions.



(a) (b)

Figure 3. Reconstructions of initial value for example 1. (a): reconstructions for different values of α. (b): reconstructions for different noise levels δ.

The solution $u_{1/2}(x,t)$ of forward problem should be expressed in terms of the series with infinite number of terms. The high frequency amplitude is small in the Fourier expansion, therefore we can approximate the solution by a finite number of terms, i.e.

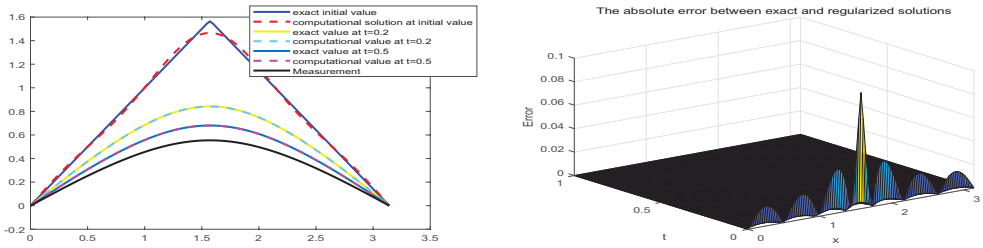
$$u_{1/2}(x,t) = \sum_{n=1}^{\infty} d_n E_{1/2,1}(-n^2 t^{1/2}) \sin nx \approx \sum_{n=1}^{100} d_n E_{1/2,1}(-n^2 t^{1/2}) \sin nx, \ t > 0 \tag{5.10}$$

with the coefficient

$$d_n = \frac{2}{\pi} \int_0^{\pi} u_0(x) \sin nx \mathrm{d}x = \frac{2(1+(-1)^{n+1})}{\pi n^2} \left[\sin \frac{n\pi}{2} - \frac{n\pi}{2} \cos \frac{n\pi}{2} \right] = \frac{4}{\pi n^2} \sin \frac{n\pi}{2} \tag{5.11}$$

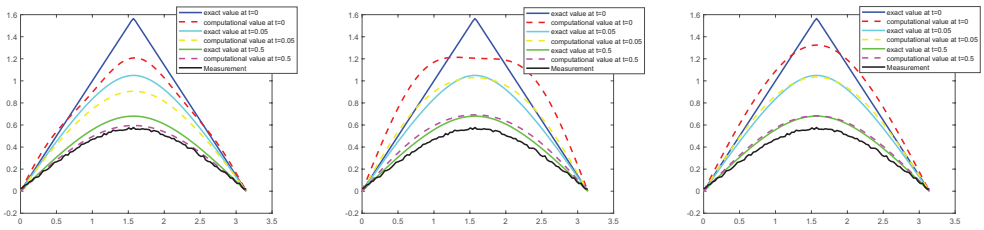
for all $n = 1, 2, \cdots$. We solve the backward problem using the final measurement data given at $T = 1$. We firstly observe that, for this smooth data at $T = 1$ with a random noise perturbation, the high frequency amplitude is almost zero in the Fourier expansion. Therefore, we can expand the solution to a finite number terms. The regularizing solution for $\gamma = 1/2$, $k = 0, 1, 2, \cdots$ is

$$\begin{aligned} v^k_{\alpha,\delta,1/2}[u^\delta(\cdot,T)](x,t) &= \sum_{n=1}^{\infty} f^k_{n,\delta} E_{1/2,1}\left(-n^2 t^{1/2}\right) \sin(nx), \\ &\approx \sum_{n=1}^{5} f^k_{n,\delta} E_{1/2,1}\left(-n^2 t^{1/2}\right) \sin(nx), \end{aligned} \tag{5.12}$$



(a) (b)

Figure 4. Reconstructions from exact input data without regularization for example 2. (a): reconstructions at $t = 0, 0.2, 0.5$ from exact data. (b): the absolute error between exact and regularized solutions.



(a) (b) (c)

Figure 5. Reconstructions for $k = 0$(a), $k = 1$(b) and $k = 2$(c) for example 2.

