5d semi-holomorphic higher Chern-Simons theory and 3d integrable field theories

Alexander Schenkel

School of Mathematical Sciences, University of Nottingham, UK.

Seminar @ Shing-Tung Yau Center of Southeast University, 16 May 2025.

Based on joint work with Benoit Vicedo [CMP 2024, arXiv:2405.08083].

A bird's-eye view on 2d Lax integrability

- A field theory on a 2d spacetime Σ is integrable if

$$\mathrm{EOM} = 0 \quad \Longleftrightarrow \quad \mathrm{d}_\Sigma A + \tfrac{1}{2}[A, A] = 0$$

for a Lax connection $A = A_t \, \mathrm{d}t + A_x \, \mathrm{d}x \in \Omega^{1,0}(\Sigma \times C, \mathfrak{g})$, constructed from the fields on Σ, depending meromorphically $\bar{\partial} A = 0$ on a Riemann surface C.

A bird's-eye view on 2d Lax integrability

- A field theory on a 2d spacetime Σ is integrable if

$$\mathrm{EOM} = 0 \quad \Longleftrightarrow \quad \mathrm{d}_\Sigma A + \tfrac{1}{2}\,[A, A] = 0$$

for a Lax connection $A = A_t\,\mathrm{d}t + A_x\,\mathrm{d}x \in \Omega^{1,0}(\Sigma \times C, \mathfrak{g})$, constructed from the fields on Σ, depending meromorphically $\bar{\partial} A = 0$ on a Riemann surface C.

- **Important consequence:** The holonomy $\mathrm{hol}_t(A)$ along Cauchy surfaces $S_t \subseteq \Sigma$ is time-independent $\partial_t \mathrm{hol}_t(A) = 0$, so its Laurent expansion

$$\mathrm{hol}_t(A) = \sum_{n\in\mathbb{Z}} Q_n(A)\, z^n \quad \text{on } C$$

gives ∞-many conserved charges, making the theory "exactly solvable".

A bird's-eye view on 2d Lax integrability

- A field theory on a 2d spacetime Σ is integrable if

$$\mathrm{EOM} = 0 \quad \Longleftrightarrow \quad \mathrm{d}_\Sigma A + \tfrac{1}{2}[A, A] = 0$$

  for a Lax connection $A = A_t \, \mathrm{d}t + A_x \, \mathrm{d}x \in \Omega^{1,0}(\Sigma \times C, \mathfrak{g})$, constructed from the fields on Σ, depending meromorphically $\bar{\partial} A = 0$ on a Riemann surface C.

- **Important consequence:** The holonomy $\mathrm{hol}_t(A)$ along Cauchy surfaces $S_t \subseteq \Sigma$ is time-independent $\partial_t \mathrm{hol}_t(A) = 0$, so its Laurent expansion

$$\mathrm{hol}_t(A) = \sum_{n \in \mathbb{Z}} Q_n(A) \, z^n \quad \text{on } C$$

  gives ∞-many conserved charges, making the theory "exactly solvable".

- What is the origin of the Lax connection? How can I find it in examples?

A bird's-eye view on 2d Lax integrability

- A field theory on a 2d spacetime Σ is integrable if

$$\mathrm{EOM} = 0 \quad \Longleftrightarrow \quad \mathrm{d}_\Sigma A + \tfrac{1}{2}\,[A,A] = 0$$

for a Lax connection $A = A_t\,\mathrm{d}t + A_x\,\mathrm{d}x \in \Omega^{1,0}(\Sigma \times C, \mathfrak{g})$, constructed from the fields on Σ, depending meromorphically $\bar{\partial} A = 0$ on a Riemann surface C.

- **Important consequence:** The holonomy $\mathrm{hol}_t(A)$ along Cauchy surfaces $S_t \subseteq \Sigma$ is time-independent $\partial_t \mathrm{hol}_t(A) = 0$, so its Laurent expansion

$$\mathrm{hol}_t(A) = \sum_{n \in \mathbb{Z}} Q_n(A)\, z^n \quad \text{on } C$$

gives ∞-many conserved charges, making the theory "exactly solvable".

- What is the origin of the Lax connection? How can I find it in examples?

≤ 2018: Mostly clever guesswork, so origin remained mysterious ☹

A bird's-eye view on 2d Lax integrability

- A field theory on a 2d spacetime Σ is integrable if

$$\mathrm{EOM} = 0 \quad \Longleftrightarrow \quad \mathrm{d}_\Sigma A + \tfrac{1}{2}\,[A, A] = 0$$

for a Lax connection $A = A_t\,\mathrm{d}t + A_x\,\mathrm{d}x \in \Omega^{1,0}(\Sigma \times C, \mathfrak{g})$, constructed from the fields on Σ, depending meromorphically $\bar{\partial} A = 0$ on a Riemann surface C.

- **Important consequence:** The holonomy $\mathrm{hol}_t(A)$ along Cauchy surfaces $S_t \subseteq \Sigma$ is time-independent $\partial_t \mathrm{hol}_t(A) = 0$, so its Laurent expansion

$$\mathrm{hol}_t(A) = \sum_{n \in \mathbb{Z}} Q_n(A)\, z^n \quad \text{on } C$$

gives ∞-many conserved charges, making the theory "exactly solvable".

- What is the origin of the Lax connection? How can I find it in examples?

≤ 2018: Mostly clever guesswork, so origin remained mysterious ☹

≥ 2019: Gauge-theoretic methods explain geometric origin of Lax connection ☺

Costello and Yamazaki's gauge-theoretic framework

- **Main idea:** Since the Lax connection is fundamental for integrability, one should develop a mother theory for Lax connections and then understand how to extract from it concrete models of 2d integrable field theories!

Costello and Yamazaki's gauge-theoretic framework

- **Main idea:** Since the Lax connection is fundamental for integrability, one should develop a mother theory for Lax connections and then understand how to extract from it concrete models of 2d integrable field theories!

