Quantum Correction of the Wilson Line and Entanglement Entropy in the AdS$_3$ Chern-Simons Gravity Theory

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Entanglement Entropy

- The entanglement entropy (EE) is

$$S_{EE} \equiv -\mathrm{Tr}_A(\rho_A \ln \rho_A), \tag{1}$$

where

$$\rho_A \equiv \mathrm{Tr}_B \rho_{AB} \tag{2}$$

is the reduced density matrix of the region A, obtained by the partial trace operation Tr_B acting on the density matrix of the region AB, ρ_{AB}.

n-sheet Manifold

$$S_{EE} = -\frac{\partial}{\partial n}\mathrm{Tr}_A \rho_A^n|_{n=1}. \tag{3}$$

We only need to calculate $\mathrm{Tr}_A \rho_A^n$, differentiate it with respect to n, and finally take the limit $n \to 1$. This is the procedure of the replica trick.

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$$\Psi\big(\phi_0(x)\big) = \int_{t_E=-\infty}^{\phi(t_E=0,x)=\phi_0(x)} D\phi \ e^{-S(\phi)}, \tag{4}$$

where $\phi(t_E, x)$ denotes the field. The value of the field at the boundary ϕ_0 depends on the spatial coordinate x.

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where $\phi(t_E, x)$ denotes the field. The value of the field at the boundary ϕ_0 depends on the spatial coordinate x. The density matrix ρ_{AB} is given by two copies of the wavefunctional

$$(\rho)_{\phi_0\phi_0'} = \Psi(\phi_0)\bar{\Psi}(\phi_0'). \tag{5}$$

The complex conjugate one $\bar{\Psi}$ can be obtained by path-integrating from $t_E = \infty$ to $t_E = 0$.

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$$\begin{aligned} & (\rho_A)_{\phi_+\phi_-} \\ = \; & (Z_1)^{-1} \int_{t_E=-\infty}^{t_E=\infty} D\phi \; e^{-S(\phi)} \prod_{x\in A} \\ & \times\delta\big(\phi(0^+,x) - \phi_+(x)\big) \cdot \delta\big(\phi(0^-,x) - \phi_-(x)\big), \end{aligned} \tag{6}$$

where Z_1 is the partition function.

To compute the $\mathrm{Tr}_A\rho_A^n$, we first prepare *n* copies of the reduced density matrix of the region A

$$(\rho_A)_{\phi_{1+}\phi_{1-}}(\rho_A)_{\phi_{2+}\phi_{2-}}\cdots(\rho_A)_{\phi_{n+}\phi_{n-}} \tag{7}$$

with the boundary condition

$$\phi_{j-}(x) = \phi_{(j+1)+}(x), \qquad j = 1, 2, \cdots, n, \tag{8}$$

where $\phi_{(n+1)+}(x) \equiv \phi_{1+}$, then integrating out ϕ_{j+} for each j, and then we take the trace.

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where $\phi_{(n+1)+}(x) \equiv \phi_{1+}$, then integrating out ϕ_{j+} for each j, and then we take the trace. The path-integral representation of the $\mathrm{Tr}_A \rho_A^n$ is:

$$\mathrm{Tr}_A \rho_A^n = (Z_1)^{-n} \int_{(t_E, x)\in\mathcal{R}_n} D\phi\ e^{-S(\phi)} \equiv \frac{Z_n}{Z_1^n}, \tag{9}$$

where $\mathcal{R}_n$ is the n-sheet manifold, and the Z_n is the n-sheet partition function.

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- Although we have the n-sheet method to avoid the conical singularity, the computation in quantum field theory is still hard.
- The holographic method used the minimum surface in the AdS$_d$ to obtain the EE in the CFT$_{d-1}$.
- The computation of the minimum surface is easier than the computation of the n-sheet method. Hence the holographic method gives a simple way to observe the exact solution in the EE.

