



Local curvature estimates along the κ -LYZ flow[☆]

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ABSTRACT

In this paper we prove a local curvature estimate for the κ -LYZ flow over Kähler manifolds introduced in Fei et al. (2019) and Li et al. (2020). In particular, we generalize the long time existence of the flow.

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1. Introduction

Let (X, ω) be a closed Kähler manifold of complex dimension n . Here and throughout the paper, a closed manifold means a compact manifold without boundary. One of the central problems in Kähler geometry is the existence of the constant scalar curvature Kähler (cscK) metrics in the Kähler class $[\omega]$. Although Chen and Cheng [1–3] made a breakthrough on the problem using the elliptic approach, the parabolic approach still remains open. In particular, the long time existence and convergence of the Calabi flow is not known in the general case (cf. references in [11] for studies of the Calabi flow). Since the Calabi flow is a fully nonlinear fourth order partial differential equation, Yuguang Zhang and the authors in the present paper introduced the following coupled flow in [11]:

$$\partial_t \omega(t) = -\text{Ric}(\omega(t)) - \omega(t) + \alpha(t), \quad \partial_t \alpha(t) = \bar{\square}_{\omega(t)} \alpha(t), \quad (\omega, \alpha)(0) = (\omega, \alpha), \quad (1.1)$$

where α is a closed Hermitian $(1, 1)$ -form, $\text{Ric}(\omega(t))$ is the Ricci form of $\omega(t)$, and $\bar{\square}_{\omega(t)}$ is the complex Hodge-Laplace operator. The motivation of defining such flow is to reduce the fully nonlinear fourth order equation to a system of better understood equations, namely, the Kähler–Ricci flow and the heat flow. Note that the stationary solution to (1.1) is the cscK metric coupled with a harmonic $(1, 1)$ -form.

In [11], the following long time existence result is obtained by using Shi type estimates. Let $(\omega(t), \alpha(t))$ be the unique solution on $[0, T_{\max})$ for some maximal time $T_{\max} > 0$.

Theorem 1.1 ([11]). *If $T_{\max} < +\infty$, then*

$$\limsup_{t \rightarrow T_{\max}} \max_X \{ |\text{Rm}(\omega(t))|_{\omega(t)}, |\alpha(t)|_{\omega(t)} \} = +\infty. \quad (1.2)$$

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Since the flow (1.1) preserves the closedness of $\omega(t)$ and $\alpha(t)$, it follows that $[\omega(t)] = [\omega]$ and $[\alpha(t)] = [\alpha]$ (cf. [11]) under the cohomological condition

$$-2\pi c_1(X) + [\alpha] = [\omega]. \tag{1.3}$$

By $\partial\bar{\partial}$ -lemma, we write

$$\omega(t) = \omega + \sqrt{-1}\partial\bar{\partial}\varphi(t), \quad \alpha(t) = \alpha + \sqrt{-1}\partial\bar{\partial}f(t) \tag{1.4}$$

for smooth functions $\varphi(t)$ and $f(t)$ on X . By (1.3), we have

$$\omega = \alpha + \sqrt{-1}\partial\bar{\partial} \ln \Omega \tag{1.5}$$

for a smooth volume form Ω on X . Therefore, the flow (1.1) is equivalent to the following parabolic complex Monge-Ampère equation coupled with the heat equation

$$\partial_t \varphi(t) = \ln \frac{(\omega + \sqrt{-1}\partial\bar{\partial}\varphi(t))^n}{\Omega} - \varphi(t) + f(t), \quad \partial_t f(t) = \Delta_{\omega(t)} f(t) + \text{tr}_{\omega(t)} \alpha, \tag{1.6}$$

with $(\varphi, f)(0) = (0, 0)$, where $\Delta_{\omega(t)}$ stands for the complex Laplacian on X . The following long time existence result is obtained by assuming (1.3).

Theorem 1.2 ([11]). *Under the condition (1.3), if α is nonnegative, and the Ricci curvature of $\omega(t)$ and $|\alpha(t)|_{\omega(t)}$ are uniformly bounded on $[0, T)$ with $T < +\infty$, then the solution $(\omega(t), \alpha(t))$ of (1.1) extends past time T .*

Inspired by M theory, Teng Fei, Bin Guo and D. H. Phong in [6] introduced the κ -LYZ flow:

$$\partial_t \omega(t) = -\text{Ric}(\omega(t)) + \lambda \omega(t) + \alpha(t), \quad \partial_t \alpha(t) = \kappa \bar{\square}_{\omega(t)} \alpha(t), \quad (\omega, \alpha)(0) = (\omega, \alpha), \tag{1.7}$$

with constants $\kappa > 0$ and $\lambda \in \mathbb{R}$. When $(\kappa, \lambda) = (1, -1)$, the flow (1.7) reduces to (1.1). Assume the cohomology condition

$$-2\pi c_1(X) + [\alpha] + \lambda[\omega] = 0. \tag{1.8}$$

For $\kappa \neq 1$, they proved that higher order estimates for $(\omega(t), \alpha(t))$ follow from C^0 estimate for (ω, α) . Moreover, they derived very interesting convergence results. For example, on the Riemann surface with negative Euler characteristic, the κ -LYZ flow converges in C^∞ topology to the cscK metric in the case of $\kappa \neq 1$. Discovered by Fei-Guo-Phong, the κ -LYZ flow, as the simplest model in the coupled Ricci flow in Kähler geometry, can also be viewed as an Abelian model for the Anomaly flow (cf. [5,7,12,13] for related studies). We believe that the κ -LYZ flow is a natural geometric flow in Kähler geometry.

We were asked by the referee of [11] whether the nonnegativity of α in Theorem 1.2 can be removed. Motivated by this question, in this paper, we consider the long time existence of the flow (1.7) without assuming the condition (1.3) and the nonnegativity of α . Our main theorem is as follows.

Theorem 1.3. *Let (X, ω) be a closed Kähler manifold of complex dimension n . If the solution $(\omega(t), \alpha(t))_{t \in [0, T)}$ of (1.7), for $T < +\infty$, satisfies*

$$\sup_{X \times [0, T)} \{ |\text{Ric}(\omega(t))|_{\omega(t)}, |\alpha(t)|_{\omega(t)} \} \leq C_1$$

for some $C_1 > 0$, then there exists $C_2 > 0$, depending only on $C_1, n, T, \kappa, \lambda$, as well as (ω, α) , such that $|\text{Rm}(\omega(t))|_{\omega(t)} \leq C_2$.

As a consequence, we can generalize Theorem 1.2.

Corollary 1.4. *Let $(\omega(t), \alpha(t))$ be the solution to (1.7) on $t \in [0, T_{\max})$ under the condition (1.8) for some maximal time $T_{\max} > 0$. If $T_{\max} < +\infty$, then*

$$\limsup_{t \rightarrow T_{\max}} \max_X \{ |\text{Ric}(\omega(t))|_{\omega(t)}, |\alpha(t)|_{\omega(t)} \} = +\infty. \tag{1.9}$$

Proof. When $\kappa \neq 1$, the statement actually follows from Theorem 1 in [6]. For $\kappa = 1$, we argue by contradiction. Suppose that $|\text{Ric}(\omega(t))|_{\omega(t)}$ and $|\alpha(t)|_{\omega(t)}$ are uniformly bounded along the flow (1.7) for $t \in [0, T_{\max})$. By Theorem 1.3 above, $|\text{Rm}(\omega(t))|_{\omega(t)}$ is also uniformly bounded. This contradicts with Theorem 1.1. \square

Remark 1.5.

- (1) Under the condition (1.8), Theorem 1.3 is already obtained in [6]. In fact, stronger estimates are obtained there.
- (2) If $\kappa = 1$, Corollary 1.4 holds without the assumption (1.8).

The strategy of proving Theorem 1.3 is to consider the local curvature estimates for (1.7), inspired from the work of Kotschwar, Munteanu, and Wang [8]. Actually, the result in Theorem 1.3 holds for a class of geometric flows introduced in (2.1)–(2.2).

Theorem 1.6 (see also [Theorem 2.1](#)). Let (M, g) be a closed Riemannian manifold of dimension n and α be a 2-form on M . If the solution $(g(t), \alpha(t))_{t \in [0, T]}$ of (2.1)–(2.2) with initial data $(g(0), \alpha(0)) = (g, \alpha)$, where T is finite, satisfies that $|\text{Ric}(g(t))|_{g(t)}$ and $|\alpha(t)|_{g(t)}$ are uniformly bounded, then $|\text{Rm}(g(t))|_{g(t)}$ is also uniformly bounded.

We expect our study in the κ -LYZ flow would be beneficial to other coupled geometric flows.

2. Basic equations

We always omit the time variable t in local computations. In [11], we have computed

$$\begin{aligned} \partial_t g_{ij} &= -R_{ij} - g_{ij} + \alpha_{ij}, \\ \partial_t \alpha_{ij} &= \Delta_{\omega_t} \alpha_{ij} + g(t)^{-1} * g(t)^{-1} * \text{Rm}(g(t)) * \alpha_{ij} - g(t)^{-1} * \text{Ric}(g(t)) * \alpha_{ij}. \end{aligned}$$

Here $A * B$ can be any linear combination of tensor products of tensors fields A and B formed by contractions on $A_{i_1 \dots i_k}$ and $B_{j_1 \dots j_\ell}$ using the metric g .

Motivated by (1.1) and (1.7), we consider the system of equations on a Riemannian manifold (M, g) of dimension n :

$$\partial_t g_{ij} = -2R_{ij} + (ag_{ij} + b\alpha_{ij}), \tag{2.1}$$

$$\partial_t \alpha_{ij} = \kappa \Delta_{g(t)} \alpha_{ij} + (g^{-1} * g^{-1} * \text{Rm} * \alpha + g^{-1} * \text{Ric} * \alpha). \tag{2.2}$$

Here a, b are two given constants, and $\kappa > 0$ is a positive constant. Let T_{\max} be the maximal time of the system (2.2), and take $T \in (0, T_{\max})$.

Theorem 2.1. Let (M, g) be a closed Riemannian manifold of dimension n and α be a 2-form on M . If the solution $(g(t), \alpha(t))_{t \in [0, T]}$ of (2.1)–(2.2) with initial data $(g(0), \alpha(0)) = (g, \alpha)$, where T is finite, satisfies that $|\text{Ric}(g(t))|_{g(t)}$ and $|\alpha(t)|_{g(t)}$ are uniformly bounded, then $|\text{Rm}(g(t))|_{g(t)}$ is also uniformly bounded.

In fact, the uniform bound for $|\text{Rm}(g(t))|_{g(t)}$ in [Theorem 2.1](#) depends only on $\max_M |\text{Ric}(g(t))|_{g(t)}$, $\max_M |\alpha(t)|_{g(t)}$, $g, \alpha, T, n, a, b, \kappa$. The explicit uniform bound will be obtained in the following proof.

Let us temporarily denote

$$\eta_{ij} := -2R_{ij} + \epsilon_{ij}, \quad \epsilon_{ij} := ag_{ij} + b\alpha_{ij}, \quad \beta_{ij} := g^{-1} * g^{-1} * \text{Rm} * \alpha + g^{-1} * \text{Ric} * \alpha.$$

We have the following general equations

$$\partial_t g_{ij} = \eta_{ij} = -2R_{ij} + \epsilon_{ij}, \quad \partial_t \alpha_{ij} = \kappa \Delta_{g(t)} \alpha_{ij} + \beta_{ij}.$$

Basic facts about evolutions of curvatures are (see for example [4]):

$$\begin{aligned} \partial_t R_{ijk}^\ell &= \frac{1}{2} g^{\ell p} (\nabla_i \nabla_j \eta_{kp} + \nabla_i \nabla_k \eta_{jp} - \nabla_i \nabla_p \eta_{jk} \\ &\quad - \nabla_j \nabla_i \eta_{kp} - \nabla_j \nabla_k \eta_{ip} + \nabla_j \nabla_p \eta_{ik}), \\ \partial_t R_{jk} &= \frac{1}{2} g^{pq} (\nabla_q \nabla_j \eta_{kp} + \nabla_q \nabla_k \eta_{jp} - \nabla_q \nabla_p \eta_{jk} - \nabla_j \nabla_k \eta_{qp}), \\ \partial_t R_{g(t)} &= -\Delta_{g(t)} \text{tr}_{g(t)} \eta(t) + \text{div}_{g(t)} (\text{div}_{g(t)} \eta(t)) - R_{ij} \eta^{ij}, \end{aligned}$$

where $(\text{div}_{g(t)} \eta(t))_j := \nabla^i \eta_{ij}$ is the divergence of $\eta(t)$. The following four lemmas can be found in [4] (slightly modified).

