Journal of Algebra 601 (2022) 54–71

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Chern character and obstructions to deforming cycles

Sen Yang [a,b,c,*]

[a] *Shing-Tung Yau Center of Southeast University, Southeast University, Nanjing, China*
[b] *School of Mathematics, Southeast University, Nanjing, China*
[c] *Yau Mathematical Sciences Center, Tsinghua University, Beijing, China*

A R T I C L E   I N F O

Article history:
Received 2 November 2021
Available online 17 March 2022
Communicated by V. Srinivas

MSC:
14C25

Keywords:
Chern character
K-theory
Perfect complex
Deformation
Obstruction
Derived category

A B S T R A C T

Green-Griffiths observed that we could eliminate obstructions to deforming divisors. Motivated by recent work of Bloch-Esnault-Kerz on deformation of algebraic cycle classes, we use Chern character to generalize Green-Griffiths' observation and to show how to eliminate obstructions to deforming cycles of codimension p.

1. Introduction

The main purpose of this paper is to study obstruction issues in deforming cycles and show how to eliminate obstructions to deforming cycles. To motivate the discussion, we recall the infinitesimal Hodge conjecture.

* Correspondence to: Shing-Tung Yau Center of Southeast University, Southeast University, Nanjing, China.

E-mail address: 101012424@seu.edu.cn.

https://doi.org/10.1016/j.jalgebra.2022.02.019

Let $\mathcal{X}/S$ be a smooth projective scheme, where $S = \mathrm{Spec}(k[[t]])$ with k a field of characteristic zero. For each integer $n \geq 0$, we write $S_n = \mathrm{Spec}\, k[t]/(t^{n+1})$ and write $X_n = \mathcal{X} \times_S S_n$. Let $K_0(X_0)$ and $H^*_{dR}(X_0/k)$ denote the Grothendieck group and de Rham cohomology respectively. There exists a Chern character ring homomorphism

$$\mathrm{ch} : K_0(X_0) \to H^*_{dR}(X_0/k).$$

For an element $\xi_0 \in K_0(X_0)_{\mathbb{Q}}$, we are interested in lifting $\mathrm{ch}(\xi_0) \in H^*_{dR}(X_0/k)$ to $\mathrm{ch}(\xi) \in H^*_{dR}(\mathcal{X}/S)$ in the sense that

$$\mathrm{ch}(\xi \mid_{X_0}) = \mathrm{ch}(\xi_0) \in H^*_{dR}(X_0/k),$$

where $\xi \in K_0(\mathcal{X})_{\mathbb{Q}}$.

Let $\nabla : H^*_{dR}(\mathcal{X}/S) \to H^*_{dR}(\mathcal{X}/S)$ denote the derivation in the parameter t given by the Gauss-Manin connection. There exists a canonical isomorphism

$$\Phi : H^*_{dR}(\mathcal{X}/S)^{\nabla} \xrightarrow{\sim} H^*_{dR}(X_0/k),$$

where $H^*_{dR}(\mathcal{X}/S)^{\nabla}$ is the kernel of ∇.

The infinitesimal Hodge conjecture predicts that

Conjecture 1.1 *(see Conjecture 1.4 of [8]). The following statements are equivalent for an element $\xi_0 \in K_0(X_0)_{\mathbb{Q}}$:*

1. $\Phi^{-1} \circ \mathrm{ch}(\xi_0) \in \bigoplus_i H^{2i}_{dR}(\mathcal{X}/S)^{\nabla} \cap F^i H^{2i}_{dR}(\mathcal{X}/S)$, *where $F^i H^{2i}_{dR}$ denotes the Hodge filtration of de Rham cohomology;*
2. *there is an element $\xi \in K_0(\mathcal{X})_{\mathbb{Q}}$ such that*

$$\mathrm{ch}(\xi \mid_{X_0}) = \mathrm{ch}(\xi_0) \in H^*_{dR}(X_0/k).$$

Some recent progress on the infinitesimal Hodge conjecture has been made by Bloch-Esnault-Kerz [7,8], Green-Griffiths [11] and Morrow [17]. Especially relevant to our study of algebraic cycles is the work by Bloch, Esnault and Kerz [8]. They proved that, in appendix A of [8], the infinitesimal Hodge conjecture is equivalent to the variational Hodge conjecture proposed by Grothendieck [14]. Moreover, motivated by the infinitesimal (variational) Hodge conjecture, they proved the following:

Theorem 1.2 *(see Theorem 1.2 of [8]). Assuming that the Chow-Künneth property (part of the standard conjecture) holds, the following statements are equivalent for an element $\xi_0 \in K_0(X_0)_{\mathbb{Q}}$:*

1. $\Phi^{-1} \circ \mathrm{ch}(\xi_0) \in \bigoplus_i H^{2i}_{dR}(\mathcal{X}/S)^{\nabla} \cap F^i H^{2i}_{dR}(\mathcal{X}/S)$;

2. *there is an element* $\hat{\xi} \in \left(\varprojlim_n K_0(X_n)\right) \otimes \mathbb{Q}$ *such that*

$$\mathrm{ch}(\hat{\xi}\mid_{X_0}) = \mathrm{ch}(\xi_0) \in H^*_{dR}(X_0/k).$$

The key point in the proof of this theorem is to eliminate obstructions to lifting $\mathrm{ch}(\xi_0)$ by using correspondences (the assumption of Chow-Künneth property guarantees enough correspondences). Moreover, if the ground field k is algebraic over $\mathbb{Q}$, without assuming the Chow-Künneth property, Bloch-Esnault-Kerz deduced that the obstructions to lifting $\mathrm{ch}(\xi_0)$ can be eliminated.

In the pioneering work [12], Green and Griffiths studied the deformation of algebraic cycles. Concretely, let X be a smooth projective variety over a field k of characteristic 0, they investigated how algebraic cycles of X deformed in X_j, which is the j-th trivial deformations of X. In particular, they studied the first order deformations of divisors and zero cycles and then defined their tangent spaces. Dribus, Hoffman and the author extended much of their theory in [9,21–23].

Let $Y \subset X$ be a subvariety of codimension 1, it is well known that the embedded deformation of the subvariety Y may be obstructed. However, by considering Y as an element of the cycle group $Z^1(X)$, Green-Griffiths predicted that we could lift the divisor Y to higher order successively. This prediction was verified by Ng in his Ph.D thesis [19] by using the semi-regularity map defined by Kodaira-Spencer [16] and Bloch [5].

We sketch Ng's idea briefly. When an infinitesimal deformation of Y is obstructed to higher order, let Z be a very ample divisor such that $H^1(O_X(Y+Z)) = 0$. According to Proposition 1.1 of [5], the semi-regularity map

$$\pi : H^1(Y \cup Z, N_{Y\cup Z/X}) \to H^2(O_X),$$

where $N_{Y\cup Z/X}$ is the normal bundle, agrees with the boundary map in the long exact sequence

$$\cdots \to H^1(O_X(Y+Z)) \to H^1(Y \cup Z, N_{Y\cup Z/X}) \xrightarrow{\pi} H^2(O_X) \to \cdots.$$

Since $H^1(O_X(Y+Z)) = 0$, the kernel of π is 0, $Y \cup Z$ is semi-regular in X. According to Kodaira-Spencer [16] (see also Theorem 1.2 of [5]), $Y \cup Z$ can be lifted to higher order successively. On the other hand, Z can be always lifted to trivial deformations $Z \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[\varepsilon]/(\varepsilon^{j+1}))$ successively.

As a cycle, Y can be written as a formal sum

$$Y = (Y+Z) - Z.$$

To deform the cycle Y is equivalent to deforming $Y \cup Z$ and Z respectively. Hence, Y lifts to higher order successively, since both $Y \cup Z$ and Z do.

The above method suggests an interesting idea to eliminate obstructions:

Idea 1.3. *When the deformation of Y is obstructed, find Z such that*

$$\begin{cases} 1.\ Z \text{ helps } Y \text{ to eliminate obstructions,} \\ 2.\ Z \text{ does not bring new obstructions.} \end{cases}$$

While the deformation of divisors are relatively well understood, it is natural to ask how to go beyond the divisor case. A very interesting work on obstructions to deforming curves on a three-fold had been done by Mukai-Nasu [18]. Inspired by a question asked by Ng in section 1.5 of [19], the author [20] used K-theory to study the deformation of 1-cycles on a three-fold. For $Y \subset X$ a subvariety of codimension p, where p is an integer such that $1 \leq p \leq \dim(X)$, Green-Griffiths [12] (page 187-190) predicted that we could lift the cycle $Y \in Z^p(X)$ to higher order successively. Their prediction has been verified in Theorem 3.11 of [23].

