Published for SISSA by Springer

Received: *September 17, 2025*
Accepted: *November 9, 2025*
Published: *December 10, 2025*

Conformal 3-point correlators in momentum space, method of subgraphs and the $1/N$ expansion

Zhijin Li [a]

[a] *Shing-Tung Yau Center and School of Physics, Southeast University, 2 Sipailow Street, Nanjing 210096, China*

E-mail: zhijin_li@seu.edu.cn

Abstract: Conformal 3-point correlators of conserved currents play important roles in numerous directions. These correlators are fixed by conformal symmetry up to a few parameters, which are known only at leading order in perturbative expansions. The major challenges come from the multi-loop Feynman integrals with three external momenta. In this work, we employ the method of subgraphs to compute the subleading order corrections to the conformal current 3-point correlators in the large N expansion. We show that the method of subgraphs generates diagrammatic expansions for the conformal 3-point correlators, and that it is closely related to the operator product expansions in momentum space. We verify the subgraph expansions of conserved current 3-point correlators using exact results in 3D. We demonstrate that multi-loop 3-point Feynman integrals can be significantly simplified by taking the subgraph expansions. Due to constraints from conformal symmetry, it suffices to keep only the first few terms in the subgraph expansions to completely fix the subleading order corrections. We apply this method to compute the $1/N$ corrections to current correlators $\langle JJJ\rangle$ in the critical $O(N)$ vector model and the Gross-Neveu-Yukawa model. We also compute the $1/N$ corrections to the coefficients in the current-current-scalar correlators $\langle JJ\sigma_T\rangle$ and $\langle JJ\sigma\rangle$ in the critical $O(N)$ vector model. We compare the perturbative results with the bootstrap data and discuss their application to conductivity near the quantum critical point.

Keywords: $1/N$ Expansion, Conformal and W Symmetry, Scale and Conformal Symmetries, Renormalization and Regularization

ArXiv ePrint: 2509.07106

https://doi.org/10.1007/JHEP12(2025)066

JHEP12(2025)066

Contents

1 Introduction

The 3-point correlators of conserved current operators in conformal field theories (CFTs) have been systematically studied in the fundamental works [1, 2]. Here the conserved current operators of interest include both the conserved spin 1 vector field operator J^a_μ and the stress tensor operator $T_{\mu\nu}$. The results in [1, 2] show that in general dimensions, the conformal 3-point correlation functions can be fixed by conformal symmetry up to a few free parameters, and the numbers of independent free parameters are further reduced by the conservation conditions of the conserved currents. These parameters are the key ingredients in the studies of the conformal current 3-point correlators. The parameters in the correlators $\langle TJJ\rangle$, $\langle TTT\rangle$ are subject to the constraints from unitarity and causality, namely the conformal collider bounds [3–6], and they play important roles in the studies of the strongly coupled systems in condensed matter physics [5, 7–11] and quantum gravity through holography [12, 13]. The conformal current 3-point correlators have also been applied to study the cosmological

JHEP12(2025)066

correlators [14–19]. For instance, the stress tensor 3-point correlator corresponds to the non-Gaussianity of gravitational waves generated during the early stage of cosmology [15].

We will be interested in the conserved current 3-point correlators $\langle J_\mu^a J_\nu^b \mathcal{O}\rangle$ and $\langle T_{\mu\nu} T_{\rho\sigma} \mathcal{O}\rangle$, in which the operators $\mathcal{O}$ are the leading terms in the operator product expansions (OPEs):

$$J_\mu^a \times J_\nu^b \sim \mathbf{1}\, \delta^{ab} + \mathcal{O}_S\, \delta^{ab} + T_{\mu\nu}\, \delta^{ab} + \cdots , \tag{1.1}$$

$$T_{\mu\nu} \times T_{\rho\sigma} \sim \mathbf{1} + \mathcal{O}_S + T_{\mu\sigma} + \cdots . \tag{1.2}$$

The above OPEs can be used to determine the charge conductivity and viscosity near quantum critical points at finite temperature [7–11], for which the dominat contributions are from the vacuum state, relevant scalars $\mathcal{O}_S$ and the stress tensor $T_{\mu\nu}$. A recent conformal bootstrap study has shown that the conserved current operators can help to improve the bootstrap results to remarkably high precision [20]. It is expected that a similar bootstrap implementation associated with the conserved currents can be used to improve the bootstrap results for the strongly coupled conformal gauge theories [21], in particular the infrared (IR) fixed point of the three-dimensional quantum electrodynamics (QED_3) coupled to $N_f = 4$ flavors electrons [22], and the perturbative results on the current 3-point correlators can provide useful input data for the bootstrap computations, reminiscent of the special roles played by the $1/N$ expansions of the QED_3 spectrum [23–25].