AdS$_3$

The spacetime interval $\left(ds_3^2\right)$ of the AdS$_3$ metric $(g_{\mu\nu})$ is given by:

$$ds_3^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = -\frac{1}{\Lambda} \frac{dt^2 + dx^2 + dz^2}{z^2}, \tag{10}$$

where $\Lambda < 0$ is the cosmological constant and the spacetime indices are labeled by μ and ν. The AdS$_3$ induced metric $(h_{\mu\nu})$ is given by:

$$ds_{3b}^2 = h_{\mu\nu} dx^\mu dx^\nu = -\frac{1}{\Lambda} \frac{1}{z^2} \left[1 + \left(\frac{dz}{dx}\right)^2\right] dx^2 \tag{11}$$

by fixing time t as a constant. Hence the area of the surface is given by

$$A_{\mathrm{AdS}_3} = \sqrt{-\frac{1}{\Lambda}} \int dx \, \frac{1}{z} \sqrt{1 + \left(\frac{dz}{dx}\right)^2}. \tag{12}$$

The minimum area satisfies the relation:

$$\begin{aligned} \frac{d}{dx}\frac{\delta A_{\mathrm{AdS}_3}}{\delta z'} &= \frac{\delta A_{\mathrm{AdS}_3}}{\delta z}, \\ \frac{d}{dx}\left[\frac{\frac{dz}{dx}}{z}\frac{1}{\sqrt{1+\left(\frac{dz}{dx}\right)^2}}\right] &= -\frac{1}{z^2}\sqrt{1+\left(\frac{dz}{dx}\right)^2}, \end{aligned} \tag{13}$$

where $z' \equiv dz/dx$. One solution is:

$$z(x) = \sqrt{L^2 - x^2}, \qquad \frac{dz}{dx} = -\frac{x}{z}. \tag{14}$$

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Hence the minimum area is given by:

$$\begin{aligned} A_{\mathrm{AdS_3}} &= \sqrt{-\frac{1}{\Lambda}}\int_{-L+\delta}^{L-\delta} dx\, \frac{1}{z}\sqrt{1+\left(\frac{dz}{dx}\right)^2} \\ &= \sqrt{-\frac{1}{\Lambda}}\ln\frac{2L-\delta}{\delta}, \end{aligned} \tag{15}$$

in which we set $L \gg \delta > 0$.

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- Hence the minimum area is given by:

$$A_{\mathrm{AdS}_3} = \sqrt{-\frac{1}{\Lambda}} \ln \frac{2L-\delta}{\delta} = \sqrt{-\frac{1}{\Lambda}} \ln \frac{L + \sqrt{L^2 - \epsilon^2}}{L - \sqrt{L^2 - \epsilon^2}}. \tag{18}$$

The holographic EE for the AdS$_3$ metric is obtained from that:

$$\begin{aligned} \frac{A_{\mathrm{AdS_3}}}{4G_3} &= \frac{1}{4\sqrt{-\Lambda}G_3} \ln \frac{L+\sqrt{L^2-\epsilon^2}}{L-\sqrt{L^2-\epsilon^2}} = \frac{c_{\mathrm{cft_2}}}{6} \ln \frac{L+\sqrt{L^2-\epsilon^2}}{L-\sqrt{L^2-\epsilon^2}} \\ &= \frac{c_{\mathrm{cft_2}}}{6} \ln \frac{4L^2}{\epsilon^2} + \cdots = \frac{c_{\mathrm{cft_2}}}{3} \ln \frac{2L}{\epsilon} + \cdots \\ &= \frac{c_{\mathrm{cft_2}}}{3} \ln \frac{L}{\epsilon} + \cdots , \end{aligned} \tag{19}$$

where G_3 is the three-dimensional gravitational constant, and the center charge of CFT$_2$ is defined by

$$c_{\mathrm{cft_2}} \equiv \frac{3}{2\sqrt{-\Lambda}G_3} \tag{20}$$

Reference of the Holographic Entanglement Entropy

- C. Holzhey, F. Larsen and F. Wilczek, “Geometric and renormalized entropy in conformal field theory,” Nucl. Phys. B **424**, 443 (1994) [hep-th/9403108].
- S. Ryu and T. Takayanagi, “Holographic derivation of entanglement entropy from AdS/CFT,” Phys. Rev. Lett. **96**, 181602 (2006) [hep-th/0603001].
- A. Lewkowycz and J. Maldacena, “Generalized gravitational entropy,” JHEP **1308**, 090 (2013) [arXiv:1304.4926 [hep-th]].

