

# Resurgence and renormalons in QFT

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# Perturbative and non-perturbative

Perturbation theory is one of the few universal tools we have in quantum physics. It allow us to compute e.g. the ground state energy of an interacting physical system as a series in the coupling constant

$$E(g) = \sum_{n \geq 0} a_n g^n$$

There are however “non-perturbative” effects which are supposed to go beyond perturbation theory. They are typically exponentially small in the coupling constant. A famous example is the superconducting energy gap

$$e^{-A/g}$$

The relation between perturbative and non-perturbative physics turns out to be quite subtle. First of all, why perturbation theory should be insufficient?

One reason is that perturbative series are not well-defined functions, since their coefficients grow in general **factorially**

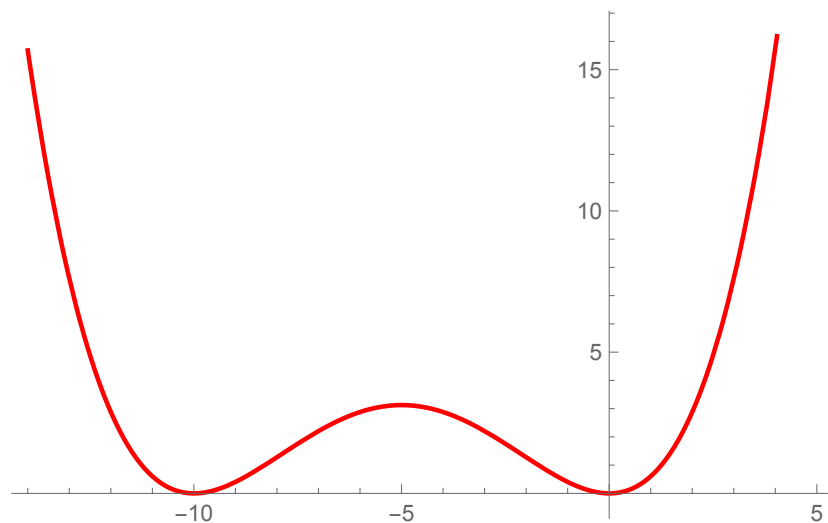
$$E(g) = \sum_{n \geq 0} a_n g^n \quad a_n \sim n!$$

Series with factorial growth have zero radius of convergence.

Sometimes one can sum up the very first terms to obtain reasonable approximations (the so-called optimal truncation), but further knowledge of the coefficients does **not** improve things

One could suspect that this factorial growth is connected to non-perturbative effects. This was pointed out by Dyson as far back as 1952. Dyson's intuition was made quantitatively precise by Lam and Bender-Wu in the early 1970s.

To illustrate this idea, let us consider one of the simplest examples of a non-perturbative effect in quantum theory: the double well potential

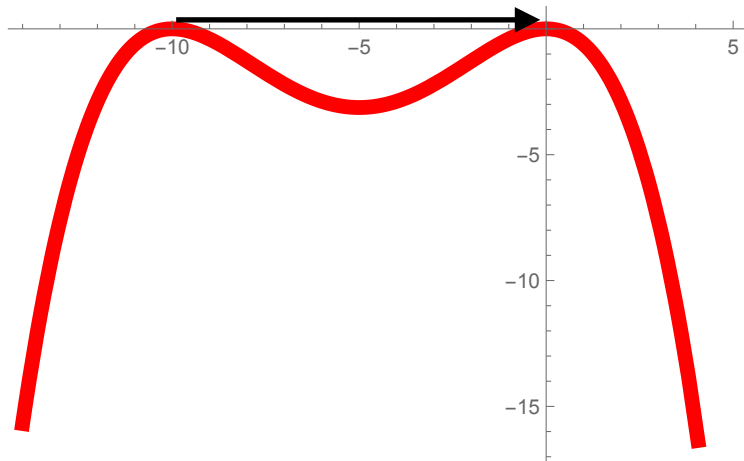


$$V(x) = \frac{1}{2}x^2(1 + gx)^2$$

$$E_0(g) = \sum_{k \geq 0} a_k g^{2k} = \frac{1}{2} - g^2 - \frac{9g^4}{2} - \frac{89g^6}{2} - \frac{5013g^8}{8} - \dots$$

Tunneling or instanton effects lead to an energy gap between the ground state and the first excited state which is exponentially small in  $g$ , i.e. non-perturbative

$$\Delta E \sim \frac{1}{g} e^{-A/g^2} \qquad A = \frac{1}{6}$$



$A$ =action of an instanton

It turns out that the characteristic scale of this gap governs the large order behavior of the perturbative series for the ground state energy

$$a_n \sim (2A)^{-n} \Gamma(n+1)$$



exponential

Conversely, one could characterize the gap by looking at the large order behavior of this series!