The goal of this work is to develop large N diagrammatic techniques to compute the $1/N$ corrections to the parameters in the conserved current 3-point correlators. The method can be applied to general conformal field theories. Here we will focus on the critical $O(N)$ vector model and the Gross-Neveu-Yukawa (GNY) model. The critical $O(N)$ vector model has been extensively studied in the past decades using $1/N$ perturbative expansions, and plenty of CFT data for this theory have been obtained, see [26] for a comprehensive review. Nevertheless, most of the perturbative studies focused on the spectrum [27–32] and the central charges [33–36] as free parameters in the conserved current 2-point correlators;[1] while the perturbative results on the conserved current 3-point correlators were computed only at the leading order in position space [1, 2] and in momentum space [5, 15]. These results are given by the free field theories. The main challenge is that the Feynman loop integrals for the 3-point correlators are more difficult to evaluate than those for the 2-point correlators. In fact, although the conformal 3-point correlators are fixed by conformal symmetry up to free parameters and have compact forms in position space, their counterparts in momentum space can only be written in complicated expressions in general dimensions.[2] Since the Feynman loop integrals are usually evaluated in momentum space, one cannot directly apply the conformal 3-point correlation functions in position space [1, 2] to the Feynman loop integrals.

The conformal 3-point correlators in momentum space, especially those for the conserved current operators have been extensively studied in the past decade [39–49], motivated by their fundamental applications in holographic cosmology and other directions. It has been shown that the conformal 3-point correlators in momentum space can be expressed as integrals of triple Bessel K functions [39] or, equivalently, as Appell functions [40]. These formulas

[1] The $1/N$ corrections to the scalar three-point correlators have been computed recently in [37, 38].

[2] In 3D the conformal conserved current 3-point correlators in momentum space are given by conformal integrals, which can be written in compact forms.

JHEP12(2025)066

provide the leading order results in the perturbative expansions of the 3-point correlators, and the Feynman loop integrals for the subleading order corrections are considerably more challenging. In this work, we apply the method of subgraphs [50–53] to decompose the original multi-loop 3-point Feynman diagrams into a series of simpler Feynman integrals. Due to the constraints from conformal symmetry, the subleading order corrections to the 3-point correlators can be completely fixed by the first few terms in the subgraph expansions, therefore it significantly simplifies the Feynman loop integrals for the $1/N$ corrections.

The method of subgraphs provides a graph-theoretical expansion for multi-scale Feynman loop integral in terms of its subgraphs. The Feynman loop integrals with multiple scales appear ubiquitously in QFTs, while in general they are hard to evaluate analytically. A natural idea is to expand the original loop integral with respect to a small/large scale parameter p. The subtlety is that such expansion does not commute with the integration and the preconditions for the p-expansion of the integrand are not satisfied in certain regions of the integration domain. This problem can be systematically resolved using the method of subgraphs or the method of regions [50–54]. The method of regions is physically motivated: one divides the integration domain into different regions $\mathcal{D}_i$, within which the integrand can be Taylor expanded with respect to parameters that are considered small in $\mathcal{D}_i$. This method has been widely applied in Feynman loop computations, see [55] for new development. Nevertheless, a strict proof of the method of regions is not available yet.

For the Feynman integrals in Euclidean spacetime, or those with off-shell momenta and masses in Minkowski spacetime, the expansion of the original integral can be represented in a graph-theoretical way, i.e., by its subgraphs. Here the role of the subgraphs is to provide graphical labels for the small/large internal momenta in the Taylor expansion of the integrand. The method of subgraphs has been proved for general Feynman diagrams in Euclidean space, see appendix B.2 in [51]. Coincidentally, the Feynman loop integrals for the CFT data, including the conformal 3-point correlators are evaluated in Euclidean spacetime, therefore the method of subgraphs can be applied as a solid approach to simplify the loop computations. Another remarkable advantage of applying the method of subgraphs to CFTs is that the conformal 3-point correlators have already been *almost* fixed by conformal symmetry. As a result, the $1/N$ corrections to the free parameters can be completely solved from the first few terms in the subgraph expansion. This is however different from the Feynman loop integrals in general QFTs, in which the method of subgraphs provides an asymptotic expansion and requires taking the subgraph expansions to high orders to obtain sufficiently precise data for the original Feynman integral [56].