Action

The action of the SL(2) Chern-Simons gravity theory is given by

$$\begin{aligned} & S_{\mathrm{G}} \\ = & \frac{k}{2\pi}\int d^3x\ \epsilon^{tr\theta}\mathrm{Tr}\bigg(A_tF_{r\theta}-\frac{1}{2}\big(A_r\partial_tA_\theta-A_\theta\partial_tA_r\big)\bigg) \\ & -\frac{k}{2\pi}\int d^3x\ \epsilon^{tr\theta}\mathrm{Tr}\bigg(\bar{A}_t\bar{F}_{r\theta}-\frac{1}{2}\big(\bar{A}_r\partial_t\bar{A}_\theta-\bar{A}_\theta\partial_t\bar{A}_r\big)\bigg) \\ & -\frac{k}{4\pi}\int dtd\theta\ \mathrm{Tr}(A_\theta^2) \\ & -\frac{k}{4\pi}\int dtd\theta\ \mathrm{Tr}(\bar{A}_\theta^2), \end{aligned} \tag{21}$$

in which we assume that the boundary conditions of the gauge fields A and $\bar{A}$ are: $A_- \equiv A_t - A_\theta = 0$ and $\bar{A}_+ = A_t + A_\theta = 0$. The variable k is defined by $l/(4G_3)$, where $1/l^2 \equiv -\Lambda$.

The gauge fields are defined by the vielbein e_μ and spin connection ω_μ:

$$A_\mu \equiv A_\mu^a J_a \equiv J_a \left(\frac{1}{l} e_\mu^a + \omega_\mu^a \right), \qquad \bar{A}_\nu \equiv \bar{A}_\nu^a \bar{J}_a \equiv \bar{J}_a \left(\frac{1}{l} e_\nu^a - \omega_\nu^a \right), \tag{22}$$

in which the Lie algebra indices are labeled by a, and the indices are raised or lowered by $\eta \equiv \mathrm{diag}(-1,1,1)$. This bulk terms in this theory are equivalent to the Chern-Simons theory up to a boundary term. The measure in this gravitation theory is $\int \mathcal{D}A\mathcal{D}\bar{A}$.

Boundary Theory

When we take the solution ($F_{r\theta} = 0$) into the action, and use the asymptotic boundary condition: $g^{-1}_{\mathrm{SL}(2)}\partial_\theta g_{\mathrm{SL}(2)}|_{r\to\infty} = A_\theta|_{r\to\infty}$ and $\bar{g}^{-1}_{\mathrm{SL}(2)}\partial_\theta \bar{g}_{\mathrm{SL}(2)}|_{r\to\infty} = \bar{A}_\theta|_{r\to\infty}$. We use the SL(2) transformations:

$$\begin{aligned} g_{\mathrm{SL}(2)} &= \begin{pmatrix} 1 & 0 \\ F & 1 \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{pmatrix} \begin{pmatrix} 1 & \Psi \\ 0 & 1 \end{pmatrix}, \\ \bar{g}_{\mathrm{SL}(2)} &= \begin{pmatrix} 1 & -\bar{F} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\bar{\lambda}} & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{\Psi} & 1 \end{pmatrix} \end{aligned} \tag{23}$$

to obtain the boundary conditions: $\lambda^2\partial_\theta F = 2r$, $\partial^2_\theta F/\partial_\theta F = -4r\Psi$, $\bar{\lambda}^2\partial_\theta\bar{F} = 2r$, and $\partial^2_\theta\bar{F}/\partial_\theta\bar{F} = -4r\bar{\Psi}$.

Finally, we obtain the boundary theory, two-dimensional Schwarzian theory

$$S_{\mathrm{G}} = \frac{k}{2\pi}\int dtd\theta\,\left(\frac{3}{2}\frac{(\partial_-\partial_\theta F)(\partial_\theta^2 F)}{(\partial_\theta F)^2} - \frac{\partial_-\partial_\theta^2 F}{\partial_\theta F}\right) - \frac{k}{2\pi}\int dtd\theta\,\left(\frac{3}{2}\frac{(\partial_+\partial_\theta \bar{F})(\partial_\theta^2 \bar{F})}{(\partial_\theta \bar{F})^2} - \frac{\partial_+\partial_\theta^2 \bar{F}}{\partial_\theta \bar{F}}\right), \qquad (24)$$

where

$$x^+ \equiv t+\theta, \qquad x^- \equiv t-\theta, \qquad (25)$$

$$\partial_+ = \frac{1}{2}\partial_t + \frac{1}{2}\partial_\theta, \qquad \partial_- = \frac{1}{2}\partial_t - \frac{1}{2}\partial_\theta. \qquad (26)$$

The measure is $\int dFd\bar{F}\,\left(1/(\partial_\theta F\partial_\theta\bar{F})\right)$.