Therefore, perturbative and non-perturbative information are **not** independent. Non-perturbative effects “**resurge**” in the large order behavior of the perturbative series

A crucial question is: how do we put **both** effects (P/NP) together?

$$\text{physical observable} = \text{perturbative series} + \text{exponentially small corrections}$$

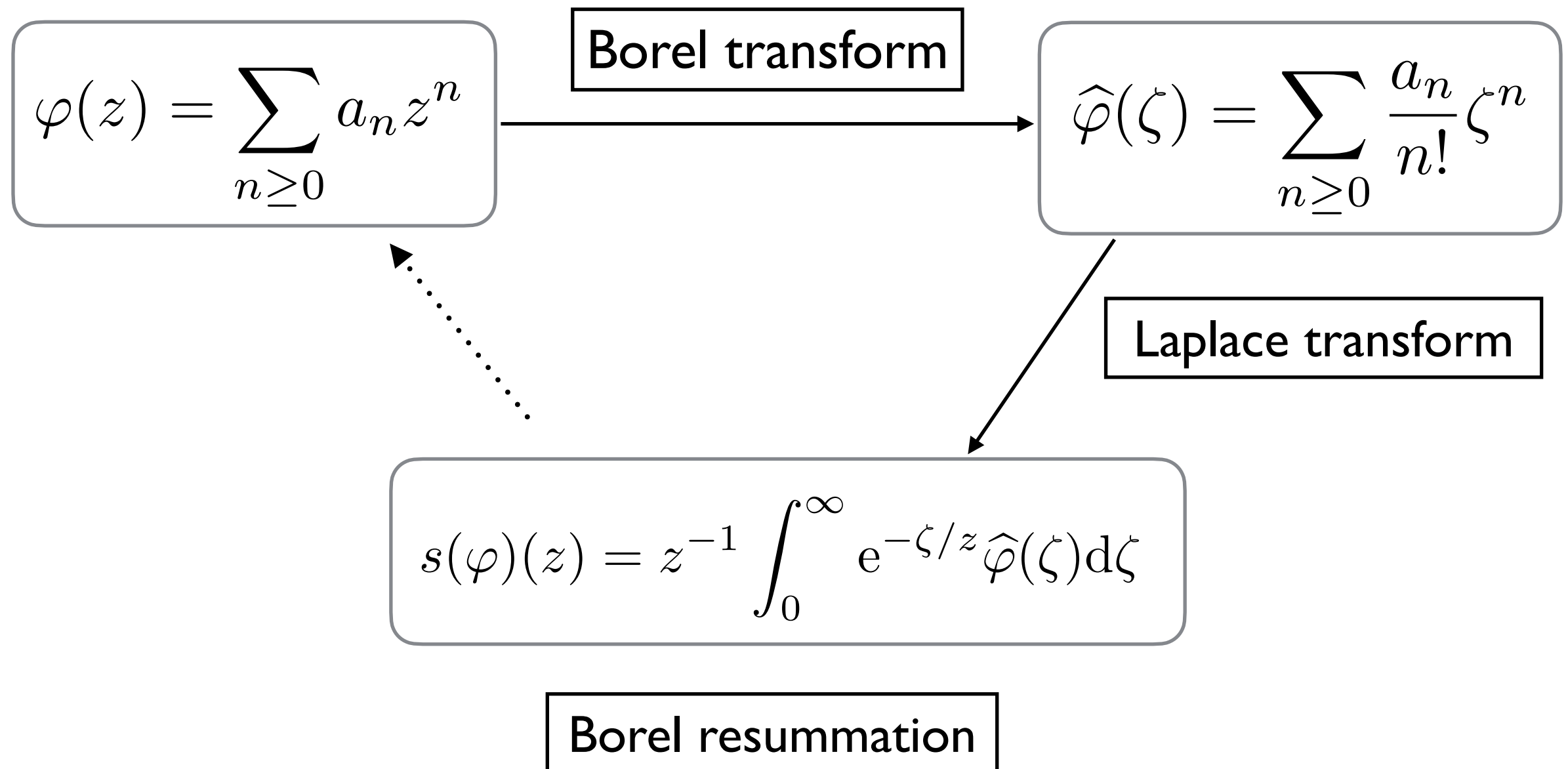
Making sense of this is very subtle (and has led to many confusions in the literature). The reason is that **the strength of an exponentially small correction is ill-defined unless we choose a resummation prescription for the perturbative series**