Lemma 2.2. For $\eta_{ij} = -2R_{ij} + (ag_{ij} + b\alpha_{ij})$, we have

$$\partial_t R_{ijk}^\ell = g^{-1} * \nabla^2 \text{Ric} + g^{-1} * \text{Rm} * \text{Ric} + (g^{-1} * \nabla^2 \alpha + g^{-1} * \text{Rm} * \alpha). \tag{2.3}$$

Lemma 2.3. For $\eta_{ij} = -2R_{ij} + (ag_{ij} + b\alpha_{ij})$, we have

$$\partial_t dV = \left(-R + \frac{an}{2} + \frac{b}{2} \text{tr} \alpha \right) dV. \tag{2.4}$$

Introduce the metric-dependent parabolic operator

$$\square \equiv \square_{g(t)} := \partial_t - \Delta_{g(t)} \equiv \partial_t - \Delta.$$

Lemma 2.4. For $\eta_{ij} = -2R_{ij} + (ag_{ij} + b\alpha_{ij})$, we have

$$\square R = |\text{Ric}|^2 + (-b \Delta \text{tr} \alpha - aR - bR_{ij} \alpha^{ij} + b \nabla^i \nabla^j \alpha_{ji}). \tag{2.5}$$

Lemma 2.5. *One has*

$$\begin{aligned} \square R_{ij} &= -2R_{ik}R^k_j + 2R_{pijq}R^{pq} \\ &+ \frac{b}{2}(-\Delta\alpha_{ij} + \nabla^k\nabla_i\alpha_{jk} + \nabla^k\nabla_j\alpha_{ik} - \nabla_i\nabla_j\text{tr}\alpha). \end{aligned} \tag{2.6}$$

In particular,

$$\begin{aligned} \square|\text{Ric}|^2 &= -2|\nabla\text{Ric}|^2 + 2R^{ij}\square R_{ij} + 2R_{ij}R^i_q\partial_t g^{jq} \\ &= -2|\nabla\text{Ric}|^2 + 4R_{pijq}R^{ij}R^{pq} - 2a|\text{Ric}|^2 \\ &+ bR^{ij}(-\Delta\alpha_{ij} + \nabla^k\nabla_i\alpha_{jk} + \nabla^k\nabla_j\alpha_{ik} - \nabla_i\nabla_j\text{tr}\alpha - 2R_i^k\alpha_{kj}) \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} |\nabla\text{Ric}|^2 &\leq -\frac{1}{2}\square|\text{Ric}|^2 + C|\text{Ric}|^2|\text{Rm}| - a|\text{Ric}|^2 + C|b||\text{Ric}|^2|\alpha| \\ &- \frac{b}{2}R^{ij}(\Delta\alpha_{ij} - \nabla^k\nabla_i\alpha_{jk} - \nabla^k\nabla_j\alpha_{ik} + \nabla_i\nabla_j\text{tr}\alpha). \end{aligned} \tag{2.8}$$

Lemma 2.6. *One has*

$$\begin{aligned} \partial_t|\text{Rm}|^2 &= \nabla^2\text{Ric} * \text{Rm} + \text{Ric} * \text{Rm} * \text{Rm} \\ &+ (\text{Rm} * \text{Rm} + \text{Rm} * \nabla^2\alpha + \text{Rm} * \text{Rm} * \alpha). \end{aligned} \tag{2.9}$$

Lemma 2.7. *For $\partial_t g_{ij} = -2R_{ij} + ag_{ij} + b\alpha_{ij}$, one has*

$$\begin{aligned} \square R_{ijkl} &= 2(B_{ijkl} - B_{ijlk} + B_{ikjl} - B_{iljk}) \\ &- (R_{ip}R^p_{jkl} + R_{jp}R_{ipk\ell} + R_k^p R_{ijp\ell} + R_\ell^p R_{ijkp}) \\ &+ C\nabla^2\alpha + \text{Rm} * \alpha + aR_{ijkl} + b\alpha_{\ell p}R^{ij}_{\ell p}. \end{aligned} \tag{2.10}$$

Here $B_{ijkl} = -R_{pijq}R^{pq}_{kl}$.

In particular,

$$\begin{aligned} |\nabla\text{Rm}|^2 &\leq -\frac{1}{2}\square|\text{Rm}|^2 + C|\text{Rm}|^3 \\ &+ C(\text{Rm} * \text{Rm} + \text{Rm} * \text{Rm} * \alpha + \text{Rm} * \nabla^2\alpha). \end{aligned} \tag{2.11}$$

Lemma 2.8. *One has*

$$\begin{aligned} (\partial_t - \kappa\Delta)|\alpha|^2 &= -2\kappa|\nabla\alpha|^2 + g^{*-2} * \text{Rm} * \alpha^{*2} + g^{-1} * \text{Ric} * \alpha^{*2} \\ &+ g^{*-4} * g * \text{Ric} * \alpha^{*2} - 2a|\alpha|^2 + g^{*-4} * g * \alpha^{*3}. \end{aligned} \tag{2.12}$$

In particular we have

$$|\nabla\alpha|^2 \leq -\frac{1}{2\kappa}(\partial_t - \kappa\Delta)|\alpha|^2 - \frac{a}{\kappa}|\alpha|^2 + C(|\text{Rm}||\alpha|^2 + |\text{Ric}||\alpha|^2 + |\alpha|^3). \tag{2.13}$$

3. The main inequality

To prove [Theorem 2.1](#), we need a local curvature estimate on $|\text{Rm}(g(t))|$. For further study, we assume in this section that the manifold M is complete, because all the estimates are purely local. Actually, the following inequality (3.29) holds on any complete manifold M provided that the geodesic ball $B_{g(0)}(x_0, \rho/\sqrt{K})$ is compact.

We consider the system (2.1)–(2.2) and the condition

$$|\text{Ric}(g(t))|_{g(t)} \leq K, \quad |\alpha(t)|_{g(t)} \leq L, \quad t \in [0, T] \text{ with } T \in (0, T_{\max}), \tag{3.1}$$

where T_{\max} is the maximal time of the system (2.1)–(2.2). Then all metrics are equivalent to $g(0)$. Moreover, from (2.4), (2.8), (2.9), (2.11), and (2.13), we get

$$|\nabla\text{Ric}|^2 \leq -\frac{1}{2}\square|\text{Ric}|^2 + CK^2|\text{Rm}| + CK^2L + \text{Ric} * \nabla^2\alpha + \text{Ric} * \nabla^2\text{tr}\alpha, \tag{3.2}$$

$$|\nabla\text{Rm}|^2 \leq -\frac{1}{2}\square|\text{Rm}|^2 + C|\text{Rm}|^3 + C|\text{Rm}|^2 + CL|\text{Rm}|^2 + \text{Rm} * \nabla^2\alpha, \tag{3.3}$$

and

$$\begin{aligned} \partial_t |\text{Rm}|^2 &= \nabla^2 \text{Ric} * \text{Rm} + \text{Ric} * \text{Rm} * \text{Rm} \\ &\quad + (\text{Rm} * \text{Rm} + \text{Rm} * \nabla^2 \alpha + \text{Rm} * \text{Rm} * \alpha), \end{aligned} \tag{3.4}$$

$$|\nabla \alpha|^2 \leq -\frac{1}{2\kappa} (\partial_t - \kappa \Delta) |\alpha|^2 + CL^2 |\text{Rm}| + CL^2 (1 + K + L), \tag{3.5}$$

$$\partial_t dV_t \leq C(1 + K + L) dV_t. \tag{3.6}$$

Now we consider the quantity

$$\frac{d}{dt} \left(\int |\text{Rm}|^p \phi^{2p} dV_t \right), \quad \int := \int_M, \quad p \geq 3, \tag{3.7}$$

where $\phi = \phi(x)$ is a cutoff function with compact support inside M . Using the identity $\partial_t |\text{Rm}|^p = \partial_t (|\text{Rm}|^2)^{p/2} = \frac{p}{2} |\text{Rm}|^{p-2} \partial_t |\text{Rm}|^2$ and (3.4) we obtain

$$\begin{aligned} &\frac{d}{dt} \left(\int |\text{Rm}|^p \phi^{2p} dV_t \right) \\ &= \int (\partial_t |\text{Rm}|^p) \phi^{2p} dV_t + \int |\text{Rm}|^p \phi^{2p} (\partial_t dV_t) \\ &= \int \frac{p}{2} |\text{Rm}|^{p-2} (\partial_t |\text{Rm}|^2) \phi^{2p} dV_t + \int |\text{Rm}|^p \phi^{2p} (\partial_t dV_t) \\ &= \int |\text{Rm}|^p \phi^{2p} (\partial_t dV_t) + \frac{p}{2} \int |\text{Rm}|^{p-2} (\nabla^2 \text{Ric} * \text{Rm} + \text{Ric} * \text{Rm} * \text{Rm} \\ &\quad + \text{Rm} * \text{Rm} + \text{Rm} * \text{Rm} * \alpha + \text{Rm} * \nabla^2 \alpha) \phi^{2p} dV_t \\ &\leq Cp(1 + K + L) \int |\text{Rm}|^p \phi^{2p} dV_t + Cp \int (\nabla^2 \text{Ric} * \text{Rm}) |\text{Rm}|^{p-2} \phi^{2p} dV_t \\ &\quad + Cp \int (\nabla^2 \alpha * \text{Rm}) |\text{Rm}|^{p-2} \phi^{2p} dV_t. \end{aligned}$$

The last two integrals can be simplified as follows.

$$\begin{aligned} &C \int (\nabla^2 \text{Ric} * \text{Rm}) |\text{Rm}|^{p-2} \phi^{2p} dV_t \\ &= C \int \nabla \text{Ric} * (\nabla \text{Rm} * |\text{Rm}|^{p-2} * \phi^{2p} + \text{Rm} * \nabla |\text{Rm}|^{p-2} * \phi^{2p} + \text{Rm} * |\text{Rm}|^{p-2} * \nabla \phi^{2p}) dV_t \\ &\leq Cp \int |\nabla \text{Ric}| |\nabla \text{Rm}| |\text{Rm}|^{p-2} \phi^{2p} dV_t + Cp \int |\nabla \text{Ric}| |\nabla \phi| |\text{Rm}|^{p-1} \phi^{2p-1} dV_t \\ &\leq Cp \int (|\nabla \text{Ric}| |\text{Rm}|^{\frac{p-1}{2}} \phi^p) (|\nabla \text{Rm}| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) dV_t \\ &\quad + Cp \int (|\nabla \text{Ric}| |\text{Rm}|^{\frac{p-1}{2}} \phi^p) (|\nabla \phi| |\text{Rm}|^{\frac{p-1}{2}} \phi^{p-1}) dV_t \\ &\leq \frac{1}{pK} \int |\nabla \text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + Cp^3 K \int |\nabla \text{Rm}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &\quad + Cp^3 K \int |\nabla \phi|^2 |\text{Rm}|^{p-1} \phi^{2p-2} dV_t. \end{aligned}$$

Similarly,

$$\begin{aligned} &C \int (\nabla^2 \alpha * \text{Rm}) |\text{Rm}|^{p-2} \phi^{2p} dV_t \\ &= C \int \nabla \alpha * (\nabla \text{Rm} * |\text{Rm}|^{p-2} * \phi^{2p} + \text{Rm} * \nabla |\text{Rm}|^{p-2} * \phi^{2p} + \text{Rm} * |\text{Rm}|^{p-2} * \nabla \phi^{2p}) dV_t \\ &\leq Cp \int |\nabla \alpha| |\nabla \text{Rm}| |\text{Rm}|^{p-2} \phi^{2p} dV_t + Cp \int |\nabla \alpha| |\nabla \phi| |\text{Rm}|^{p-1} \phi^{2p-1} dV_t \\ &\leq \frac{1}{pK} \int |\nabla \alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + Cp^3 K \int |\nabla \text{Rm}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &\quad + Cp^3 K \int |\nabla \phi|^2 |\text{Rm}|^{p-1} \phi^{2p-2} dV_t. \end{aligned}$$

