A Physical Interpretation of the TTbar Deformation of 2d Field Theory

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Outline

- what is “TTbar” and why is it interesting/peculiar?
- review of magnetoelasticity
- fixed strain vs fixed stress ensemble for the torus
- mathematical interlude: TTbar deformed modular forms
- Laplace *vs.* Legendre transform
- fluid analogy: shocks and collapse
- simply connected domains

What is TTbar?

- near critical short-range lattice systems have a *scaling limit* as the lattice spacing $\to 0$ which is a local euclidean field theory
- observables like magnetization, energy density, etc. correspond to local fields whose correlation functions have power law behavior $|x_1 - x_2|^{-2\Delta}$ on scales $\ll$ correlation length, described by a conformal field theory (CFT)
- one of these fields is the *stress-energy tensor* $T_{ij}(x)$
  - response of the free energy to an infinitesimal change in the metric
  - conserved Noether current of translational symmetry
  - dimension $\Delta = d$

- from this we can form scalar bilinears

$$T_{ij}T_{ij}\,, \quad T_{ii}T_{jj}\ (= 0 \text{ at critical point})$$

- dominant irrelevant terms in many 2d lattice models
- Zamolodchikov (2004) showed that in 2d the combination "TTbar"

$$\lambda\int(T_{11}T_{22}-T_{12}T_{21})d^2x = \lambda\int(\det T)d^2x = \tfrac{1}{2}\lambda\int\epsilon_{ik}\epsilon_{jl}T_{ij}T_{kl}d^2x$$

is special, in that many features of the deformed theory are finite and solvable in terms of the original theory

- example of a non-local theory with a UV length scale $\propto \sqrt{\lambda}$
- in holography $\lambda < 0$ corresponds to 'going into the bulk'
- in massive theories it gives particles a hardcore width $\sim \lambda m$

TTbar peculiarities

- in 1+1 dimensions, if space $= [0, R]$, eigenvalues of hamiltonian obey

$$\partial_\lambda E_n^\lambda(R) = E_n^\lambda(R)\,\partial_R E_n^\lambda(R)$$

- if undeformed theory is critical (a CFT), $E_n^0(R) = C_n/R$,

$$E_n^\lambda(R) = \frac{R}{2\lambda}\left(\sqrt{1+\frac{4\lambda C_n}{R^2}}-1\right)$$

$$E_n^0(R) = E_n^\lambda(R)\left(1+\lambda E_n^\lambda(R)/R\right) \quad \text{where } \rho(E^0) \sim e^{\text{ct.}\sqrt{E^0}}$$

  - $\lambda > 0$: fast growth in dos; maximum temperature
  - $\lambda < 0$: maximum in dos; finite entropy density at infinite temperature

- one aim of this work is to explain these features in a different physical setting

Magnetoelasticity

- coupling of magnetic degrees of freedom to displacement field $u(x)$ of elastic solid
  - eg magnetostriction [Joule 1842]
- other ‘matter’ internal degrees of freedom can be similarly coupled as stress$\times$strain:

$$\int T_{ij}^{\text{total}}\,\varepsilon_{ij}\,d^2x \quad \text{where strain} \quad \varepsilon_{ij} = \tfrac{1}{2}(\partial_i u_j + \partial_j u_i)$$

$$T_{ij}^{\text{total}} = T_{ij}^{\text{elastic}} + \left(T_{ij}^{\text{matter}} + T_{ij}^{\text{coupling}}\right)$$

$$\text{where} \quad T_{ij}^{\text{elastic}} = \Lambda_{ij;kl}\varepsilon_{kl} = 2\mu\,\varepsilon_{ij} + \bar{\lambda}\,\delta_{ij}\,\varepsilon_{kk} \qquad \text{(Hooke's law)}$$

- Lamé constants $(\mu, \bar{\lambda})$: normally $\mu > 0$, $\bar{\lambda} + \mu > 0$

$$Z = \int d[\text{matter}] \int d\varepsilon_{ij}\, e^{-T^{\text{matter}}\cdot\varepsilon - \varepsilon\cdot\Lambda\cdot\varepsilon}$$

