Large Charge 't Hooft limit of $\mathcal{N} = 4$ SYM

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Based on 2306.00929 with Shota Komatsu and Yifan Wang

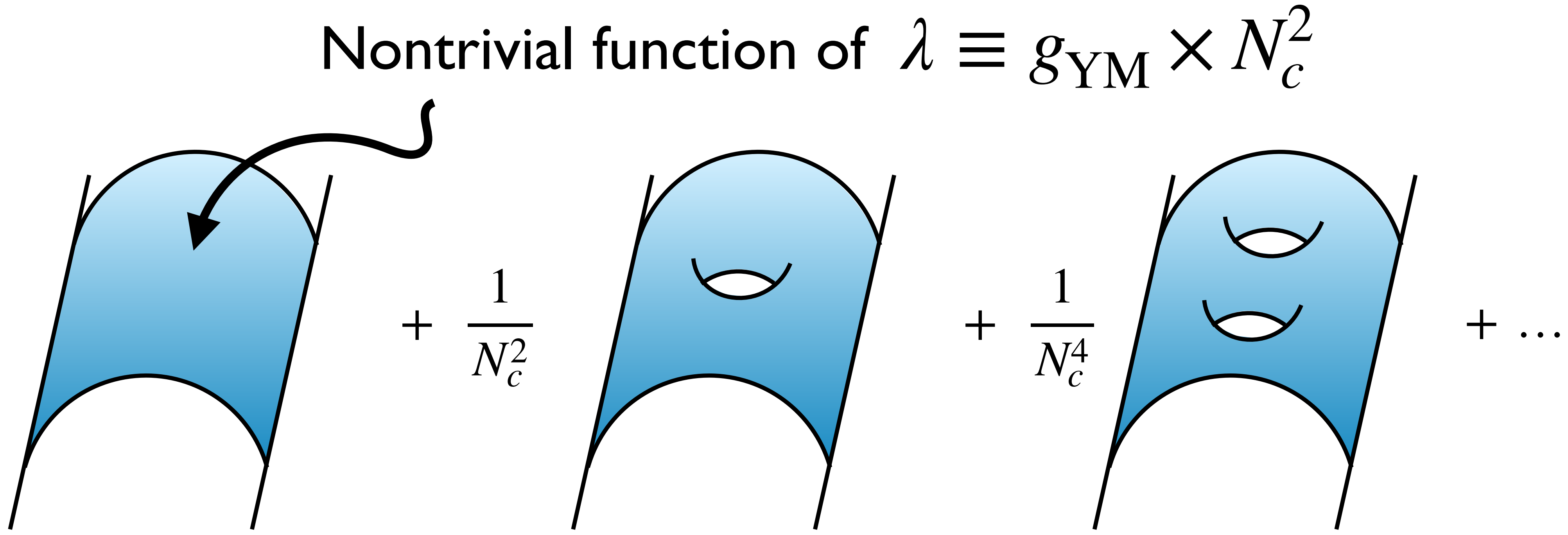
This talk is about systems with a large number of degrees of freedom

One way of considering large # d.o.f.: family of theories containing a parameter N_c that 'counts' them and take $N_c \to \infty$

In gauge theories, $N_c =$ number of colors

By considering the 't Hooft limit $N_c \to \infty$ $\lambda \equiv g_{\rm YM} \times N_c^2$ fixed

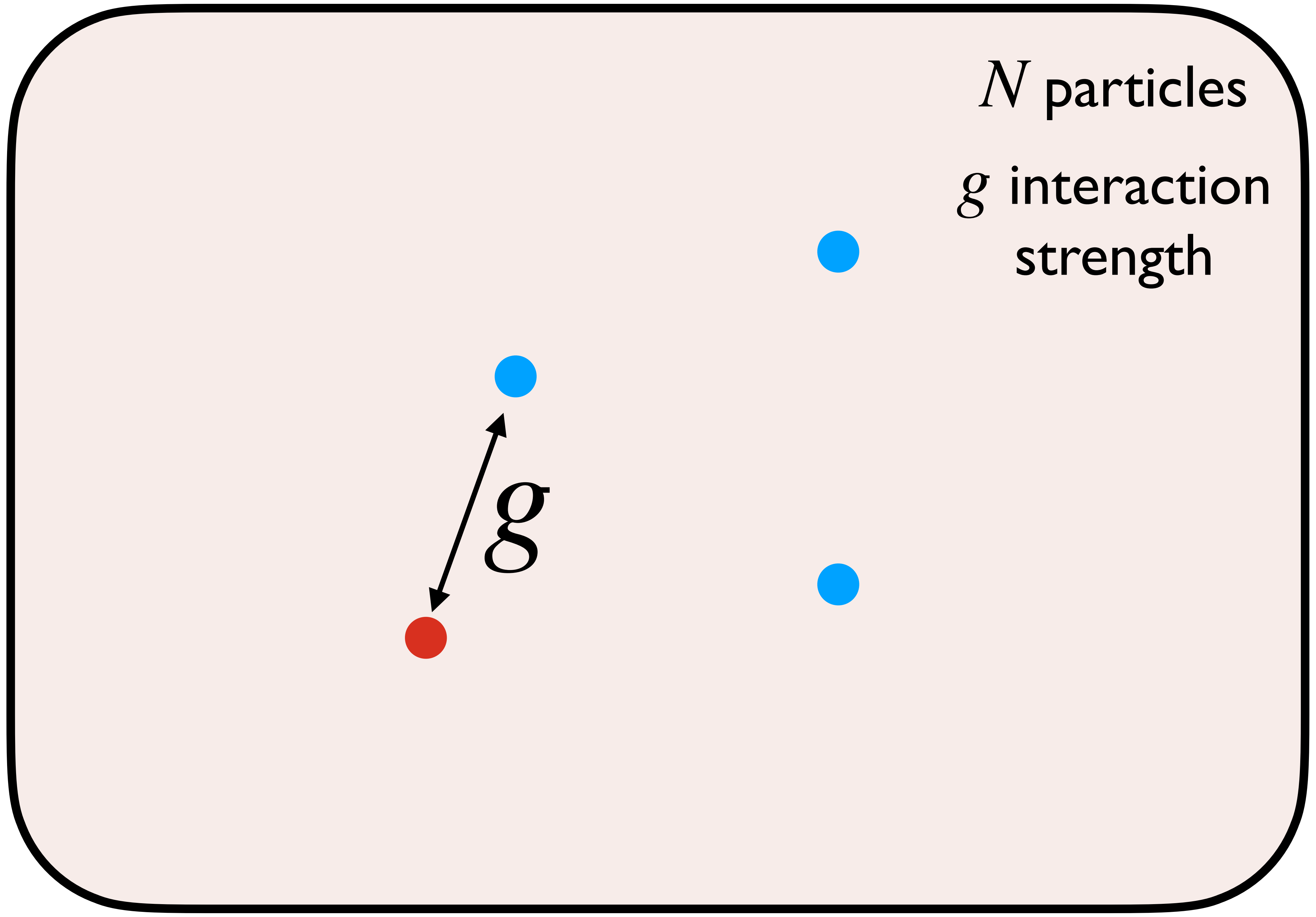
emergent structures appear:



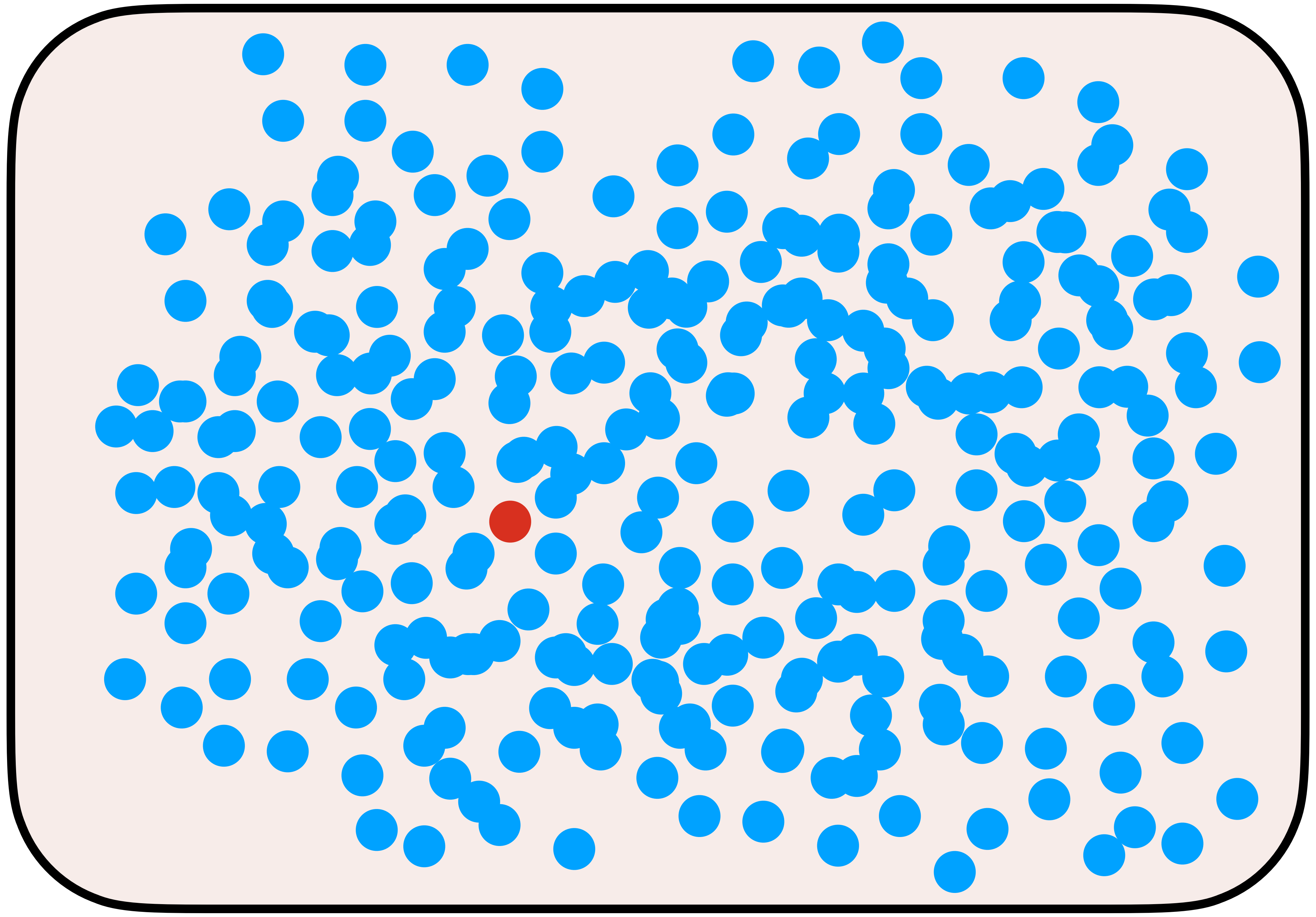
This talk is about systems with a large number of degrees of freedom

Another way of considering large # d.o.f.: state in a theory with a large number of excitations

Large # of d.o.f



For $g \ll 1$ and N small:
perturbation around free system



Effective interaction strength
$\lambda_{\text{eff}} \sim g\,N$

Large # of d.o.f

$$\lambda_{\text{eff}} \sim g\, N$$

Suggests a double scaling $N \to \infty$ and λ_{eff} fixed
(e.g. mean field theory)

Formally looks like a 't Hooft limit…

Is there any similarity with the standard 't Hooft limit....?

Large # of d.o.f

Gauge theory with a global charge J

Physics of this large charge 't Hooft limit $\overset{?}{\longleftrightarrow}$ Standard large N_c 't Hooft limit

$g_{\text{YM}}^2 J$ $\qquad$ $g_{\text{YM}}^2 N_c$

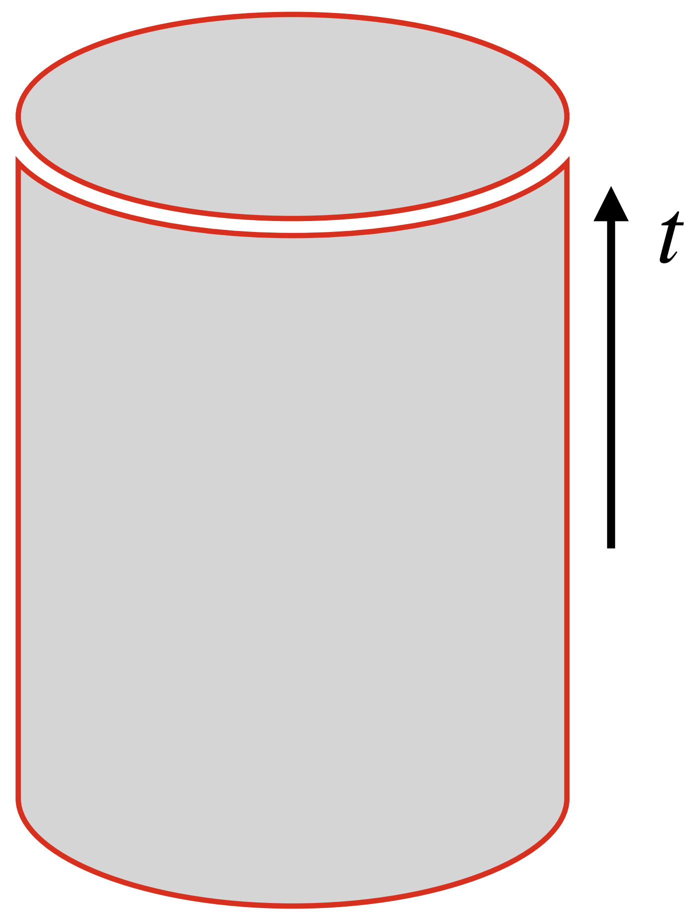
Setup: $\mathcal{N} = 4$ SYM

Outline

- Standard large charge limit vs large charge 't Hooft limit
- Spectral problem in $\mathcal{N} = 4$ SYM in the leading large charge
- Corrections to the large charge limit
- Higher-point functions
- Conclusions & future directions