Hence

$$\begin{aligned} & \frac{d}{dt} \left(\int |\text{Rm}|^p \phi^{2p} dV_t \right) \\ & \leq \frac{1}{K} \int |\nabla \text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + Cp^4 K \int |\nabla \text{Rm}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ & \quad + Cp^4 K \int |\text{Rm}|^{p-1} |\nabla \phi|^2 \phi^{2p-2} dV_t + Cp(1+K+L) \int |\text{Rm}|^p \phi^{2p} dV_t + \frac{1}{K} \int |\nabla \alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t. \end{aligned}$$

Introduce five **bad terms**, which involve derivatives of Rm and α ,

$$\begin{aligned} B_1 & := \frac{1}{K} \int |\nabla \text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t, \\ B_2 & := \int |\nabla \text{Rm}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t, \\ B_3 & := \frac{1}{K} \int |\nabla \alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t, \\ B_4 & := \frac{1}{K} \int |\nabla \text{tr} \alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \leq CB_3, \\ B_5 & := \int |\nabla \alpha|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t. \end{aligned}$$

Define also four **good terms**

$$\begin{aligned} A_1 & := \int |\text{Rm}|^p \phi^{2p} dV_t, \\ A_2 & := \int |\text{Rm}|^{p-1} |\nabla \phi|^2 \phi^{2p-1} dV_t, \\ A_3 & := \int |\text{Rm}|^{p-1} \phi^{2p} dV_t, \\ A_4 & := \int |\text{Rm}|^{p-1} |\nabla \phi|^2 \phi^{2p-2} dV_t. \end{aligned}$$

Then the previous calculation can be written equivalently as

$$\frac{d}{dt} A_1 \leq B_1 + Cp^4 KB_2 + B_3 + Cp^4 KA_4 + Cp(1+K+L)A_1. \tag{3.8}$$

3.1. Auxiliary lemmas

We start with the following five lemmas.

Lemma 3.1. *We have*

$$\begin{aligned} B_1 & \leq Cp^2 KB_2 + Cp^2 B_3 + p^2 CB_4 \\ & \quad + Cp^2 KA_1 + Cp^2 K(1+K+L)A_2 + Cp^2 KA_4 \\ & \quad - \frac{1}{2K} \frac{d}{dt} \left(\int |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \right). \end{aligned} \tag{3.9}$$

Proof. According to (3.2) we obtain

$$\begin{aligned} B_1 & \leq \frac{1}{K} \int \left[\frac{1}{2} (\Delta - \partial_t) |\text{Ric}|^2 + CK^2 |\text{Rm}| + CK^2(1+L) + \text{Ric} * \nabla^2 \alpha + \text{Ric} * \nabla^2 \text{tr} \alpha \right] |\text{Rm}|^{p-1} \phi^{2p} dV_t \\ & = \frac{1}{2K} \int [(\Delta - \partial_t) |\text{Ric}|^2] |\text{Rm}|^{p-1} \phi^{2p} dV_t + CK A_1 + CK(1+L)A_2 \\ & \quad + \frac{1}{K} \int (\text{Ric} * \nabla^2 \alpha + \text{Ric} * \nabla^2 \text{tr} \alpha) |\text{Rm}|^{p-1} \phi^{2p} dV_t. \end{aligned}$$

We first estimate the last two terms involving higher derivatives of α as follows.

$$\begin{aligned} & \frac{1}{K} \int (\text{Ric} * \nabla^2 \alpha) |\text{Rm}|^{p-1} \phi^{2p} dV_t \\ &= \frac{1}{K} \int \nabla \alpha * (\nabla \text{Ric} * |\text{Rm}|^{p-1} * \phi^{2p} + \text{Ric} * \nabla |\text{Rm}|^{p-1} * \phi^{2p} + \text{Ric} * |\text{Rm}|^{p-1} * \nabla \phi^{2p}) dV_t \\ &\leq \frac{C}{K} \int |\nabla \alpha| |\nabla \text{Ric}| |\text{Rm}|^{p-1} \phi^{2p} dV_t + Cp \int |\nabla \text{Rm}| |\nabla \alpha| |\text{Rm}|^{p-2} \phi^{2p} dV_t \\ &\quad + Cp \int |\nabla \alpha| |\nabla \phi| |\text{Rm}|^{p-1} \phi^{2p-1} dV_t \\ &\leq \frac{C}{K} \int (|\nabla \alpha| |\text{Rm}|^{\frac{p-1}{2}} \phi^p) (|\nabla \text{Ric}| |\text{Rm}|^{\frac{p-1}{2}} \phi^p) dV_t + Cp \int (|\nabla \alpha| |\text{Rm}|^{\frac{p-1}{2}} \phi^p) (|\nabla \text{Rm}| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) dV_t \\ &\quad + Cp \int (|\nabla \alpha| |\text{Rm}|^{\frac{p-1}{2}} \phi^p) (|\nabla \phi| |\text{Rm}|^{\frac{p-1}{2}} \phi^{p-1}) dV_t \\ &\leq \frac{1}{100} B_1 + CpB_3 + CpKB_2 + CpKA_4. \end{aligned}$$

The second one can be similarly computed:

$$\frac{1}{K} \int (\text{Ric} * \nabla^2 \text{tr} \alpha) |\text{Rm}|^{p-1} \phi^{2p} dV_t \leq \frac{1}{100} B_1 + CpB_4 + CpKB_2 + CpKA_4.$$

To deal with the term

$$\frac{1}{2K} \int [(\Delta - \partial_t) |\text{Ric}|^2] |\text{Rm}|^{p-1} \phi^{2p} dV_t,$$

we calculate

$$\begin{aligned} & \frac{1}{2K} \int [(\Delta - \partial_t) |\text{Ric}|^2] |\text{Rm}|^{p-1} \phi^{2p} dV_t \\ &= \frac{1}{2K} \int (\Delta |\text{Ric}|^2) |\text{Rm}|^{p-1} \phi^{2p} dV_t - \frac{1}{2K} \int \left[\partial_t (|\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t) - |\text{Ric}|^2 (\partial_t |\text{Rm}|^{p-1}) \phi^{2p} dV_t \right. \\ &\quad \left. - |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} \left(-R + \frac{a}{2} n + \frac{b}{2} \text{tr} \alpha \right) dV_t \right] \\ &\leq -\frac{1}{2K} \left[\int \langle \nabla |\text{Ric}|^2, \nabla |\text{Rm}|^{p-1} \rangle \phi^{2p} dV_t + \int \langle \nabla |\text{Ric}|^2, \nabla \phi^{2p} \rangle |\text{Rm}|^{p-1} dV_t \right] \\ &\quad - \frac{1}{2K} \frac{d}{dt} \left(\int |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \right) + \frac{1}{2K} \int |\text{Ric}|^2 (\partial_t |\text{Rm}|^{p-1}) \phi^{2p} dV_t \\ &\quad + CK(1 + K + L)A_2, \end{aligned}$$

where

$$\begin{aligned} & -\frac{1}{2K} \int \langle \nabla |\text{Ric}|^2, \nabla |\text{Rm}|^{p-1} \rangle \phi^{2p} dV_t \\ &= -\frac{1}{2K} \int \left(2\text{Ric} * \nabla \text{Ric} * \frac{p-1}{2} |\text{Rm}|^{p-3} * 2\text{Rm} * \nabla \text{Rm} \right) \phi^{2p} dV_t \\ &\leq \frac{Cp}{K} \int |\text{Ric}| |\nabla \text{Ric}| |\text{Rm}|^{p-2} |\nabla \text{Rm}| \phi^{2p} dV_t \\ &\leq Cp \int (|\nabla \text{Ric}| |\text{Rm}|^{\frac{p-1}{2}} \phi^p) (|\nabla \text{Rm}| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) dV_t \\ &\leq \frac{1}{100} B_1 + Cp^2KB_2 \end{aligned}$$

and

$$\begin{aligned} & -\frac{1}{2K} \int \langle \nabla |\text{Ric}|^2, \nabla \phi^{2p} \rangle |\text{Rm}|^{p-1} dV_t \\ &= -\frac{1}{2K} \int (2\text{Ric} * \nabla \text{Ric} * 2p\phi^{2p-1} \nabla \phi) |\text{Rm}|^{p-1} dV_t \end{aligned}$$

$$\begin{aligned} &\leq Cp \int |\nabla \text{Ric}| |\nabla \phi| |\text{Rm}|^{p-1} \phi^{2p-1} dV_t \\ &\leq \frac{1}{100} B_1 + Cp^2 KA_4 \end{aligned}$$

and, using (3.4),

$$\begin{aligned} &\frac{1}{2K} \int |\text{Ric}|^2 (\partial_t |\text{Rm}|^{p-1}) \phi^{2p} dV_t \\ &= \frac{p-1}{4K} \int |\text{Ric}|^2 (|\text{Rm}|^{p-3} \partial_t |\text{Rm}|^2) \phi^{2p} dV_t \\ &= \frac{Cp}{K} \int |\text{Ric}|^2 |\text{Rm}|^{p-3} \phi^{2p} [\nabla^2 \text{Ric} * \text{Rm} + \text{Ric} * \text{Rm} * \text{Rm} \\ &\quad + \text{Rm} * \text{Rm} + \text{Rm} * \text{Rm} * \alpha + \text{Rm} * \nabla^2 \alpha] dV_t \\ &\leq \frac{Cp}{K} \int (\nabla^2 \text{Ric} * \text{Rm}) |\text{Ric}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t + CpK(1 + K + L)A_2 \\ &\quad + \frac{Cp}{K} \int (\text{Rm} * \nabla^2 \alpha) |\text{Ric}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t. \end{aligned}$$

Here, under the same method,

$$\begin{aligned} &\frac{Cp}{K} \int (\text{Rm} * \nabla^2 \alpha) |\text{Ric}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &= \frac{Cp}{K} \int \nabla \alpha * (\nabla \text{Rm} * |\text{Ric}|^2 |\text{Rm}|^{p-3} \phi^{2p} + \text{Rm} * \nabla |\text{Ric}|^2 * |\text{Rm}|^{p-3} * \phi^{2p} \\ &\quad + \text{Rm} * |\text{Ric}|^2 * \nabla |\text{Rm}|^{p-3} * \phi^{2p} + \text{Rm} * |\text{Ric}|^2 * |\text{Rm}|^{p-3} * \nabla \phi^{2p}) dV_t \\ &\leq Cp \int |\nabla \alpha| |\nabla \text{Rm}| |\text{Rm}|^{p-2} \phi^{2p} dV_t + \frac{Cp}{K} \int |\nabla \alpha| |\nabla \text{Ric}| |\text{Rm}|^{p-1} \phi^{2p} dV_t \\ &\quad + Cp \int |\nabla \alpha| |\nabla \phi| |\text{Rm}|^{p-1} \phi^{2p-1} dV_t \\ &\leq \frac{1}{100} B_1 + Cp^2 B_3 + Cp^2 KB_2 + Cp^2 KA_4 \end{aligned}$$

and

$$\begin{aligned} &\frac{Cp}{K} \int |\text{Ric}|^2 |\text{Rm}|^{p-3} \phi^{2p} (\nabla^2 \text{Ric} * \text{Rm}) dV_t \\ &= \frac{Cp}{K} \nabla \text{Ric} * \nabla (|\text{Ric}|^2 |\text{Rm}|^{p-3} \phi^{2p} * \text{Rm}) dV_t \\ &= \frac{Cp}{K} \int \nabla \text{Ric} * (\text{Ric} * \nabla \text{Ric} * |\text{Rm}|^{p-3} \phi^{2p} * \text{Rm} + |\text{Ric}|^2 |\text{Rm}|^{p-3} * \phi^{2p} \nabla \text{Rm} \\ &\quad + |\text{Ric}|^2 |\text{Rm}|^{p-3} \phi^{2p-1} \nabla \phi * \text{Rm} + |\text{Ric}|^2 |\text{Rm}|^{p-4} \nabla \text{Rm} * \phi^{2p} \text{Rm}) dV_t \\ &\leq Cp \int |\nabla \text{Ric}|^2 |\text{Rm}|^{p-2} \phi^{2p} dV_t + Cp \int |\nabla \text{Ric}| |\nabla \text{Rm}| |\text{Ric}| |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &\quad + \frac{Cp}{K} \int |\nabla \text{Ric}| |\nabla \phi| |\text{Ric}|^2 |\text{Rm}|^{p-2} \phi^{2p-1} dV_t \\ &\leq \frac{1}{100} B_1 + Cp^2 KB_2 + Cp^2 KA_4. \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{1}{2K} \int |\text{Ric}|^2 (\partial_t |\text{Rm}|^{p-1}) \phi^{2p} dV_t \\ &\leq \frac{1}{100} B_1 + Cp^2 KB_2 + Cp^2 KA_4 + CpK(1 + K + L)A_2 + \frac{1}{100} B_1 \\ &\quad + Cp^2 B_3 + Cp^2 KB_2 + Cp^2 KA_4 \\ &\leq \frac{2}{100} B_1 + Cp^2 KB_2 + Cp^2 B_3 + CpK(1 + K + L)A_2 + Cp^2 KA_4. \end{aligned}$$