Reference of the AdS$_3$ Chern-Simons Gravity Theory

- E. Witten, "(2+1)-Dimensional Gravity as an Exactly Soluble System," Nucl. Phys. B **311**, 46 (1988).
- E. Witten, "Three-Dimensional Gravity Revisited," arXiv:0706.3359 [hep-th].
- O. Coussaert, M. Henneaux and P. van Driel, "The Asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant," Class. Quant. Grav. **12**, 2961 (1995) [gr-qc/9506019].
- J. Cotler and K. Jensen, "A theory of reparameterizations for AdS$_3$ gravity," JHEP **1902**, 079 (2019) [arXiv:1808.03263 [hep-th]].

Boundary Effective Action on the Sphere Manifold

The bulk Euclidean AdS$_3$ metric can be asymptotically written as

$$ds^2_{3a} = r^2 ds^2_{\mathrm{s}} + \frac{dr^2}{r^2}, \tag{27}$$

where

$$ds^2_{\mathrm{s}} = d\psi^2 + \sin^2\psi d\theta^2, \qquad 0 \le \psi < \pi, \qquad 0 \le \theta < 2\pi. \tag{28}$$

The ds^2_{s} is the spacetime interval for the sphere with a unit radius.

The asymptotic behavior of the gauge fields are:

$$A = \begin{pmatrix} \frac{dr}{2r} & 0 \\ rE^{+} & -\frac{dr}{2r} \end{pmatrix}, \qquad \bar{A} = \begin{pmatrix} -\frac{dr}{2r} & -rE^{-} \\ 0 & \frac{dr}{2r} \end{pmatrix}, \tag{29}$$

where

$$E^{+} \equiv E^{\theta} + E^{t}, \qquad E^{-} \equiv E^{\theta} - E^{t}. \tag{30}$$

The $E^{\pm}$ is the boundary zweibein. Then we can find the below boundary condition by

$$\lambda = \sqrt{\frac{2rE_{\theta}^{+}}{\partial_{\theta}F}}, \qquad \Psi = -\frac{1}{4rE_{\theta}^{+}}\frac{\partial_{\theta}^{2}F}{\partial_{\theta}F},$$
$$\bar{\lambda} = \sqrt{\frac{2rE_{\theta}^{-}}{\partial_{\theta}\bar{F}}}, \qquad \bar{\Psi} = -\frac{1}{4rE_{\theta}^{-}}\frac{\partial_{\theta}^{2}\bar{F}}{\partial_{\theta}\bar{F}}. \tag{31}$$

For the sphere manifold, we have

$$E^{\psi}=d\psi,\qquad E^{\theta}=\sin\psi d\theta. \tag{32}$$

Because we did the Wick rotation ($t=-i\psi$), we use the following coordinates:

$$\begin{aligned} x^{+} &= -i\psi+\theta,\qquad x^{-}=-i\psi-\theta,\\ \psi &= \frac{i}{2}(x^{+}+x^{-}),\qquad \theta=\frac{x^{+}-x^{-}}{2}. \end{aligned} \tag{33}$$

The θ-component of the boundary zweibein is defined by the $E_{\theta}^{\pm}$. Therefore, we have $E_{\theta}^{+}=E_{\theta}^{-}=\sin\psi$. The boundary gauge-field in the Lorentzian manifold satisfies the conditions:

$$E_{\theta}^{+}A^{t}-E_{t}^{+}A^{\theta}=0,\qquad E_{\theta}^{-}\bar{A}^{t}-E_{t}^{-}\bar{A}^{\theta}=0. \tag{34}$$