- integrate out strain field ε

$$\varepsilon = -\tfrac{1}{2}\Lambda^{-1}\cdot T^{\text{matter}} + \text{gaussian fluctuations}$$
$$\to e^{\frac{1}{4}T^{\text{matter}}\cdot\Lambda^{-1}\cdot T^{\text{matter}}}$$

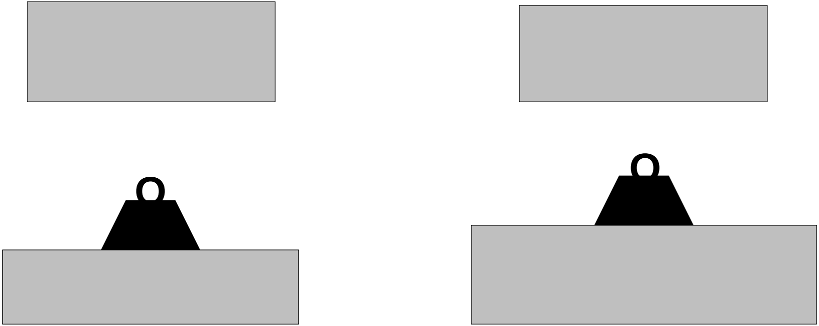
- to get TTbarite we need $\varepsilon_{ij} = -\lambda\epsilon_{ik}\epsilon_{jl}T_{kl}$

$$\varepsilon_{11} = -\lambda T_{22}\,, \quad \varepsilon_{22} = -\lambda T_{11}\,, \quad \varepsilon_{12} = \lambda T_{12}$$

- infinite Poisson's ratio

Normal material $\nu>0$

TTbarite $\nu=\infty$ $\lambda>0$



- more interesting to consider a protocol where initially the sample is in equilibrium with $\lambda = 0$, with the only stresses due to finite-size Casimir-type forces, then λ is turned on adiabatically

Why TTbar is solvable

- many ways to see this, but –
- under $\lambda \to \lambda + \delta\lambda$, $\delta\varepsilon_{ij} = -(\delta\lambda)\epsilon_{ik}\epsilon_{jl} T^{\lambda}{}_{kl}$

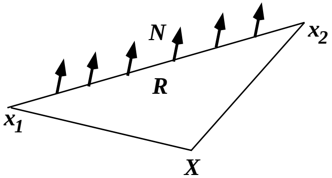
$$\partial_j[\delta u_i^{\lambda}(x)] = -(\delta\lambda)\epsilon_{ik}\epsilon_{jl} T^{\lambda}{}_{kl}(x)$$

- note that we use updated T^{λ}: this is a flow not a simple perturbation

$$\text{Integrate wrt } x\text{:} \quad \partial_{\lambda} u_i^{\lambda}(x) = -\epsilon_{ik} \oint_X^x T^{\lambda}{}_{kl}(x')\epsilon_{jl} dx'_j$$

$= -\epsilon_{ik} \times$flux $N_k(X, x)$ of conserved current $T^{\lambda}{}_{kl}$ across $[X, x]$

- independent of contour $C[X, x]$
- for other values of Poisson's ratio this would not happen



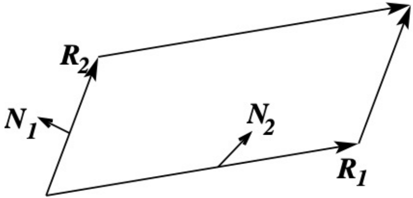
$$\vec{R}^{\lambda}_{1,2} = \vec{x}_2 + \vec{u}^{\lambda}(x_2) - \vec{x}_1 - \vec{u}^{\lambda}(x_1)$$

- $x \to x + u(x)$ is *not* a diffeomorphism: $\vec{x}$ is an absolute frame of reference and $u(x)$ is a physical field; no requirement of general covariance

$$\partial_\lambda R_i^{\lambda} = -\epsilon_{ik} N_k^{\lambda}(C^{\lambda}) = -\epsilon_{ik} \times (\text{force acting across } C)_k$$

- in many cases C is macroscopic and we can choose an ensemble where $N_k^{\lambda}(C^{\lambda})$ is non-fluctuating and moreover independent of λ. R^{λ} then evolves *linearly*