Combining all estimates (3.9) follows. \square

Lemma 3.2. *We have*

$$B_2 \leq Cp^2A_1 + Cp^2(1 + K + L)A_2 + Cp^2A_4 + Cp^2B_5 - \frac{1}{p-1} \frac{d}{dt} \left(\int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right). \tag{3.10}$$

Proof. From (3.3) we obtain

$$\begin{aligned} B_2 &= \int |\nabla \text{Rm}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &\leq \int \left[\frac{1}{2} (\Delta - \partial_t) |\text{Rm}|^2 + C |\text{Rm}|^3 + C |\text{Rm}|^2 + CL |\text{Rm}|^2 + \text{Rm} * \nabla^2 \alpha \right] |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &= \frac{1}{2} \int (\Delta |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t - \frac{1}{2} \int (\partial_t |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &\quad + CA_1 + C(1 + L)A_2 + \int (\text{Rm} * \nabla^2 \alpha) |\text{Rm}|^{p-3} \phi^{2p} dV_t. \end{aligned}$$

Since

$$\begin{aligned} &\int (\text{Rm} * \nabla^2 \alpha) |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &= \int \nabla \alpha * (\nabla \text{Rm} * |\text{Rm}|^{p-3} * \phi^{2p} + \text{Rm} * \nabla |\text{Rm}|^{p-3} * \phi^{2p} + \text{Rm} * |\text{Rm}|^{p-3} * \nabla \phi^{2p}) dV_t \\ &\leq C \int |\nabla \alpha| |\nabla \text{Rm}| |\text{Rm}|^{p-3} \phi^{2p} dV_t + Cp \int |\nabla \alpha| |\nabla \text{Rm}| |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &\quad + Cp \int |\nabla \alpha| |\nabla \phi| |\text{Rm}|^{p-2} \phi^{2p-1} dV_t \\ &\leq Cp \int (|\nabla \alpha| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) (|\nabla \text{Rm}| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) dV_t \\ &\quad + Cp \int (|\nabla \alpha| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) (|\nabla \phi| |\text{Rm}|^{\frac{p-1}{2}} \phi^{p-1}) dV_t \\ &\leq \frac{1}{100} B_2 + Cp^2 B_5 + CA_4 \end{aligned}$$

and (since $p \geq 3$)

$$\begin{aligned} &\frac{1}{2} \int (\Delta |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &= -\frac{1}{2} \int \langle \nabla |\text{Rm}|^2, \nabla (|\text{Rm}|^{p-3} \phi^{2p}) \rangle dV_t \\ &= -\frac{1}{2} \int 2\text{Rm} * \nabla \text{Rm} * (|\text{Rm}|^{p-3} * 2p\phi^{2p-1} * \nabla \phi + \phi^{2p} * \frac{p-3}{2} |\text{Rm}|^{p-5} * 2\text{Rm} \nabla \text{Rm}) dV_t \\ &\leq Cp \int |\nabla \text{Rm}| |\nabla \phi| |\text{Rm}|^{p-2} \phi^{2p-1} dV_t - (p-3) \int |\nabla \text{Rm}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &\leq Cp \int |\nabla \text{Rm}| |\nabla \phi| |\text{Rm}|^{p-2} \phi^{2p-1} dV_t \\ &= Cp \int (|\nabla \text{Rm}| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) (|\nabla \phi| |\text{Rm}|^{\frac{p-1}{2}} \phi^{p-1}) dV_t \\ &\leq \frac{1}{100} B_2 + Cp^2 A_4, \end{aligned}$$

we arrive at

$$\begin{aligned} B_2 &\leq \frac{1}{100} B_2 + Cp^2 A_4 + CA_1 + C(1 + L)A_2 \\ &\quad + \frac{1}{100} B_2 + Cp^2 B_5 + CA_4 - \frac{1}{2} \int (\partial_t |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t. \end{aligned}$$

Therefore

$$B_2 \leq CA_1 + C(1 + L)A_2 + Cp^2A_4 + Cp^2B_5 - \frac{1}{2} \int (\partial_t |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t.$$

We now estimate the last integral. Direct computation shows

$$\begin{aligned} & -\frac{1}{2} \int (\partial_t |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t \\ &= -\frac{1}{2} \int \left[\partial_t (|\text{Rm}|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t) - |\text{Rm}|^2 (\partial_t |\text{Rm}|^{p-3}) \phi^{2p} dV_t - |\text{Rm}|^{p-1} \phi^{2p} (\partial_t dV_t) \right] \\ &= -\frac{1}{2} \frac{d}{dt} \left(\int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right) + \frac{p-3}{4} \int |\text{Rm}|^{p-3} (\partial_t |\text{Rm}|^2) \phi^{2p} dV_t \\ & \quad + \frac{1}{2} \int |\text{Rm}|^{p-1} \phi^{2p} \left(-R + \frac{an}{2} + \frac{b}{2} \text{tr}\alpha \right) dV_t. \end{aligned}$$

Thus

$$-\frac{1}{2} \int (\partial_t |\text{Rm}|^2) |\text{Rm}|^{p-3} \phi^{2p} dV_t \leq \frac{C}{p-1} (1 + K + L)A_2 - \frac{1}{p-1} \frac{d}{dt} \left(\int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right).$$

This implies (3.10). \square

Lemma 3.3. *We have*

$$\begin{aligned} B_3 &\leq C \frac{L^2}{K} A_1 + Cp \frac{L^2}{K} (1 + K + L)A_2 + Cp^2 \frac{L^2}{K} A_4 \\ & \quad + Cp^2 \frac{L^2}{K} B_2 + Cp^2 \frac{L^2}{K} B_5 - \frac{1}{2K} \frac{d}{dt} \left(\int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \right). \end{aligned} \tag{3.11}$$

Proof. Using (3.5) implies

$$\begin{aligned} B_3 &\leq \frac{1}{K} \int |\text{Rm}|^{p-1} \phi^{2p} \left[-\frac{1}{2\kappa} (\partial_t - \kappa \Delta) |\alpha|^2 + CL^2 |\text{Rm}| + CL^2 (1 + K + L) \right] dV_t \\ &= \frac{1}{2K\kappa} \int |\text{Rm}|^{p-1} \phi^{2p} (\kappa \Delta - \partial_t) |\alpha|^2 dV_t + C \frac{L^2}{K} A_1 + C \frac{L^2}{K} (1 + K + L)A_2. \end{aligned}$$

The integral involving $\Delta |\alpha|^2$ can be simplified as

$$\begin{aligned} & \frac{1}{2K} \int |\text{Rm}|^{p-1} \phi^{2p} (\Delta |\alpha|^2) dV_t \\ &= -\frac{1}{2K} \int \langle \nabla |\alpha|^2, \nabla (|\text{Rm}|^{p-1} \phi^{2p}) \rangle dV_t \\ &= -\frac{1}{2K} \int \alpha * \nabla \alpha * [(p-1) |\text{Rm}|^{p-3} * \text{Rm} * \nabla \text{Rm} * \phi^{2p} + 2p |\text{Rm}|^{p-1} \phi^{2p-1} \nabla \phi] dV_t \\ &\leq Cp \frac{L}{K} \int |\nabla \alpha| |\nabla \text{Rm}| |\text{Rm}|^{p-2} \phi^{2p} dV_t + Cp \frac{L}{K} \int |\nabla \alpha| |\nabla \phi| |\text{Rm}|^{p-1} \phi^{2p-1} dV_t \\ &\leq Cp \frac{L}{K} \int \left(|\nabla \text{Rm}| |\text{Rm}|^{\frac{p-3}{2}} \phi^p \right) \left(|\nabla \alpha| |\text{Rm}|^{\frac{p-1}{2}} \phi^p \right) dV_t \\ & \quad + Cp \frac{L}{K} \int \left(|\nabla \phi| |\text{Rm}|^{\frac{p-1}{2}} \phi^{p-1} \right) \left(|\nabla \alpha| |\text{Rm}|^{\frac{p-1}{2}} \phi^p \right) dV_t \\ &\leq \frac{2}{100} B_3 + Cp \frac{L^2}{K} (B_2 + A_4). \end{aligned}$$

The integral involving $\partial_t |\alpha|^2$ can be simplified as

$$\begin{aligned} & \frac{1}{2K\kappa} \int |\text{Rm}|^{p-1} \phi^{2p} (-\partial_t |\alpha|^2) dV_t \\ &= -\frac{1}{2K\kappa} \int \left[\partial_t (|\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t) - |\alpha|^2 (\partial_t |\text{Rm}|^{p-1}) \phi^{2p} dV_t - |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} (\partial_t dV_t) \right] \\ &= -\frac{1}{2K\kappa} \frac{d}{dt} \left(\int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \right) + \frac{p-1}{4K\kappa} \int |\alpha|^2 |\text{Rm}|^{p-3} (\partial_t |\text{Rm}|^2) \phi^{2p} dV_t \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{2K\kappa} \int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} \left(-R + \frac{an}{2} + \frac{b}{2} \text{tr}\alpha \right) dV_t \\
 \leq & -\frac{1}{2K\kappa} \frac{d}{dt} \left(\int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \right) + C \frac{L^2}{K} (1 + K + L) A_2 \\
 & + \frac{p-1}{4K\kappa} \int |\alpha|^2 |\text{Rm}|^{p-3} (\nabla^2 \text{Ric} * \text{Rm} + \text{Ric} * \text{Rm} * \text{Rm} \\
 & + \text{Rm} * \text{Rm} + \text{Rm} * \text{Rm} * \alpha + \text{Rm} * \nabla^2 \alpha) \phi^{2p} dV_t \\
 \leq & -\frac{1}{2K\kappa} \frac{d}{dt} \left(\int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \right) + C_p \frac{L^2}{K} (1 + K + L) A_2 \\
 & + \frac{p-1}{4K\kappa} \int (\text{Rm} * \nabla^2 \text{Ric} + \text{Rm} * \nabla^2 \alpha) |\alpha|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t
 \end{aligned}$$

using (3.4). As before, we should estimate the integrals involving higher derivatives. Because $p \geq 3$,

$$\begin{aligned}
 & \frac{p-1}{4K\kappa} \int |\alpha|^2 |\text{Rm}|^{p-3} (\text{Rm} * \nabla^2 \text{Ric}) \phi^{2p} dV_t \\
 = & \frac{p-1}{4K\kappa} \int \nabla \text{Ric} * \nabla (|\alpha|^2 |\text{Rm}|^{p-3} * \text{Rm} * \phi^{2p}) dV_t \\
 = & \frac{p-1}{4K\kappa} \int \nabla \text{Ric} * (2\alpha * \nabla \alpha * |\text{Rm}|^{p-3} * \text{Rm} * \phi^{2p} + |\alpha|^2 * \frac{p-3}{2} |\text{Rm}|^{p-5} * 2\text{Rm} * \nabla \text{Rm} * \text{Rm} * \phi^{2p} \\
 & + |\alpha|^2 |\text{Rm}|^{p-3} * \nabla \text{Rm} * \phi^{2p} + |\alpha|^2 |\text{Rm}|^{p-3} * \text{Rm} * 2p\phi^{2p-1} \nabla \phi) dV_t \\
 \leq & C_p \frac{L}{K} \int |\nabla \text{Ric}| |\nabla \alpha| |\text{Rm}|^{p-2} \phi^{2p} dV_t + C_p^2 \frac{L^2}{K} \int |\nabla \text{Ric}| |\nabla \text{Rm}| |\text{Rm}|^{p-3} \phi^{2p} dV_t \\
 & + C_p^2 \frac{L^2}{K} \int |\nabla \text{Ric}| |\nabla \phi| |\text{Rm}|^{p-2} \phi^{2p-1} dV_t \\
 \leq & C_p \frac{L}{K} \int (|\nabla \alpha| |\text{Rm}|^{\frac{p-1}{2}} \phi^p) (|\nabla \text{Ric}| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) dV_t \\
 & + C_p^2 \frac{L^2}{K} \int (|\nabla \text{Rm}| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) (|\nabla \text{Ric}| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) dV_t \\
 & + C_p^2 \frac{L^2}{K} \int (|\nabla \phi| |\text{Rm}|^{\frac{p-1}{2}} \phi^{p-1}) (|\nabla \text{Ric}| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) dV_t \\
 \leq & \frac{1}{100} B_3 + C_p^2 \frac{L^2}{K} B_2 + C_p^2 \frac{L^2}{K} A_4.
 \end{aligned}$$