- Lax connections live on a 4d manifold $X = \Sigma \times C$, with Σ a 2d spacetime and C a Riemann surface, and they take the form of gauge fields

$$A \,=\, A_t \, \mathrm{d}t + A_x \, \mathrm{d}x + A_{\bar{z}} \, \mathrm{d}\bar{z} \,\in\, \overline{\Omega}^1(X, \mathfrak{g}) \,\subseteq\, \Omega^1(X, \mathfrak{g})$$

modeled on the de Rham-Dolbeault complex $\overline{\Omega}^\bullet(X) = \Omega^\bullet(\Sigma) \,\hat{\otimes}\, \Omega^{0,\bullet}(C)$ and taking values in the Lie algebra $\mathfrak{g}$ of some structure group G.

Costello and Yamazaki's gauge-theoretic framework

- **Main idea:** Since the Lax connection is fundamental for integrability, one should develop a mother theory for Lax connections and then understand how to extract from it concrete models of 2d integrable field theories!

- Lax connections live on a 4d manifold $X = \Sigma \times C$, with Σ a 2d spacetime and C a Riemann surface, and they take the form of gauge fields

$$A \,=\, A_t \,\mathrm{d}t + A_x \,\mathrm{d}x + A_{\bar{z}} \,\mathrm{d}\bar{z} \,\in\, \overline{\Omega}^1(X, \mathfrak{g}) \,\subseteq\, \Omega^1(X, \mathfrak{g})$$

  modeled on the de Rham-Dolbeault complex $\overline{\Omega}^\bullet(X) = \Omega^\bullet(\Sigma) \,\hat{\otimes}\, \Omega^{0,\bullet}(C)$ and taking values in the Lie algebra $\mathfrak{g}$ of some structure group G.

- Their dynamics is governed by the 4d Chern-Simons action

$$S_\omega(A) \,=\, \frac{\mathrm{i}}{2\pi} \int_X \omega \wedge \mathsf{CS}(A) \,=\, \frac{\mathrm{i}}{2\pi} \int_X \omega \wedge \big\langle A, \tfrac{1}{2}\,\mathrm{d}A + \tfrac{1}{3!}\,[A, A]\big\rangle \quad ,$$

  where $\omega = \omega_z \,\mathrm{d}z \in \Omega^{1,0}(C)$ is a meromorphic 1-form $\bar{\partial}\omega = 0$ and $\langle\cdot,\cdot\rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is a non-degenerate Ad-invariant symmetric form.

How to extract 2d integrable field theories? (Sketch!)

- **Key point:** The action S_ω is *not* invariant under gauge transformations $g : A \to {}^g A := g\, A\, g^{-1} - \mathrm{d}g\, g^{-1}$, for all $g \in C^\infty(X, G)$, with the violations localized at 2d surface defects located at the poles $\hat{\boldsymbol{z}} \subseteq C$ of ω

$$\boldsymbol{j} \,:\, \widehat{D} \,=\, \Sigma \times \hat{\boldsymbol{z}} \longrightarrow X \,=\, \Sigma \times C \quad .$$

How to extract 2d integrable field theories? (Sketch!)

◇ **Key point:** The action S_ω is *not* invariant under gauge transformations $g : A \to {}^g A := g\,A\,g^{-1} - \mathrm{d}g\,g^{-1}$, for all $g \in C^\infty(X, G)$, with the violations localized at 2d surface defects located at the poles $\hat{\boldsymbol{z}} \subseteq C$ of ω

$$\boldsymbol{j} \,:\, \widehat{D} \,=\, \Sigma \times \hat{\boldsymbol{z}} \longhookrightarrow X \,=\, \Sigma \times C \quad .$$

$\rightsquigarrow$ Impose suitable boundary conditions at $\widehat{D}$ to obtain gauge invariance!

How to extract 2d integrable field theories? (Sketch!)

- ◇ **Key point:** The action S_ω is *not* invariant under gauge transformations $g : A \to {}^g A := g\,A\,g^{-1} - \mathrm{d}g\,g^{-1}$, for all $g \in C^\infty(X, G)$, with the violations localized at 2d surface defects located at the poles $\hat{\boldsymbol{z}} \subseteq C$ of ω

$$\boldsymbol{j} : \widehat{D} = \Sigma \times \hat{\boldsymbol{z}} \hookrightarrow X = \Sigma \times C \quad .$$

⇝ Impose suitable boundary conditions at $\widehat{D}$ to obtain gauge invariance!

- ◇ Boundary conditions in gauge theory are additional structure (edge modes)

$$\underbrace{\boldsymbol{j}^*(\Phi) = \phi_{\mathrm{bdy}}}_{\text{equality}} \qquad \text{vs} \qquad \underbrace{\boldsymbol{j}^*(A) \xrightarrow{h} \alpha_{\mathrm{bdy}}}_{\text{gauge transformation}}$$

and the boundary conditioned action $S(A, h) = S_\omega(A) + S_{\mathrm{bdy}}(A, h)$ receives a boundary term for the edge mode field $h \in C^\infty(\widehat{D}, G) \cong C^\infty(\Sigma, G^{\hat{\boldsymbol{z}}})$.

How to extract 2d integrable field theories? (Sketch!)

◇ **Key point:** The action S_ω is *not* invariant under gauge transformations $g : A \to {}^g A := g\,A\,g^{-1} - \mathrm{d}g\,g^{-1}$, for all $g \in C^\infty(X, G)$, with the violations localized at 2d surface defects located at the poles $\hat{\boldsymbol{z}} \subseteq C$ of ω

$$\boldsymbol{j} \,:\, \widehat{D} \,=\, \Sigma \times \hat{\boldsymbol{z}} \longhookrightarrow X \,=\, \Sigma \times C \quad .$$

⇝ Impose suitable boundary conditions at $\widehat{D}$ to obtain gauge invariance!