[WKB: Dingle, Voros, Silverstone; QFT: Parisi, David]

Therefore, it is mandatory to use resummations

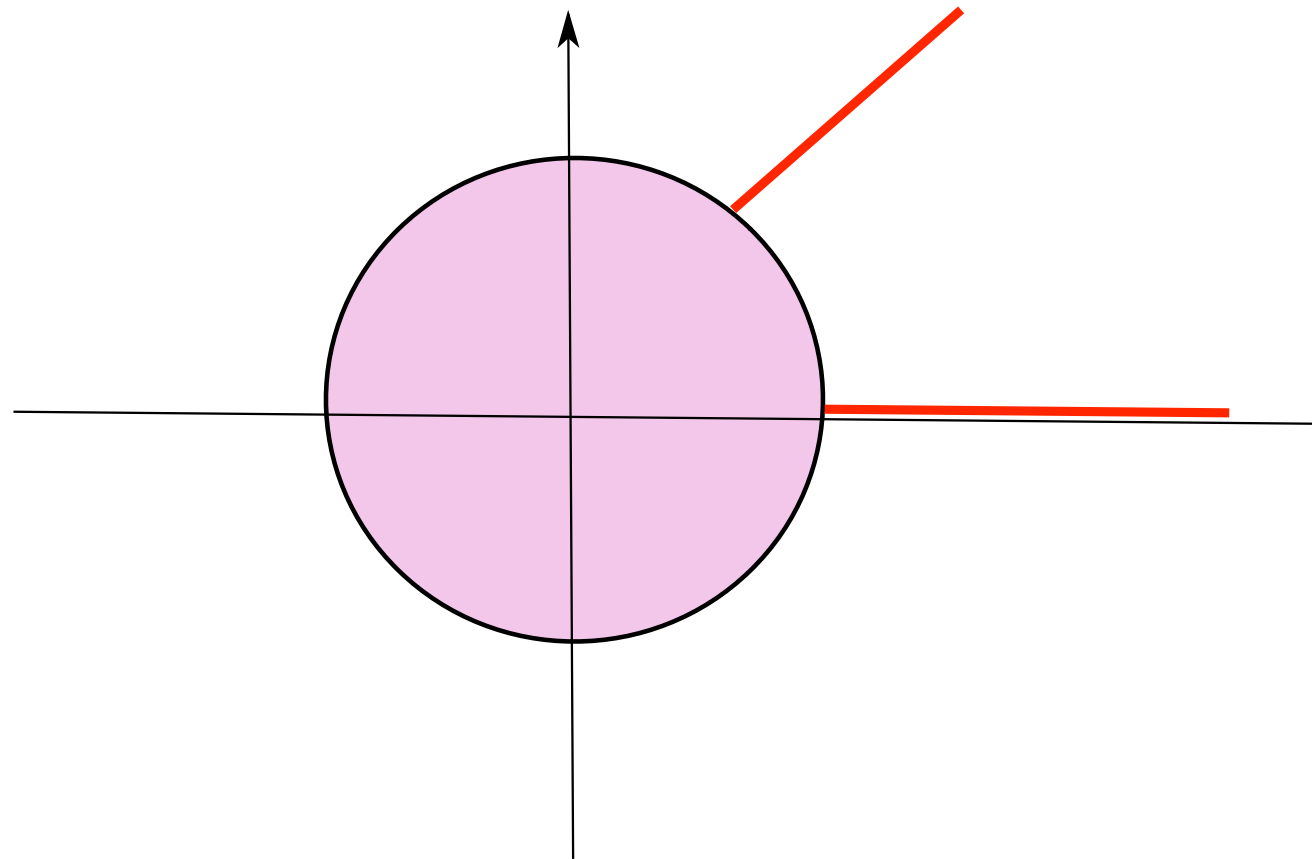
# The Borel triangle

The Borel method is a systematic way of making sense of factorially divergent formal power series