Similarly,

$$\begin{aligned}
 & \frac{p-1}{4K\kappa} \int |\alpha|^2 |\text{Rm}|^{p-3} (\text{Rm} * \nabla^2 \alpha) \phi^{2p} dV_t \\
 = & \frac{p-1}{4K\kappa} \int \nabla \alpha * (2\alpha * \nabla \alpha * \\
 & |\text{Rm}|^{p-3} * \text{Rm} * \phi^{2p} + |\alpha|^2 * \frac{p-3}{2} |\text{Rm}|^{p-5} * 2\text{Rm} * \nabla \text{Rm} * \text{Rm} * \phi^{2p} \\
 & + |\alpha|^2 |\text{Rm}|^{p-3} * \nabla \text{Rm} * \phi^{2p} + |\alpha|^2 |\text{Rm}|^{p-3} * \text{Rm} * 2p\phi^{2p-1} \nabla \phi) dV_t \\
 \leq & C_p \frac{L}{K} \int |\nabla \alpha|^2 |\text{Rm}|^{p-2} \phi^{2p} dV_t + C_p^2 \frac{L^2}{K} \int |\nabla \alpha| |\nabla \text{Rm}| |\text{Rm}|^{p-3} \phi^{2p} dV_t \\
 & + C_p^2 \frac{L^2}{K} \int |\nabla \alpha| |\nabla \phi| |\text{Rm}|^{p-2} \phi^{2p-1} dV_t \\
 \leq & C_p \frac{L}{K} \int (|\nabla \alpha| |\text{Rm}|^{\frac{p-1}{2}} \phi^p) (|\nabla \alpha| |\text{Rm}|^{\frac{p-3}{2}} \phi^p) dV_t
 \end{aligned}$$

$$\begin{aligned}
 &+ Cp^2 \frac{L^2}{K} \int \left(|\nabla\alpha| |\text{Rm}|^{\frac{p-3}{2}} \phi^p \right) \left(|\nabla\text{Rm}| |\text{Rm}|^{\frac{p-3}{2}} \phi^p \right) dV_t \\
 &+ Cp^2 \frac{L^2}{K} \int \left(|\nabla\alpha| |\text{Rm}|^{\frac{p-3}{2}} \phi^p \right) \left(|\nabla\phi| |\text{Rm}|^{\frac{p-1}{2}} \phi^{p-1} \right) dV_t \\
 &\leq \frac{1}{100} B_3 + Cp^2 \frac{L^2}{K} B_5 + Cp^2 \frac{L^2}{K} B_2 + Cp^2 \frac{L^2}{K} A_4.
 \end{aligned}$$

Those estimates yield (3.11). \square

Lemma 3.4. We have

$$B_4 \leq CB_3. \tag{3.12}$$

Proof. It follows from the definitions. \square

Finally, we estimate the term B_5 .

Lemma 3.5. We have

$$\begin{aligned}
 B_5 &\leq -\frac{d}{dt} \left[\frac{1}{Cp^4L^2} \int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{1}{p^2(p-1)} \int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right] \\
 &- \frac{d}{dt} \left[\frac{C^{\frac{p-1}{2}} p^{2(p-3)} L^{p-3}}{K^{\frac{p-3}{2}}} \int |\alpha|^2 \phi^{2p} \right] + CA_1 + C \left(1 + K + L + \frac{K}{p^4L^2} \right) A_2 \\
 &+ \frac{C^{\frac{p-1}{2}} p^{2(p-3)} L^{p-1}}{K^{\frac{p-3}{2}}} \int |\nabla\phi|^2 \phi^{2p-2} dV_t + CA_4 \\
 &+ \frac{C^{\frac{(p-1)^2}{2(p-2)}} p^{2\frac{p^2-4p+5}{p-2}} (1+K+L)}{(K \wedge 1)^{\frac{1}{2}(p-2+\frac{1}{p-2})}} (L \vee 1)^{p+2+\frac{1}{p-2}} \int \phi^{2p} dV_t.
 \end{aligned} \tag{3.13}$$

Here

$$K \wedge 1 := \min\{K, 1\}, \quad L \vee 1 := \max\{L, 1\}.$$

Proof. For any $\eta > 0$, we have

$$\begin{aligned}
 B_5 &= \int |\nabla\alpha|^2 |\text{Rm}|^{p-3} \phi^{2p} dV_t \\
 &\leq \eta \int |\nabla\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{\left(\frac{p-3}{p-1}\right)^{\frac{p-1}{2}}}{\eta^{\frac{p-3}{2}}} \int |\nabla\alpha|^2 \phi^{2p} dV_t \\
 &= \eta B_3 + \frac{\left(\frac{p-3}{p-1}\right)^{\frac{p-1}{2}}}{\eta^{\frac{p-3}{2}}} \int |\nabla\alpha|^2 \phi^{2p} dV_t \\
 &\leq \eta B_3 + \frac{1}{\eta^{\frac{p-3}{2}}} \int |\nabla\alpha|^2 \phi^{2p} dV_t
 \end{aligned}$$

because, $0 \leq \frac{p-3}{p-1} < 1$, and for any $\epsilon > 0$,

$$\begin{aligned}
 |\text{Rm}|^{p-3} \phi^{2p} &= \left(\epsilon |\text{Rm}|^{p-3} \phi^{\frac{p-3}{p-1} 2p} \cdot \frac{1}{\epsilon} \phi^{\frac{2}{p-1} 2p} \right) \\
 &\leq \frac{p-3}{p-1} \left(\epsilon |\text{Rm}|^{p-3} \phi^{\frac{p-3}{p-1} 2p} \right)^{\frac{p-1}{p-3}} + \frac{2}{p-1} \left(\frac{1}{\epsilon} \phi^{\frac{2}{p-1} 2p} \right)^{\frac{p-1}{2}} \\
 &\leq \frac{p-3}{p-1} \epsilon^{\frac{p-1}{p-3}} |\text{Rm}|^{p-1} \phi^{2p} + \frac{2}{p-1} \frac{1}{\epsilon^{\frac{p-1}{2}}} \phi^{2p}.
 \end{aligned}$$

Using (2.13) yields

$$\begin{aligned} B_5 &\leq \eta B_3 + \frac{1}{\eta^{\frac{p-3}{2}}} \int \phi^{2p} \left[\frac{1}{2\kappa} (\kappa \Delta - \partial_t) |\alpha|^2 + CL^2 |\text{Rm}| + CL^2 (1 + K + L) \right] dV_t \\ &\leq \eta B_3 + \frac{C}{\eta^{\frac{p-3}{2}}} \int L^2 (1 + K + L) \phi^{2p} dV_t \\ &\quad + \frac{C}{\eta^{\frac{p-3}{2}} 2\kappa} \int \phi^{2p} (\kappa \Delta - \partial_t) |\alpha|^2 dV_t + \frac{CL^2}{\eta^{\frac{p-3}{2}}} \int |\text{Rm}| \phi^{2p} dV_t. \end{aligned}$$

To estimate the integral involving $|\text{Rm}| \phi^{2p}$, we observe for $p \geq 3$ that

$$\begin{aligned} |\text{Rm}| \phi^{2p} &= \left(\epsilon |\text{Rm}| \phi^{\frac{2p}{p-1}} \right) \left(\frac{1}{\epsilon} \phi^{2p \frac{p-2}{p-1}} \right) \\ &\leq \frac{1}{p-1} \left(\epsilon |\text{Rm}| \phi^{\frac{2p}{p-1}} \right)^{p-1} + \frac{p-2}{p-1} \left(\frac{1}{\epsilon} \phi^{2p \frac{p-2}{p-1}} \right)^{\frac{p-1}{p-2}} \\ &= \frac{\epsilon^{p-1}}{p-1} |\text{Rm}|^{p-1} \phi^{2p} + \frac{p-2}{p-1} \frac{1}{\epsilon^{\frac{p-1}{p-2}}} \phi^{2p}. \end{aligned}$$

Letting $\eta' := \epsilon^{p-1}/(p-1)$, it follows that

$$\begin{aligned} |\text{Rm}| \phi^{2p} &\leq \eta' |\text{Rm}|^{p-1} \phi^{2p} + \frac{p-2}{p-1} \frac{1}{(p-1)^{\frac{1}{p-2}}} \frac{1}{\eta'^{\frac{1}{p-2}}} \phi^{2p} \\ &\leq \eta' |\text{Rm}|^{p-1} \phi^{2p} + \frac{1}{\eta'^{\frac{1}{p-2}}} \phi^{2p}. \end{aligned} \tag{3.14}$$

Therefore

$$\begin{aligned} \frac{CL^2}{\eta^{\frac{p-3}{2}}} \int |\text{Rm}| \phi^{2p} dV_t &\leq \frac{CL^2}{\eta^{\frac{p-3}{2}}} \left(\eta' \int |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{1}{\eta'^{\frac{1}{p-2}}} \int \phi^{2p} dV_t \right) \\ &\leq \frac{CL^2 \eta'}{\eta^{\frac{p-3}{2}}} \int |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{CL^2}{\eta^{\frac{p-3}{2}} \eta'^{\frac{1}{p-2}}} \int \phi^{2p} dV_t. \end{aligned}$$

Choosing particularly $\eta' := \eta^{\frac{p-1}{2}}/CL^2$, it follows that

$$\frac{CL^2}{\eta^{\frac{p-3}{2}}} \int |\text{Rm}| \phi^{2p} dV_t \leq \eta \int |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{(CL^2)^{\frac{p-1}{p-2}}}{\eta^{\frac{p-3}{2} + \frac{p-1}{2(p-2)}}} \int \phi^{2p} dV_t.$$

Since $2 < 2 \frac{p-1}{p-2} \leq 4$ ($p \geq 3$), it follows that

$$\begin{aligned} B_5 &\leq \eta B_3 + \eta A_2 + \frac{CL^2(1+K+L)}{\eta^{\frac{p-3}{2}}} \int \phi^{2p} dV_t \\ &\quad + \frac{(CL^2)^{\frac{p-1}{p-2}}}{\eta^{\frac{p-3}{2} + \frac{p-1}{2(p-2)}}} \int \phi^{2p} dV_t + \frac{C}{\eta^{\frac{p-3}{2}} 2\kappa} \int \phi^{2p} (\Delta - \partial_t) |\alpha|^2 dV_t \\ &\leq \eta B_3 + \eta A_2 + \frac{CL^2(1+K+L)}{\eta^{\frac{p-3}{2}}} \int \phi^{2p} dV_t + \frac{C(L \vee 1)^4}{\eta^{\frac{p-3}{2} + \frac{p-1}{2(p-2)}}} \int \phi^{2p} dV_t \\ &\quad + \frac{C}{\eta^{\frac{p-3}{2}} 2\kappa} \int \phi^{2p} (\kappa \Delta - \partial_t) |\alpha|^2 dV_t. \end{aligned}$$