◇ Boundary conditions in gauge theory are additional structure (edge modes)

$$\underbrace{\boldsymbol{j}^*(\Phi) \,=\, \phi_{\mathrm{bdy}}}_{\text{equality}} \qquad \text{vs} \qquad \underbrace{\boldsymbol{j}^*(A) \,\xrightarrow{\;h\;}\, \alpha_{\mathrm{bdy}}}_{\text{gauge transformation}}$$

and the boundary conditioned action $S(A, h) = S_\omega(A) + S_{\mathrm{bdy}}(A, h)$ receives a boundary term for the edge mode field $h \in C^\infty(\widehat{D}, G) \cong C^\infty(\Sigma, G^{\hat{\boldsymbol{z}}})$.

◇ Picking a suitable solution of the bulk equation of motion gives a 2d integrable field theory for h on Σ, together with its Lax connection $A = A(h)$.

How to extract 2d integrable field theories? (Sketch!)

- **Key point:** The action S_ω is *not* invariant under gauge transformations $g : A \to {}^g A := g\,A\,g^{-1} - \mathrm{d}g\,g^{-1}$, for all $g \in C^\infty(X, G)$, with the violations localized at 2d surface defects located at the poles $\hat{\boldsymbol{z}} \subseteq C$ of ω

$$\boldsymbol{j} : \widehat{D} = \Sigma \times \hat{\boldsymbol{z}} \longhookrightarrow X = \Sigma \times C \quad .$$

$\rightsquigarrow$ Impose suitable boundary conditions at $\widehat{D}$ to obtain gauge invariance!

- Boundary conditions in gauge theory are additional structure (edge modes)

$$\underbrace{\boldsymbol{j}^*(\Phi) = \phi_{\mathrm{bdy}}}_{\text{equality}} \qquad \text{vs} \qquad \underbrace{\boldsymbol{j}^*(A) \xrightarrow{h} \alpha_{\mathrm{bdy}}}_{\text{gauge transformation}}$$

  and the boundary conditioned action $S(A, h) = S_\omega(A) + S_{\mathrm{bdy}}(A, h)$ receives a boundary term for the edge mode field $h \in C^\infty(\widehat{D}, G) \cong C^\infty(\Sigma, G^{\hat{\boldsymbol{z}}})$.

- Picking a suitable solution of the bulk equation of motion gives a 2d integrable field theory for h on Σ, together with its Lax connection $A = A(h)$.

- The details of this construction are somewhat technical and can be found in [Benini, AS, Vicedo: CMP 2022, arXiv:2008.01829].

How about integrability in $d + 1$ dimensions?

- **Long-standing problem:** What are good foundations for, and examples of, integrable field theories on a $(d + 1)$-dimensional spacetime M, for $d > 1$?

How about integrability in $d+1$ dimensions?

- **Long-standing problem:** What are good foundations for, and examples of, integrable field theories on a $(d+1)$-dimensional spacetime M, for $d>1$?
- Taking the Lax formalism seriously, one should be looking for d-dimensional holonomies $\mathrm{hol}_t(\mathcal{A})$ along Cauchy surfaces $S_t \subseteq M$ which are
  1. time-independent $\partial_t \mathrm{hol}_t(\mathcal{A}) = 0$
  2. meromorphic on some Riemann surface C (maybe also higher-dimensional?)

  such that its Laurent expansion gives ∞-many conserved charges.

How about integrability in $d+1$ dimensions?

- **Long-standing problem:** What are good foundations for, and examples of, integrable field theories on a $(d+1)$-dimensional spacetime M, for $d > 1$?
- Taking the Lax formalism seriously, one should be looking for d-dimensional holonomies $\mathrm{hol}_t(\mathcal{A})$ along Cauchy surfaces $S_t \subseteq M$ which are
  1. time-independent $\partial_t \mathrm{hol}_t(\mathcal{A}) = 0$
  2. meromorphic on some Riemann surface C (maybe also higher-dimensional?)

  such that its Laurent expansion gives ∞-many conserved charges.
- This can be realized in terms of connections $\mathcal{A}$ on higher principal bundles $\mathcal{G} \curvearrowright \mathcal{P} \to M \times C$ for a Lie d-group $\mathcal{G}$. (Higher gauge theory!)

How about integrability in $d+1$ dimensions?

- **Long-standing problem:** What are good foundations for, and examples of, integrable field theories on a $(d+1)$-dimensional spacetime M, for $d > 1$?
- Taking the Lax formalism seriously, one should be looking for d-dimensional holonomies $\mathrm{hol}_t(\mathcal{A})$ along Cauchy surfaces $S_t \subseteq M$ which are
  1. time-independent $\partial_t \mathrm{hol}_t(\mathcal{A}) = 0$
  2. meromorphic on some Riemann surface C (maybe also higher-dimensional?)

  such that its Laurent expansion gives ∞-many conserved charges.
- This can be realized in terms of connections $\mathcal{A}$ on higher principal bundles $\mathcal{G} \curvearrowright \mathcal{P} \to M \times C$ for a Lie d-group $\mathcal{G}$. (Higher gauge theory!)
- **Aim of the rest of the talk:**

  Focusing on the simplest non-trivial case, I will present a semi-holomorphic 2-Chern-Simons theory on 5d manifolds $X = M \times C$ which generates $3d$ integrable field theories on M and their higher Lax connections.

Lie 2-groups and Lie 2-algebras

◇ Strict Lie 2-groups $\mathcal{G} \simeq$ crossed modules of Lie groups (G, H, t, α) where

- G and H are ordinary Lie groups,
- $t : H \to G$ is a Lie group homomorphism, and
- $\alpha : G \to \mathrm{Aut}(H)$ is a G-action on H,

such that, for all $g \in G$ and $h, h' \in H$,

$$t\big(\alpha(g,h)\big) \,=\, g\, t(h)\, g^{-1} \quad , \quad \alpha\big(t(h), h'\big) \,=\, h\, h'\, h^{-1} \quad .$$

Lie 2-groups and Lie 2-algebras

◇ Strict Lie 2-groups $\mathcal{G} \simeq$ crossed modules of Lie groups (G, H, t, α) where

- G and H are ordinary Lie groups,
- $t : H \to G$ is a Lie group homomorphism, and
- $\alpha : G \to \mathrm{Aut}(H)$ is a G-action on H,

such that, for all $g \in G$ and $h, h' \in H$,

$$t\big(\alpha(g,h)\big) \,=\, g\, t(h)\, g^{-1} \quad , \quad \alpha\big(t(h), h'\big) \,=\, h\, h'\, h^{-1} \quad .$$

◇ The associated Lie 2-algebra is the crossed module of Lie algebras $(\mathfrak{g}, \mathfrak{h}, t_*, \alpha_*)$ where

- $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, and
- $t_* = dt|_{1_H} : \mathfrak{h} \to \mathfrak{g}$ and $\alpha_* = d\alpha|_{1_G} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ via differentiation.