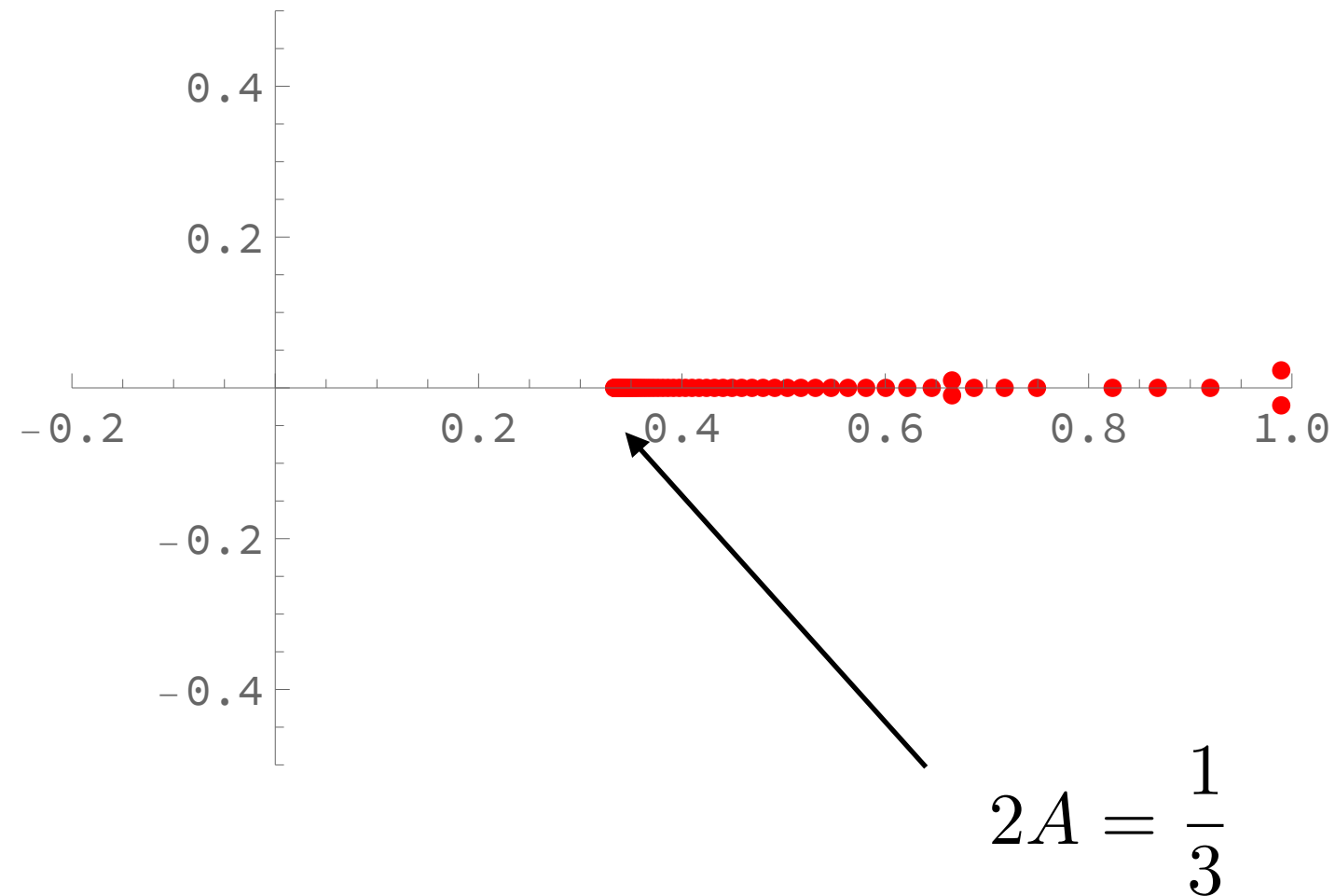


The Borel transform  $\hat{\varphi}(\zeta)$  is **analytic** at the origin. Very often it can be analytically continued to the complex plane, displaying **singularities** (poles, branch cuts).