The purpose of this paper is to generalize Idea 1.3 to the study of deformations of cycles codimension p. In the second section, we recall background on K-theory and Milnor K-theoretic cycles. In section 3, we show how to eliminate obstructions to deforming cycles of codimension p.

We summarize the main result of this paper as follows. In notation of Setting 2.1 below, let Y^1 be a first order infinitesimal deformation of Y, which is generically given by $f_1 + \varepsilon g_1, f_2, \cdots, f_p$ with $g_1 \in O_{X,y}$. By Definition 2.2 below, we attach two elements $\mu_Y(Y^1)$ and $\mu_Y(Y)$ to Y^1 and Y respectively. Using the isomorphism $O_{X,y} = (O_{X,w})_{Q_1}$, we write $g_1 = \dfrac{a_1}{b_1}$, where $a_1, b_1 \in O_{X,w}$ and $b_1 \notin Q_1$.

To avoid heavy notations, we state the main result in an informal way:

Theorem 1.4 *(cf. Lemma 3.4 and Theorem 3.6). With notation as above, b_1 is either in or not in the maximal idea $(f_1, \cdots, f_p, f_{p+1}) \subset O_{X,w}$, then there are two cases as follows.*

Case 1: *If $b_1 \notin (f_1, \cdots, f_p, f_{p+1})$, then $\mu_Y(Y^1)$ lifts $\mu_Y(Y)$ and it can be lifted to higher order successively in the sense of Definition 2.11.*

Case 2: *If $b_1 \in (f_1, \cdots, f_p, f_{p+1})$, then $\mu_Y(Y^1)$ may not be a lifting of $\mu_Y(Y)$ and obstructions to lifting $\mu_Y(Y)$ occur. In this case, we could find another irreducible closed subscheme $Z \subset X$ of codimension p and attach it an element $\mu_Z(Z)$ (see Remark 2.4) such that $\mu_Z(Z)$ helps to eliminate obstructions to lifting $\mu_Y(Y)$.*

We remark that Theorem 3.6 of this paper is different from Theorem 3.11 of [23]. This is mainly because we do not know whether the map μ_Y of Definition 2.2 is surjective or not.[1]

Notation.

(1). K-theory in this paper is Thomason-Trobaugh non-connective K-theory, if not stated otherwise.
(2). For any abelian group M, $M_{\mathbb{Q}}$ denotes $M \otimes_{\mathbb{Z}} \mathbb{Q}$.
(3). If not stated otherwise, X is a smooth projective variety over a field k of characteristic 0. For each integer $j \geq 0$, X_j denotes the j-th infinitesimally trivial deformation of X, i.e., $X_j = X \times_{\mathrm{Spec(k)}} \mathrm{Spec}(k[\varepsilon]/\varepsilon^{j+1})$.

2. K-theory and deformation of cycles

The following setting is used below.

Setting 2.1. *Let $Y \subset X$ be an irreducible closed subvariety of codimension p, with generic point y. Let $W \subset Y$ be an irreducible closed subvariety of codimension 1 in Y, with generic point w.*

We assume that W is generically defined by $f_1, f_2, \cdots, f_p, f_{p+1}$ and Y is generically defined by $f_1, f_2, \cdots, f_p$. It follows that $O_{X,y} = (O_{X,w})_{Q_1}$, where Q_1 is the ideal $(f_1, f_2, \cdots, f_p) \subset O_{X,w}$.

For each integer $j \geq 0$, we denote by $K_0(O_{X_j,y}$ on $y)$ the Grothendieck group of the triangulated category $D^b(O_{X_j,y}$ on $y)$, which is the derived category of perfect complexes of $O_{X_j,y}$-modules with homology supported on the closed point $y \in \mathrm{Spec}(O_{X_j,y})$.

A first order infinitesimal embedded deformation $Y^1 \subset X_1$ is generically given by a regular sequence $\{f_1 + \varepsilon g_1, f_2 + \varepsilon g_2, \cdots, f_p + \varepsilon g_p\}$, where $g_1, \cdots, g_p \in O_{X,y}$, see [21] (page 711-712) for related discussions if necessary.

Let $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$ denote the Koszul complex associated to the regular sequence $\{f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p\}$, which defines an element $[F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)] \in K_0(O_{X_1,y}$ on $y)_{\mathbb{Q}}$. We recall a map from the Zariski tangent space $\mathrm{T}_Y\mathrm{Hilb}(X)$ to the Hilbert scheme at the point Y to the Grothendieck group $K_0(O_{X_1,y}$ on $y)_{\mathbb{Q}}$.

Definition 2.2 *(Definition 2.4 of [21]).* With notation as above, we define a map

$$\mu_Y : \mathrm{T}_Y\mathrm{Hilb}(X) \to K_0(O_{X_1,y} \text{ on } y)_{\mathbb{Q}}$$
$$Y^1 \longrightarrow [F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)].$$

For $Y \in \mathrm{T}_Y\mathrm{Hilb}(X)$, $\mu_Y(Y) = [F_\bullet(f_1, \cdots, f_p)]$, where $F_\bullet(f_1, \cdots, f_p)$ is the Koszul complex associated to the sequence $\{f_1, \cdots, f_p\}$.

[1] The author thanks Spencer Bloch [6] for discussions on this issue.

In notation of Setting 2.1, let z be the point defined by the prime ideal $Q_2 = (f_{p+1}, f_2, \cdots, f_p) \subset O_{X,w}$, then $z \in X^{(p)}$.

Definition 2.3. With notation as above, we define a subscheme $Z \subset X$ to be the Zariski closure of z with closed reduced structure

$$Z := \overline{\{z\}}.$$

Remark 2.4. We can similarly define a map

$$\mu_Z : \mathrm{T}_Z\mathrm{Hilb}(X) \to K_0(O_{X_1,z} \text{ on } z)_{\mathbb{Q}}$$

as in Definition 2.2. Let $F_\bullet(f_{p+1}, f_2, \cdots, f_p)$ be the Koszul complex of the sequence $\{f_{p+1}, f_2, \cdots, f_p\}$. For $Z \in \mathrm{T}_Z\mathrm{Hilb}(X)$, $\mu_Z(Z) = [F_\bullet(f_{p+1}, f_2, \cdots, f_p)]$.

Recall that Milnor K-groups with support are rationally defined as certain eigenspaces of K-groups in [22].

Definition 2.5 *(Definition 3.2 of [22]).* Let X be a finite equi-dimensional noetherian scheme and $y \in X^{(p)}$. For each $l \in \mathbb{Z}$, Milnor K-group with support $K_l^M(O_{X,y} \text{ on } y)$ is rationally defined to be

$$K_l^M(O_{X,y} \text{ on } y) := K_l^{(l+p)}(O_{X,y} \text{ on } y)_{\mathbb{Q}},$$

where $K_l^{(l+p)}$ is the eigenspace of $\psi^m = m^{l+p}$ and ψ^m is the Adams operations.

Adams operations ψ^m for K-theory of perfect complexes has the following property.

Lemma 2.6 *(Prop 4.12 of [10]). Let $L(x_1, \cdots, x_p)$ be the Koszul complex of a regular sequence $\{x_1, \cdots, x_p\}$, then Adams operations ψ^m on $L(x_1, \cdots, x_p)$ satisfy that*

$$\psi^m(L(x_1, \cdots, x_p)) = m^p L(x_1, \cdots, x_p).$$

It follows that $[F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)] \in K_0(O_{X_1,y} \text{ on } y)_{\mathbb{Q}}$ lies in the eigenspace space $K_0^{(p)}(O_{X_1,y} \text{ on } y)_{\mathbb{Q}}$. In other words, $[F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)]$ lies in the Milnor K-group with support

$$[F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)] \in K_0^M(O_{X_1,y} \text{ on } y).$$

Theorem 2.7 *(Theorem 3.14 of [22]). For each integer $j > 0$, there exists the following commutative diagram in which the morphisms* Ch *from K-groups to local cohomology groups are induced by Chern character from K-theory to negative cyclic homology*

$$\begin{array}{ccc}
\bigoplus_{y\in X^{(p)}} H_y^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) & \xleftarrow{\mathrm{Ch}} & \bigoplus_{y\in X^{(p)}} K_0^M(O_{X_j,y} \text{ on } y) \\
\Big\downarrow \partial_1^{p,-p} & & \Big\downarrow d_{1,X_j}^{p,-p} \\
\bigoplus_{w\in X^{(p+1)}} H_w^{p+1}((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) & \xleftarrow[\cong]{\mathrm{Ch}} & \bigoplus_{w\in X^{(p+1)}} K_{-1}^M(O_{X_j,w} \text{ on } w).
\end{array} \tag{2.1}$$

Tensor triangular Chow groups of a tensor triangulated category were defined by Balmer [3], and they were further explored by Klein [15]. By slight modifying Balmer's definition, we proposed Milnor K-theoretic cycles.