The structure identities differentiate to, for all $x \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$,

$$t_*\big(\alpha_*(x,y)\big) \,=\, [x, t_*(y)] \quad , \quad \alpha_*\big(t_*(y), y'\big) \,=\, [y, y'] \quad .$$

Lie 2-groups and Lie 2-algebras

◇ Strict Lie 2-groups $\mathcal{G} \simeq$ crossed modules of Lie groups (G, H, t, α) where

- G and H are ordinary Lie groups,
- $t : H \to G$ is a Lie group homomorphism, and
- $\alpha : G \to \mathrm{Aut}(H)$ is a G-action on H,

such that, for all $g \in G$ and $h, h' \in H$,

$$t\big(\alpha(g,h)\big) \,=\, g\, t(h)\, g^{-1} \quad , \quad \alpha\big(t(h), h'\big) \,=\, h\, h'\, h^{-1} \quad .$$

◇ The associated Lie 2-algebra is the crossed module of Lie algebras $(\mathfrak{g}, \mathfrak{h}, t_*, \alpha_*)$ where

- $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of G and H, and
- $t_* = dt|_{1_H} : \mathfrak{h} \to \mathfrak{g}$ and $\alpha_* = d\alpha|_{1_G} : \mathfrak{g} \to \mathrm{Der}(\mathfrak{h})$ via differentiation.

The structure identities differentiate to, for all $x \in \mathfrak{g}$ and $y, y' \in \mathfrak{h}$,

$$t_*\big(\alpha_*(x,y)\big) \,=\, [x, t_*(y)] \quad , \quad \alpha_*\big(t_*(y), y'\big) \,=\, [y, y'] \quad .$$

! All of this is quite explicit, so with some practice one can do computations!

Connections on trivial principal 2-bundles

- Fix a manifold X and strict Lie 2-group (G, H, t, α). A connection is a pair

$$\mathcal{A} = (A, B) \in \Omega^1(X, \mathfrak{g}) \times \Omega^2(X, \mathfrak{h}) \quad .$$

Connections on trivial principal 2-bundles

- Fix a manifold X and strict Lie 2-group (G, H, t, α). A connection is a pair

$$\mathcal{A} = (A, B) \in \Omega^1(X, \mathfrak{g}) \times \Omega^2(X, \mathfrak{h}) \quad .$$

- A gauge transformation is a pair $(g, \gamma) \in C^\infty(X, G) \times \Omega^1(X, \mathfrak{h})$ and it transforms connections $(g, \gamma) : (A, B) \longrightarrow {}^{(g,\gamma)}(A, B)$ according to

$$\begin{aligned} {}^{(g,\gamma)}A &= g\, A\, g^{-1} - \mathrm{d}g\, g^{-1} - t_*(\gamma) \quad , \\ {}^{(g,\gamma)}B &= \alpha_*(g, B) - F(\gamma) - \alpha_*\big({}^{(g,\gamma)}A, \gamma\big) \quad , \end{aligned}$$

where $F(\gamma) := \mathrm{d}\gamma + \frac{1}{2}\,[\gamma, \gamma]$.

Connections on trivial principal 2-bundles

- Fix a manifold X and strict Lie 2-group (G,H,t,α). A connection is a pair
$$\mathcal{A} \,=\, (A,B) \,\in\, \Omega^1(X,\mathfrak{g})\times\Omega^2(X,\mathfrak{h}) \quad .$$

- A gauge transformation is a pair $(g,\gamma)\in C^\infty(X,G)\times\Omega^1(X,\mathfrak{h})$ and it transforms connections $(g,\gamma) : (A,B)\longrightarrow {}^{(g,\gamma)}(A,B)$ according to
$$\begin{aligned} {}^{(g,\gamma)}A \,&=\, g\,A\,g^{-1} - \mathrm{d}g\,g^{-1} - t_\ast(\gamma) \quad ,\\ {}^{(g,\gamma)}B \,&=\, \alpha_\ast(g,B) - F(\gamma) - \alpha_\ast\big({}^{(g,\gamma)}A,\gamma\big) \quad ,\end{aligned}$$
where $F(\gamma) := \mathrm{d}\gamma + \frac{1}{2}\,[\gamma,\gamma]$.

Rem: A useful perspective on connections is as 1-cochains in the dg-Lie algebra
$$L \,:=\, \mathrm{Tot}\left(\begin{array}{ccccccc} \overset{(0,0)}{\Omega^0(X,\mathfrak{g})} & \xrightarrow{\mathrm{d}} & \overset{(0,1)}{\Omega^1(X,\mathfrak{g})} & \xrightarrow{\mathrm{d}} & \cdots & \xrightarrow{\mathrm{d}} & \overset{(0,n)}{\Omega^n(X,\mathfrak{g})} \\ {\scriptstyle t_\ast}\big\uparrow & & {\scriptstyle t_\ast}\big\uparrow & & & & {\scriptstyle t_\ast}\big\uparrow \\ \overset{(-1,0)}{\Omega^0(X,\mathfrak{h})} & \xrightarrow{\mathrm{d}} & \overset{(-1,1)}{\Omega^1(X,\mathfrak{h})} & \xrightarrow{\mathrm{d}} & \cdots & \xrightarrow{\mathrm{d}} & \overset{(-1,n)}{\Omega^n(X,\mathfrak{h})} \end{array}\right) \quad .$$

2-Chern-Simons 4-form and action

- There exist general techniques (buzzword: Maurer-Cartan theory) to extract from a dg-Lie algebra $(L, \mathrm{d}_L, [\cdot,\cdot]_L)$ a Lagrangian and action.