In addition, the location of the singularities in the Borel plane gives us information about the non-perturbative effects

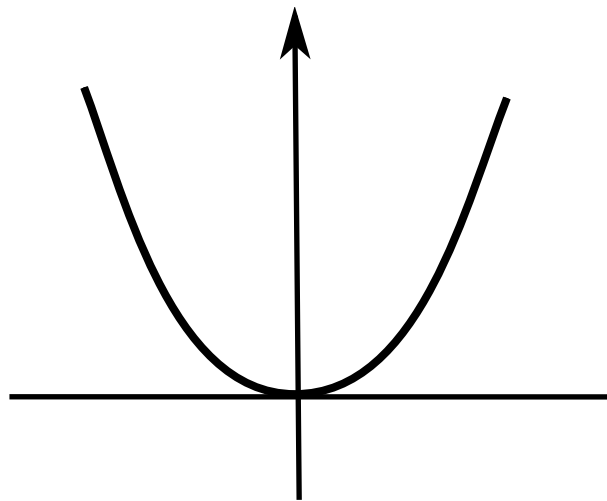
In the case of the double-well potential one finds



This uses the Bender-Wu code [Sulejmanpasic-Unsal] to calculate many terms of the series

$$E_0(g)$$

In some cases Borel resummation works remarkably well, as in the energies of the quartic oscillator in quantum mechanics



$$H = \frac{p^2}{2} + \frac{x^2}{2} + gx^4$$

$$E_0(g) \sim \frac{1}{2} + \frac{3}{4}g - \frac{21}{8}g^2 + \dots$$

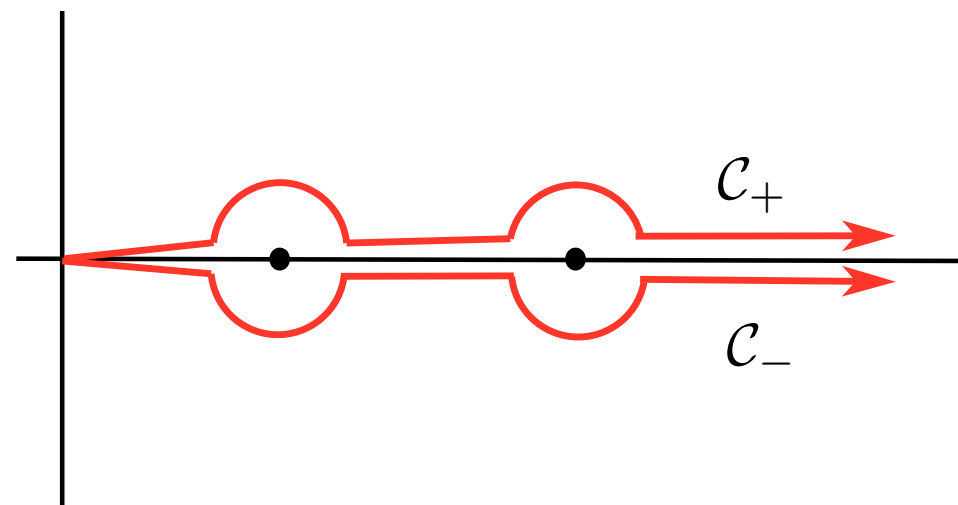
Here, Borel resummation of the perturbative series in the coupling constant reproduces the **exact** spectrum (level by level)

However, if the Borel transform has singularities along the positive real axis, as in the double-well potential, the integral giving the resummation is ill-defined, and the series is not Borel summable.

One can however perform so-called **lateral Borel resummations** by deforming the integration contour

$$s_{\pm}(\varphi)(z) = z^{-1} \int_{c_{\pm}} e^{-\zeta/z} \widehat{\varphi}(\zeta) d\zeta$$

Borel plane



If e.g. the Borel transform has a simple pole at  $\zeta = A$ , the two lateral resummations differ by an imaginary, **exponentially small quantity**

$$s_+(\varphi)(z) - s_-(\varphi)(z) = \frac{2\pi i}{z} e^{-A/z}$$

This is an example of the **non-perturbative ambiguity** in non-Borel summable theories. Note that singularities in the Borel plane lead to exponentially small corrections!