Definition 2.8 *(Definition 3.4 and 3.15 of [22]).* The p-th Milnor K-theoretic cycle group of X is defined to be[2]

$$Z_p^M(D^{\mathrm{Perf}}(X)) := \bigoplus_{y\in X^{(p)}} K_0^M(O_{X,y} \text{ on } y).$$

For each integer $j > 0$, the p-th Milnor K-theoretic cycle group of X_j is defined to be[3]

$$Z_p^M(D^{\mathrm{Perf}}(X_j)) := \mathrm{Ker}(d_{1,X_j}^{p,-p}),$$

where $d_{1,X_j}^{p,-p}$ is the differential in the commutative diagram (2.1).

The elements of $Z_p^M(D^{\mathrm{Perf}}(X))$ and $Z_p^M(D^{\mathrm{Perf}}(X_j))$ are called Milnor K-theoretic cycles.

By Lemma 2.6, both $\mu_Y(Y)$ and $\mu_Z(Z)$ have eigenweight p. This shows that

Corollary 2.9. *Both $\mu_Y(Y)$ and $\mu_Z(Z)$ are Milnor K-theoretic cycles*

$$\mu_Y(Y) \in Z_p^M(D^{\mathrm{Perf}}(X)),\ \mu_Z(Z) \in Z_p^M(D^{\mathrm{Perf}}(X)).$$

Remark 2.10. It is obvious that $K_0^M(O_{X,y} \text{ on } y)$ is a direct summand of $K_0^M(O_{X_j,y} \text{ on } y)$ and its image under $d_{1,X_j}^{p,-p}$ is zero, so $Z_p^M(D^{\mathrm{Perf}}(X))$ is a direct summand of $Z_p^M(D^{\mathrm{Perf}}(X_j))$.

Milnor K-theoretic cycles can detect nilpotents, which is important in the study of deformation of cycles. For each integer $j > 0$, the natural map $g_j : X_{j-1} \to X_j$ induces a commutative diagram (see section 3.1 of [23]),

[2] It was proved in Theorem 3.16 of [22] that $Z_p^M(D^{\mathrm{Perf}}(X))$ agreed with the classical cycle group $Z^p(X)_{\mathbb{Q}}$.

[3] The reason why we use the kernel of $d_{1,X_j}^{p,-p}$ to define Milnor K-theoretic cycles $Z_p^M(D^{\mathrm{Perf}}(X_j))$ is explained in section 2.2 of [23].

$$\begin{array}{ccc}
\bigoplus_{y\in X^{(p)}} K_0^M(O_{X_j,y} \text{ on } y) & \xrightarrow{g_j^*} & \bigoplus_{y\in X^{(p)}} K_0^M(O_{X_{j-1},y} \text{ on } y) \\
d^{p,-p}_{1,X_j}\downarrow & & d^{p,-p}_{1,X_{j-1}}\downarrow \\
\bigoplus_{w\in X^{(p+1)}} K_{-1}^M(O_{X_j,w} \text{ on } w) & \xrightarrow{g_j^*} & \bigoplus_{w\in X^{(p+1)}} K_{-1}^M(O_{X_{j-1},w} \text{ on } w).
\end{array}$$

This further induces

$$g_j^*: Z_p^M(D^{\text{perf}}(X_j)) \to Z_p^M(D^{\text{perf}}(X_{j-1})). \tag{2.2}$$

Definition 2.11 *(Definition 3.3 of [23]).* Given $\xi_{j-1} \in Z_p^M(D^{\text{perf}}(X_{j-1}))$, an element $\xi_j \in Z_p^M(D^{\text{perf}}(X_j))$ is called a deformation (or lift) of ξ_{j-1}, if $g_j^*(\xi_j) = \xi_{j-1}$.

The elements ξ_{j-1} and ξ_j can be formally written as finite sums

$$\xi_{j-1} = \sum_{y\in X^{(p)}} \lambda_{j-1}\cdot\overline{\{y\}},\ \xi_j = \sum_{y\in X^{(p)}} \lambda_j\cdot\overline{\{y\}},$$

where $\overline{\{y\}}$ is with closed reduced structure and λ_j's are perfect complexes such that $\sum_y \lambda_j \in \text{Ker}(d^{p,-p}_{1,X_j}) \subset \bigoplus_{y\in X^{(p)}} K_0(O_{X_j,y} \text{ on } y)_{\mathbb{Q}}$. When we lift from ξ_{j-1} to ξ_j, we lift the coefficients from $\sum_y \lambda_{j-1}$ to $\sum_y \lambda_j$.

For later purpose, we want to describe the map Ch in Theorem 2.7

$$\text{Ch}: \bigoplus_{y\in X^{(p)}} K_0^M(O_{X_j,y} \text{ on } y) \to \bigoplus_{y\in X^{(p)}} H_y^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}).$$

When $j=1$, this map Ch has been described by using a construction of Angéniol and Lejeune-Jalabert [1], see Lemma 3.8 of [21]. For general j, it can still be described by their construction. For readers' convenience, we sketch the description of Ch below.

An element $M \in K_0^M(O_{X_j,y} \text{ on } y) \subset K_0(O_{X_j,y} \text{ on } y)_{\mathbb{Q}}$ is represented by a strict perfect complex $L_\bullet$ supported at y

$$0 \longrightarrow F_n \xrightarrow{M_n} F_{n-1} \xrightarrow{M_{n-1}} \dots \xrightarrow{M_2} F_1 \xrightarrow{M_1} F_0 \longrightarrow 0,$$

where each $F_i = O^{r_i}_{X_j,y}$ and M_i's are matrices with entries in $O_{X_j,y}$.

Definition 2.12 *(page 24 of [1]).* The local fundamental class attached to this perfect complex is defined to be the following collection

$$[L_\bullet]_{loc} = \{\frac{1}{p!} dM_i \circ dM_{i+1} \circ \dots \circ dM_{i+p-1}\}, i = 1, 2, \cdots,$$

where $d = d_{\mathbb{Q}}$ and each dM_i is the matrix of absolute differentials. In other words,

$$dM_i \in \mathrm{Hom}(F_i, F_{i-1} \otimes \Omega^1_{O_{X_j,y}/\mathbb{Q}}).$$

By Lemme 3.1.1 (on page 24) and Definition 3.4 (on page 29) of [1], the class $[L_\bullet]_{loc}$ above is a cycle in $\mathcal{H}om(L_\bullet, \Omega^p_{O_{X_j,y}/\mathbb{Q}} \otimes L_\bullet)$, and the image of $[L_\bullet]_{loc}$ in $H^p(\mathcal{H}om(L_\bullet, \Omega^p_{O_{X_j,y}/\mathbb{Q}} \otimes L_\bullet))$ does not depend on the choice of the basis of $L_\bullet$.

Since

$$H^p(\mathcal{H}om(L_\bullet, \Omega^p_{O_{X_j,y}/\mathbb{Q}} \otimes L_\bullet)) = \mathcal{E}XT^p(L_\bullet, \Omega^p_{O_{X_j,y}/\mathbb{Q}} \otimes L_\bullet),$$

the above local fundamental class $[L_\bullet]_{loc}$ defines an element (still denoted $[L_\bullet]_{loc}$) of $\mathcal{E}XT^p(L_\bullet, \Omega^p_{O_{X_j,y}/\mathbb{Q}} \otimes L_\bullet)$.

Since $L_\bullet$ is supported on y, by discussions after Definition 2.3.1 on page 98-99 of [1], there exists the following trace map

$$\mathrm{Tr} : \mathcal{E}XT^p(L_\bullet, \Omega^p_{O_{X_j,y}/\mathbb{Q}} \otimes L_\bullet) \longrightarrow H^p_y(\Omega^p_{X_j/\mathbb{Q}}).$$

Definition 2.13 *(Definition 2.3.2 on page 99 of [1]).* The image of $[L_\bullet]_{loc}$ under the above trace map, denoted $\mathcal{V}^p_{L_\bullet}$, is called Newton class.