Example: torus



- may think of R_a, N_a as $\in \mathbb{R}^2$ or $\in \mathbb{C}$
- different ensembles:
  - fixed strain (R_1, R_2) [$\sim$canonical (volume, temperature)]
  - fixed stress (N_1, N_2) [$\sim$(pressure, energy)]
  - mixed (R_1, N_2) [$\sim$microcanonical (volume, energy)]
- related by Laplace or Legendre transforms

Laplace transforms

$$Z^0(R_1, R_2) = \int e^{-N_2.R_2} \rho^0(R_1, N_2) d^2 N_2 = \int_{\mathcal{C}} e^{s_2.R_2} \, \omega^0(R_1, s_2) \frac{d^2 s_2}{(2\pi i)^2}$$

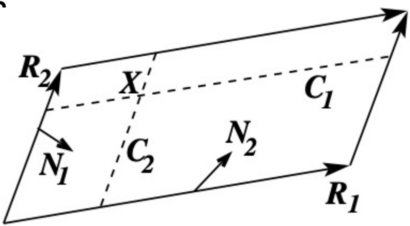
$$\text{where} \quad \omega^0(R_1, s_2) = \int_{R_1 \wedge R_2' > 0} e^{-s_2.R_2'} Z^0(R_1, R_2') dR_2'$$

OR

$$Z^0(R_1, R_2) = \int_{\mathcal{C}} \int_{\mathcal{C}} e^{s_1.R_1 + s_2.R_2} \, \Omega^0(s_1, s_2) \frac{d^2 s_1}{(2\pi i)^2} \frac{d^2 s_2}{(2\pi i)^2}$$

$$\text{where} \quad \Omega^0(s_1, s_2) = \int_{R_1' \wedge R_2' > 0} e^{-s_1.R_1' - s_2.R_2'} Z^0(R_1', R_2') d^2 R_1' d^2 R_2'$$

TTbar evolution



$$\partial_\lambda R^\lambda_{ai} = -\epsilon_{ik}\epsilon_{ab} N_{bk}$$

However, since there is a marked point where $u(X) = 0$, we should be considering $Z^\lambda_X(R_1, R_2) = Z^\lambda(R_1, R_2)/(R_1 \wedge R_2)$.

In a CFT,

$$Z_X(R_1, R_2) = |R_1|^{-2} z^0(\tau = R_2/R_1) \quad \text{so} \quad z^0(-1/\tau) = |\tau|^2 z^0(\tau)$$

z^0 transforms like the absolute value of a modular form of weight 2.

In the mixed (R_1, N_2) ensemble, $R_1^\lambda = R_1^0 - \lambda i N_2$, so

$$Z_X^\lambda(R_1, R_2) = \int_{\mathcal{C}} e^{s.R_2}\, \omega^0(R_1 - \lambda i s, s) \frac{d^2 s}{(2\pi i)^2}$$

where $s.R_2 = \mathrm{Re}\,(s R_2^*)$ and

$$\omega^0(R_1, s) = \int_{R_1 \wedge R_2' > 0} e^{-s.R_2'} Z_X^0(R_1, R_2') d^2 R_2'$$

$$= \int_{R_1 \wedge R_2' > 0} e^{-s.R_2'} |R_1|^{-2} z^0(\tau' = R_2'/R_1) d^2 R_2' = \int_{\mathbb{H}} e^{-s.\tau' R_1} z^0(\tau') d^2 \tau'$$

$$Z_X^\lambda(R_1, R_2) = \int_{\mathcal{C}} e^{s.\tau R_1} \int_{\mathbb{H}} e^{-s.\tau'(R_1 - \lambda i s)} z^0(\tau') d^2\tau' \frac{d^2 s}{(2\pi i)^2}$$

Setting $\alpha = \lambda/(\text{area})$ and rescaling s, $Z_X^\lambda(R_1, R_2) = |R_1|^{-2} z^\alpha(\tau)$ where

$$\boxed{z^\alpha(\tau) = (1/4\pi\alpha) \int_{\mathbb{H}} e^{-|\tau - \tau'|^2/4\alpha\tau_2\tau_2'} (\tau_2'/\tau_2) z^0(\tau') \frac{d^2\tau'}{{\tau_2'}^2}}$$