To estimate the last integral we start with

$$\begin{aligned} \int (\Delta |\alpha|^2) \phi^{2p} dV_t &= - \int \langle \Delta |\alpha|^2, \nabla \phi^{2p} \rangle dV_t \\ &= - \int \langle 2\alpha \nabla \alpha, 2p \phi^{2p-1} \nabla \phi \rangle dV_t \\ &= -4p \int \alpha \phi^{2p-1} \langle \nabla \alpha, \nabla \phi \rangle dV_t \end{aligned}$$

$$\begin{aligned} &\leq Cp \int |\alpha| |\nabla \alpha| \phi^p |\nabla \phi| \phi^{p-1} dV_t \\ &\leq \epsilon \int |\nabla \alpha|^2 \phi^{2p} dV_t + \frac{Cp^2}{\epsilon} \int |\alpha|^2 |\nabla \phi|^2 \phi^{2p-2} dV_t \\ &\leq \frac{Cp^2 L^2}{\epsilon} \int |\nabla \phi|^2 \phi^{2p-2} dV_t + \epsilon \int \phi^{2p} \left[\frac{1}{2\kappa} (\kappa \Delta - \partial_t) |\alpha|^2 \right. \\ &\quad \left. + CL^2 |\text{Rm}| + CL^2 (1 + K + L) \right] dV_t. \end{aligned}$$

Together with

$$\begin{aligned} \int (-\partial_t |\alpha|^2) \phi^{2p} dV_t &= -\frac{d}{dt} \left(\int |\alpha|^2 \phi^{2p} dV_t \right) + \int |\alpha|^2 \phi^{2p} \partial_t dV_t \\ &\leq C(1 + K + L) \int |\alpha|^2 \phi^{2p} dV_t - \frac{d}{dt} \left[\int |\alpha|^2 \phi^{2p} dV_t \right], \end{aligned}$$

we arrive at

$$\begin{aligned} \frac{1}{\kappa} \int [(\kappa \Delta - \partial_t) |\alpha|^2] \phi^{2p} dV_t &\leq \frac{\epsilon}{2\kappa} \int \phi^{2p} [(\kappa \Delta - \partial_t) |\alpha|^2] dV_t \\ &\quad + \frac{Cp^2 L^2}{\epsilon} \int |\nabla \phi|^2 \phi^{2p-2} dV_t \\ &\quad + C(1 + \epsilon) L^2 (1 + K + L) \int \phi^{2p} dV_t \\ &\quad + C\epsilon L^2 \int \phi^{2p} |\text{Rm}| dV_t \\ &\quad - \frac{d}{dt} \left[\frac{1}{\kappa} \int |\alpha|^2 \phi^{2p} dV_t \right]. \end{aligned}$$

Therefore (taking $\epsilon = 1$)

$$\begin{aligned} &\int [(\kappa \Delta - \partial_t) |\alpha|^2] \phi^{2p} dV_t \\ &\leq Cp^2 L^2 \int |\nabla \phi|^2 \phi^{2p-2} dV_t + CL^2 (1 + K + L) \int \phi^{2p} dV_t \\ &\quad + CL^2 \int |\text{Rm}| \phi^{2p} dV_t - \frac{d}{dt} \left[2 \int |\alpha|^2 \phi^{2p} dV_t \right] \\ &\leq Cp^2 L^2 \int |\nabla \phi|^2 \phi^{2p-2} dV_t + CL^2 (1 + K + L) \int \phi^{2p} dV_t \\ &\quad + CL^2 \bar{\eta} A_2 + \frac{CL^2}{\bar{\eta}^{\frac{1}{p-2}}} \int \phi^{2p} dV_t - \frac{d}{dt} \left[2 \int |\alpha|^2 \phi^{2p} dV_t \right] \end{aligned}$$

using (3.14), where $\bar{\eta}$ is any positive number. Consequently

$$\begin{aligned} B_5 &\leq \eta B_3 + \eta A_2 + \left[\frac{CL^2 (1 + K + L)}{\eta^{\frac{p-3}{2}}} + \frac{C(L \vee 1)^4}{\eta^{\frac{p-3}{2} + \frac{p-1}{2(p-2)}}} \right] \int \phi^{2p} dV_t + \frac{CL^2 \bar{\eta}}{\eta^{\frac{p-3}{2}}} A_2 \\ &\quad + \frac{Cp^2 L^2}{\eta^{\frac{p-3}{2}}} \int |\nabla \phi|^2 \phi^{2p-2} dV_t + \frac{CL^2}{\eta^{\frac{p-3}{2}} \bar{\eta}^{\frac{1}{p-2}}} \int \phi^{2p} dV_t \\ &\quad - \frac{d}{dt} \left[\frac{C}{\eta^{\frac{p-3}{2}}} \int |\alpha|^2 \phi^{2p} dV_t \right]. \end{aligned}$$

If we take $CL^2 \bar{\eta} / \eta^{\frac{p-3}{2}} = \eta$, then $\bar{\eta} = \eta^{\frac{p-1}{2}} / CL^2$ and

$$\frac{CL^2}{\eta^{\frac{p-3}{2}} \bar{\eta}^{\frac{1}{p-2}}} = \frac{(CL^2)^{\frac{p-1}{p-2}}}{\eta^{\frac{p-3}{2} + \frac{p-1}{2(p-2)}}}.$$

Hence

$$\begin{aligned}
 B_5 \leq & \eta B_3 + 2\eta A_2 - \frac{d}{dt} \left[\frac{C}{\eta^{\frac{p-3}{2}}} \int |\alpha|^2 \phi^{2p} dV_t \right] \\
 & + \frac{Cp^2 L^2}{\eta^{\frac{p-3}{2}}} \int |\nabla \phi|^2 \phi^{2p-2} dV_t \\
 & + \left[\frac{CL^2(1+K+L)}{\eta^{\frac{p-3}{2}}} + \frac{C(L \vee 1)^4}{\eta^{\frac{p-3}{2} + \frac{p-1}{2(p-2)}}} \right] \int \phi^{2p} dV_t.
 \end{aligned} \tag{3.15}$$

From (3.10) and (3.11) we obtain

$$\begin{aligned}
 B_3 \leq & -\frac{1}{2K} \frac{d}{dt} \left[\int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \right] \\
 & - \frac{p^2}{p-1} \frac{CL^2}{K} \frac{d}{dt} \left[\int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right] \\
 & + \frac{Cp^4 L^2}{K} A_1 + \frac{Cp^4 L^2}{K} (1+K+L) A_2 + \frac{Cp^4 L^2}{K} A_4 + \frac{Cp^4 L^2}{K} B_5.
 \end{aligned} \tag{3.16}$$

Plugging (3.16) into (3.15), it follows that

$$\begin{aligned}
 B_5 \leq & -\frac{\eta}{2K} \frac{d}{dt} \left[\int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \right] - \frac{p^2}{p-1} \frac{C\eta L^2}{K} \frac{d}{dt} \left[\int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right] \\
 & - \frac{d}{dt} \left[\frac{C}{\eta^{\frac{p-3}{2}}} \int |\alpha|^2 \phi^{2p} dV_t \right] \\
 & + \frac{Cp^4 \eta L^2}{K} [A_1 + (1+K+L)A_2 + A_4] + \frac{Cp^4 \eta L^2}{K} B_5 \\
 & + 2\eta A_2 + \frac{CL^2}{\eta^{\frac{p-3}{2}}} \int |\nabla \phi|^2 \phi^{2p-2} dV_t \\
 & + \left[\frac{Cp^2 L^2(1+K+L)}{\eta^{\frac{p-3}{2}}} + \frac{C(L \vee 1)^4}{\eta^{\frac{p-3}{2} + \frac{p-1}{2(p-2)}}} \right] \int \phi^{2p} dV_t.
 \end{aligned}$$

Taking $\eta := K/2Cp^4L^2$, we have

$$\frac{Cp^4 \eta L^2}{K} = \frac{1}{2}, \quad \frac{\eta}{2K} = \frac{1}{4Cp^4 L^2},$$

and then ($C \geq 1$)

$$\begin{aligned}
 B_5 \leq & -\frac{d}{dt} \left[\frac{1}{Cp^4 L^2} \int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{1}{p^2(p-1)} \int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right] \\
 & - \frac{d}{dt} \left[\frac{C^{\frac{p-1}{2}} p^{2(p-3)} L^{p-3}}{K^{\frac{p-3}{2}}} \int |\alpha|^2 \phi^{2p} \right] + [A_1 + (1+K+L)A_2 + A_4] \\
 & + \frac{4K}{Cp^4 L^2} A_2 + \frac{C^{\frac{p-1}{2}} p^{2(p-3)} L^{p-1}}{K^{\frac{p-3}{2}}} \int |\nabla \phi|^2 \phi^{2p-2} dV_t + \int \phi^{2p} dV_t \\
 & \cdot \left[\frac{C^{\frac{p-1}{2}} p^{2(p-2)} L^{p-1} (1+K+L)}{K^{\frac{p-3}{2}}} + \frac{C^{\frac{(p-1)^2}{2(p-2)}} p^{2\frac{p^2-4p+5}{p-2}} (L \vee 1)^{\frac{p^2-3}{p-2}}}{K^{\frac{p-3}{2} + \frac{p-1}{2(p-2)}}} \right] \\
 \leq & -\frac{d}{dt} \left[\frac{1}{Cp^4 L^2} \int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{1}{p^2(p-1)} \int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right] \\
 & - \frac{d}{dt} \left[\frac{C^{\frac{p-1}{2}} p^{2(p-3)} L^{p-3}}{K^{\frac{p-3}{2}}} \int |\alpha|^2 \phi^{2p} \right] + CA_1 + C \left(1+K+L + \frac{K}{p^4 L^2} \right) A_2
 \end{aligned}$$

$$\begin{aligned}
 &+ CA_4 + \frac{C^{\frac{p-1}{2}} p^{2(p-3)} L^{p-1}}{K^{\frac{p-3}{2}}} \int |\nabla\phi|^2 \phi^{2p-2} dV_t \\
 &+ \frac{C^{\frac{(p-1)^2}{2(p-2)}} p^{\frac{p^2-4p+5}{p-2}} (1+K+L)}{(K \wedge 1)^{\frac{1}{2}(p-2+\frac{1}{p-2})}} (L \vee 1)^{p+2+\frac{1}{p-2}} \int \phi^{2p} dV_t.
 \end{aligned}$$

Here $K \wedge 1 := \min\{K, 1\}$ and $L \vee 1 := \max\{L, 1\}$. \square

For convenience, let

$$\alpha_1 := 1 + K + L + \frac{K}{p^4 L^2}, \quad \alpha_2 := \frac{L^{p-1}}{K^{\frac{p-3}{2}}}, \quad \alpha_3 := \frac{(1+K+L)(L \vee 1)^{p+2+\frac{1}{p-2}}}{(K \wedge 1)^{\frac{1}{2}(p-2+\frac{1}{p-2})}}. \tag{3.17}$$

Then

$$\begin{aligned}
 B_5 \leq &-\frac{d}{dt} \left[\frac{1}{Cp^4 L^2} \int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{1}{p^2(p-1)} \int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right] \\
 &-\frac{d}{dt} \left[\frac{C^{\frac{p-1}{2}} p^{2(p-3)} \alpha_2}{L^2} \int |\alpha|^2 \phi^{2p} \right] + CA_1 + C\alpha_1 A_2 + CA_4 \\
 &+ C^{\frac{p-1}{2}} p^{2(p-3)} \alpha_2 \int |\nabla\phi|^2 \phi^{2p-2} dV_t + C^{\frac{(p-1)^2}{2(p-2)}} p^{\frac{p^2-4p+5}{p-2}} \alpha_3 \int \phi^{2p} dV_t.
 \end{aligned}$$