Theorem 2.14 *(Proposition 4.3.1 on page 113 of [1]). The Newton class $\mathcal{V}^p_{L_\bullet}$ is well-defined on $K_0(O_{X_j,y}$ on $y)$.*

The truncation map $\rfloor \dfrac{\partial}{\partial \varepsilon} : \Omega^p_{X_j/\mathbb{Q}} \to \Omega^{p-1}_{X/\mathbb{Q}} \otimes k[\varepsilon]/(\varepsilon^j)$ induces a map

$$\rfloor \frac{\partial}{\partial \varepsilon} : H^p_y(\Omega^p_{X_j/\mathbb{Q}}) \longrightarrow H^p_y((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}),$$

where we identify $\Omega^{p-1}_{X/\mathbb{Q}} \otimes k[\varepsilon]/(\varepsilon^j)$ with $(\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}$.

Lemma 2.15 *(cf. Lemma 3.8 of [21]). With notation as above, the map*

$$\mathrm{Ch} : K_0^M(O_{X_j,y} \text{ on } y) \to H^p_y((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j})$$

can be described as a composition

$$\begin{array}{ccccc} K_0^M(O_{X_j,y} \text{ on } y) & \to & H^p_y(\Omega^p_{X_j/\mathbb{Q}}) & \to & H^p_y((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) \\ L_\bullet & \longrightarrow & \mathcal{V}^p_{L_\bullet} & \longrightarrow & \mathcal{V}^p_{L_\bullet} \rfloor \dfrac{\partial}{\partial \varepsilon}. \end{array}$$

We use this Lemma to describe $\mathrm{Ch}(L_\bullet)$ below, where $L_\bullet$ is represented by some Koszul complexes. When $j = 1$, such descriptions were given in [21] (page 715-716).

In notation of Setting 2.1, let $g_1, \cdots, g_j$ be arbitrary elements of $O_{X,y}$. The Koszul resolution of $O_{X_j,y}/(f_1 + \varepsilon g_1 + \cdots + \varepsilon^j g_j, f_2, \cdots, f_p)$, denoted $F^j_\bullet$, has the form

$$0 \longrightarrow F_p^j \longrightarrow F_{p-1}^j \longrightarrow \cdots \longrightarrow F_0^j \longrightarrow 0,$$

where each $F_i^j = \bigwedge^i((O_{X_j,y}))^{\oplus p}$. This complex defines an element $[F_\bullet^j] \in K_0^M(O_{X_j,y} \text{ on } y)$ whose image under the Ch map can be described by Lemma 2.15. Concretely, the following diagram

$$\begin{cases} F_\bullet^j & \longrightarrow \quad O_{X_j,y}/(f_1 + \varepsilon g_1 + \cdots + \varepsilon^j g_j, f_2, \cdots, f_p) \\ F_p^j(\cong O_{X_j,y}) & \xrightarrow{[F_\bullet^j]_{loc}} \quad F_0^j \otimes \Omega^p_{O_{X_j,y}/\mathbb{Q}}(\cong \Omega^p_{O_{X_j,y}/\mathbb{Q}}), \end{cases}$$

where $[F_\bullet^j]_{loc}$ is the local fundamental class attached to $F_\bullet^j$, gives an element in $Ext^p(O_{X_j,y}/(f_1 + \varepsilon g_1 + \cdots + \varepsilon^j g_j, f_2, \cdots, f_p), \Omega^p_{O_{X_j,y}/\mathbb{Q}})$. This further gives Newton class $\mathcal{V}^p_{F_\bullet^j} \in H_y^p(\Omega^p_{X_j/\mathbb{Q}})$.

Let $F_\bullet(f_1, f_2, \cdots, f_p)$ be the Koszul resolution of $O_{X,y}/(f_1, f_2, \cdots, f_p)$, which has the form

$$0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0,$$

where each $F_i = \bigwedge^i O_{X,y}^{\oplus p}$. The image $\mathrm{Ch}([F_\bullet^j])$, which is the truncation of Newton class $\mathcal{V}^p_{F_\bullet^j}$, is represented by the following diagram,

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p) & \longrightarrow \quad O_{X,y}/(f_1, f_2, \cdots, f_p) \\ F_p(\cong O_{X,y}) & \xrightarrow{[F_\bullet^j]_{loc}\rfloor \frac{\partial}{\partial \varepsilon}} F_0 \otimes (\Omega^{p-1}_{O_{X,y}/\mathbb{Q}})^{\oplus j}(\cong (\Omega^{p-1}_{O_{X,y}/\mathbb{Q}})^{\oplus j}), \end{cases} \tag{2.3}$$

where the truncation $[F_\bullet^j]_{loc}\rfloor \dfrac{\partial}{\partial \varepsilon} = (-1)^{p-1}(g_1 + \cdots + jg_j)df_2 \wedge \cdots \wedge df_p$ with $d = d_{\mathbb{Q}}$. To be precise, the above diagram gives an element α in $Ext^p(O_{X,y}/(f_1, f_2, \cdots, f_p), (\Omega^{p-1}_{O_{X,y}/\mathbb{Q}})^{\oplus j})$. Since

$$H_y^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j}) = \varinjlim_{n\to\infty} Ext^p(O_{X,y}/(f_1, f_2, \cdots, f_p)^n, (\Omega^{p-1}_{O_{X,y}/\mathbb{Q}})^{\oplus j}),$$

the image $[\alpha]$ of α under the limit is in $H_y^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j})$ and it is $\mathrm{Ch}([F_\bullet^j])$.

3. Chern character and obstructions

Let $D^{\mathrm{perf}}(X_j)$ denote the derived category obtained from the exact category of perfect complexes on X_j and $\mathcal{L}_{(i)}(X_j)$ is defined to be

$$\mathcal{L}_{(i)}(X_j) := \{E \in D^{\mathrm{perf}}(X_j) \mid \mathrm{codim}(\mathrm{supph(E)}) \geq -i\},$$

where the closed subset $\mathrm{supph}(E) \subset X$ is the support of the total homology of the perfect complex E and the codimension of $\mathrm{supph}(E)$ is no less than $-i$.

Let $(\mathcal{L}_{(i)}(X_j)/\mathcal{L}_{(i-1)}(X_j))^{\#}$ denote the idempotent completion of Verdier quotient $\mathcal{L}_{(i)}(X_j)/\mathcal{L}_{(i-1)}(X_j)$ in the sense of Balmer-Schlichting [4].

Theorem 3.1 *([2]). For each $i \in \mathbb{Z}$, localization induces an equivalence*

$$(\mathcal{L}_{(i)}(X_j)/\mathcal{L}_{(i-1)}(X_j))^{\#} \simeq \bigsqcup_{x \in X^{(-i)}} D_x^{\mathrm{perf}}(X_j)$$

between the idempotent completion of the quotient $\mathcal{L}_{(i)}(X_j)/\mathcal{L}_{(i-1)}(X_j)$ and the coproduct over $x \in X^{(-i)}$ of the derived category of perfect complexes of $O_{X_j,x}$-modules with homology supported on the closed point $x \in \mathrm{Spec}(O_{X,x})$. Consequently, localization induces an isomorphism

$$K_0((\mathcal{L}_{(i)}(X_j)/\mathcal{L}_{(i-1)}(X_j))^{\#}) \simeq \bigoplus_{x \in X^{(-i)}} K_0(O_{X_j,x} \text{ on } x). \tag{3.1}$$

We keep the notation of Setting 2.1 below. For each non-negative integer j, let $a_1, \cdots, a_j$ be arbitrary elements of $O_{X,w}$. We denote by C_j the Koszul resolution of $O_{X,w}/(f_1 f_{p+1} + \varepsilon a_1 + \cdots + \varepsilon^j a_j, f_2, \cdots, f_p)$. Since the support of the Koszul complex C_j has codimension p, we consider C_j as an element of $\mathcal{L}_{(-p)}(X_j)$ which defines an element of $K_0((\mathcal{L}_{(-p)}(X_j)/\mathcal{L}_{(-p-1)}(X_j))^{\#})_{\mathbb{Q}}$, denoted $[C_j]$.