[Dubovsky *et al*, 2018; Datta & Jiang, 2020]

The kernel $e^{-|\tau-\tau'|^2/4\alpha\tau_2\tau_2'}(d^2\tau'/{\tau_2'}^2)$ is invariant under $(\tau,\tau')\to(-1/\tau,-1/\tau')$, so that

$$z^\alpha(-1/\tau)=|\tau|^2 z^\alpha(\tau)\,,\quad\text{and}\quad z^\alpha(\tau+1)=z^\alpha(\tau)$$

so $Z^\lambda(R_1,R_2)$ is SL$(2,\mathbb{Z})$ invariant as expected, as long as the integrals converge.

Since $z^0(\tau')\sim e^{\pi c/6\tau_2'}$ as $\tau_2'\to 0$, this requires $\tau_2/4\alpha>\pi c/6$. This corresponds to the 'Hagedorn' maximum temperature $\sim\tau_2^{-1}$. Similarly $1/4\alpha\tau_2>\pi c/6$ as $\tau_2'\to\infty$.

$$Z_X^\lambda(R_1, R_2) = \int_{\mathcal{C}} e^{s.\tau R_1} \int_{\mathbb{H}} e^{-s.\tau'(R_1-\lambda is)} z^0(\tau') d^2\tau' \frac{d^2 s}{(2\pi i)^2}$$

If $z^0(\tau')$ is a sum of terms of the form $(1/{\tau_2'}^2) e^{-2\pi\Delta\tau_2' + 2\pi i p \tau_1'}$, doing the τ' integration sets $s_1 = 2\pi p$ gives $\log(s_2 - s_-) + \log(s_2 - s_+)$ where

$$s_2^\pm = (1/2\alpha)\big(1 \pm \sqrt{1 + 4\pi\Delta\alpha + 4\pi^2 p^2 \alpha^2}\big)$$

and pulling back the contour to wrap around the branch cut at $s_2 = s^-$, we recover the deformed Zamolodchikov spectrum for states with $p \neq 0$.

[Remarks about $\alpha < 0$]

A mathematical diversion

These results extend straightforwardly to 1-point functions on the torus

$$\langle \Phi(X) \rangle^{\lambda} = |R_1|^{-\Delta_\Phi} f^{\alpha}(\tau)$$

In fact we can play this game with any modular or Jacobi form: if $F^0(\tau)$ is such a form of weight k so that $|F^0(-1/\tau)|^2 = |\tau|^{2k} |F^0(\tau)|^2$, and

$$|F^0(\tau)|^2 = \sum_{n \geq 0} \sum_p a_{n,p} e^{-2\pi(\Delta+n)\tau_2 + 2\pi i p \tau_1}$$

then

$$\sum_{n \geq 0} \sum_p a_{n,p} \frac{(1 + \sqrt{1 + 4\pi\Delta\alpha\tau_2 + 4\pi^2 p^2 \alpha^2 \tau_2^2})^{2-2k}}{\sqrt{1 + 4\pi\Delta\alpha\tau_2 + 4\pi^2 p^2 \alpha^2 \tau_2^2}} \times e^{-(1/2\alpha)\left(\sqrt{1+4\pi\Delta\alpha\tau_2+4\pi^2 p^2\alpha^2\tau_2^2}-1\right)+2\pi i p \tau_1}$$

has the same modular properties.

Legendre transforms

- Legendre is a steepest descent approximation to Laplace
- simpler, but usually valid only in the thermodynamic limit
- however, if $Z^0 \sim (\cdots)^c$ (a 'holographic' CFT), it is valid as $c \to \infty$ with λc fixed
- fixed (R_1, R_2) ensemble:

$$F^\lambda(R_1, R_2) = -\log Z^\lambda(R_1, R_2)\,, \quad N_a^\lambda = \partial_{R_a} F^\lambda$$

- fixed (N_1, N_2) ensemble: solve for (R_1, R_2) in terms of (N_1, N_2)