To solve this ambiguity, we have to enlarge the original data and consider **trans-series**

# Trans-series

We have to incorporate the exponentially small terms from the very beginning:

$$\Phi(z) = \varphi(z) + \sum_{\ell=1}^{\infty} C_{\ell} e^{-\ell A/z} \varphi_{\ell}(z)$$

↑  
trans-series parameters

$$\varphi(z) = \sum_{n \geq 0} a_n z^n$$

$$\varphi_{\ell}(z) = z^{-b_{\ell}} \sum_{n=0}^{\infty} a_{\ell,n} z^n$$

factorially  
divergent formal  
power series

Physically, this corresponds to adding non-perturbative sectors

# The double-well, revisited

The perturbative series for the ground state energy is **not** Borel summable. If we do lateral resummations, we obtain a physically unacceptable imaginary piece (exponentially small in the coupling)

To solve this problem, we have to promote this series to a trans-series [Zinn-Justin, Alvarez, ...]:


$$E_0(g) + E^{1\text{-inst}}(g) + E^{2\text{-inst}}(g) + \dots$$

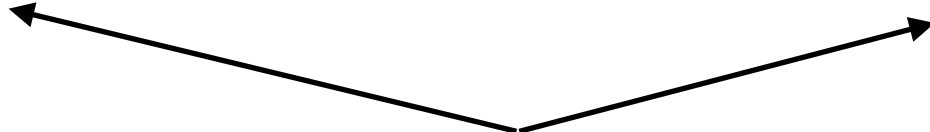
$$E^{1\text{-inst}} = -\frac{1}{\sqrt{\pi}g} e^{-\frac{1}{6g^2}} \left( 1 - \frac{71g^2}{12} - \frac{6299g^4}{288} - \frac{2691107g^6}{10368} - \dots \right)$$

$$E^{2\text{-inst}} = \frac{i}{g^2} e^{-\frac{1}{3g^2}} \left( 1 - \frac{53g^2}{6} - \frac{1277g^4}{72} - \frac{336437g^6}{1296} - \dots \right) + \text{real}$$

Physically, this trans-series indicates the presence of **multi-instanton sectors**, or equivalently, contributions from tunneling

The **exact** energy is given by the lateral resummation of the trans-series:



$$s_{\pm}(E_0)(g) + s_{\pm}(E^{1\text{-inst}})(g) \pm s_{\pm}(E^{2\text{-inst}})(g) + \dots$$


This turns out to be real, as it should, since e.g. an **explicitly complex** two-instanton amplitude **cancels** the imaginary part of the lateral resummation! [Voros, Bogomolny, Zinn-Justin...]



The research program of resurgence is based on the  
idea that

**in a quantum theory, observables which have an asymptotic expansion can be obtained exactly as (lateral) Borel resummations of trans-series**

A stronger version of this program is that

**all series in the trans-series can be obtained from the Borel singularities of the perturbative series**

# Where do trans-series come from?

As in quantum mechanics, a typical source of trans-series are expansions around different saddle points (i.e. **instanton sectors**) of the path integral.

Resurgence implies that these instanton sectors control the behavior of perturbation theory at large order. It is believed that instantons encode the factorial growth of perturbation theory due to the growth in the **total number of diagrams** [Bender-Wu]

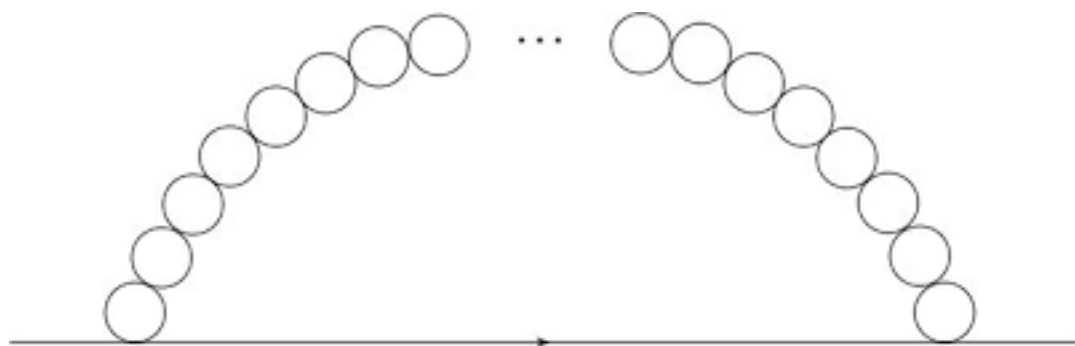
For a while, people had the hope that trans-series in QFTs would come just from perturbative and instanton sectors. Sometimes this is the case, like the  $\lambda\Phi^4$  theory in two dimensions [Serone et al.]