When $p = 1$ and $j = 1$, for X a surface, it was proved in Theorem 2.18 of [23] that the Koszul complex of $f_1 f_2 + \varepsilon a_1$ defined a Milnor K-theoretic cycle.[4]

It is interesting to generalize Theorem 2.18 of [23] and find more Milnor K-theoretic cycles. Let $Q_1 = (f_1, f_2, \cdots, f_p)$ as in Setting 2.1 and let z be point given by $Q_2 = (f_{p+1}, f_2, \cdots, f_p)$ as in Definition 2.3, we denote by C_j^1 and C_j^2 the Koszul resolutions of $(O_{X_j,w})_{Q_1}/(f_1 + \varepsilon \frac{a_1}{f_{p+1}} + \cdots + \varepsilon^j \frac{a_j}{f_{p+1}}, f_2, \cdots, f_p)$ and $(O_{X_j,w})_{Q_2}/(f_{p+1} + \varepsilon \frac{a_1}{f_1} + \cdots + \varepsilon^j \frac{a_j}{f_1}, f_2, \cdots, f_p)$ respectively.

Using isomorphisms $O_{X_j,y} = (O_{X_j,w})_{Q_1}$ and $O_{X_j,z} = (O_{X_j,w})_{Q_2}$, one sees that C_j^1 and C_j^2 gives elements of $K_0(O_{X_j,y} \text{ on } y)$ and $K_0(O_{X_j,z} \text{ on } z)$, denoted $[C_j^1]$ and $[C_j^2]$ respectively.

Under the isomorphism (3.1) (let $i = -p$)

$$K_0((\mathcal{L}_{(-p)}(X_j)/\mathcal{L}_{(-p-1)}(X_j))^{\#}) \simeq \bigoplus_{y \in X^{(p)}} K_0(O_{X_j,y} \text{ on } y),$$

the element $[C_j]$ decomposes into the direct sum of $[C_j^1]$ and $[C_j^2]$,[5]

[4] For the geometric meaning of Theorem 2.18 of [23], we refer to page 103-104 and the summary on page 119 of Green-Griffiths [12]. See also page 316-318 of [23] for a summary.

[5] Since f_{p+1}^{-1} exists in $(O_{X_j,w})_{Q_1}$, the complex C_j^1 and the Koszul resolution of $(O_{X_j,w})_{Q_1}/(f_1 + \varepsilon \frac{a_1}{f_{p+1}} + \cdots + \varepsilon^j \frac{a_j}{f_{p+1}}, \frac{f_2}{f_{p+1}}, \cdots, \frac{f_p}{f_{p+1}})$ define the same element of Grothendieck group. There is a similar explanation for $[C_j^2]$.

$$[C_j] = [C_j^1] + [C_j^2].$$

By Lemma 2.6, one sees that $[C_j^1] \in K_0^M(O_{X_j,y} \text{ on } y)$ and $[C_j^2] \in K_0^M(O_{X_j,z} \text{ on } z)$. In particular, when $j = 0$, C_0^1 and C_0^2 are Koszul complexes of sequences $\{f_1, f_2, \cdots, f_p\}$ and $\{f_{p+1}, f_2, \cdots, f_p\}$ respectively. It is obvious that $[C_0^1] = \mu_Y(Y)$ and $[C_0^2] = \mu_Z(Z)$, where $\mu_Y(Y)$ and $\mu_Z(Z)$ are defined in Definition 2.2 and Remark 2.4.

The following theorem gives a generalization of Theorem 2.18 of [23].

Theorem 3.2. *With notation as above, $[C_j] = [C_j^1] + [C_j^2]$ is a Milnor K-theoretic cycle in the sense of Definition 2.8*

$$[C_j] = [C_j^1] + [C_j^2] \in Z_p^M(D^{\mathrm{Perf}}(X_j)).$$

The strategy of proving this theorem is to use the commutative diagram (2.1) in Theorem 2.7. Concretely, we describe the images $\mathrm{Ch}([C_j^1])$ and $\mathrm{Ch}([C_j^2])$, and then show that $\mathrm{Ch}([C_j^1]) + \mathrm{Ch}([C_j^2])$ lies in the kernel of the differential $\partial_1^{p,-p}$. This implies that $[C_j]$ lies in the kernel of the differential $d_{1,X_j}^{p,-p}$.

Proof. The images $\mathrm{Ch}([C_j^1])$ and $\mathrm{Ch}([C_j^2])$ can be described by Lemma 2.15. In fact, they can be represented by diagrams as (2.3) on page 10. Concretely, let Q_1 be the ideal $(f_1, f_2, \cdots, f_p)$ as in Setting 2.1. Let $F_\bullet(f_1, f_2, \cdots, f_p)$ be the Koszul resolution of $(O_{X,w})_{Q_1}/(f_1, f_2, \cdots, f_p)$, which has the form

$$0 \longrightarrow F_p \longrightarrow F_{p-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0,$$

where each $F_i = \bigwedge^i((O_{X,w})_{Q_1})^{\oplus p}$. The image $\mathrm{Ch}([C_j^1]) \in H_y^p((\Omega^{p-1}_{X/\mathbb{Q}})^{\oplus j})$ is represented by the following diagram,

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p) \longrightarrow \qquad (O_{X,w})_{Q_1}/(f_1, f_2, \cdots, f_p) \\ F_p(\cong (O_{X,w})_{Q_1}) \xrightarrow{\omega_1} F_0 \otimes (\Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}})^{\oplus j}(\cong (\Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}})^{\oplus j}), \end{cases}$$

where $\omega_1 = (-1)^{p-1}(\frac{a_1}{f_{p+1}} + \cdots + \frac{ja_j}{f_{p+1}})df_2 \wedge \cdots \wedge df_p$.

Recall that Q_2 is the ideal $(f_{p+1}, f_2, \cdots, f_p)$ of $O_{X,w}$. The Koszul resolution of $(O_{X,w})_{Q_2}/(f_{p+1}, f_2, \cdots, f_p)$, denoted $F_\bullet(f_{p+1}, f_2, \cdots, f_p)$, has the form

$$0 \longrightarrow F'_p \longrightarrow F'_{p-1} \longrightarrow \cdots \longrightarrow F'_0 \longrightarrow 0,$$

where each $F'_i = \bigwedge^i((O_{X,w})_{Q_2})^{\oplus p}$. Let $\omega_2 = (-1)^{p-1}(\frac{a_1}{f_1} + \cdots + \frac{ja_j}{f_1})df_2 \wedge \cdots \wedge df_p$, the image $\mathrm{Ch}([C_j^2])$ is represented by the following diagram

$$\begin{cases} F_\bullet(f_{p+1}, f_2, \cdots, f_p) \longrightarrow \qquad (O_{X,w})_{Q_2}/(f_{p+1}, f_2, \cdots, f_p) \\ F'_p(\cong (O_{X,w})_{Q_2}) \xrightarrow{\omega_2} F'_0 \otimes (\Omega^{p-1}_{(O_{X,w})_{Q_2}/\mathbb{Q}})^{\oplus j}(\cong (\Omega^{p-1}_{(O_{X,w})_{Q_2}/\mathbb{Q}})^{\oplus j}). \end{cases}$$

Let $F_\bullet(f_1, f_2, \cdots, f_p, f_{p+1})$ and $F_\bullet(f_{p+1}, f_2, \cdots, f_p, f_1)$ be Koszul resolutions of $O_{X,w}/(f_1, f_2, \cdots, f_p, f_{p+1})$ and $O_{X,w}/(f_{p+1}, f_2, \cdots, f_p, f_1)$ respectively. The image $\partial_1^{p,-p}(\mathrm{Ch}([C_j^1]))$ is represented by the following diagram (denoted β_1)

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p, f_{p+1}) \longrightarrow O_{X,w}/(f_1, f_2, \cdots, f_p, f_{p+1}) \\ F_{p+1}(\cong O_{X,w}) \xrightarrow{(-1)^{p-1}(a_1+\cdots+ja_j)df_2\wedge\cdots\wedge df_p} F_0 \otimes (\Omega^{p-1}_{O_{X,w}/\mathbb{Q}})^{\oplus j}(\cong (\Omega^{p-1}_{O_{X,w}/\mathbb{Q}})^{\oplus j}), \end{cases}$$

and $\partial_1^{p,-p}(\mathrm{Ch}([C_j^2]))$ is represented by the following diagram (denoted β_2)

$$\begin{cases} F_\bullet(f_{p+1}, f_2, \cdots, f_p, f_1) \longrightarrow O_{X,w}/(f_{p+1}, f_2, \cdots, f_p, f_1) \\ F_{p+1}(\cong O_{X,w}) \xrightarrow{(-1)^{p-1}(a_1+\cdots+ja_j)df_2\wedge\cdots\wedge df_p} F_0 \otimes (\Omega^{p-1}_{O_{X,w}/\mathbb{Q}})^{\oplus j}(\cong (\Omega^{p-1}_{O_{X,w}/\mathbb{Q}})^{\oplus j}). \end{cases}$$