Hence from (3.16) we have

$$\begin{aligned}
 B_3 \leq &-\frac{d}{dt} \left[\frac{C}{K} \int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{p^2}{p-1} \frac{CL^2}{K} \int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right] \\
 &-\frac{d}{dt} \left[\frac{C^{\frac{p+1}{2}} p^{2(p-1)} \alpha_2}{K} \int |\alpha|^2 \phi^{2p} \right] + \frac{CL^2}{K} A_1 + \frac{Cp^4 L^2}{K} \alpha_1 A_2 + \frac{Cp^4 L^2}{K} A_4 \\
 &+ \frac{C^{\frac{p+1}{2}} p^{2(p-1)} L^2}{K} \alpha_2 \int |\nabla\phi|^2 \phi^{2p-2} dV_t + \frac{C^{\frac{p^2-3}{2(p-2)}} p^{\frac{(p-1)^2}{p-2}} L^2}{K} \alpha_3 \int \phi^{2p} dV_t,
 \end{aligned}$$

and from (3.10) we have

$$\begin{aligned}
 B_2 \leq &-\frac{d}{dt} \left[\frac{C}{p^2 L^2} \int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + C \frac{p^2+1}{p-1} \int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right] \\
 &-\frac{d}{dt} \left[\frac{C^{\frac{p+1}{2}} p^{2(p-2)} \alpha_2}{L^2} \int |\alpha|^2 \phi^{2p} \right] + Cp^2 A_1 + Cp^2 \alpha_1 A_2 + Cp^2 A_4 \\
 &+ C^{\frac{p+1}{2}} p^{2(p-2)} \alpha_2 \int |\nabla\phi|^2 \phi^{2p-2} dV_t + C^{\frac{p^2-3}{2(p-2)}} p^{\frac{p^2-3p+3}{p-2}} \alpha_3 \int \phi^{2p} dV_t.
 \end{aligned}$$

From (3.9) we see that

$$\begin{aligned}
 B_1 \leq &-\frac{d}{dt} \left[\frac{1}{2K} \int |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{Cp^4(K^2+L^2)}{K(p-1)} \int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right. \\
 &+ \left. \frac{Cp^2(K^2+L^2)}{KL^2} \int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{C^{\frac{p+3}{2}} p^{2p} \alpha_2}{K} \int |\alpha|^2 \phi^{2p} dV_t \right] \\
 &+ \frac{Cp^4(K^2+L^2)}{K} A_1 + \frac{Cp^6(K^2+L^2)}{K} \alpha_1 A_2 + \frac{Cp^6(K^2+L^2)}{K} A_4 \\
 &+ \frac{C^{\frac{p+3}{2}} p^{2p} (K^2+L^2)}{K} \alpha_2 \int |\nabla\phi|^2 \phi^{2p-2} dV_t \\
 &+ \frac{C^{\frac{p^2+2p-7}{2(p-2)}} p^{\frac{p^2-p-1}{p-2}} (K^2+L^2)}{K} \alpha_3 \int \phi^{2p} dV_t.
 \end{aligned}$$

Finally, (3.8) yields

$$\begin{aligned}
 A'_1 \leq & -\frac{d}{dt} \left[\frac{1}{2K} \int |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{Cp^6(K^2 + L^2)}{K(p-1)} \int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right. \\
 & + \frac{Cp^2(K^2 + L^2)}{KL^2} \int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \\
 & \left. + \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{KL^2} \int |\alpha|^2 \phi^{2p} dV_t \right] \\
 & + \frac{Cp^6(K^2 + L^2)}{K} \alpha_1 A_1 + \frac{Cp^6(K^2 + L^2)}{K} \alpha_1 A_2 + \frac{Cp^6(K^2 + L^2)}{K} A_4 \\
 & + \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{K} \alpha_2 \int |\nabla \phi|^2 \phi^{2p-2} dV_t + \frac{C^{\frac{p^2+2p-7}{2(p-2)}} p^{2\frac{p^2-p-1}{p-2}} (K^2 + L^2)}{K} \alpha_3 \int \phi^{2p} dV_t.
 \end{aligned}$$

Because

$$\frac{p^2 + 2p - 7}{2(p-2)} \leq \frac{p}{2}, \quad \frac{p^2 - p - 1}{p-2} \leq 2p, \quad p \geq 3,$$

we arrive at

$$\begin{aligned}
 A'_1 \leq & -\frac{d}{dt} \left[\frac{1}{2K} \int |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{Cp^6(K^2 + L^2)}{K(p-1)} \int |\text{Rm}|^{p-1} \phi^{2p} dV_t \right. \\
 & + \frac{Cp^2(K^2 + L^2)}{KL^2} \int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t + \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{KL^2} \int |\alpha|^2 \phi^{2p} dV_t \left. \right] \\
 & + \frac{Cp^6(K^2 + L^2)}{K} \alpha_1 A_1 + \frac{Cp^6(K^2 + L^2)}{K} \alpha_1 A_2 + \frac{Cp^6(K^2 + L^2)}{K} A_4 \\
 & + \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{K} \alpha_2 \int |\nabla \phi|^2 \phi^{2p-2} dV_t + \frac{C^{\frac{p}{2}} p^p(K^2 + L^2)}{K} \alpha_3 \int \phi^{2p} dV_t.
 \end{aligned}$$

3.2. Local curvature estimates

To prove Theorem 2.1, we introduce the following quantity

$$\begin{aligned}
 U(t) := & \int |\text{Rm}|^p \phi^{2p} dV_t + \frac{1}{2K} \int |\text{Ric}|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \\
 & + \frac{Cp^6(K^2 + L^2)}{K(p-1)} \int |\text{Rm}|^{p-1} \phi^{2p} dV_t \\
 & + \frac{Cp^2(K^2 + L^2)}{KL^2} \int |\alpha|^2 |\text{Rm}|^{p-1} \phi^{2p} dV_t \\
 & + \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{KL^2} \int |\alpha|^2 \phi^{2p} dV_t.
 \end{aligned} \tag{3.18}$$

Then

$$\begin{aligned}
 U' \leq & \frac{Cp^6(K^2 + L^2)}{K} (\alpha_1 U + A_4 + \alpha_1 A_2) \\
 & + \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{K} \alpha_2 \int |\nabla \phi|^2 \phi^{2p-2} dV_t + \frac{C^{\frac{p}{2}} p^p(K^2 + L^2)}{K} \alpha_3 \int \phi^{2p} dV_t.
 \end{aligned} \tag{3.19}$$

In the following we will prove a local curvature estimate so we may without loss of generality assume that M is a complete manifold (M may not be compact). Assume now that

$$|\text{Ric}_{g(t)}|_{g(t)} \leq K, \quad |\alpha(t)|_{g(t)} \leq L \quad \text{on } \Omega = B_{g(0)}(x_0, \rho/\sqrt{K}) \tag{3.20}$$

on $\Omega \times [0, T]$, where ρ, K, L are positive constants and $x_0 \in M$, and Ω is compactly contained in M . Consider the cutoff function

$$\phi := \left(\frac{\rho/\sqrt{K} - d_{g(0)}(x_0, \cdot)}{\theta \rho/\sqrt{K}} \right)_+, \tag{3.21}$$

where $\theta \geq 1$ is any positive constant. Then

$$e^{-2(K+|a|+|b|L)t}g(0) \leq g(t) \leq e^{2(K+|a|+|b|L)t}g(0)$$

and

$$|\nabla_{g(t)}\phi|_{g(t)} \leq e^{(K+|a|+|b|L)T}|\nabla_{g(0)}\phi|_{g(0)} \leq \frac{\sqrt{K}}{\theta\rho}e^{(K+|a|+|b|L)T}$$

for any $t \in [0, T]$. Set

$$K' := K + |a| + |b|L. \tag{3.22}$$

Then

$$e^{-2K't}g(0) \leq g(t) \leq e^{2K't}g(0), \quad |\nabla_{g(t)}\phi|_{g(t)} \leq \frac{\sqrt{K'}}{\theta\rho}e^{K'T}. \tag{3.23}$$

By Young's inequality we have

$$\begin{aligned} A_4 &= \int_{\mathcal{M}} |\text{Rm}|^{p-1} |\nabla\phi|^2 \phi^{2p-2} dV_t \\ &\leq \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |\text{Rm}|^{p-1} \phi^{2p-2} \frac{K'}{\theta^2 \rho^2} e^{2K'T} dV_t \\ &\leq \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} \left[\frac{(|\text{Rm}|^{p-1} \phi^{2p-2})^{\frac{p}{p-1}}}{\frac{p}{p-1}} + \frac{(K'(\theta\rho)^{-2} e^{2K'T})^p}{p} \right] dV_t \\ &\leq A_1 + \frac{K'^p e^{2K'pT}}{p} (\theta\rho)^{-2p} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \\ &\leq U + K'^p (\theta\rho)^{-2p} e^{2K'pT} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right). \end{aligned}$$

The inequality (3.23) yields

$$\int_{\mathcal{M}} |\nabla\phi|^2 \phi^{2p-2} dV_t \leq K'\theta^{-2p} \rho^{-2} e^{2K'T} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right)$$

and

$$\int_{\mathcal{M}} \phi^{2p} dV_t \leq \theta^{-2p} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right).$$

Therefore from (3.19) and $A_2 \leq \theta^{-1}A_4 \leq A_4$, we obtain

$$\begin{aligned} U' &\leq \frac{Cp^6(K^2 + L^2)}{K} a_1 U + \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{\theta^{2p} K} \\ &\quad \times \left(a_1 K'^p \rho^{-2p} e^{2K'pT} + a_2 K' \rho^{-2} e^{2K'T} + a_3 \right) \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right). \end{aligned}$$

According to Young's inequality, we have

$$\begin{aligned} a_2 K' \rho^{-2} e^{2K'T} &= \left(p^{1/p} K' \rho^{-2} e^{2K'T} \right) \left(\frac{a_2}{p^{1/p}} \right) \\ &\leq K'^p \rho^{-2p} e^{2K'pT} + \frac{(a_2/p^{1/p})^q}{q} \\ &\leq K'^p \rho^{-2p} e^{2K'pT} + a_2^q \\ &\leq a_1 K'^p \rho^{-2p} e^{2K'pT} + (a_2 \vee 1)^2, \end{aligned}$$

where $q = \frac{p}{p-1} \in (1, 2)$ (because $p \geq 3$). Consequently

$$\begin{aligned} \frac{d}{dt} U(t) &\leq \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{\theta^{2p} K} \left[a_1 K'^p \rho^{-2p} e^{2K'pT} + (a_2 \vee 1)^2 + a_3 \right] \\ &\quad \times \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) + \frac{Cp^6(K^2 + L^2)}{K} a_1 U(t). \end{aligned} \tag{3.24}$$

For convenience, we also introduce

$$A := Cp^6 \left(1 + \frac{K^2 + L^2}{K} \right) \mathbf{a}_1,$$

$$B := \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{\theta^{2p} K} \left[\mathbf{a}_1 K'^p \rho^{-2p} e^{2K'pT} + (\mathbf{a}_2 \vee 1)^2 + \mathbf{a}_3 \right].$$

Now the inequality (3.24) becomes

$$U'(t) \leq AU(t) + B \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right)$$

and then

$$e^{-At} U(t) \leq U(0) + \int_0^t B e^{-A\tau} \text{Vol}_{g(\tau)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) d\tau. \tag{3.25}$$

Using (3.23) we have that for any $\tau \in [0, t]$,

$$g(\tau) \leq e^{2K'\tau} g(0) \leq e^{2K'\tau} e^{2K't} g(t) \leq e^{4K'T} g(t)$$

and hence

$$\text{Vol}_{g(\tau)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \leq e^{2nk'T} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right).$$

Plugging into (3.25), it follows that

$$\begin{aligned} U(t) &\leq e^{At} \left[U(0) + B e^{2nk'T} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \int_0^t e^{-A\tau} d\tau \right] \\ &\leq e^{At} \left[U(0) + \frac{B}{A} e^{2nk'T} (1 - e^{-At}) \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \right] \\ &\leq e^{At} \left[U(0) + \frac{B}{A} e^{2nk'T} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \right]. \end{aligned} \tag{3.26}$$

Moreover we get

$$U(t) \leq e^{AT} \left[U(0) + \frac{B}{A} e^{4nk'T} \text{Vol}_{g(T)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \right], \quad t \in [0, T].$$