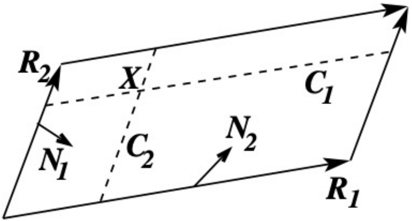
$$G^\lambda(N_1, N_2) \equiv F^\lambda(R_1, R_2) - R_1.N_1 - R_2.N_2\,, \quad R_a^\lambda = -\partial_{N_a} G^\lambda$$

Then

$$\boxed{G^\lambda(N_1, N_2) = G^0(N_1, N_2) + \lambda N_1 \wedge N_2}$$

- evolution in this ensemble is simple, and invariant under $S : (N_1, N_2) \to (N_2, -N_1)$ and $T : (N_1, N_2) \to (N_1, N_2 + N_1)$
- however the passage $F^0 \to G^0 \to G^\lambda \to F^\lambda$ fails if the map $(R_1^0, R_2^0) \to (R_1^\lambda, R_2^\lambda)$ is singular: either
  - $|\partial R_i^\lambda / \partial R_j^0| = 0$: formation of a *shock*; or
  - area $A^\lambda = R_1^\lambda \wedge R_2^\lambda \to 0$: *collapse* of the elastic sample

Fluid analogy



$$\partial_\lambda R^\lambda_{ai} = -\epsilon_{ab}\epsilon_{ij} N_{bj}$$

are the equations of motion of a 4d fluid in the Lagrangian (particle) picture, where $\lambda =$ time and velocity $v_{ai} = -\epsilon_{ab}\epsilon_{ij} N_{bj}$.

$$\text{Euler equations:} \quad \partial_\lambda N^\lambda_{ck} = \epsilon_{ab}\epsilon_{ij} N^\lambda_{bj} \partial_{R_{ai}} N^\lambda_{ck}$$

generalize Zamolodchikov's equation and may become singular, but in the fixed strain ensemble the evolution is linear

$$R^\lambda_{ai} = R^0_{ai} - \lambda \epsilon_{ab}\epsilon_{ij} N_{bj}(R^0)$$

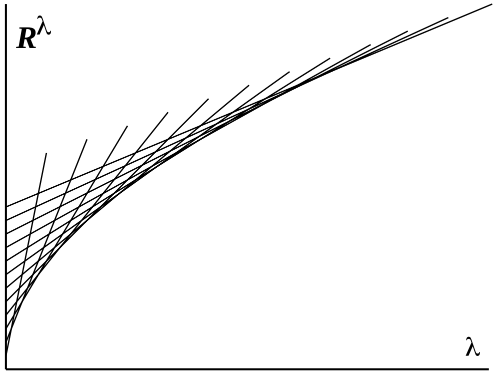
where $N_{bj}(R^0) = \partial_{R_{bj}} F^0(R^0) = -\partial_{R_{bj}} \log Z^0(R^0)$

$R_2^0 \gg R_1^0$

$$F^0 \sim -C(R_2^0/R_1^0) \quad (C = \pi c/6) \quad \text{so}$$

$$R_1^\lambda = R_1^0 + \lambda \frac{C}{R_1^0}\,, \quad R_2^\lambda = R_2^0 - \lambda \frac{CR_2^0}{(R_1^0)^2}\,, \quad \partial R_a^\lambda / \partial R_a^0 = 1 - \lambda \frac{C}{(R_1^0)^2}$$

- $\lambda > 0$: sample expands in 1-direction and shrinks in 2-direction; shock forms when $\lambda \sim (R_1^0)^2$; minimum value for R_1^λ [= maximum temperature, Hagedorn point]
- $\lambda < 0$: expands in 2-direction and shrinks in 1-direction; collapses when $\lambda \sim -(R_1^0)^2$; maximum stress N_1/R_2^λ [= maximum energy density]



Formation of a shock or caustic: particles from smaller R^0 move faster and overtake those from larger R^0

- more generally,

$$\partial_\lambda R^\lambda_{ai} = -\epsilon_{ab}\epsilon_{ij} N_{bj} = -\epsilon_{ij}\epsilon_{kl} T^0_{jk}(R^0) R^0_{al} = T^0_{ij}(R^0) R^0_{aj}$$