Physics of large charged states in CFT$_d$

Two-point function $\langle O_J O_J \rangle \longrightarrow$ map to cylinder $R_t \times S_L^{d-1}$



Operator with minimal conformal dimension $\Delta_{\min}$ with charge J

$$E_{\text{state}} = \frac{\Delta_{\min}}{L} \qquad \longrightarrow \qquad \epsilon_{\text{state}} \sim \frac{\Delta_{\min}}{L^d} \qquad j_{\text{state}} \sim \frac{J}{L^{d-1}}$$

Large charge limit: $J \to \infty$, $L \to \infty$, j_{state} fixed:

Lowest energy state in flat space with fixed charge density

Physics of large states in CFT_d

$$\epsilon_{\text{state}} \sim \frac{\Delta_{\text{min}}}{L^d} \qquad j_{\text{state}} \sim \frac{J}{L^{d-1}}$$

- Generic (non-supersymmetric) CFTs, non-zero charge density state carries finite ϵ_{state}.

With both $\epsilon_{\text{state}}, j_{\text{state}}$ finite:

$$\Delta_{\text{min}} \sim J^{\frac{d}{d-1}}$$

[Alvarez-Gaume, Cuomo, Dondi, Giombi, Hellerman, Kalogerakis, Loukas, Monin, Orlando, Pirtskhalava, Rattazzi, Reffert, Sannino, Watanabe etc.]

- CFTs with moduli space of vacua, $\epsilon_{\text{state}} = 0$.

  E.g. in SCFT, BPS operators:

$$\Delta_{\text{min}} \sim J$$

[Arias-Tamargo, Beccaria, Bourget, Hellerman, Maeda, Orlando, Rodriguez-Gomez, Reffert, Russo, Watanabe etc.]

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Large charge limit vs Large charge 't Hooft limit (in SCFTs)

Example: $\mathcal{N} = 2$ SCFTs in $4D$

$$O_J \sim \mathrm{tr}\, \phi^J$$

(Coulomb branch operators)

Scalar of vector multiplet

Intuition:

$$\langle O_J(x_1)\bar{O}_J(x_2)\rangle = \int \mathcal{D}\phi\mathcal{D}\bar{\phi}\, \exp\left(-S + \underbrace{J\log(\phi)\,\delta(x-x_1) + J\log(\bar{\phi})\,\delta(x-x_2)}_{\text{Source terms}}\right)$$

$\rightarrow$ nontrivial profile of ϕ

$$\langle \phi \rangle \neq 0$$

Similar to the Coulomb Branch

Large charge limit vs
Large charge 't Hooft limit (in SCFTs)

In a theory with a marginal coupling $g_{\rm YM}$: $\langle \phi \rangle \sim g_{\rm YM} \sqrt{J}$

On this background, BPS W-bosons acquire $m_W \sim g_{\rm YM} \sqrt{J} \rightarrow \infty \quad \text{as} \quad J \rightarrow \infty$

Integrating them out → higher-derivatives of massless fields $\sim (1/J)^n$ ⇔ EFT

Powerful limit! However it looses some information about massive particles…

Large charge limit vs Large charge 't Hooft limit (in SCFTs)

Large charge 't Hooft limit:

$$J \to \infty$$
$$g_{\text{YM}} \to 0$$

$\lambda_J \equiv g_{\text{YM}}^2 J$ fixed

[Bourget, Rodriguez-Gomez, Russo'18]

- $m_W \sim g_{\text{YM}} \sqrt{J}$ are now finite → contribute to observables
- Different Physics from the standard large charge limit

$\mathcal{N} = 4$ SYM with SU(2) gauge group

Goal: Determining the spectrum of non-BPS operators in the large charge limit

Typically these operators are constructed as follows:

- Start with a protected 'vacuum' state $|Z\ldots Z\rangle \Leftrightarrow O_{\text{vac}} = \text{Tr}\,(Z\,Z)^{J/2}$

  "Coulomb branch operator"

  Complex scalar

  $\Delta = J$

- Add other fields to the vacuum $|Z\ldots Z\chi Z\ldots Z\chi Z\ldots Z\rangle$

  $\Delta = J + \text{corrections}$

$\mathcal{N} = 4$ SYM with SU(2) gauge group

As before, vacuum is very heavy: sources a non-trivial profile for Z

$$\left\langle O_{\rm vac}(t_1,\Omega)\bar{O}_{\rm vac}(t_2,\Omega)\right\rangle = \int [\mathscr{D}\Phi]\, \exp\left[-\frac{1}{g_{\rm YM}^2}\left(S_{\rm SYM}+\frac{\lambda}{2}\log({\rm Tr}\,ZZ)\,\delta(t-t_1)+\frac{\lambda}{2}\log({\rm Tr}\,\bar{Z}\bar{Z})\,\delta(t-t_2)\right)\right]$$

$\sim J$

Since these are protected operators, $S_{\rm SYM} \to S_{\rm free}$ and:

$$\langle Z\rangle = \begin{pmatrix} \frac{g_{\rm YM}\sqrt{J}}{2\pi}e^{i\varphi} & 0 \\ 0 & -\frac{g_{\rm YM}\sqrt{J}}{2\pi}e^{i\varphi} \end{pmatrix}$$

gauge partially fixed :

SU(2) →U(1)

$\mathcal{N} = 4$ SYM with SU(2) gauge group

$$|Z\ldots Z\chi Z\ldots Z\chi Z\ldots Z\rangle$$

Excitations are classified by irreps of the symmetry preserved by the vacuum

Symmetry preserved by the vacuum: $SO(4) \times SO(4) \longrightarrow \mathfrak{psu}(2|2)^2 \times \mathbb{R}$

- $SO(4)$: Rotational symmetry around origin
- $SO(4)$: Rotation of scalars that are not Z nor $\bar{Z}$

Spectrum from algebra

$\mathfrak{psu}(2|2)^2 \times \mathbb{R}$ not enough to constrain the dynamics (no g_{YM} dependence)

Actual symmetry is larger: centrally extended $\mathfrak{psu}(2|2)^2 \times \mathbb{R}$!