Therefore, the AdS$_3$ gravitation action with the sphere asymptotic boundary condition is

$$\begin{aligned} & S_{\mathrm{GS}} \\ = \quad & \frac{k}{2\pi}\int d^3x\ \epsilon^{tr\theta}\mathrm{Tr}\Big(A_t F_{r\theta} - \frac{1}{2}\big(A_r\partial_t A_\theta - A_\theta\partial_t A_r\big)\Big) \\ & -\frac{k}{2\pi}\int d^3x\ \epsilon^{tr\theta}\mathrm{Tr}\Big(\bar{A}_t \bar{F}_{r\theta} - \frac{1}{2}\big(\bar{A}_r\partial_t \bar{A}_\theta - \bar{A}_\theta\partial_t \bar{A}_r\big)\Big) \\ & +\frac{k}{4\pi}\int dtd\theta\ \mathrm{Tr}\Big(\frac{E_t^+}{E_\theta^+}A_\theta^2\Big) \\ & -\frac{k}{4\pi}\int dtd\theta\ \mathrm{Tr}\Big(\frac{E_t^-}{E_\theta^-}\bar{A}_\theta^2\Big). \end{aligned} \tag{35}$$

Then we use the conditions $\lambda^2\partial_\theta F = 2E_\theta^+ r$ and $\bar{\lambda}^2\partial_\theta\bar{F} = 2E_\theta^- r$ to obtain the boundary effective action on the sphere manifold

$$S_{\mathrm{GS}} = \frac{k}{\pi}\int dtd\theta\,\left(\frac{(\partial_\theta\lambda)(D_-\lambda)}{\lambda^2} - \frac{(\partial_\theta\bar{\lambda})(D_+\bar{\lambda})}{\bar{\lambda}^2}\right), \qquad (36)$$

where

$$D_+ \equiv \frac{1}{2}\partial_t + \frac{1}{2}\frac{E_t^-}{E_\theta^-}\partial_\theta, \qquad D_- \equiv \frac{1}{2}\partial_t + \frac{1}{2}\frac{E_t^+}{E_\theta^+}\partial_\theta. \qquad (37)$$

From the field redefinition:

$$\mathcal{F} \equiv \frac{F}{E_\theta^+}, \qquad \bar{\mathcal{F}} \equiv \frac{\bar{F}}{E_\theta^-}. \tag{38}$$

the gravitation action on the sphere manifold becomes:

$$\begin{aligned} & S_{\mathrm{GS}} \\ = \;& \frac{k}{4\pi}\int dtd\theta\,\left(\frac{(\partial_\theta^2\mathcal{F})(D_-\partial_\theta\mathcal{F})}{(\partial_\theta\mathcal{F})^2}-\frac{(\partial_\theta^2\bar{\mathcal{F}})(D_+\partial_\theta\bar{\mathcal{F}})}{(\partial_\theta\bar{\mathcal{F}})^2}\right) \\ = \;& \frac{k}{4\pi}\int dtd\theta\,\left[\frac{(\partial_\theta^2\phi)(D_-\partial_\theta\phi)}{(\partial_\theta\phi)^2}-(\partial_\theta\phi)(D_-\phi)\right] \\ & -\frac{k}{4\pi}\int dtd\theta\,\left[\frac{(\partial_\theta^2\bar{\phi})(D_+\partial_\theta\bar{\phi})}{(\partial_\theta\bar{\phi})^2}-(\partial_\theta\bar{\phi})(D_+\bar{\phi})\right], \end{aligned} \tag{39}$$

in which we used

$$\mathcal{F} \equiv \tan\left(\frac{\phi}{2}\right), \qquad \bar{\mathcal{F}} \equiv \tan\left(\frac{\bar{\phi}}{2}\right). \tag{40}$$

When we take the scale transformation on the the boundary zweibein, this theory is invariant. Therefore, we can use the conformal transformation to compute the EE as in the CFT.

EE for One-Interval

We first perform the coordinate transformation to get $ds_s^2 = \text{sech}^2(y)(dy^2 + d\theta^2)$, in which we used $\text{sech}\, y = \sin\psi$. In the n-sheet manifold, the range of the θ is $0 < \theta \leq 2\pi n$. The periodicity of this theory with respect to the θ is $2\pi n$. When we do the computation, we need to regularize the range of the y-direction. The range of the y-direction is $-\ln(L/\epsilon) < y \leq \ln(L/\epsilon)$. The periodicity of this theory with respect to the y is $4\ln(L/\epsilon)$ because we assume the Dirichlet boundary condition in the y-direction. The L is the length of an interval, and ϵ is the cut-off on the ending point of the interval.