2-Chern-Simons 4-form and action

- There exist general techniques (buzzword: Maurer-Cartan theory) to extract from a dg-Lie algebra $(L, \mathrm{d}_L, [\cdot,\cdot]_L)$ a Lagrangian and action.

- These require the choice of a cyclic structure on L, which in our example amounts to a non-degenerate pairing $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{h}\to\mathbb{C}$ satisfying

$$\big\langle g\,x\,g^{-1}, \alpha_*(g,y)\big\rangle = \langle x,y\rangle \quad , \quad \langle t_*(y),y'\rangle = \langle t_*(y'),y\rangle \quad .$$

2-Chern-Simons 4-form and action

- There exist general techniques (buzzword: Maurer-Cartan theory) to extract from a dg-Lie algebra $(L, \mathrm{d}_L, [\cdot,\cdot]_L)$ a Lagrangian and action.
- These require the choice of a cyclic structure on L, which in our example amounts to a non-degenerate pairing $\langle\cdot,\cdot\rangle : \mathfrak{g} \otimes \mathfrak{h} \to \mathbb{C}$ satisfying

$$\big\langle g\, x\, g^{-1}, \alpha_*(g,y)\big\rangle = \langle x, y\rangle \quad , \quad \langle t_*(y), y'\rangle = \langle t_*(y'), y\rangle \quad .$$

Def: The 2-Chern-Simons 4-form associated to a connection $\mathcal{A} = (A, B)$ is

$$\begin{aligned}\mathsf{CS}(\mathcal{A}) \,:&= \big\langle \mathcal{A}, \tfrac{1}{2}\, \mathrm{d}_L \mathcal{A} + \tfrac{1}{3!}\, [\mathcal{A},\mathcal{A}]_L \big\rangle_L \\ &= \big\langle F(A) - \tfrac{1}{2}\, t_*(B), B\big\rangle - \tfrac{1}{2}\, \mathrm{d}\langle A, B\rangle \,\in\, \Omega^4(X) \quad .\end{aligned}$$

2-Chern-Simons 4-form and action

- ◇ There exist general techniques (buzzword: Maurer-Cartan theory) to extract from a dg-Lie algebra $(L, \mathrm{d}_L, [\cdot,\cdot]_L)$ a Lagrangian and action.

- ◇ These require the choice of a cyclic structure on L, which in our example amounts to a non-degenerate pairing $\langle\cdot,\cdot\rangle : \mathfrak{g}\otimes\mathfrak{h}\to\mathbb{C}$ satisfying

$$\big\langle g\,x\,g^{-1}, \alpha_*(g,y)\big\rangle = \langle x,y\rangle \quad , \quad \langle t_*(y), y'\rangle = \langle t_*(y'), y\rangle \quad .$$

Def: The 2-Chern-Simons 4-form associated to a connection $\mathcal{A} = (A,B)$ is

$$\begin{aligned} \mathsf{CS}(\mathcal{A}) \,:&= \big\langle \mathcal{A}, \tfrac{1}{2}\,\mathrm{d}_L\mathcal{A} + \tfrac{1}{3!}\,[\mathcal{A},\mathcal{A}]_L\big\rangle_L \\ &= \big\langle F(A) - \tfrac{1}{2}\,t_*(B), B\big\rangle - \tfrac{1}{2}\,\mathrm{d}\langle A, B\rangle \,\in\, \Omega^4(X) \quad . \end{aligned}$$

- ◇ In the 5d semi-holomorphic case $X = M\times C$, choosing a meromorphic 1-form $\omega = \omega_z\,\mathrm{d}z \in \Omega^{1,0}(C)$ allows us to define the action

$$S_\omega(\mathcal{A}) = \frac{\mathrm{i}}{2\pi}\int_X \omega\wedge\mathsf{CS}(\mathcal{A}) \quad ,$$

generalizing Costello-Yamazaki from 4d to 5d.

Behavior near the poles $\hat{\boldsymbol{z}} \subseteq C$ of ω

- At the 3d volume defects $\boldsymbol{j} : \widehat{D} = M \times \hat{\boldsymbol{z}} \longhookrightarrow X = M \times C$ there are interesting phenomena captured by the holomorphic jet expansions

$$\boldsymbol{j}^*(\cdot) \,=\, \bigg(\sum_{p=0}^{n_x - 1} \tfrac{1}{p!}\, \iota_x^* \partial_z^p(\cdot) \otimes \epsilon_x^p \bigg)_{x \in \text{poles}} \qquad (n_x = \text{order of pole } x) \quad .$$

Behavior near the poles $\hat{\boldsymbol{z}} \subseteq C$ of ω

- At the 3d volume defects $\boldsymbol{j} : \widehat{D} = M \times \hat{\boldsymbol{z}} \longrightarrow X = M \times C$ there are interesting phenomena captured by the holomorphic jet expansions

$$\boldsymbol{j}^*(\cdot) \,=\, \bigg(\sum_{p=0}^{n_x - 1} \tfrac{1}{p!}\, \iota_x^* \partial_z^p(\cdot) \otimes \epsilon_x^p \bigg)_{x \in \mathrm{poles}} \qquad (n_x = \text{order of pole } x) \quad .$$

- The defect data live in the jet Lie groups

$$G^{\hat{\boldsymbol{z}}} \,=\, \prod_{x \in \mathrm{poles}} J^{n_x - 1} G \quad , \quad H^{\hat{\boldsymbol{z}}} \,=\, \prod_{x \in \mathrm{poles}} J^{n_x - 1} H$$

and their Lie algebras $\mathfrak{g}^{\hat{\boldsymbol{z}}}$ and $\mathfrak{h}^{\hat{\boldsymbol{z}}}$.