[Beisert]

Same symmetry as in the large N limit!

Central extension of $\mathfrak{psu}(2|2)$

In general: $$\{Q, Q\} = 0$$ for any two supercharges Q

This is true when acting on gauge invariant operators.

But on individual fields, this **does not** have to be the case!

$$\{Q, Q\}\chi \sim [Z, \chi]$$

Field-dependent gauge transformation!

↓

Carries information about the interaction terms of the Lagrangian

Central extension of $\mathfrak{psu}(2|2)$

The central extended algebra is then:

$$\{Q^a{}_\alpha, Q^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta} P\,, \quad \{S^\alpha{}_a, S^\beta{}_b\} = \epsilon_{ab}\epsilon^{\alpha\beta} K$$

$$P \cdot \chi \equiv [\mathrm{Z}, \chi]\,, \quad K \cdot \chi \equiv [\mathrm{Z}^{-1}, \chi]$$

The action of P and K are determined by the vev of Z!

Spectrum from central extension

Every field χ is an adjoint field of SU(2)

$$\chi = \begin{pmatrix} m^0 & m^{+} \\ m^{-} & -m^{0} \end{pmatrix}$$

U(1) gauge indices

Central extension

$$\{Q, Q\}\, m^{\pm} \sim \pm 2\lambda m^{\pm} \qquad \{Q, Q\}\, m^0 = 0$$

By demanding closure of the central extended algebra on a state $|Z \ldots m^{\pm} \ldots Z\rangle$

Single-particle state

$$(\hat{D} - \hat{J})^2 - 4PK = 1$$

Spectrum from central extension

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Single-particle state

$$(D - \hat{J})\, |Z \ldots m^{\pm} \ldots Z\rangle = \sqrt{1 + 16\,\lambda}\, |Z \ldots m^{\pm} \ldots Z\rangle$$

$$(D - \hat{J})\, |Z \ldots m^{0} \ldots Z\rangle = |Z \ldots m^{0} \ldots Z\rangle$$

Spectrum from central extension

Now to construct a gauge invariant state, demand

$$\# m^+ = \# m^-$$

(also states are parity even!)

e.g. $$|Z \ldots Z\, m^- m^- m^0 Z \ldots Z\, m^+ m^+ Z \ldots Z\rangle$$

Energy = $\sum$ individual energies

(interactions $\sim 1/J$)

$$\Delta - J = 1 + 4\sqrt{1 + 16\,\lambda}$$

Small λ expansion matches direct perturbative computation with Feynman diagrams (2-loop check)

Spectrum from central extension

Spectrum of excitations include spinning letters, i.e.

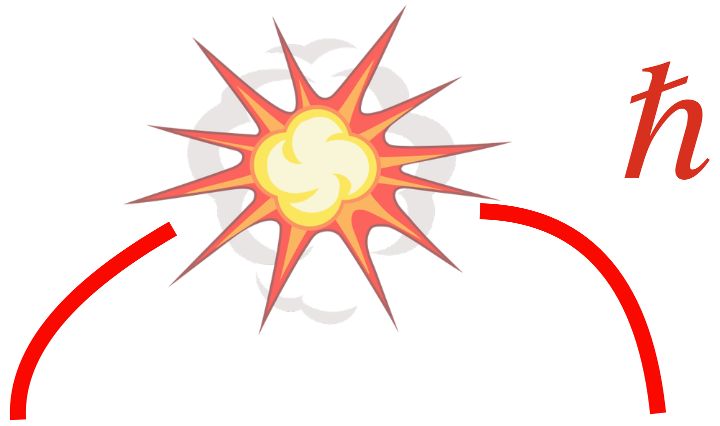
$$|Z \ldots Z\, D^n Z\, Z \ldots Z\rangle$$

They sit in (anti-symmetric) “bound state” BPS representation:

$$(\hat{D} - \hat{J})^2 - 4PK = (n+1)^2$$

$$E = \sqrt{n^2 + 16\,\lambda}$$

Spectrum at order $1/J$



$$\hbar \sim \frac{1}{J}$$

$$|Z \ldots Z\, m^- m^- m^0\, Z \ldots Z\, m^+\, Z \ldots Z\, m^+\, Z \ldots Z\rangle$$

At $1/J$, only two-body interactions.

Particles are commutative (no color indices at this point) $\longrightarrow$ every particle interacts with every other

Distinct from spin chain picture of large N!

Spectrum at order $1/J$

Algebraic constraints on the dilatation operator? $|Z\ldots Z m^- m^- m^0 Z\ldots Z m^+ Z\ldots Z m^+ Z\ldots Z\rangle$

E.g. $\mathfrak{su}(2|2)$ subsector:

U(1) gauge indices

$$\{ |\phi^m_{a=1,2}\rangle, |\psi^m_{\alpha=3,4}\rangle \}$$

Most general ansatz compatible with (bosonic) symmetries:

E.g. $$D_{1-\text{loop}}|\phi_a^{m_1}\phi_b^{m_2}\rangle_J = \alpha_{m_1m_2}(\lambda)|\phi_a^{m_1}\phi_b^{m_2}\rangle_J + \beta_{m_1m_2}(\lambda)|\phi_b^{m_1}\phi_a^{m_2}\rangle_J + \gamma_{m_1m_2}(\lambda)\epsilon^{\alpha\beta}\epsilon_{ab}|\psi_\alpha^{m_1}\psi_\beta^{m_2}\rangle_{J-1}$$

charge of the state (omitting vacuum fields)

Unknown coefficients

similarly to other fields…

Spectrum at order $1/J$

$$D_{1-\text{loop}}|\phi_a^{m_1}\phi_b^{m_2}\rangle_J = \alpha_{m_1m_2}(\lambda)|\phi_a^{m_1}\phi_b^{m_2}\rangle_J + \beta_{m_1m_2}(\lambda)|\phi_b^{m_1}\phi_a^{m_2}\rangle_J + \gamma_{m_1m_2}(\lambda)\,\epsilon^{\alpha\beta}\epsilon_{ab}|\psi_\alpha^{m_1}\psi_\beta^{m_2}\rangle_{J-1}$$

- We allow for one-loop corrections to the supercharges as well
- Impose closure of $\mathfrak{su}(2|2)$ algebra

Outcome:

- Coefficients α, β, γ uniquely fixed up to a global normalization!
- Product of two (short) fundamental representations $\rightarrow$ unique long-multiplet