The two complexes $F_\bullet(f_1, f_2, \cdots, f_p, f_{p+1})$ and $F_\bullet(f_{p+1}, f_2, \cdots, f_p, f_1)$ are related by the following commutative diagram (see page 691 of [13])

$$\begin{array}{ccccccccc} O_{X,w} & \xrightarrow{D_{p+1}} & \wedge^p O_{X,w}^{\oplus p+1} & \xrightarrow{D_p} & \cdots & \longrightarrow & O_{X,w}^{\oplus p+1} & \xrightarrow{D_1} & O_{X,w} \\ \downarrow{\scriptstyle \det A_1} & & \downarrow{\scriptstyle \wedge^p A_1} & & \downarrow & & \downarrow{\scriptstyle A_1} & & \downarrow{\scriptstyle =} \\ O_{X,w} & \xrightarrow{E_{p+1}} & \wedge^p O_{X,w}^{\oplus p+1} & \xrightarrow{E_p} & \cdots & \longrightarrow & O_{X,w}^{\oplus p+1} & \xrightarrow{E_1} & O_{X,w}, \end{array}$$

where each D_i and E_i are defined as usual. In particular, $D_1 = (f_1, f_2, \cdots, f_p, f_{p+1})$, $E_1 = (f_{p+1}, f_2, \cdots, f_p, f_1)$, and A_1 is the matrix

$$\begin{pmatrix} 0 & 0 & 0 & \cdots 1 \\ 0 & 1 & 0 & \cdots 0 \\ 0 & 0 & 1 & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots 0 \end{pmatrix}.$$

Since the determinant $\det A_1 = -1$, one has

$$\beta_1 = -\beta_2 \in Ext^{p+1}(O_{X,w}/(f_1, f_2, \cdots, f_p, f_{p+1}), (\Omega^{p-1}_{O_{X,w}/\mathbb{Q}})^{\oplus j}).$$

Consequently, $\partial_1^{p,-p}(\mathrm{Ch}([C_j^1])) + \partial_1^{p,-p}(\mathrm{Ch}([C_j^2])) = 0 \in H_w^{p+1}(\Omega^{p-1}_{O_{X,w}/\mathbb{Q}})$. This implies that $d_{1,X_j}^{p,-p}([C_j^1] + [C_j^2]) = 0$ because of the commutative diagram (2.1)

$$\begin{array}{ccc} \mathrm{Ch}([C_j^1]) + \mathrm{Ch}([C_j^2]) & \xleftarrow{\mathrm{Ch}} & [C_j^1] + [C_j^2] \\ \downarrow{\scriptstyle \partial_1^{p,-p}} & & \downarrow{\scriptstyle d_{1,X_j}^{p,-p}} \\ \partial_1^{p,-p}(\mathrm{Ch}([C_j^1]) + \mathrm{Ch}([C_j^2])) = 0 & \underset{\cong}{\xleftarrow{\mathrm{Ch}}} & d_{1,X_j}^{p,-p}([C_j^1] + [C_j^2]). \end{array}$$

In conclusion, $[C_j^1]+[C_j^2]$ is a Milnor K-theoretic cycle in the sense of Definition 2.8. □

For each integer j, $g_j^*([C_j]) = [C_{j-1}]$, where g_j^* is the map (2.2). When $j = 1$, $g_1^*([C_1]) = g_1^*([C_1^1] + [C_1^2]) = \mu_Y(Y) + \mu_Z(Z)$. This shows that

Corollary 3.3. *With notation as above, $[C_1] \in Z_p^M(D^{\mathrm{Perf}}(X_1))$ is a first order deformation of $\mu_Y(Y)+\mu_Z(Z)$ and it can be successively lifted to higher order $[C_j] \in Z_p^M(D^{\mathrm{Perf}}(X_j))$.*

For a first order infinitesimal deformation Y^1 of Y, by Definition 2.2, $\mu_Y(Y^1) \in K_0(O_{X_1,y} \text{ on } y)_{\mathbb{Q}}$ is given by the Koszul complex $F_\bullet(f_1 + \varepsilon g_1, \cdots, f_p + \varepsilon g_p)$. We want to check whether $\mu_Y(Y^1)$ is a Milnor K-theoretic cycles or not.

For simplicity, we assume that $g_2 = \cdots = g_p = 0$ in the following. In notation of Setting 2.1, for $g_1 \in O_{X,y} = (O_{X,w})_{Q_1}$, we write $g_1 = \dfrac{a_1}{b_1}$, where $a_1, b_1 \in O_{X,w}$ and $b_1 \notin Q_1$, then b_1 is either in or not in the maximal idea $(f_1, \cdots, f_p, f_{p+1}) \subset O_{X,w}$.

Lemma 3.4. *With notation as above, in the case $b_1 \notin (f_1, \cdots, f_p, f_{p+1})$, then $\mu_Y(Y^1)$ is a Milnor K-theoretic cycle which lifts $\mu_Y(Y)$ and it can be lifted to higher order in $Z_p^M(D^{\mathrm{Perf}}(X_j))$ successively in the sense of Definition 2.11.*

Proof. If $b_1 \notin (f_1, \cdots, f_p, f_{p+1})$, then b_1 is a unit in $O_{X,w}$, this says $g_1 = \dfrac{a_1}{b_1} \in O_{X,w}$. Let T^1 denote Koszul resolution of $(O_{X_1,w})_{Q_1}/(f_1 + \varepsilon g_1, f_2, \cdots, f_p)$. For each integer $j \geq 2$, let T^j denote Koszul resolution of $(O_{X_j,w})_{Q_1}/(f_1 + \varepsilon g_1 + \varepsilon^2 h_2 + \cdots + \varepsilon^j h_j, f_2, \cdots, f_p)$, where $h_2, \cdots, h_j$ are arbitrary elements of $O_{X,w}$.

For each $j \geq 1$, T^j gives an element $[T^j] \in K_0(O_{X_j,y} \text{ on } y)_{\mathbb{Q}}$. Moreover, by Lemma 2.6, $[T^j] \in K_0^M(O_{X_j,y} \text{ on } y)$. We use the same strategy of proving Theorem 3.2 to prove that $[T^j] \in Z_p^M(D^{\mathrm{Perf}}(X_j))$. The image of $[T^j]$ under the Ch map, $\mathrm{Ch}([T^j])$, can be represented by the following diagram (cf. (2.3) on page 10)

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p) \longrightarrow (O_{X,w})_{Q_1}/(f_1, f_2, \cdots, f_p) \\ F_p(\cong (O_{X,w})_{Q_1}) \xrightarrow{\eta} F_0 \otimes (\Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}})^{\oplus j} (\cong (\Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}})^{\oplus j}). \end{cases}$$

Here $\eta = (-1)^{p-1}(g_1 + 2h_2 + \cdots + jh_j)df_2 \wedge \cdots \wedge df_p$ and $F_\bullet(f_1, f_2, \cdots, f_p)$ is the Koszul resolution of $(O_{X,w})_{Q_1}/(f_1, f_2, \cdots, f_p)$. Since $f_{p+1} \notin Q_1$, f_{p+1}^{-1} exists in $(O_{X,w})_{Q_1}$, η can be rewritten as $\eta = \dfrac{f_{p+1}\eta}{f_{p+1}}$.