Behavior near the poles $\hat{\boldsymbol{z}} \subseteq C$ of ω

- At the 3d volume defects $\boldsymbol{j} : \widehat{D} = M \times \hat{\boldsymbol{z}} \longrightarrow X = M \times C$ there are interesting phenomena captured by the holomorphic jet expansions

$$\boldsymbol{j}^*(\cdot) \;=\; \bigg(\sum_{p=0}^{n_x - 1} \tfrac{1}{p!}\, \iota_x^* \partial_z^p(\cdot) \otimes \epsilon_x^p \bigg)_{x \in \mathrm{poles}} \qquad (n_x = \text{order of pole } x) \quad .$$

- The defect data live in the jet Lie groups

$$G^{\hat{\boldsymbol{z}}} \;=\; \prod_{x \in \mathrm{poles}} J^{n_x - 1} G \quad , \quad H^{\hat{\boldsymbol{z}}} \;=\; \prod_{x \in \mathrm{poles}} J^{n_x - 1} H$$

and their Lie algebras $\mathfrak{g}^{\hat{\boldsymbol{z}}}$ and $\mathfrak{h}^{\hat{\boldsymbol{z}}}$.

Prop: Under gauge transformations $(g, \gamma) \in C^\infty(X, G) \times \Omega^1(X, \mathfrak{h})$:

$$S_\omega\big({}^{(g,\gamma)}(A,B)\big) = S_\omega(A,B) + \frac{1}{2} \int_M \Big(\big\langle\!\big\langle \boldsymbol{j}^*(g)\, \boldsymbol{j}^*(A)\, \boldsymbol{j}^*(g)^{-1}, F_M\big(\boldsymbol{j}^*(\gamma)\big) \big\rangle\!\big\rangle_\omega$$

$$+ \big\langle\!\big\langle \boldsymbol{j}^*\big(t_*(\gamma)\big), \mathrm{d}_M \boldsymbol{j}^*(\gamma) + \tfrac{1}{3}\big[\boldsymbol{j}^*(\gamma), \boldsymbol{j}^*(\gamma)\big] \big\rangle\!\big\rangle_\omega$$

$$- \big\langle\!\big\langle \mathrm{d}_M \boldsymbol{j}^*(g)\, \boldsymbol{j}^*(g)^{-1} + \boldsymbol{j}^*\big(t_*(\gamma)\big), \boldsymbol{j}^*\big(\alpha_*(g,B)\big) + F_M\big(\boldsymbol{j}^*(\gamma)\big) \big\rangle\!\big\rangle_\omega \Big)$$

Boundary conditions, edge modes and boundary action

- From this specific form of gauge symmetry violation of S_ω, one observes that a suitable class of boundary conditions is given by isotropic Lie 2-subgroups

$$(G^\diamond, H^\diamond, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}}) \subseteq (G^{\hat{\boldsymbol{z}}}, H^{\hat{\boldsymbol{z}}}, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}}) \quad \text{such that} \quad \langle\!\langle \cdot, \cdot \rangle\!\rangle_\omega \big|_{\mathfrak{g}^\diamond \otimes \mathfrak{h}^\diamond} = 0 \quad .$$

Boundary conditions, edge modes and boundary action

- From this specific form of gauge symmetry violation of S_ω, one observes that a suitable class of boundary conditions is given by isotropic Lie 2-subgroups

$$(G^\diamond, H^\diamond, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}}) \subseteq (G^{\hat{\boldsymbol{z}}}, H^{\hat{\boldsymbol{z}}}, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}}) \quad \text{such that} \quad \langle\!\langle \cdot, \cdot \rangle\!\rangle_\omega \big|_{\mathfrak{g}^\diamond \otimes \mathfrak{h}^\diamond} = 0 \quad .$$

- This yields edge modes $(k, \kappa) \in C^\infty(M, G^{\hat{\boldsymbol{z}}}) \times \Omega^1(M, \mathfrak{h}^{\hat{\boldsymbol{z}}})$ implementing

$$^{(k,\kappa)}\boldsymbol{j}^*(A, B) \in \Omega^1(M, \mathfrak{g}^\diamond) \times \Omega^2(M, \mathfrak{h}^\diamond) \quad .$$

Boundary conditions, edge modes and boundary action

- From this specific form of gauge symmetry violation of S_ω, one observes that a suitable class of boundary conditions is given by isotropic Lie 2-subgroups

$$(G^\diamond, H^\diamond, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}}) \subseteq (G^{\hat{\boldsymbol{z}}}, H^{\hat{\boldsymbol{z}}}, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}}) \quad \text{such that} \quad \langle\!\langle \cdot, \cdot \rangle\!\rangle_\omega \big|_{\mathfrak{g}^\diamond \otimes \mathfrak{h}^\diamond} = 0 \quad .$$

- This yields edge modes $(k, \kappa) \in C^\infty(M, G^{\hat{\boldsymbol{z}}}) \times \Omega^1(M, \mathfrak{h}^{\hat{\boldsymbol{z}}})$ implementing

$${}^{(k,\kappa)}\boldsymbol{j}^*(A,B) \in \Omega^1(M, \mathfrak{g}^\diamond) \times \Omega^2(M, \mathfrak{h}^\diamond) \quad .$$

- The induced action on bulk fields (A, B) and edge modes (k, κ) is

$$\begin{aligned} S\big((A,B),(k,\kappa)\big) \,=\, &\frac{\mathrm{i}}{2\pi} \int_X \omega \wedge \big\langle F(A) - \tfrac{1}{2} t_*(B), B \big\rangle \\ &+ \frac{1}{2} \int_M \Big(\big\langle\!\big\langle {}^{(k,\kappa)}\boldsymbol{j}^*(A), \alpha^{\hat{\boldsymbol{z}}}_*\big({}^{(k,\kappa)}\boldsymbol{j}^*(A), \kappa\big) + 2F_M(\kappa) \big\rangle\!\big\rangle_\omega \\ &\qquad + \big\langle\!\big\langle t^{\hat{\boldsymbol{z}}}_*(\kappa), \mathrm{d}_M \kappa + \tfrac{1}{3}[\kappa,\kappa] \big\rangle\!\big\rangle_\omega \Big) \quad . \end{aligned}$$

A recipe for constructing 3d integrable field theories

1. Fix input data of the 5d theory, i.e.
    - meromorphic 1-form $\omega \in \Omega^{1,0}(C)$,
    - Lie 2-group (G, H, t, α) with cyclic structure $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{h} \to \mathbb{C}$, and
    - isotropic Lie 2-subgroup $(G^{\diamond}, H^{\diamond}, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}}) \subseteq (G^{\hat{\boldsymbol{z}}}, H^{\hat{\boldsymbol{z}}}, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}})$.