A short summary of the results. The gist of this work is to show that the method of subgraphs, together with the constraints from conformal symmetry provides an efficient approach to compute the higher order corrections of the conformal 3-point correlators. Our results can be summarized into three parts.

a. In section 2 we apply the method of subgraphs to study the general scalar conformal 3-point correlators in momentum space and demonstrate that it provides a graph-theoretical expansion for the complicated correlation functions. A particularly important property is that the subgraph expansion automatically captures the singularities in the

JHEP12(2025)066

OPE limit in the momentum space. We compare the subgraph expansions for the scalar 3-point functions $\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle$ with the solutions from the conformal Ward identities, and find perfect agreement. In section 3 we present the subgraph expansions of the conserved current 3-point correlators $\langle JJJ \rangle$, $\langle TJJ \rangle$ and $\langle TTT \rangle$, and verify the results using the exact correlation functions in 3D.

b. We compute the $1/N$ corrections to the conserved current 3-point correlators $\langle JJJ \rangle$ in the critical $O(N)$ vector model and GNY model. In both theories the 3-point correlators can be expanded in terms of the free field theories

$$\langle JJJ \rangle = n_s \langle JJJ \rangle_{\text{free scalar}} + n_f \langle JJJ \rangle_{\text{free fermion}}. \tag{1.3}$$

The parameters n_s and n_f, up to the order $O(1/N)$, are given by

$$n_s = 1 - \frac{2^{D+2}(3D-4)\sin\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{\pi^{3/2}(D-2)^2 D^2 \Gamma\left(\frac{D}{2}-2\right) N}, \quad n_f = -\frac{2^{D+3}\sin\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{\pi^{3/2}(D-2)^2 D^2 \Gamma\left(\frac{D}{2}-2\right) N}, \tag{1.4}$$

for the critical $O(N)$ vector model and

$$n_s = \frac{2^{D+2}\sin\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{\pi^{3/2} D^2 \Gamma\left(\frac{D}{2}-1\right) N}, \quad n_f = 1 + \frac{2(D+1)\sin\left(\frac{\pi D}{2}\right)\Gamma(D-1)}{\pi \Gamma\left(\frac{D}{2}+1\right)^2 N}, \tag{1.5}$$

for the GNY model. Details of the computations and the special role of the method of subgraphs are presented in section 4.

c. We compute the $1/N$ corrections to the correlators $\langle JJ\mathcal{O} \rangle$ in the critical $O(N)$ vector model motivated by their applications in the conductivity at finite temperature. We focus on the lowest $O(N)$ traceless symmetric scalar σ_T and the $O(N)$ singlet scalar σ. Up to the subleading order, the results are

$$\lambda_{JJ\sigma_T} = D - 2 + \frac{((D-12)D+16)(\cos(\pi D)-1)\Gamma\left(2-\frac{D}{2}\right)\Gamma(D-2)}{\pi^2 D \Gamma\left(\frac{D}{2}+1\right) N}, \tag{1.6}$$

and

$$\lambda_{JJ\sigma}|_{D=3} = \frac{4}{\pi\sqrt{N}} - \frac{368}{9\pi^3\sqrt{N}N}. \tag{1.7}$$

The normalization for above OPE coefficients is introduced in (5.6), which follows [57]. Details of the $1/N$ loop computations and renormalization are explained in section 5.

We expect the method developed in this work can be employed to provide a perturbative study for the conserved current 3-point correlators in many physically interesting CFTs. We discuss some future works in section 6.

JHEP12(2025)066

2 Method of subgraphs and conformal 3-point correlators in momentum space

Feynman diagrams are usually evaluated in momentum space. Therefore, to study conformal correlation functions using this method, one needs to know the momentum-space form of the correlators. The conformal 2-point correlation functions in momentum space can be obtained using Fourier transformations, and the results are given in simple forms. However, for the conformal 3-point correlators, the Fourier transformations of the correlation functions in position space lead to complicated expressions [39, 40]. Besides, there are extra contributions from the contact terms of the correlation functions. In this section, we first review the results on the conformal 3-point correlators in momentum space obtained from the Fourier transformations and the conformal Ward identities [39, 40, 47], then we propose that the method of subgraphs provides a useful graph-theoretical expansion for the conformal 3-point correlation function in momentum space.