The image $\partial_1^{p,-p}(\mathrm{Ch}([T^j]))$ is represented by the following diagram (denoted γ),

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p, f_{p+1}) \longrightarrow O_{X,w}/(f_1, f_2, \cdots, f_p, f_{p+1}) \\ F_{p+1}(\cong O_{X,w}) \xrightarrow{f_{p+1}\eta} F_0 \otimes (\Omega^{p-1}_{O_{X,w}/\mathbb{Q}})^{\oplus j} (\cong (\Omega^{p-1}_{O_{X,w}/\mathbb{Q}})^{\oplus j}), \end{cases}$$

where the complex $F_\bullet(f_1, f_2, \cdots, f_p, f_{p+1})$ is of the form

$$0 \longrightarrow \bigwedge^{p+1}(O_{X,w})^{\oplus p+1} \xrightarrow{M_{p+1}} \bigwedge^{p}(O_{X,w})^{\oplus p+1} \longrightarrow \cdots.$$

Let $\{e_1, \cdots, e_{p+1}\}$ be a basis of $(O_{X,w})^{\oplus p+1}$, the map M_{p+1} is

$$e_1 \wedge \cdots \wedge e_{p+1} \to \sum_{j=1}^{p+1} (-1)^j f_j e_1 \wedge \cdots \wedge \hat{e_j} \wedge \cdots e_{p+1},$$

where $\hat{e_j}$ means to omit e_j.

Since f_{p+1} appears in M_{p+1}, the diagram γ defines a trivial element of $Ext^{p+1}(O_{X,w}/(f_1, \cdots, f_p, f_{p+1}), (\Omega^{p-1}_{O_{X,w}/\mathbb{Q}})^{\oplus j})$. Hence, $\partial_1^{p,-p}(\mathrm{Ch}([T^j])) = 0$. It follows from the commutative diagram (2.1) that $d^{p,-p}_{1,X_j}([T^j]) = 0$. This proves that $[T^j] \in Z^M_p(D^{\mathrm{Perf}}(X_j))$.

It is obvious that $g_j^*([T^j]) = [T^{j-1}]$, where g_j^* is the map (2.2). In particular, $[T^1] = \mu_Y(Y^1)$ and $g_1^*([T^1]) = g_1^*(\mu_Y(Y^1)) = \mu_Y(Y)$.

In conclusion, $[T^1] = \mu_Y(Y^1)$ is a Milnor K-theoretic cycle and it lifts $\mu_Y(Y)$. Moreover, $[T^1]$ lifts to higher order $[T^j] \in Z^M_p(D^{\mathrm{Perf}}(X_j))$ successively. $\square$

Now, we consider the case $b_1 \in (f_1, f_2, \cdots, f_p, f_{p+1})$. Since $b_1 \notin (f_1, f_2, \cdots, f_p)$, we can write $b_1 = \sum_{i=1}^p l_i f_i^{n_i} + l_{p+1} f_{p+1}^{n_{p+1}}$, where l_{p+1} is a unit in $O_{X,w}$ and each n_j is some integer. For simplicity, we assume that each $n_j = 1$ and $l_{p+1} = 1$.

Let $K_0^M(O_{X_1,y} \text{ on } y, \varepsilon)$ denote the kernel of the natural projection

$$K_0^M(O_{X_1,y} \text{ on } y) \xrightarrow{\varepsilon=0} K_0^M(O_{X,y} \text{ on } y).$$

There exists the following isomorphism (see Corollary 9.5 in [9] or Corollary 3.11 in [22])

$$K_0^M(O_{X_1,y} \text{ on } y, \varepsilon) \cong H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}).$$

It follows that there is an isomorphism

$$(\mathrm{P}, \mathrm{Ch}) : K_0^M(O_{X_1,y} \text{ on } y) \cong K_0^M(O_{X,y} \text{ on } y) \oplus H^p_y(\Omega^{p-1}_{X/\mathbb{Q}}), \tag{3.2}$$

where P is induced by the map $\varepsilon \to 0$ and Ch is the map induced by Chern character from K-theory to negative cyclic homology (see Theorem 2.7).

For $\mu_Y(Y^1) = [F_\bullet(f_1 + \varepsilon g_1, f_2, \cdots, f_p)] \in K_0^M(O_{X_1,y} \text{ on } y)$, where $g_1 = \dfrac{a_1}{b_1}$, the image $\mathrm{P}(\mu_Y(Y^1)) = \mu_Y(Y) \in K_0^M(O_{X,y} \text{ on } y)$. The image $\mathrm{Ch}(\mu_Y(Y^1))$ can be described by Lemma 2.15 (cf. (2.3) on page 10). Concretely, let $F_\bullet(f_1, f_2, \cdots, f_p)$ be the Koszul resolution of $(O_{X,w})_{Q_1}/(f_1, f_2, \cdots, f_p)$, which is of the form

$$0 \longrightarrow F_p \xrightarrow{M_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow 0,$$

where each $F_i = \bigwedge^i ((O_{X,w})_{Q_1})^{\oplus p}$. The map M_p is

$$e_1 \wedge \cdots \wedge e_p \to \sum_{j=1}^{p} (-1)^j f_j e_1 \wedge \cdots \wedge \hat{e_j} \wedge \cdots e_p, \tag{3.3}$$

where $\{e_1, \cdots, e_p\}$ is a basis of $((O_{X,w})_{Q_1})^{\oplus p}$ and $\hat{e_j}$ means to omit e_j.

The following diagram (denoted γ_1)

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p) \longrightarrow (O_{X,w})_{Q_1}/(f_1, f_2, \cdots, f_p) \\ F_p(\cong (O_{X,w})_{Q_1}) \xrightarrow{(-1)^{p-1}\frac{a_1}{b_1} df_2\wedge\cdots\wedge df_p} F_0 \otimes \Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}} (\cong \Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}}), \end{cases}$$

defines an element of $Ext^p((O_{X,w})_{Q_1}/(f_1, f_2, \cdots, f_p), \Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}})$. The limit $[\gamma_1] \in H^p_y(\Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}})$ of γ_1 is $\mathrm{Ch}(\mu_Y(Y^1))$.

By Lemma 2.6, the Koszul resolution of $(O_{X_1,w})_{Q_1}/(f_1 + \varepsilon\frac{a_1}{f_{p+1}}, f_2, \cdots, f_p)$ gives an element $[F(f_1 + \varepsilon\frac{a_1}{f_{p+1}}, f_2, \cdots, f_p)] \in K^M_0(O_{X_1,y} \text{ on } y)$ whose image under the map P is $\mu_Y(Y)$. By Lemma 2.15, the image of $[F(f_1 + \varepsilon\frac{a_1}{f_{p+1}}, f_2, \cdots, f_p)]$ under the map Ch is the limit $[\gamma_2] \in H^p_y(\Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}})$, where γ_2 is the following diagram (cf. (2.3) on page 10)

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p) \longrightarrow (O_{X,w})_{Q_1}/(f_1, f_2, \cdots, f_p) \\ F_p(\cong (O_{X,w})_{Q_1}) \xrightarrow{(-1)^{p-1}\frac{a_1}{f_{p+1}} df_2\wedge\cdots\wedge df_p} F_0 \otimes \Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}} (\cong \Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}}). \end{cases}$$

It follows from the isomorphism (3.2) that

$$\mu_Y(Y^1) = \mu_Y(Y) + [\gamma_1],\ [F(f_1 + \varepsilon\frac{a_1}{f_{p+1}}, f_2, \cdots, f_p)] = \mu_Y(Y) + [\gamma_2].$$

Since $\frac{a_1}{b_1} - \frac{a_1}{f_{p+1}} = \frac{a_1(-\sum_{i=1}^p l_i f_i)}{b_1 f_{p+1}}$ and each f_i $(i = 1, \cdots, p)$ appears in the map M_p (3.3), $[\gamma_1] = [\gamma_2] \in H^p_y(\Omega^{p-1}_{(O_{X,w})_{Q_1}/\mathbb{Q}})$. This shows that

Lemma 3.5. *The element $\mu_Y(Y^1)$ agrees with $[F(f_1 + \varepsilon\frac{a_1}{f_{p+1}}, f_2, \cdots, f_p)]$.*

It is sufficient to assume that $\mu_Y(Y^1)$ is represented by the Koszul resolution of $(O_{X_1,w})_{Q_1}/(f_1 + \varepsilon\frac{a_1}{f_{p+1}}, f_2, \cdots, f_p)$ in the following. The image $\partial_1^{p,-p}(\mathrm{Ch}(\mu_Y(Y^1)))$ is represented by the following diagram

$$\begin{cases} F_\bullet(f_1, f_2, \cdots, f_p, f_{p+1}) \longrightarrow O_{X,w}/(f_1, f_2, \cdots, f_p, f_{p+1}) \\ F_{p+1}(\cong O_{X,w}) \xrightarrow{(-1)^{p-1} a_1 df_2\wedge\cdots\wedge df_p} F_0 \otimes \Omega^{p-1}_{O_{X,w}/\mathbb{Q}} (\cong \Omega^{p-1}_{O_{X,w}/\mathbb{Q}}), \end{cases}$$

which is not necessarily to be trivial. It follows from the commutative diagram (2.1) that $\mu_Y(Y^1)$ is not a Milnor K-theoretic cycles. Hence, $\mu_Y(Y^1)$ is not a deformation of $\mu_Y(Y)$. In this way, obstructions to deforming $\mu_Y(Y)$ arise.