Finally, we identify the sphere from the torus to determine the complex structure τ on the sphere. The coordinates of torus $z \equiv (\theta + iy)/n$ satisfy the identification: $z \sim z + 2\pi$ and $z \sim z + 2\pi\tau$. The boundary condition of the fields, ϕ and $\bar{\phi}$ is given by

$$\begin{aligned} \phi(y/n, \theta/n + 2\pi) &= \phi(y/n, \theta/n) + 2\pi, \\ \phi\big(y/n + 2\pi \cdot \mathrm{Im}(\tau), \theta/n + 2\pi \cdot \mathrm{Re}(\tau)\big) &= \phi(y/n, \theta/n), \\ \bar{\phi}(y/n, \theta/n + 2\pi) &= \bar{\phi}(y/n, \theta/n) + 2\pi, \\ \bar{\phi}\big(y/n + 2\pi \cdot \mathrm{Im}(\tau), \theta/n + 2\pi \cdot \mathrm{Re}(\tau)\big) &= \bar{\phi}(y/n, \theta/n). \end{aligned} \tag{41}$$

Therefore, we can quickly find that the complex structure on the sphere is

$$\tau = \frac{2i}{n\pi} \ln \frac{L}{\epsilon}. \tag{42}$$

When we take this complex structure, we can obtain the periodicity $4\ln(L/\epsilon)$. The fields on the sphere can be expanded from the way:

$$\phi = \frac{\theta}{n} + \epsilon(y,\theta), \qquad \bar{\phi} = \frac{\theta}{n} + \bar{\epsilon}(y,\theta), \tag{43}$$

where

$$\begin{aligned} \epsilon(y,\theta) &\equiv \sum_{j,k} \epsilon_{j,k} e^{i\frac{j}{n}\theta - \frac{k}{\tau}y}, \qquad \epsilon^*_{j,k} \equiv \epsilon_{-j,-k}, \\ \bar{\epsilon}(y,\theta) &\equiv \sum_{j,k} \bar{\epsilon}_{j,k} e^{i\frac{j}{n}\theta - \frac{k}{\tau}y}, \qquad \bar{\epsilon}^*_{j,k} \equiv \bar{\epsilon}_{-j,-k}. \end{aligned} \tag{44}$$

Because this theory has the SL(2) redundancy, the variables has the constraints:

$$\epsilon_{j,k} = 0, \qquad \bar{\epsilon}_{j,k} = 0 \qquad \text{when } j = -1, 0, 1. \tag{45}$$

To compute the partition function on the sphere, we need to do the Wick rotation $t = -i\psi$. Now we consider the expansion from the $\epsilon(y,\theta)$ and $\bar{\epsilon}(y,\theta)$ to obtain the one-loop effect. Therefore, we obtain the Rényi entropy

$$S_n = \frac{(c+26)(n+1)}{6n} \ln \frac{L}{\epsilon} \tag{46}$$

and the entanglement entropy is

$$S_{EE} = \frac{c+26}{3} \ln \frac{L}{\epsilon}. \tag{47}$$

Wilson Line

The EE in the two-dimensional Schwarzian theory gives the non-conformal deformation from the quantum correction. Here we want to obtain a bulk description of the EE. Since the Wilson lines

$$W(P,Q) \equiv \mathrm{Tr}\Big[\mathcal{P}\exp\Big(\int_Q^P \bar{A}\Big)\mathcal{P}\exp\Big(\int_Q^P A\Big)\Big], \tag{48}$$

can provide the EE in the CFT$_2$, we begin from this operator to study. The $\mathcal{P}$ denotes the path ordering, P and Q are the two-ending points of the Wilson lines at a time slice. Here the trace operation acts on the representation, which has the Casimir (c_2) $\sqrt{2c_2} = c(1/n-1)/6$.