- for a given R^0, we may rotate to a basis where $T^{jl}(R^0) = \mathrm{diag}(T^0, -T^0)$, so

$$R^\lambda_{a1} = (1 + \lambda T^0) R^0_{a1} \,, \qquad R^\lambda_{a2} = (1 - \lambda T^0) R^0_{a2}$$

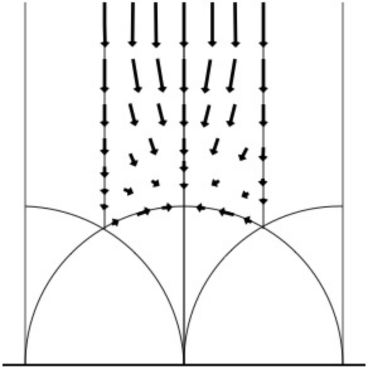
  so the torus always contracts in one direction and expands in the orthogonal direction

- area $A^\lambda = R^\lambda_1 \wedge R^\lambda_2 = R^\lambda_{11} R^\lambda_{22}$ obeys

$$A^\lambda = (1 - \lambda^2 (T^0)^2) A^0$$

  so for either sign of λ is always decreasing unless $T^0 = 0$.

- in general the principal axes of $T^{ij}(R^0)$ do not line up simply with $R^0_{1,2}$, unless the torus has more symmetry, either
  - $R^0_1 \perp R^0_2$ ($\mathrm{Re}\,\tau = 0$, tiling of $\mathbb{R}^2$ by rectangles)
  - $|R^0_1| = |R^0_2|$ ($|\tau| = 1$, tiling by rhombi)
  - $\tau = i$, tiling by squares, $T_{ij} \propto \delta_{ij}$ but it is traceless so in fact it vanishes
  - $\tau = e^{\pm i\pi/3}$, tiling by equilateral triangles, similarly T_{ij} vanishes
- in the last 2 cases, the torus does not evolve, corresponding to stagnation points in the flow
- these allow the construction of the general features of the flow pattern:



Flows projected onto the principal region of the $\tau = i\delta$ plane

- in this region a shock forms for $\alpha = \lambda/$(area)$> \alpha_c^+(\tau) > 0$ where $\alpha_c^+(\tau) \to \infty$ at the stagnation points
- similarly collapse occurs for $\alpha < \alpha_c^-(\tau) < 0$
- this suggests that for $c \to \infty$ modular invariance in fact holds in a wider domain $\alpha_c^-(\tau) < \alpha < \alpha_c^+(\tau)$

Simply connected domains

- *caveat:* displacement $u(x)$ no longer uniform in general
- polygon: if no shear forces acting at edges, angles stay the same, shape changes
  - $\lambda > 0$: short edges grow faster than long ones: becomes more symmetrical; minimum size $\sim \lambda^{1/2}$
  - $\lambda < 0$: short edges shrink faster than long ones: becomes less symmetrical, eventually collapses along narrowest axis
  - for rectangle Z^0 known exactly and complete analysis possible; results similar to torus
- disc:
  - expands as $\lambda \uparrow$: minimum radius $\sim \lambda^{1/2}$; at fixed $\lambda > 0$ the free energy $F^\lambda(R)$ has a singularity at this radius
  - for $\lambda < 0$ collapses at $\lambda \sim -R_0^2$: remains symmetric but any instabilities grow

Summary

- TTbar deformation of a 2d euclidean field theory may be understood as a coupling to an elastic medium, where the displacement $u(x)$ is a physical field rather than a coordinate change
- medium has infinite Poisson's ratio, which gives it unusual properties
- evolution is linear in the fixed stress ensemble, where the equations are those of simple fluid flow, but known results at fixed strain can be derived
- for $\lambda > 0$, Hagedorn-type singularities at fixed strain correspond to formation of shocks in the fluid
- for $\lambda < 0$, finite entropy density infinite temperature corresponds to collapse of the sample
- for large c a simpler thermodynamic treatment is possible
- modular invariance is explicit in the fixed stress ensemble, and true in a restricted sense but more difficult to show at fixed strain
- reproduces known results for the torus, new ones for other domains
- similar methods apply to TTbar deformed modular forms, suggesting new mathematics