2.1 Conformal 3-point correlators in momentum space

We use the scalar 3-point correlator to exhibit the complexity of the expressions for the conformal 3-point correlation functions in momentum space. The 3-point correlators of spinning operators can be decomposed into scalar correlation functions associated with specific tensor structures [39].

The conformal 3-point correlator in momentum space can be obtained through Fourier transformations of the correlators in position space. Consider a 3-point correlator of scalars $\mathcal{O}_i$ with scaling dimensions Δ_i. In position space, the correlation function is given by

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}x_{23}^{\Delta_2+\Delta_3-\Delta_1}x_{31}^{\Delta_1+\Delta_3-\Delta_2}}, \tag{2.1}$$

where $x_{ij} \equiv |x_i - x_j|$. We take the Fourier transformation of the above formula to get the same conformal 3-point correlation function in momentum space $\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(p_3)\rangle$, up to the contributions from contact terms.[3] The Fourier transformation of the 3-point correlation function (2.1) is proportional to the one-loop integral

$$\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(p_3)\rangle \propto \mathcal{I}(\nu_1,\nu_2,\nu_3) \equiv \int \frac{d^Dk}{(2\pi)^D} \frac{1}{\left(k^2\right)^{\nu_3}\left((p_1-k)^2\right)^{\nu_2}\left((p_2+k)^2\right)^{\nu_1}}, \tag{2.2}$$

where we have implicitly assumed momentum conservation condition $p_3^\mu = -p_1^\mu - p_2^\mu$ and

$$\Delta_t = \Delta_1+\Delta_2+\Delta_3, \quad \nu_i = \frac{D-\Delta_t}{2} + \Delta_i.$$

The integral $\mathcal{I}(\nu_1,\nu_2,\nu_3)$ can be studied using recursive relations derived from either conformal symmetry constraints or integration-by-parts relations [40, 58]. Using standard techniques

[3] The conformal 3-point correlation functions in position space like (2.1) are valid when the three coordinates x_i do not coincide. The contact terms appear when the coordinates x_i coincide, and their Fourier transformations correspond to analytic terms of the external momenta [15].

JHEP12(2025)066

for Feynman integrals, the scalar conformal 3-point correlation function in momentum space can be rewritten in an integral form [39]

$$\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(p_3)\rangle = c_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}\delta^d(p_1+p_2+p_3)p_1^{\Delta_1-\frac{d}{2}}p_2^{\Delta_2-\frac{d}{2}}p_3^{\Delta_3-\frac{d}{2}} \times \int_0^\infty dx x^{\frac{d}{2}-1}K_{\Delta_1-\frac{d}{2}}(p_1x)K_{\Delta_2-\frac{d}{2}}(p_2x)K_{\Delta_3-\frac{d}{2}}(p_3x), \tag{2.3}$$

where $c_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ is a constant related to $\lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}$ in (2.1) and $K_a(z)$ is the modified Bessel function of the second kind. For general parameters, the above triple-K integral cannot be solved in a compact form. In fact, it may be divergent and a suitable regularization is needed, see [39] for more details.

The conformal 3-point correlators can be derived directly in momentum space by solving the conformal Ward identities, without referring to their position-space counterparts [15, 39, 40]. Poincaré invariance enforces momentum conservation and implies that the scalar 3-point function depends only on the magnitudes of the external momenta, $p_i \equiv |p_i^\mu|$. The additional conformal Ward identities are derived from dilatation and special conformal transformations, which can be written in momentum space:

$$\left(2D+\sum_{i=1}^3\left(p_i\frac{\partial}{\partial p_i}-\Delta_i\right)\right)\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(p_3)\rangle = 0, \tag{2.4}$$

$$\sum_{i=1}^3\left(\frac{\partial^2}{\partial p_i^2}+\frac{D+1-2\Delta_i}{p_i}\frac{\partial}{\partial p_i}\right)p_i^\mu\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(p_3)\rangle = 0. \tag{2.5}$$

The dilatation symmetry (2.4) implies that the scalar 3-point function is a homogeneous function of the external momenta p_i with degree $\Delta_t - 2D$:

$$\langle \mathcal{O}_1(p_1)\mathcal{O}_2(p_2)\mathcal{O}_3(p_3)\rangle = p_3^{\Delta_t-2D}f\left(\frac{p_1}{p_3},\frac{p_2}{p_3}\right). \tag{2.6}$$

Substituting this ansatz into the special conformal Ward identity (2.5) yields a system of partial differential equations for the function f. The general solutions to these equations are expressed in terms of the generalized hypergeometric function of two variables, the Appell F_4 functions. Among these, the physical solution corresponds precisely to the momentum-space expression given in (2.3).