Spectrum at order $1/J$

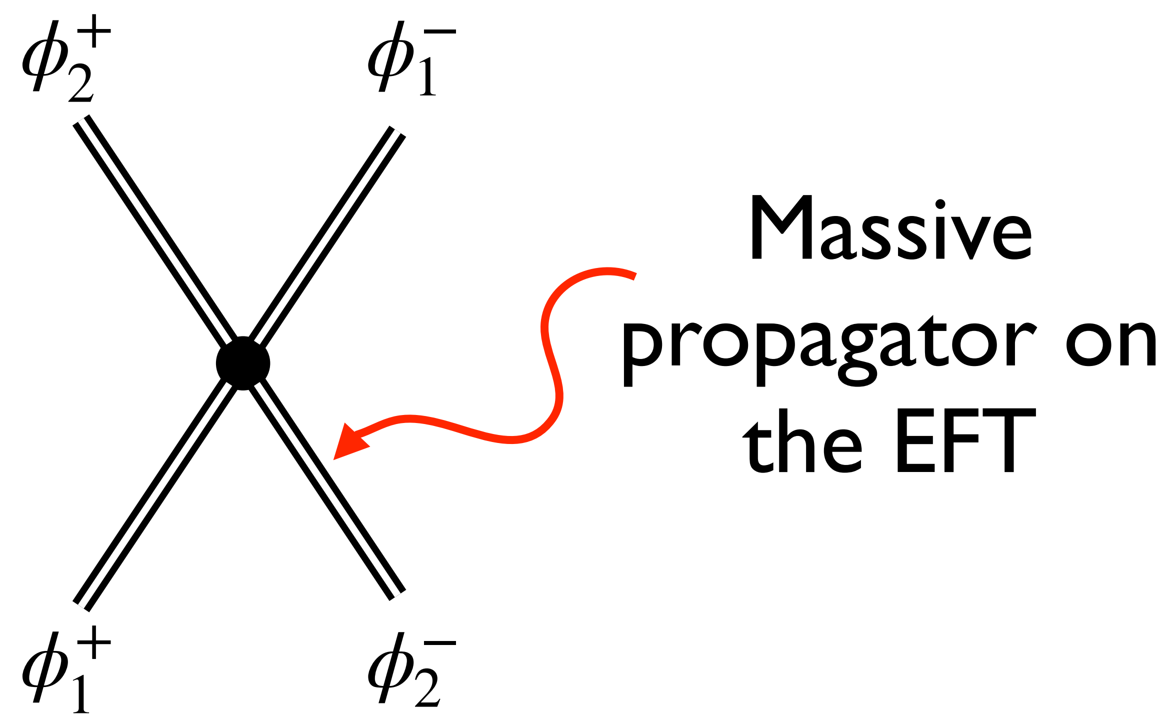
- Normalization from semiclassic perturbation theory:

$$Z = Z_{\text{classical}} + \delta Z$$

$$\mathscr{L}_{\text{EFT}} = \mathscr{L}_{\mathcal{N}=4\,\text{SYM}}\,[Z = Z_{\text{classical}} + \delta Z]$$

- The missing normalization: 1-loop Feynman diagrams in the EFT

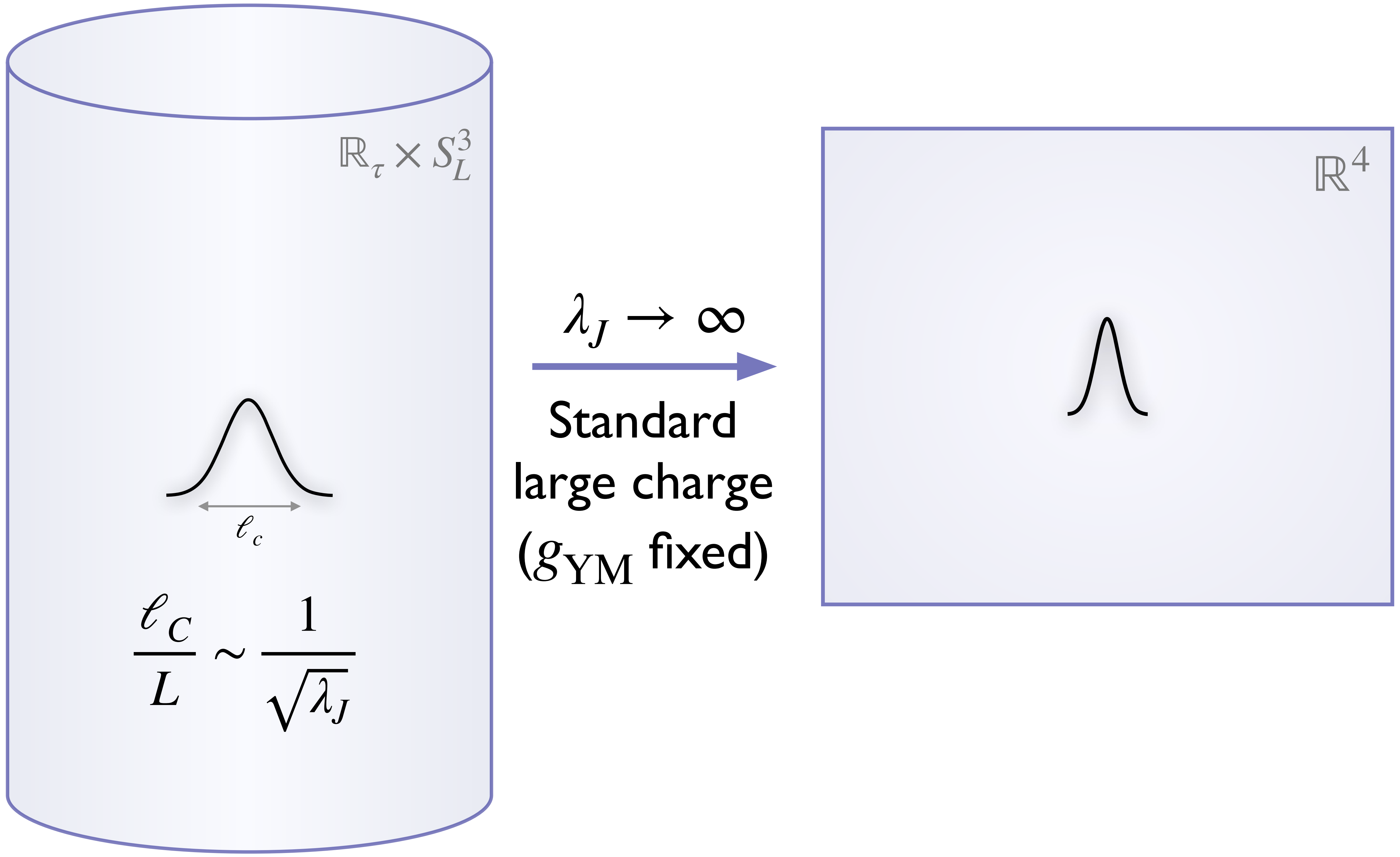
E.g. $\beta_{+-}(\lambda)$ arises from



$$\beta_{+-}(\lambda) = -\frac{8\lambda}{1+16\lambda} \simeq -8\,\lambda + 128\,\lambda^2 + \ldots$$

We determined a few of these coefficients by direct computation in the EFT but not all of them. We can compute the full spectrum to order $1/J$ up to two unknown functions of λ.