We extend the Wilson line to the following form

$$\begin{aligned} & W_{\mathcal{R}}(C) \\ = & \int DUDPD\lambda \\ & \times \exp\left[\int_C ds\ \left(\mathrm{Tr}(PU^{-1}D_sU) + \lambda(s)\big(\mathrm{Tr}(P^2) - c_2\big)\right)\right], \end{aligned} \tag{49}$$

where U is an SL(2) element, P is its conjugate momentum, and the covariant derivative is defined as that:

$$D_sU \equiv \frac{d}{ds}U + A_sU + U\bar{A}_s, \qquad A_s \equiv A_\mu \frac{dx^\mu}{ds}. \tag{50}$$

The equations of motion are

$$
\begin{aligned}
i\frac{k}{2\pi}F_{\mu_1\mu_2} &= -\int ds\ \frac{dx^{\mu_3}}{ds}\epsilon_{\mu_1\mu_2\mu_3}\delta^3\big(x-x(s)\big)\,UPU^{-1}, \\
i\frac{k}{2\pi}\bar{F}_{\mu_1\mu_2} &= \int ds\ \frac{dx^{\mu_3}}{ds}\epsilon_{\mu_1\mu_2\mu_3}\delta^3\big(x-x(s)\big)\,P.
\end{aligned} \tag{51}
$$

The solution can be expressed as that:

$$\begin{aligned} A &= L^{-1}aL + LdL^{-1}, \qquad L = \exp(-\rho L_0)\exp(-L_1 z), \\ \bar{A} &= -R^{-1}aR - R^{-1}dR, \qquad R = \exp(-L_{-1}\bar{z})\exp(-\rho L_0), \end{aligned} \tag{52}$$

where the gauge fields are given as that:

$$a = \sqrt{\frac{c_2}{2}}\frac{1}{k}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right)L_0, \tag{53}$$

The SL(2) algebra is defined by that:

$$[L_m, L_n] = (n - m)L_{m+n}, \qquad m, n = 0, \pm 1, \tag{54}$$

$$\mathrm{Tr}(L_0^2) = \frac{1}{2}, \qquad \mathrm{Tr}(L_{-1}L_1) = -1, \tag{55}$$

and the traces of other bilinears vanish. Here we choose $z = r\exp(i\theta)$. Then the spacetime interval is

$$ds_3^2 = d\rho^2 + \exp(2\rho)(dr^2 + n^2 r^2 d\theta^2). \tag{56}$$

With the $r = \exp(t)$ and a scale transformation, the n-sheet cylinder appears at the boundary ($\rho \to \infty$). This solution corresponds to

$$U = 1, \qquad P = \sqrt{2c_2}L_0 \tag{57}$$

with the curve

$$\rho(s) = s, \qquad z(s) = 0. \tag{58}$$

Reference of the Solution

- M. Ammon, A. Castro and N. Iqbal, "Wilson Lines and Entanglement Entropy in Higher Spin Gravity," JHEP **1310**, 110 (2013) doi:10.1007/JHEP10(2013)110 [arXiv:1306.4338 [hep-th]].

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- Since we should choose the smooth fluctuation, we still obtain the two-dimensional Schwarzian theory by integrating out the time-component gauge fields. The n-sheet geometry can be used in the smooth region $\rho \neq 0$. Hence computing the Wilson line $W_{\mathcal{R}}$ in the Chern-Simons gravity theory is equivalent to computing the Z_n/Z_1^n in the two-dimensional Schwarzian theory.
- In other words, the entanglement entropy is

$$S_{EE} = -\lim_{n\to 1}\frac{1}{1-n}\ln\langle W_{\mathcal{R}}\rangle, \tag{59}$$

where $\langle W_{\mathcal{R}}\rangle$ is the expectation value of the Wilson line.

- When we take the classical solution into the Wilson line, the EE gives the CFT$_2$ result. This implies that the Wilson line can be seen as the geodesic line at the on-shell level. The equivalence between the Wilson line and the EE is exact, not only restricted to the one-loop order. Hence the Wilson line can be seen as the suitable operator for the quantum deformation of the minimum surface.

- We compute the EE at the one-loop order in the boundary theory. This gives the non-CFT effect from the one-loop correction.

- We compute the EE at the one-loop order in the boundary theory. This gives the non-CFT effect from the one-loop correction.
- We show that the Wilson line is the suitable operator for doing the quantum deformation of the minimum surface. This result shows the AdS/non-CFT correspondence in the EE and also the interesting proposal, "Minimum Surface=EE".