We will be particularly interested in the zero-momentum limit, where

$$p_1^\mu = p^\mu,\ p_2^\mu = -p^\mu - q^\mu,\ p_3^\mu = q^\mu,\quad q\equiv|q^\mu| \ll p\equiv|p^\mu|,$$

and the Bessel function $K_a(p_3x)$ in (2.3) admits the following series expansion

$$K_a(p_3x) = 2^{a-1}\Gamma(a)p_3^{-a}x^{-a}\left(1+O(p_3^2)\right)+2^{-a-1}\Gamma(-a)p_3^a x^a\left(1+O(p_3^2)\right).$$

By applying the above expansion to (2.3), the triple-K integral for the 3-point correlation function (2.3) can be simplified to

$$\begin{aligned}&\langle \mathcal{O}_1(p)\mathcal{O}_2(-p-q)\mathcal{O}_3(q)\rangle \\ &\propto p^{\Delta_t-2D}2^{\Delta_3-\frac{D}{2}-1}\Gamma\left(\Delta_3-\frac{D}{2}\right)R\left(D-\Delta_3,\Delta_1-\frac{D}{2},\Delta_2-\frac{D}{2}\right)\left(1+O(q^2)\right) \\ &\quad+p^{\Delta_t-2\Delta_3-D}q^{2\Delta_3-D}2^{\frac{D}{2}-\Delta_3-1}\Gamma\left(\frac{D}{2}-\Delta_3\right)R\left(\Delta,\Delta_1-\frac{D}{2},\Delta_2-\frac{D}{2}\right)\left(1+O(q^2)\right),\end{aligned} \tag{2.7}$$

JHEP12(2025)066

in which we have applied the double-K integral [59]

$$R(a,b,c) \equiv \int_0^\infty dx x^{a-1} K_b(x) K_c(x) \tag{2.8}$$
$$= \frac{2^{a-3}\Gamma\left(\frac{1}{2}(a-b-c)\right)\Gamma\left(\frac{1}{2}(a+b-c)\right)\Gamma\left(\frac{1}{2}(a-b+c)\right)\Gamma\left(\frac{1}{2}(a+b+c)\right)}{\Gamma(a)}.$$

An important property in (2.7) is that, for general Δ_i, the small q expansion separates into two parts: a regular part whose leading term is independent of q and a second part with leading term proportional to $q^{2\Delta_3-D}$. For $\Delta_3 > D/2$ the entire correlation function is regular in the zero-momentum limit, while for $\Delta_3 < D/2$ the 3-point function is dominated by a singular term proportional to $q^{2\Delta_3-D}$ in the zero-momentum limit. Such behavior of the conformal 3-point correlators has been studied in [10, 11] to evaluate the dynamical response functions near quantum critical points. They also play an important role in analyzing the relation between the scale and conformal invariance [41, 60].

The small q expansion of the scalar 3-point correlation function (2.7) can also be obtained from the properties of the Appell function F_4 [40]. In the following section, we show that the two terms in the zero-momentum limit can be reconstructed through the subgraph expansion of the one-loop integral (2.2). The advantage of the subgraph expansion is that it provides a simplified diagrammatic representation of the asymptotic behaviors of the correlation functions, without using the mathematical properties of the complicated expressions for the original integrals.

2.2 Method of subgraphs for the 2-scale loop integrals

The conformal 3-point correlation functions in momentum space (2.2) depend on the one-loop integral $\mathcal{I}(\nu_1,\nu_2,\nu_3)$, which contains two scales p_1 and p_2. The Feynman loop diagrams with multiple scales are usually difficult to evaluate analytically. For the multi-scale Feynman loop integrals in Euclidean space, the method of subgraphs provides an efficient approach to evaluate the original integral by decomposing it into its subgraphs with lower scales [50–52]. In this section, we briefly review the subgraph expansions of Feynman diagrams, and then apply the method of subgraphs to compute the conformal 3-point correlators in momentum space.