Recall that Z is the subscheme defined in Definition 2.3 and $\mu_Z(Z) \in Z^M_p(D^{\mathrm{Perf}}(X))$ is represented by the Koszul complex of the sequence $\{f_{p+1}, f_2, \cdots, f_p\}$. Inspired by Idea 1.3, we use $\mu_Z(Z)$ to eliminate obstructions to deforming $\mu_Y(Y)$.

Since $\mu_Y(Y)$ can be written as a formal sum

$$\mu_Y(Y) = (\mu_Y(Y) + \mu_Z(Z)) - \mu_Z(Z) \in Z^M_p(D^{\mathrm{Perf}}(X)),$$

to lift $\mu_Y(Y)$ is equivalent to lifting $\mu_Y(Y) + \mu_Z(Z)$ and $\mu_Z(Z)$ respectively.

By Corollary 3.3, the element $[C_1]$ is a Milnor K-theoretic cycle and it is a first order deformation of $\mu_Y(Y)+\mu_Z(Z)$. By Remark 2.10, $\mu_Z(Z) \in Z^M_p(D^{\mathrm{Perf}}(X_1))$, so the formal sum $[C_1]-\mu_Z(Z) \in Z^M_p(D^{\mathrm{Perf}}(X_1))$. Since $g_1^*([C_1]-\mu_Z(Z)) = (\mu_Y(Y)+\mu_Z(Z))-\mu_Z(Z) = \mu_Y(Y)$, where g_1^* is the map (2.2), $[C_1] - \mu_Z(Z)$ is a first order deformation of $\mu_Y(Y)$.

The Milnor K-theoretic cycle $[C_1] - \mu_Z(Z)$ lies in the direct sum of K-groups

$$[C_1] - \mu_Z(Z) \in Z^M_p(D^{\mathrm{Perf}}(X_1)) \subset \bigoplus_{y \in X^{(p)}} K^M_0(O_{X_1,y} \text{ on } y).$$

Let $([C_1]-\mu_Z(Z))|_Y$ denote the direct summand corresponding to Y (with generic point y) of $[C_1]-\mu_Z(Z)$, one sees that $([C_1]-\mu_Z(Z))|_Y = [F(f_1+\varepsilon\frac{a_1}{f_{p+1}}, f_2, \cdots, f_p)] = \mu_Y(Y^1)$.

By Remark 2.10, for each integer $j > 1$, $\mu_Z(Z) \in Z^M_p(D^{\mathrm{Perf}}(X_j))$. According to Corollary 3.3, the element $[C_1] \in Z^M_p(D^{\mathrm{Perf}}(X_1))$ lifts to $[C_j] \in Z^M_p(D^{\mathrm{Perf}}(X_j))$ successively. It follows that $[C_1] - \mu_Z(Z) \in Z^M_p(D^{\mathrm{Perf}}(X_1))$ lifts to $[C_j] - \mu_Z(Z) \in Z^M_p(D^{\mathrm{Perf}}(X_j))$ successively. In summary,

Theorem 3.6. *With notation as above, in the case $b_1 \in (f_1, \cdots, f_p, f_{p+1})$, there exists a Milnor K-theoretic cycle $\mu_Z(Z) \in Z^M_p(D^{\mathrm{Perf}}(X))$, where $Z \subset X$ is another irreducible closed subscheme of codimension p, and a Milnor K-theoretic cycle $[C_1] \in Z^M_p(D^{\mathrm{Perf}}(X_1))$, which is a first order deformation of $\mu_Y(Y) + \mu_Z(Z)$ such that*

1. $([C_1] - \mu_Z(Z))|_Y = \mu_Y(Y^1)$;
2. *$[C_1] - \mu_Z(Z)$ is a first order deformation of $\mu_Y(Y)$;*
3. *$[C_1] - \mu_Z(Z)$ lifts to higher order successively.*

Acknowledgments

This paper is a revision of [20]. The main result of this paper, Theorem 3.6 and Lemma 3.4, is a straightforward generalization of Theorem 3.7 of [20].

The author thanks Spencer Bloch, Jerome William Hoffman and Christophe Soulé for discussions, and thanks Shiu-Yuen Cheng and Congling Qiu for help. He also thanks referee(s) for professional suggestions that greatly improved this paper.

The idea to eliminate obstructions to deforming divisors had been known to Mark Green and Phillip Griffiths.

References

[1] B. Angéniol, M. Lejeune-Jalabert, Calcul différentiel et classes caractéristiques en géométrie algébrique (French) [Differential calculus and characteristic classes in algebraic geometry], With an English summary, Travaux en Cours [Works in Progress], vol. 38, Hermann, Paris, 1989.

[2] P. Balmer, Supports and filtrations in algebraic geometry and modular representation theory, Am. J. Math. 129 (5) (2007) 1227–1250.

[3] P. Balmer, Tensor triangular Chow groups, J. Geom. Phys. 72 (2013) 3–6.

[4] P. Balmer, M. Schlichting, Idempotent completion of triangulated categories, J. Algebra 236 (2) (2001) 819–834.

[5] S. Bloch, Semi-regularity and de Rham cohomology, Invent. Math. 17 (1972) 51–66.

[6] S. Bloch, Private discussions at Tsinghua University, Beijing, Spring semester (2015) and Fall semester (2016).

[7] S. Bloch, H. Esnault, M. Kerz, p-adic deformation of algebraic cycle classes, Invent. Math. 195 (3) (2014) 673–722.

[8] S. Bloch, H. Esnault, M. Kerz, Deformation of algebraic cycle classes in characteristic zero, Algebr. Geom. 1 (3) (2014) 290–310.

[9] B. Dribus, J.W. Hoffman, S. Yang, Tangents to Chow groups: on a question of Green-Griffiths, Boll. UMI 11 (2018) 205–244.

[10] H. Gillet, C. Soulé, Intersection theory using Adams operations, Invent. Math. 90 (2) (1987) 243–277.

[11] M. Green, P. Griffiths, Formal deformation of Chow groups, in: The Legacy of Niels Henrik Abel, Springer, Berlin, 2004, pp. 467–509.

[12] M. Green, P. Griffiths, On the Tangent Space to the Space of Algebraic Cycles on a Smooth Algebraic Variety, Annals of Math Studies, vol. 157, Princeton University Press, Princeton, NJ, 2005, vi+200 pp., ISBN 0-681-12044-7.

[13] P. Griffiths, J. Harris, Principles of Algebraic Geometry, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York, ISBN 0-471-32792-1, 1978, xii+813 pp.

[14] G. Grothendieck, On the de Rham cohomology of algebraic varieties, Publ. Math. IHÉS (29) (1966) 95–103.

[15] S. Klein, Chow groups of tensor-triangulated categories, J. Pure Appl. Algebra 220 (4) (2016) 1343–1381.

[16] K. Kodaira, D.C. Spencer, A theorem of completeness of characteristic systems of complete continuous systems, Am. J. Math. 81 (1959) 477–500.

[17] M. Morrow, A case of the deformational Hodge conjecture via a pro Hochschild-Kostant-Rosenberg theorem, C. R. Math. Acad. Sci. Paris 352 (3) (2014) 173–177.

[18] S. Mukai, H. Nasu, Obstructions to deforming curves on a 3-fold, I: a generalization of Mumford's example and an application to Hom schemes, J. Algebraic Geom. 18 (4) (2009) 691–709.

[19] T.F. Ng, Geometry of algebraic varieties, Ph.D. thesis, Princeton University, 2004.

[20] S. Yang, Eliminating obstructions: curves on a 3-fold, arXiv:1611.07279, not published.

[21] S. Yang, K-theory, local cohomology and tangent spaces to Hilbert schemes, Ann. K-Theory 3 (4) (2018) 709–722.

[22] S. Yang, On extending Soulé's variant of Bloch-Quillen identification, Asian J. Math. 23 (1) (2019) 049.

[23] S. Yang, Deformation of K-theoretic cycles, Asian J. Math. 24 (2) (2020) 303–330.