This “**instanton dominance**” seems to hold as well in some string theories, where instanton sectors correspond to D-branes, and control the large order behavior of string perturbation theory [Shenker,...]

# Renormalons

However, the dream of instanton dominance was shattered in the 1970s-1980s by the discovery of a new and mysterious source of singularities in the Borel plane: **renormalon singularities**

One manifestation of these singularities are bubble diagrams in renormalizable theories, which grow factorially due to the integration over momenta. This is a different source of growth than the overall proliferation of diagrams!



$$\int_0^1 (-\log(k))^n dk = n!$$



Parisi argued that in asymptotically free (AF) theories at infinite volume there are Borel singularities at

$$\zeta = \frac{2d}{|\beta_0|} \quad d = 1, 2, \dots$$

They obstruct Borel summability and lead to exponentially small non-perturbative ambiguities.

(note however that we have recently found evidence for additional singularities in the Borel plane)

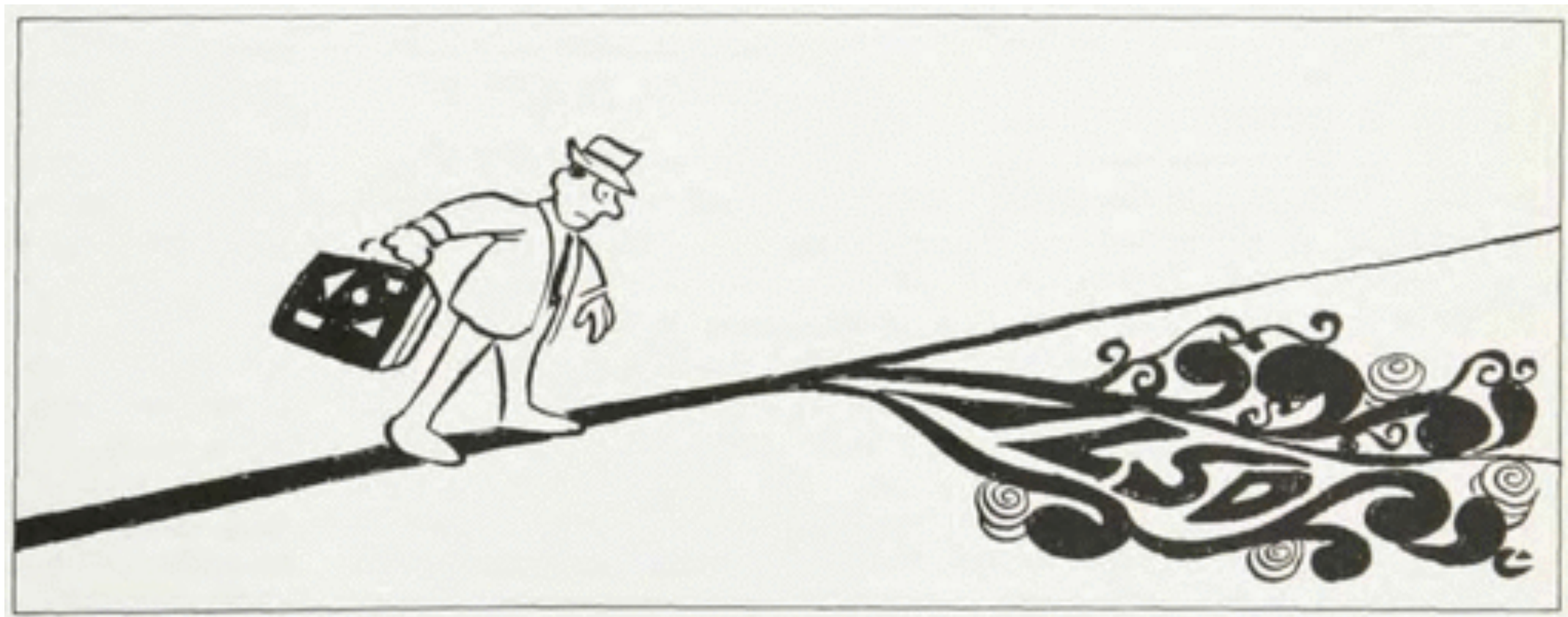
# Do renormalons actually exist?

Parisi's prediction of a Borel singularity due to renormalons in asymptotically free theories has been questioned. It was speculated that there might be cancellations which make the perturbative series better behaved, and some calculations of these series failed to show renormalon behavior.