A recipe for constructing 3d integrable field theories

1. Fix input data of the 5d theory, i.e.
   - meromorphic 1-form $\omega \in \Omega^{1,0}(C)$,
   - Lie 2-group (G, H, t, α) with cyclic structure $\langle \cdot, \cdot \rangle : \mathfrak{g} \otimes \mathfrak{h} \to \mathbb{C}$, and
   - isotropic Lie 2-subgroup $(G^\diamond, H^\diamond, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}}) \subseteq (G^{\hat{\boldsymbol{z}}}, H^{\hat{\boldsymbol{z}}}, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}})$.
2. Work in gauge $A_{\bar{z}} = 0$ and $B_{i\bar{z}} = 0$, so that the bulk equations of motion become $\omega \wedge \bar{\partial} A = 0$ and $\omega \wedge \bar{\partial} B = 0$. Choose admissible bulk solution (A, B) and solve boundary conditions for higher Lax connection:

$$^{(k,\kappa)}\boldsymbol{j}^*(A, B) \in \Omega^1(M, \mathfrak{g}^\diamond) \times \Omega^2(M, \mathfrak{h}^\diamond) \quad \overset{\text{solve}}{\rightsquigarrow} \quad (A, B) = \big(A(k, \kappa), B(k, \kappa)\big)$$

A recipe for constructing 3d integrable field theories

1. Fix input data of the 5d theory, i.e.
   - meromorphic 1-form $\omega \in \Omega^{1,0}(C)$,
   - Lie 2-group (G, H, t, α) with cyclic structure $\langle\cdot,\cdot\rangle : \mathfrak{g} \otimes \mathfrak{h} \to \mathbb{C}$, and
   - isotropic Lie 2-subgroup $(G^\diamond, H^\diamond, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}}) \subseteq (G^{\hat{\boldsymbol{z}}}, H^{\hat{\boldsymbol{z}}}, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}})$.
2. Work in gauge $A_{\bar{z}} = 0$ and $B_{i\bar{z}} = 0$, so that the bulk equations of motion become $\omega \wedge \bar{\partial} A = 0$ and $\omega \wedge \bar{\partial} B = 0$. Choose admissible bulk solution (A, B) and solve boundary conditions for higher Lax connection:

$$^{(k,\kappa)}\boldsymbol{j}^*(A,B) \,\in\, \Omega^1(M, \mathfrak{g}^\diamond) \times \Omega^2(M, \mathfrak{h}^\diamond) \quad \overset{\text{solve}}{\rightsquigarrow} \quad (A,B) \,=\, \big(A(k,\kappa), B(k,\kappa)\big)$$

3. Inserting back into action yields 3d integrable field theory

$$S_{3d}(k,\kappa) \,= \frac{1}{2}\int_M \bigg(\Big\langle\!\Big\langle {}^{(k,\kappa)}\boldsymbol{j}^*(A), \alpha_*^{\hat{\boldsymbol{z}}}\big({}^{(k,\kappa)}\boldsymbol{j}^*(A), \kappa\big) + 2F_M(\kappa) \Big\rangle\!\Big\rangle_\omega + \Big\langle\!\Big\langle t_*^{\hat{\boldsymbol{z}}}(\kappa), \mathrm{d}_M\kappa + \tfrac{1}{3}[\kappa,\kappa] \Big\rangle\!\Big\rangle_\omega \bigg) \quad .$$

A recipe for constructing 3d integrable field theories

1. Fix input data of the 5d theory, i.e.
   - meromorphic 1-form $\omega \in \Omega^{1,0}(C)$,
   - Lie 2-group (G, H, t, α) with cyclic structure $\langle\cdot,\cdot\rangle : \mathfrak{g} \otimes \mathfrak{h} \to \mathbb{C}$, and
   - isotropic Lie 2-subgroup $(G^\diamond, H^\diamond, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}}) \subseteq (G^{\hat{\boldsymbol{z}}}, H^{\hat{\boldsymbol{z}}}, t^{\hat{\boldsymbol{z}}}, \alpha^{\hat{\boldsymbol{z}}})$.
2. Work in gauge $A_{\bar{z}} = 0$ and $B_{i\bar{z}} = 0$, so that the bulk equations of motion become $\omega \wedge \bar{\partial} A = 0$ and $\omega \wedge \bar{\partial} B = 0$. Choose admissible bulk solution (A, B) and solve boundary conditions for higher Lax connection:

$$ {}^{(k,\kappa)}\boldsymbol{j}^*(A,B) \in \Omega^1(M,\mathfrak{g}^\diamond) \times \Omega^2(M,\mathfrak{h}^\diamond) \quad \overset{\text{solve}}{\rightsquigarrow} \quad (A,B) = \big(A(k,\kappa), B(k,\kappa)\big) $$

3. Inserting back into action yields 3d integrable field theory

$$ S_{3d}(k,\kappa) = \frac{1}{2}\int_M \bigg(\Big\langle\!\Big\langle {}^{(k,\kappa)}\boldsymbol{j}^*(A), \alpha_*^{\hat{\boldsymbol{z}}}\big({}^{(k,\kappa)}\boldsymbol{j}^*(A),\kappa\big) + 2F_M(\kappa) \Big\rangle\!\Big\rangle_\omega + \Big\langle\!\Big\langle t_*^{\hat{\boldsymbol{z}}}(\kappa), \mathrm{d}_M\kappa + \tfrac{1}{3}[\kappa,\kappa] \Big\rangle\!\Big\rangle_\omega \bigg) \quad . $$

! By construction, the EOM from S_{3d} is equivalent to flatness of (A, B)!