Relation to Poincaré SUSY



Centrally extended $\mathfrak{psu}(2|2)^2$ $\longrightarrow$ Centrally extended Poincaré SUSY

BPS rep of $\mathfrak{psu}(2|2)^2$ $\longrightarrow$ BPS particle of Poincaré SUSY

Relation to Poincaré SUSY

$\mathfrak{psu}(2|2)$:

$$\{S^\alpha{}_a, Q^b{}_\beta\} = \delta^b_a L^\alpha{}_\beta + \delta^\alpha_\beta R^b{}_a + \delta^b_a \delta^\alpha_\beta \frac{(\hat{D}-\hat{J})}{2},$$
$$\{Q^a{}_\alpha, Q^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta} P, \quad \{S^\alpha{}_a, S^\beta{}_b\} = \epsilon_{ab}\epsilon^{\alpha\beta} K.$$

In the standard large charge limit: $\hat{D}-\hat{J}, K, P \propto \sqrt{J}$

$$Q^a{}_\alpha = J^{1/4}\mathbf{Q}^a{}_\alpha, \quad S^\alpha{}_a = J^{1/4}\bar{\mathbf{Q}}^a{}_\alpha, \quad \frac{\hat{D}-\hat{J}}{2} = \sqrt{J}\,\mathbf{P}^0, \quad P = \sqrt{J}\mathbf{Z}, \quad K = \sqrt{J}\,\bar{\mathbf{Z}}.$$

$$\{\mathbf{Q}^a{}_\alpha, \bar{\mathbf{Q}}^b{}_{\dot\beta}\} = \delta^{a,b}\delta_{\alpha,\dot\beta}\mathbf{P}_0,$$
$$\{\mathbf{Q}^a{}_\alpha, \mathbf{Q}^b{}_\beta\} = \epsilon^{ab}\epsilon_{\alpha\beta}\mathbf{Z}, \qquad \{\bar{\mathbf{Q}}^a{}_{\dot\alpha}, \bar{\mathbf{Q}}^b{}_{\dot\beta}\} = \epsilon^{ab}\epsilon_{\dot\alpha\dot\beta}\bar{\mathbf{Z}},$$

$$(\hat{D}-\hat{J})^2 - 4PK = (n+1)^2 \quad \mapsto \quad \mathbf{P}_0 = |\mathbf{Z}|$$

BPS condition becomes the massive BPS condition in flat space extended SUSY

This ends the analysis of the spectrum in the large charge limit.

Other interesting observables are the higher point functions.

Higher-point functions

- Consider higher-point functions involving also light-operators

$$\langle O_J(0) \underbrace{O_{i_1}(x_1)\dots O_{i_n}(x_n)}_{\text{Light-operators}} O_J(\infty)\rangle = \langle J|\, O_{i_1}(x_1)\dots O_{i_n}(x_n)\,|J\rangle$$

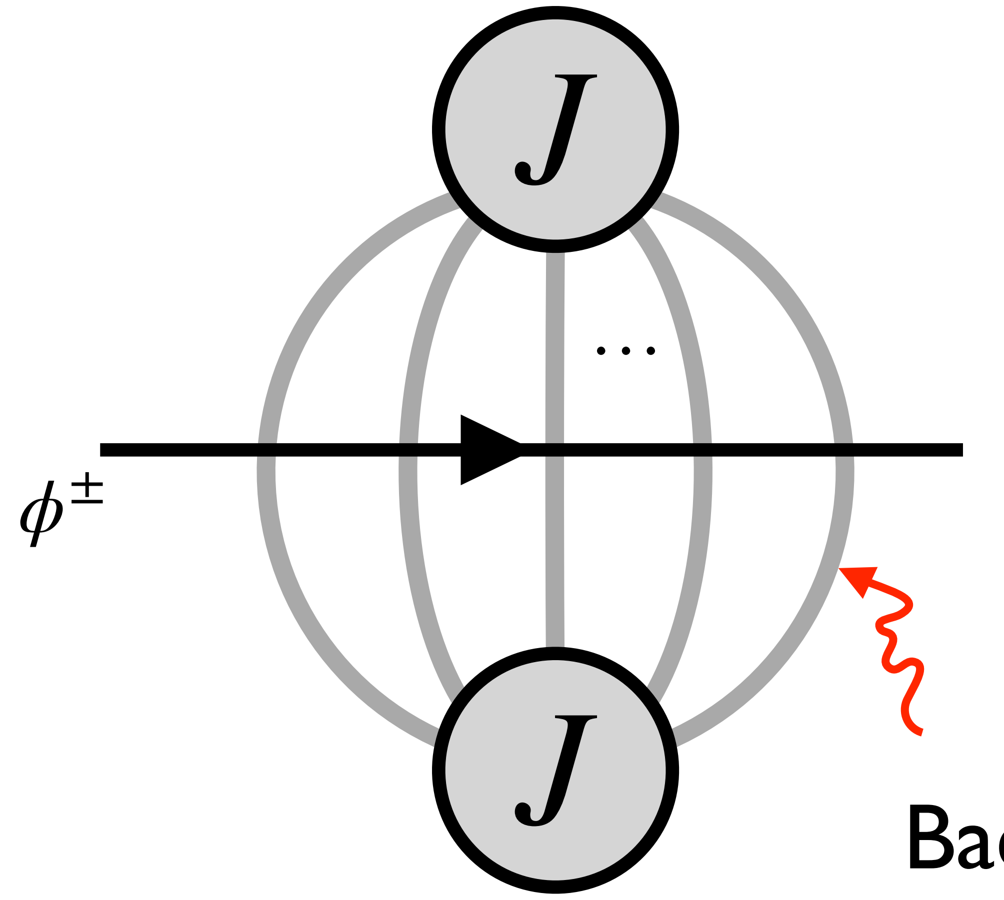
- Example: light half-BPS operators $O_2 \sim \mathrm{tr}(\mathrm{Y}^{\mathrm{I}}\phi_{\mathrm{I}})^2$ Protected operators, defined by polarization null vector Y_I

- Four-point function $\langle J|\, O_2(x_1) O_2(x_2)\,|J\rangle \overset{\text{'t Hooft}}{=} \langle O_2(x_1) O_2(x_2)\rangle_{\mathrm{LC\,backg}}$