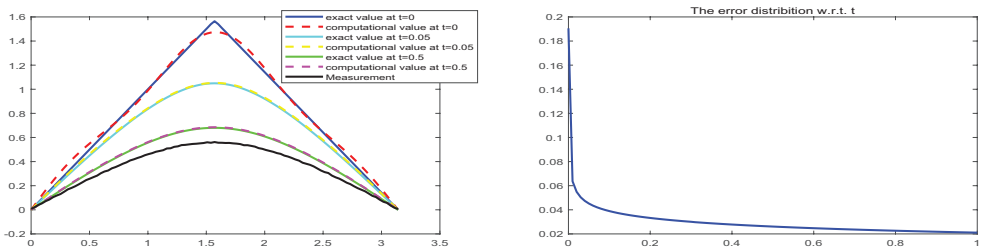
with the coefficients

$$f_{n,\delta}^{k} = \frac{1}{\alpha(-1)^k n^{2k} + E_{1/2,1}\left(-n^2 T^{1/2}\right)} \frac{2}{\pi} \int_0^{\pi} u_{1/2}^{\delta}(x,T) \sin nx \mathrm{d}x$$
$$\approx \frac{1}{\alpha(-1)^k n^{2k} + E_{1/2,1}\left(-n^2 T^{1/2}\right)} \frac{2}{100} \sum_{i=1}^{100} u_{1/2}^{\delta}(x_i,T) \sin nx_i \tag{5.13}$$

with $0 = x_1 < \cdots < x_{101} = \pi$ the grids dividing the interval $[0, \pi]$.

Firstly, it is easy to see that the reconstruction will be satisfactory for exact or almost exact input data, even if we do not apply the regularizing scheme. This phenomena, showing the mild ill-posedness of the backward problem, are shown numerically in figure 4, where the computational results for $t = 0, 0.2, 0.5$ from the exact data given at $T = 1$ using the explicit expressions (5.12) for the regularizing parameter $\alpha = 0$ are shown in (a), while the absolute errors between exact and regularized solutions are shown in (b). From these results, the numerical approximations are quite accurate with exact input data except for some boundary points. We can find the reconstruction for $u(x, 0)$ near to $x = \frac{\pi}{2}$ with exact input data is not so good. The reason is that $u(x, 0)$ is not smooth at $x = \frac{\pi}{2}$, while our regularizing scheme has some smooth effect.

For this non-smooth initial distribution, we also check the performances of the proposed regularizing scheme with different values $k = 0, 1, 2$ from noisy data with $\delta = 0.03$ by regularized parameter from our choice strategy. In figure 5, we present the reconstructions for different k. We can see that the numerical results are not very well at $t = 0$, while the numerical results are relatively good at the other time points. Similarly to the performances shown for



(a) (b)

Figure 6. Reconstruction performances for example 2 with non-smooth initial distribution. (a): reconstructions at $t = 0, 0.05, 0.5$ from noise data. (b): error distribution with respect to time by L^2 norm with $k = 2$ for spatial variable.

example 1, the results with $k = 2$ (see (c)) generate more satisfactory results. However, our reconstructions are smooth due to the smoothing effect by αA^k for $k = 0, 1, 2$ introduced in our regularizing scheme.

In figure 6(a), the inversion results for $t = 0, 0.05, 0.5$ from the noise data at $T = 1$ with $\delta = 0.01$ are shown using the explicit expressions (5.12) for the regularizing parameter $\alpha = 1e - 5$, while in figure 6(b), the spatial error distributions in L^2 norm in time interval $[0, 1]$ for $k = 2$ are shown, with the distributions at different instants t_m for $m = 1, 2, \cdots, 101$ defined by

$$err(t_m) := \sqrt{\frac{\pi}{100}\sum_{j=1}^{101}\left(v^k_{\alpha,\delta,1/2}[u^\delta_{1/2}(\cdot, T)](x_j, t_m) - u_{1/2}(x_j, t_m)\right)^2}.$$

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