Some first concrete examples with $C = \mathbb{C}P^1$

◇ **Example 1:** (3d Chern-Simons theory)

- Choose Lie 2-group $(G, G, \mathrm{id}, \mathrm{Ad})$ and

$$\omega = \tfrac{1-z}{z}\,\mathrm{d}z \quad , \quad G^{\diamond} = G \times \big(1_G \ltimes \mathfrak{g}\big) \quad , \quad H^{\diamond} = 1_G \times \big(1_G \ltimes \mathfrak{g}\big) \quad .$$

Some first concrete examples with $C = \mathbb{C}P^1$

◇ **Example 1:** (3d Chern-Simons theory)

- Choose Lie 2-group $(G, G, \mathrm{id}, \mathrm{Ad})$ and
$$\omega = \tfrac{1-z}{z}\,\mathrm{d}z \quad , \quad G^\diamond = G \times \big(1_G \ltimes \mathfrak{g}\big) \quad , \quad H^\diamond = 1_G \times \big(1_G \ltimes \mathfrak{g}\big) \quad .$$
- Gauge fixing the edge mode $(k, \kappa) = \big((1_G, (1_G, 0)), (0, (\alpha, 0))\big)$ gives
$$S_{3d}(\alpha) = -\int_M \big\langle \alpha, \tfrac{1}{2}\,\mathrm{d}_M \alpha + \tfrac{1}{3!}\,[\alpha, \alpha]\big\rangle \quad \text{with} \quad (A, B) = \Big(\alpha, \tfrac{z}{z-1}\,F_M(\alpha)\Big) \quad .$$

Some first concrete examples with $C = \mathbb{C}P^1$

◇ **Example 1:** (3d Chern-Simons theory)

- Choose Lie 2-group $(G, G, \mathrm{id}, \mathrm{Ad})$ and

$$\omega = \tfrac{1-z}{z}\,\mathrm{d}z \quad , \quad G^\diamond = G \times \big(1_G \ltimes \mathfrak{g}\big) \quad , \quad H^\diamond = 1_G \times \big(1_G \ltimes \mathfrak{g}\big) \quad .$$

- Gauge fixing the edge mode $(k,\kappa) = \big((1_G,(1_G,0)),(0,(\alpha,0))\big)$ gives

$$S_{3d}(\alpha) = -\int_M \big\langle \alpha, \tfrac{1}{2}\,\mathrm{d}_M\alpha + \tfrac{1}{3!}\,[\alpha,\alpha]\big\rangle \quad \text{with} \quad (A,B) = \Big(\alpha, \tfrac{z}{z-1}\,F_M(\alpha)\Big) \quad .$$

◇ **Example 2:** (3d Ward equation)

- Choose Lie 2-group $T[1]G = (G, \mathfrak{g}, 1_G, \mathrm{Ad})$ and

$$\omega = \frac{z\prod_{i=1}^{3}(z-a_i)}{(z-r)^2\,(z-s)^2}\,\mathrm{d}z \quad , \quad \begin{cases} G^\diamond = (1_G \ltimes \mathfrak{g}) \times (G \ltimes \mathfrak{g}) \times (1_G \ltimes \mathfrak{g}) & , \\ H^\diamond = (0 \times \mathfrak{g}) \times (0 \times 0) \times (0 \times \mathfrak{g}) & . \end{cases}$$

Some first concrete examples with $C = \mathbb{C}P^1$

◇ **Example 1:** (3d Chern-Simons theory)

- Choose Lie 2-group $(G, G, \mathrm{id}, \mathrm{Ad})$ and

$$\omega = \tfrac{1-z}{z}\,\mathrm{d}z \quad , \quad G^\diamond = G \times \big(1_G \ltimes \mathfrak{g}\big) \quad , \quad H^\diamond = 1_G \times \big(1_G \ltimes \mathfrak{g}\big) \quad .$$

- Gauge fixing the edge mode $(k,\kappa) = \big((1_G,(1_G,0)),(0,(\alpha,0))\big)$ gives

$$S_{3d}(\alpha) = -\int_M \langle \alpha, \tfrac{1}{2}\,\mathrm{d}_M\alpha + \tfrac{1}{3!}\,[\alpha,\alpha]\rangle \quad \text{with} \quad (A,B) = \Big(\alpha, \tfrac{z}{z-1}\,F_M(\alpha)\Big) \quad .$$

◇ **Example 2:** (3d Ward equation)

- Choose Lie 2-group $T[1]G = (G, \mathfrak{g}, 1_G, \mathrm{Ad})$ and

$$\omega = \frac{z\prod_{i=1}^3 (z-a_i)}{(z-r)^2\,(z-s)^2}\,\mathrm{d}z \quad , \quad \begin{cases} G^\diamond = (1_G \ltimes \mathfrak{g}) \times (G \ltimes \mathfrak{g}) \times (1_G \ltimes \mathfrak{g}) \quad , \\ H^\diamond = (0 \times \mathfrak{g}) \times (0 \times 0) \times (0 \times \mathfrak{g}) \quad . \end{cases}$$

- Gauge fixing the edge mode

$$k = \big((q,0),(1_G,0),(1_G,0)\big) \quad , \quad \kappa = \big((\alpha,0),(0,\beta),(\gamma,0)\big)$$

yields EOM which, for suitable 1-forms α,β,γ, generalizes Ward's equation

$$\big(\eta^{\mu\nu} + v_\rho\,\epsilon^{\rho\mu\nu}\big)\partial_\mu\big(q^{-1}\partial_\nu q\big) = 0 \quad \text{with} \quad \eta^{\mu\nu}\,v_\mu v_\nu = 1 \quad .$$