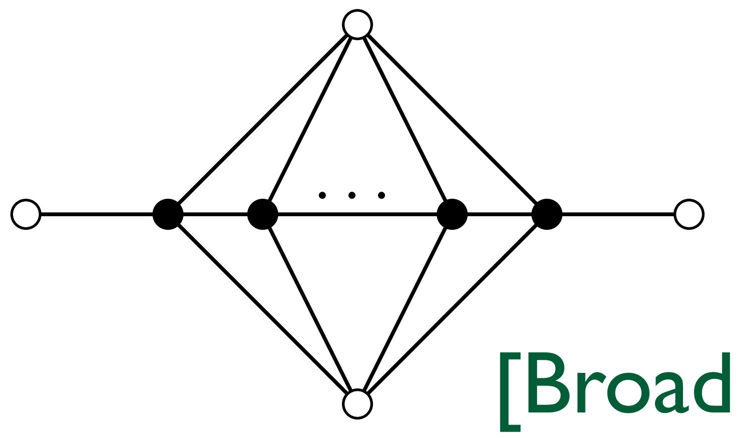
Higher-point functions

- At leading order in J: tree level computation in the EFT (exact in λ)
- Basic building block: propagator in the large charge background

$$\langle \phi\phi \rangle_{\text{LC backg}} =$$



$$\propto \sum_k \lambda^k F^{(k)}(z, \bar{z})$$



Background (=vacuum)

Conformal ladder integrals

[Broadhurst, Davydychev ‘ 10]

Higher-point functions

- We found a refined formula:

$$\langle\phi\phi\rangle_{\text{LC backg}} = \frac{(1-z)(1-\bar{z})}{\sqrt{z\bar{z}}}\sum_{n=0}^{\infty} W(\varphi+2\pi n) + W(2\pi-\varphi+2\pi n)$$

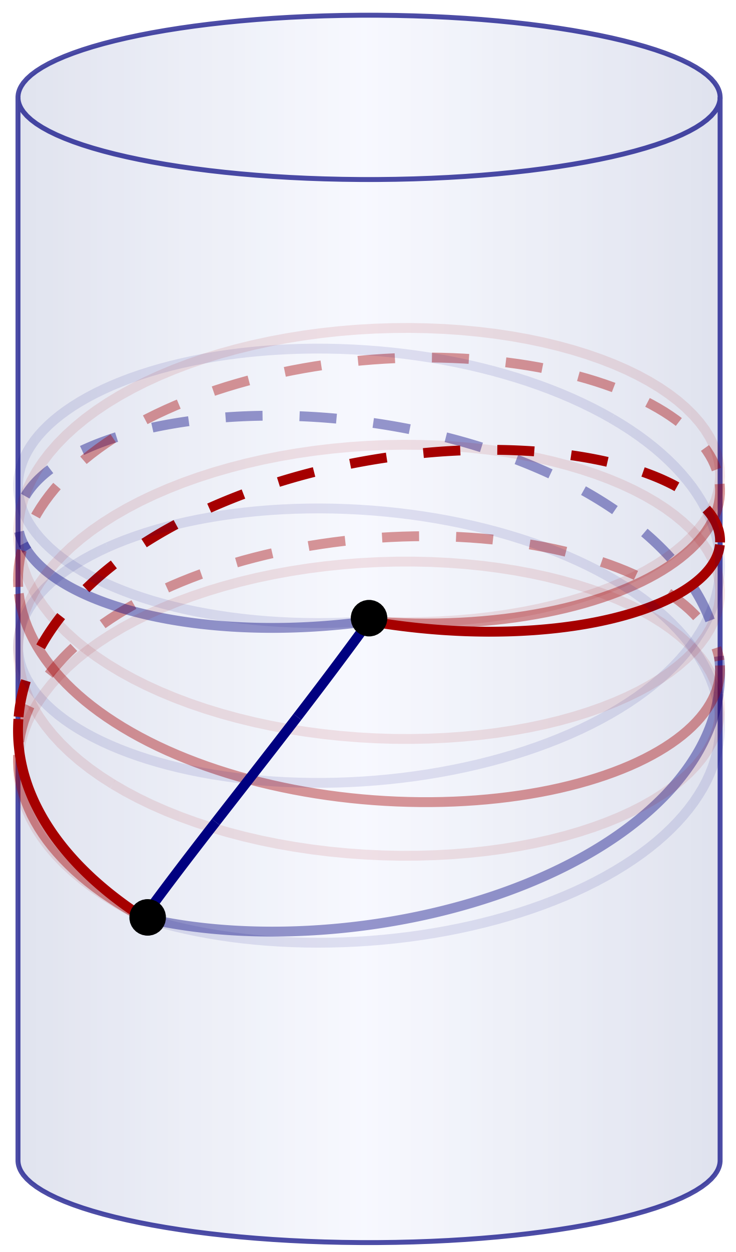
cross-ratio

$$W(x) \sim K_1(\sqrt{\lambda}x)$$

$\lambda \gg 1:$

$$\langle\phi\phi\rangle_{\text{LC backg}} \sim \sum_n \# e^{-4\sqrt{\lambda}\sqrt{\sigma^2+(2\pi n+\varphi)^2}} + \# e^{-4\sqrt{\lambda}\sqrt{\sigma^2+(2\pi(n+1)-\varphi)^2}}$$

Strong coupling limit of the mass of particles

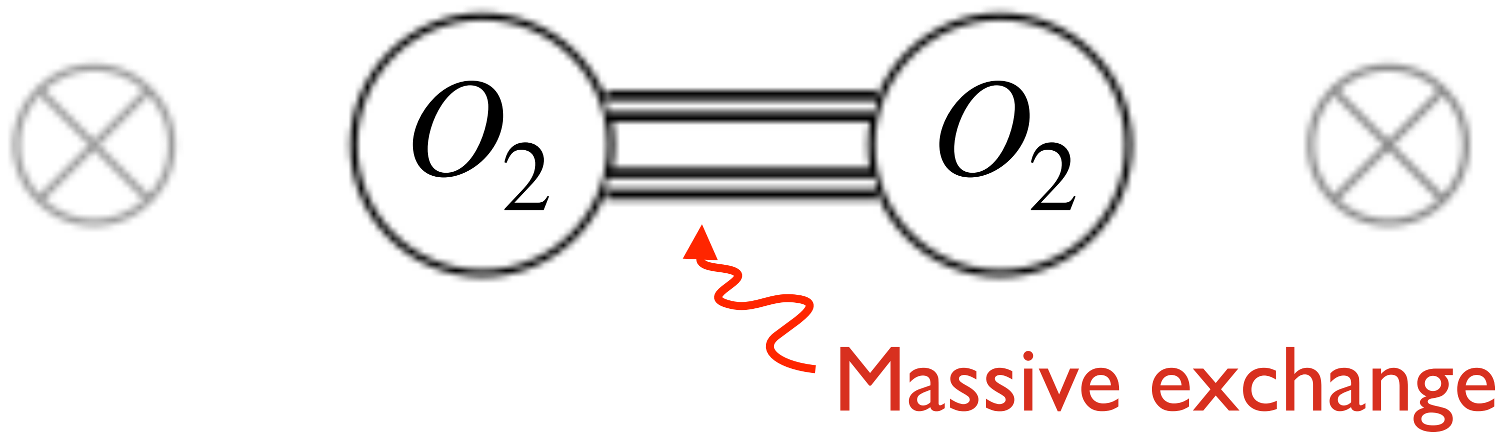


Infinite sum of massive propagator: worldline instanton wrapping a great circle of S^3 n times, in 2 different orientations