However, there is growing evidence for Parisi's picture since we need **long perturbative series**. One impressive example is the numerical calculation of perturbative series at very large order in pure YM, with lattice techniques [Bauer-Bali-Pineda]

Calculations in QCD are difficult, so what should we do?

“A (wo)man grows stale if (s)he works all the time on insoluble problems, and a trip to the beautiful world of one dimension will refresh his/her imagination better than a dose of LSD” (Freeman Dyson, 1967)



# Renormalons and integrable 2d QFTs

There are many asymptotically free theories in 2d that are integrable, i.e. their S-matrix is exactly known: the non-linear sigma model, the Gross-Neveu model, the principal chiral field

One can use Bethe ansatz techniques to calculate the ground state energy of these theories, once they are coupled to a “chemical potential”  $h$  through a global conserved charge  $Q$

$$\mathcal{F}(h) = - \lim_{V, \beta \rightarrow \infty} \frac{1}{V\beta} \log \text{Tr} e^{-\beta(H-hQ)}$$



On one hand, this quantity can be computed exactly from the Bethe ansatz. On the other hand, it can be computed in perturbation theory when  $h \gg \Lambda$  and leads to a perturbative series in the effective coupling  $g=g(h)$  at the scale set by  $h$ :

$$\mathcal{F}(h) \sim \sum_n a_n g^n$$

It turns out that one can extract this series from the Bethe ansatz (in a slightly different scheme) and to very high orders by using a method due to D.Volin. Analytically one can get 40-50 terms [Volin, M.M.-Reis], and numerically one can get more than 2000 terms in some cases [Abbott et al.].

# Main results

In this way one can verify the presence of a leading singularity in the Borel plane [Volin, M.M.-Reis], and of the expected type (which is determined by the second coefficient of the beta function)

Due to these Borel singularities, the observable  $\mathcal{F}(h)$  is given by a trans-series

$$\mathcal{F}(h) = \sum_{n \geq 0} a_n g^n + C g^{-b} e^{-\frac{2}{|\beta_0|g}} \sum_{n \geq 0} a_{1,n} g^n + \dots$$

Information on these non-perturbative corrections can be obtained numerically from the perturbative series [M.M.-Reis, Abbott et al.]

There has been some recent progress in obtaining **analytic results** for the trans-series. For example, one can focus on the perturbative series obtained order by order in the  $1/N$  expansion, since renormalon singularities (in contrast to instanton singularities) survive in the large  $N$  limit [Fateev-Kazakov-Wiegmann, M.M.-Miravitllas-Reis, Di Pietro-M.M.-Serone-Sberveglieri]

In many examples, the exact result for  $\mathcal{F}(h)$  is obtained by using the so-called “median resummation” of the trans-series, so the resurgence program is fully vindicated!

However, in the context of the  $1/N$  expansion, it has been shown in various examples that the trans-series can not be obtained just from the perturbative series.

This might be an accident of the  $1/N$  expansion, or it might be an obstruction to the “strong” version of the resurgence program

# Renormalon calculus?

Tran-series are a far-reaching generalization of conventional perturbation theory, and the rules to obtain them are not systematic.

This problem was addressed in calculating vevs of products of currents in QCD. The ITEP group invented a generalization of perturbation theory by using the OPE and assuming the existence of “condensates”:

$$\mathcal{O}(Q^2) = C_0(Q^2) + C_1(Q^2) \text{Tr } F^2 + \dots$$

$$\langle \mathcal{O}(Q^2) \rangle = \sum_{n \geq 0} a_n g^n + \frac{\Lambda^4}{Q^4} \sum_{n \geq 0} a_{n,1} g^n \dots$$

However, more work is needed to understand and compute renormalon trans-series in full generality, and from first principles. This is in my view a crucial open problem in this subject (and in QFT in general).

# Conclusions

Conventional perturbation theory should be replaced by “resurgent perturbation theory”, in which perturbative series are replaced by trans-series. This framework seems to be powerful enough to include realistic QFTs (as well as condensed matter examples).

The most important source of non-perturbative effects in realistic theories are renormalon contributions, which do not have a description in terms of saddle-points of the path integral. They can however be characterized quite generally (Parisi) and computed in detail in some simple models.

Most important open problem: the calculation from first principles of trans-series associated to renormalons. These are largely unexplored new sectors of the path integral!

**Thank you for your attention**