The last step is to estimate the initial data $U(0)$:

$$\begin{aligned} U(0) &\leq \int |\text{Rm}_{g(0)}|_{g(0)}^p \phi^{2p} dV_{g(0)} + \frac{1}{2K} \int |\text{Ric}_{g(0)}|_{g(0)}^2 |\text{Rm}_{g(0)}|_{g(0)}^{p-1} \phi^{2p} dV_{g(0)} \\ &\quad + \frac{Cp^6(K^2 + L^2)}{K(p-1)} \int |\text{Rm}_{g(0)}|_{g(0)}^{p-1} \phi^{2p} dV_{g(0)} + \frac{Cp^2(K^2 + L^2)}{KL^2} \int |\alpha(0)|_{g(0)}^2 |\text{Rm}_{g(0)}|_{g(0)}^{p-1} \phi^{2p} dV_{g(0)} \\ &\quad + \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{KL^2} \int |\alpha(0)|^2 \phi^{2p} dV_{g(0)} \\ &\leq \int |\text{Rm}_{g(0)}|_{g(0)}^p \phi^{2p} dV_{g(0)} + \frac{Cp^6(K^2 + L^2)}{K} \int |\text{Rm}_{g(0)}|_{g(0)}^{p-1} \phi^{2p} dV_{g(0)} \\ &\quad + \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{K} \int \phi^{2p} dV_{g(0)} \\ &\leq \int |\text{Rm}_{g(0)}|_{g(0)}^p \phi^{2p} dV_{g(0)} + \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{K} \int \phi^{2p} dV_{g(0)} \\ &\quad + \frac{Cp^6(K^2 + L^2)}{K} \left[\frac{p-1}{p} \int |\text{Rm}_{g(0)}|_{g(0)}^p \phi^{2p} dV_{g(0)} + \frac{1}{p} \int \phi^{2p} dV_{g(0)} \right] \\ &\leq Cp^6 \left(1 + \frac{K^2 + L^2}{K} \right) \int |\text{Rm}_{g(0)}|_{g(0)}^p \phi^{2p} dV_{g(0)} \\ &\quad + \frac{C^{\frac{p+3}{2}} p^{2p}(K^2 + L^2)}{K\theta^{2p}} \text{Vol}_{g(0)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \end{aligned}$$

$$\begin{aligned} &\leq Cp^6 \left(1 + \frac{K^2 + L^2}{K}\right) \int |\text{Rm}_{g(0)}|_{g(0)}^p \phi^{2p} dV_{g(0)} \\ &\quad + \frac{C^{\frac{p+3}{2}} p^{2p} (K^2 + L^2)}{K\theta^{2p}} e^{nk'T} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right). \end{aligned}$$

Together with (3.26), we arrive at

$$\begin{aligned} U(t) &\leq e^{AT} \left[A \int |\text{Rm}_{g(0)}|_{g(0)}^p \phi^{2p} dV_{g(0)} + \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \right. \\ &\quad \times \left. \left(\frac{C^{\frac{p+3}{2}} p^{2p} (K^2 + L^2)}{K\theta^{2p}} e^{nk'T} + \frac{B}{A} e^{2nk'T} \right) \right] \\ &\leq e^{AT} \left[Cp^6 \left(1 + \frac{K^2 + L^2}{K}\right) a_1 \int |\text{Rm}_{g(0)}|_{g(0)}^p \phi^{2p} dV_{g(0)} \right. \\ &\quad \left. + \left(\frac{B}{A} + \frac{C^{\frac{p+3}{2}} p^{2p} (K^2 + L^2)}{\theta^{2p} K} \right) e^{2nk'T} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \right]. \end{aligned} \tag{3.27}$$

In particular, for any $\tau > 1$, one has

$$\begin{aligned} &\int_{B_{g(0)}(x_0, \rho/\tau\sqrt{K})} |\text{Rm}_{g(t)}|_{g(t)}^p dV_{g(t)} \\ &\leq \left(\frac{\tau}{\tau - 1} \theta \right)^{2p} e^{AT} \left[\frac{Cp^6}{\theta^{2p}} \left(1 + \frac{K^2 + L^2}{K}\right) a_1 \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |\text{Rm}_{g(0)}|_{g(0)}^p dV_{g(0)} \right. \\ &\quad \left. + \left(\frac{B}{A} + \frac{C^{\frac{p+3}{2}} p^{2p} (K^2 + L^2)}{\theta^{2p} K} \right) e^{2nk'T} \text{Vol}_{g(t)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \right]. \end{aligned} \tag{3.28}$$

Fixing the volume of the ball $B_{g(0)}(x_0, \rho/\sqrt{K})$, we have

$$\begin{aligned} &\int_{B_{g(0)}(x_0, \rho/\tau\sqrt{K})} |\text{Rm}_{g(t)}|_{g(t)}^p dV_{g(t)} \\ &\leq \left(\frac{\tau}{\tau - 1} \theta \right)^{2p} e^{AT} \left[\frac{Cp^6}{\theta^{2p}} \left(1 + \frac{K^2 + L^2}{K}\right) a_1 \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |\text{Rm}_{g(0)}|_{g(0)}^p dV_{g(0)} \right. \\ &\quad \left. + \left(\frac{B}{A} + \frac{C^{\frac{p+3}{2}} p^{2p} (K^2 + L^2)}{\theta^{2p} K} \right) e^{4nk'T} \text{Vol}_{g(T)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \right]. \end{aligned} \tag{3.29}$$

3.3. Proof of Theorem 2.1

Letting $\tau \rightarrow +\infty$ in (3.29), it follows that

$$\begin{aligned} |\text{Rm}_{g(t)}|_{g(t)}^p(x_0) &\leq e^{AT} \left[Cp^6 \left(1 + \frac{K^2 + L^2}{K}\right) a_1 \int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |\text{Rm}_{g(0)}|_{g(0)}^p dV_{g(0)} \right. \\ &\quad \left. + \left(\frac{B\theta^{2p}}{A} + C^{\frac{p+3}{2}} p^{2p} \frac{K^2 + L^2}{K} \right) e^{4nk'T} \text{Vol}_{g(T)} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \right] \end{aligned}$$

and then

$$\begin{aligned} |\text{Rm}_{g(t)}|_{g(t)}(x_0) &\leq e^{Cp^5(1+(K^2+L^2)/K)a_1} 2^{p-1} \left\{ C^{1/p} p^{6/p} \left(1 + \frac{K^2 + L^2}{K}\right)^{1/p} a_1^{1/p} \right. \\ &\quad \times \left. \left(\int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |\text{Rm}_{g(0)}|_{g(0)}^p dV_{g(0)} \right)^{1/p} \right. \\ &\quad \left. + 2^{p-1} \left[\left(\frac{B\theta^{2p}}{A} \right)^{\frac{1}{p}} + C^{\frac{p+3}{2p}} p^2 \left(\frac{K^2 + L^2}{K} \right)^{\frac{1}{p}} \right] e^{4nk'T/p} \text{Vol}_{g(T)}^{1/p} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \right\} \\ &\leq e^{Cp^5(1+(K^2+L^2)/K)a_1} 2^{p-1} \left\{ C^{1/p} p^{6/p} \left(1 + \frac{K^2 + L^2}{K}\right)^{1/p} a_1^{1/p} \right. \end{aligned}$$

$$\begin{aligned} & \times \left(\int_{B_{g(0)}(x_0, \rho/\sqrt{K})} |\text{Rm}_{g(0)}|_{g(0)}^p dV_{g(0)} \right)^{1/p} \\ & + 2^{p-1} \left[\frac{C^{\frac{p+1}{2p}} p^{\frac{2(p-3)}{p}}}{\alpha_1^{1/p}} \left(\alpha_1 K'^p \rho^{-2p} e^{2K'T} + (\alpha_2 \vee 1)^2 + \alpha_3 \right)^{\frac{1}{p}} \right. \\ & \left. + C^{\frac{p+3}{2p}} p^2 \left(\frac{K^2 + L^2}{K} \right)^{\frac{1}{p}} \right] e^{4nK'T/p} \text{Vol}_{g(T)}^{1/p} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \}. \end{aligned}$$

Because the above inequality holds for any $p \geq 3$, we particularly obtain

$$\begin{aligned} |\text{Rm}_{g(t)}|_{g(t)}(x_0) & \leq C e^{C(1+\frac{K^2+L^2}{K})\alpha'_1} \left\{ \left[\left(1 + \frac{K^2 + L^2}{K} \right) \alpha'_1 \right]^{1/4} \right. \\ & \times \|\text{Rm}_{g(0)}\|_{L^4(g(0), B_{g(0)}(x_0, \rho/\sqrt{K}))} + \\ & \left[(\alpha'_1 (K'/\rho^2)^4 e^{8K'T} + (\alpha'_2 \vee 1)^2 + \alpha'_3)^{1/4} + \left(\frac{K^2 + L^2}{K} \right)^{1/4} \right] \\ & \left. \times e^{nK'T} \text{Vol}_{g(T)}^{1/4} \left(B_{g(0)} \left(x_0, \frac{\rho}{\sqrt{K}} \right) \right) \right\} \end{aligned} \tag{3.30}$$

where

$$\alpha'_1 := 1 + K + L + \frac{K}{L^2}, \quad \alpha'_2 := \frac{L^3}{K^{1/2}}$$

and

$$\alpha'_3 := \frac{(1 + K + L)(L \vee 1)^{7/2}}{(K \wedge 1)^{5/4}}.$$

Now the estimate (3.30) implies [Theorem 2.1](#).

Remark 3.6. In [10], the first author applies the method of proving [Theorem 2.1](#) to the Ricci-harmonic flow and weakens the condition of [Theorem 2.7](#) in [9].

References

[1] Xiuxiong Chen, Jingrui Cheng, On the constant scalar curvature Kähler metrics, apriori estimates, [arXiv:1712.06697](#).
 [2] Xiuxiong Chen, Jingrui Cheng, On the constant scalar curvature Kähler metrics, existence results, [arXiv:1801.00656](#).
 [3] Xiuxiong Chen, Jingrui Cheng, On the constant scalar curvature Kähler metrics, general automorphism group, [arXiv:1801.05907](#).
 [4] Bennett Chow, Peng Lu, Lei Ni, Hamilton'S Ricci Flow, Graduate Studies in Mathematics, Vol. 77, American Mathematical Society, Providence, RI; Science Press, Beijing, New York, ISBN: 978-0-8218-4231-7, 2006, p. xxxvi+608, 0-8218-4231-5, MR2274812.
 [5] Teng Fei, Bin Guo, Duong H. Phong, Parabolic dimensional reductions of 11D supergravity, [arXiv:1806.00583](#).
 [6] Teng Fei, Bin Guo, Duong H. Phong, On convergence criteria for the coupled flow of Li-Yuan-Zhang, *Math. Z.* 292 (1–2) (2019) 473–497, MR3968911.
 [7] Bin Guo, Zhijie Huang, Duong H. Phong, Pseudo-locality for a coupled Ricci flow, *Comm. Anal. Geom.* 26 (3) (2018) 585–626, MR3844116.
 [8] Brett Kotschwar, Ovidiu Munteanu, Jiaping Wang, A local curvature estimate for the Ricci flow, *J. Funct. Anal.* 271 (9) (2016) 2604–2630, MR3545226.
 [9] Yi Li, Local curvature estimates for the Ricci-harmonic flow, [arXiv:1810.09760](#).
 [10] Yi Li, Local curvature estimates for the Ricci-harmonic flow II, 2021, in preparation.
 [11] Yi Li, Yuan Yuan, Yuguang Zhang, On a new geometric flow over Kähler manifolds, *Comm. Anal. Geom.* 28 (6) (2020) 1251–1288, MR4184819.
 [12] Duong H. Phong, Sebastien Picard, Xiangwen Zhang, Geometric flows and Strominger systems, *Math. Z.* 288 (1–2) (2018) 101–113, MR3774405.
 [13] Duong H. Phong, Sebastien Picard, Xiangwen Zhang, New curvature flows in complex geometry, in: *Surveys in differential geometry 2017. Celebrating the 50th anniversary of the Journal of Differential Geometry*, *Surv. Differ. Geom.*, Vol. 22, Int. Press, Somerville, MA, 2018, pp. 331–364, MR3838124.