Four-point functions HHLL

At leading order in $1/J$, tree level computation in the EFT:

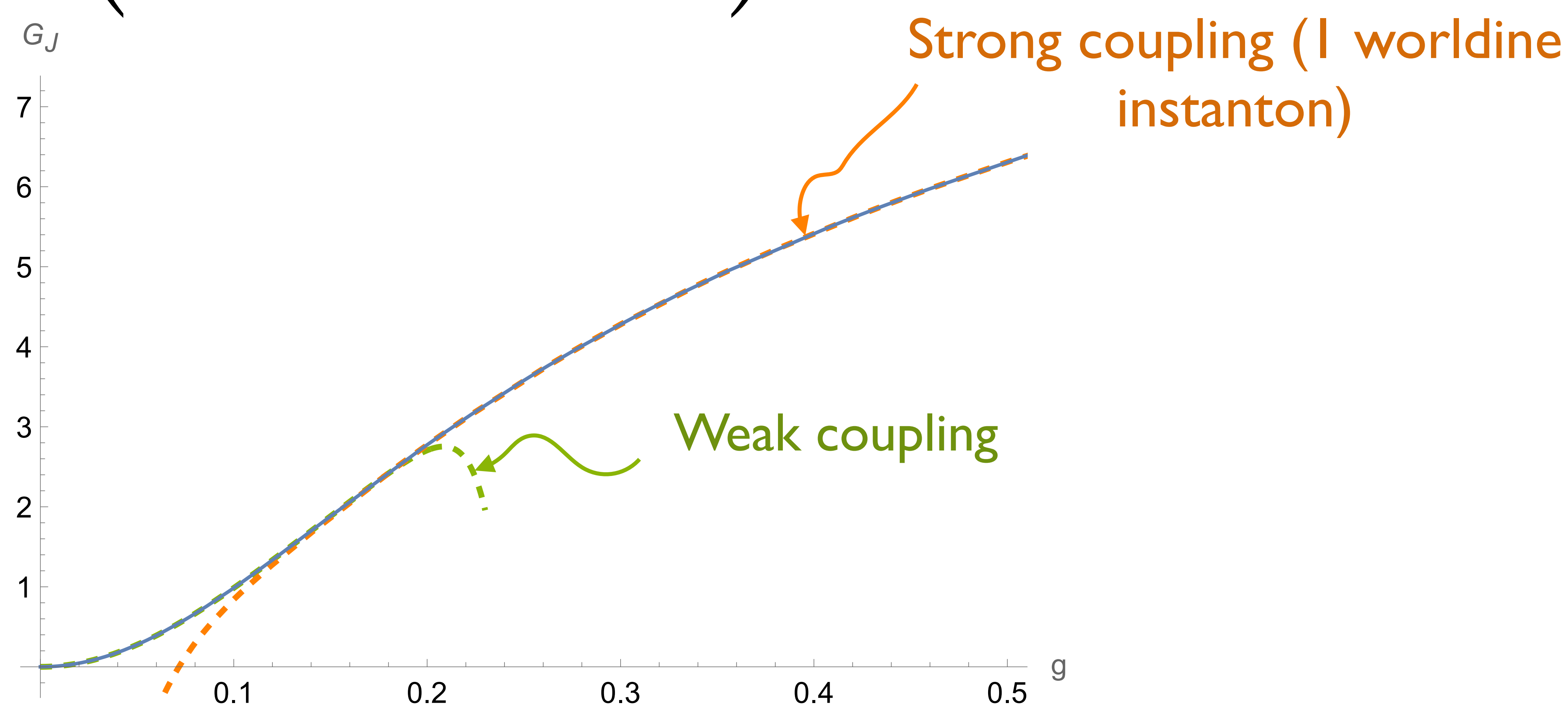
$$\langle O_2(x_1) O_2(x_2)\rangle_{\text{EFT}} \sim$$



$$\sim \left(\langle \phi\phi\rangle_{\text{LC backg}}\right)^2$$

Check: integrated correlator $\displaystyle\int d\mu(z,\bar{z})\ \left(\langle\phi\phi\rangle_{\text{LC backg}}\right)^2$

Similar in spirit to: [Grassi, Komargodski, Tizzano' 19]



Can be matched with the result of supersymmetric localization: "emergent" matrix model of size of order J.

Non-BPS ops: Three-point function HHK

- Three-point with Konishi $\quad K(x) = \mathrm{Tr}\,\Phi_{\mathrm{I}}\Phi_{\mathrm{I}} = 2\Phi^0_{\mathrm{I}}\Phi^0_{\mathrm{I}} + 2\Phi^+_{\mathrm{I}}\Phi^-_{\mathrm{I}}$

$$\langle J \,|\, K(x_1) \,|\, J \rangle \overset{\text{'t Hooft}}{=} \langle K(x_1) \rangle_{\mathrm{EFT}}$$

- Interesting part (coupling dependent) arises from massive contraction

$$2\Phi^0_I\Phi^0_I + 2\underbrace{\Phi^+_I\Phi^-_I} \xrightarrow{\text{Summing ladders}} C_{KJJ} = -8\lambda + 4\sqrt{\lambda}\int_0^\infty dw \frac{4\sqrt{\lambda}w - J_1(8\sqrt{\lambda}w)}{\sinh^2(w)}$$

(Beyond localization…)

Conclusions & future directions

- Large charge 't Hooft limit provides another solvable corner of $\mathcal{N} = 4$ SYM
- Same underlying algebra, and equally powerful
- Other observables like correlators can also be determined in this limit without much effort

- Topological expansion interpretation?
- Higher rank gauge group?
- More general three-point functions?
- Less supersymmetric states? E.g. semiclassics around 1/16 BPS states?
- Combining large J with large N? Semiclassics from multi-scaling limit?
- Large spin 't Hooft limit? E.g. $g_{\rm YM}^2 \log S \equiv \text{fixed}$

Thank you!