The multi-scale Feynman integrals in the limits with small or large external momenta have been studied for a long time [61]. This problem is subtle as the limit of the integral cannot be obtained simply by taking the same limit on the integrand. Take the one-loop integral $\mathcal{I}(1,1,1)$ for example:

$$\mathcal{I}(1,1,1) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 (p+k)^2 (k-q)^2}. \tag{2.9}$$

This is an integral with two external momenta. In this work, the integrals with n external momenta are dubbed n-scale integral. For $n = 0$, the integral becomes scaleless and vanishes

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{\left(k^2\right)^a} = 0. \tag{2.10}$$

The 2-point correlation functions are given by the 1-scale integrals, and their one-loop diagrams can be exactly solved through

$$\int \frac{d^d k}{(2\pi)^D} \frac{1}{\left(k^2\right)^a \left((k-p)^2\right)^b} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma\left(\frac{D}{2}-a\right)\Gamma\left(\frac{D}{2}-b\right)\Gamma\left(a+b-\frac{D}{2}\right)}{\Gamma(a)\Gamma(b)\Gamma(D-a-b)} p^{D-2a-2b}. \quad (2.11)$$

The integrals for the 2-point correlators with tensor structures can be reduced to the scalar integrals (2.11) through tensor reductions. The one-loop integrals will be extensively employed throughout this work. In contrast, there are no compact solutions for the one-loop 2-scale integrals like (2.9), and we take asymptotic expansions for these integrals.

We now show the subtlety in the small q expansion of the 2-scale integrals. Consider the Taylor expansion with small q ($\mathcal{T}_q$) for the integrand in (2.9):

$$\frac{1}{k^2(p+k)^2(k-q)^2} = \frac{1}{k^2(p+k)^2}\mathcal{T}_q \frac{1}{(k-q)^2} = \frac{1}{(k^2)^2(p+k)^2} + \frac{2k\cdot q}{(k^2)^3(p+k)^2} + \cdots . \quad (2.12)$$

The integral of each term in the expansion is regular in q

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{\left(k^2\right)^2 (p+k)^2} + \frac{2k\cdot q}{(k^2)^3(p+k)^2} = \frac{2^{-D}\pi^{-\frac{D}{2}}\Gamma\left(3-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(\frac{D}{2}-1\right)}{\Gamma(D-3)} p^{D-6}$$
$$- \frac{2^{-D}\pi^{-\frac{D}{2}}\Gamma\left(4-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(\frac{D}{2}-1\right)}{\Gamma(D-3)} p^{D-8}\, p\cdot q. \quad (2.13)$$

With $D=3$, the two leading terms vanish. Actually the integral (2.13) is IR divergent in 3D, and the right-hand side of (2.13) gives the integral in dimensional regularization. The result in (2.13) can be compared with the integral (2.9) in 3D, which becomes a conformal integral and can be evaluated exactly:

$$\mathcal{I}(1,1,1)|_{D=3} = \frac{1}{8pq|p+q|} = \frac{1}{8p^2 q} - \frac{p\cdot q}{8p^4 q} + \cdots . \quad (2.14)$$

The original integral has a pole in the zero-momentum limit $q \to 0$, which is not shown in (2.13). The small q expansion of the integrand failed to capture the zero-momentum limit of the one-loop integral (2.9). The reason is that the Taylor expansion (2.12) is broken down in the integration domain $k < q$. In this region the integrand of $\mathcal{I}(1,1,1)$ can be alternatively expanded with small k:

$$\frac{1}{k^2(p+k)^2(k-q)^2} = \frac{1}{k^2(k-q)^2}\mathcal{T}_k \frac{1}{(p+k)^2} = \frac{1}{p^2k^2(k-q)^2} - \frac{2k\cdot p}{(p^2)^2k^2(k-q)^2} + \cdots , \quad (2.15)$$

and its integral is given by

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{p^2k^2(k-q)^2} - \frac{2k\cdot p}{(p^2)^2k^2(k-q)^2} \quad (2.16)$$
$$= \frac{2^{-D}\pi^{-\frac{D}{2}}\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)^2}{\Gamma(D-2)p^2} q^{D-4} - \frac{2^{1-D}\pi^{-\frac{D}{2}}\Gamma\left(2-\frac{D}{2}\right)\Gamma\left(\frac{D}{2}-1\right)\Gamma\left(\frac{D}{2}\right)p_\mu}{\Gamma(D-1)p^4} q^{D-4} q\cdot p,$$

which produces the exact asymptotic behavior (2.14) of $\mathcal{I}(1,1,1)$ in 3D.

JHEP12(2025)066

This example demonstrates that the integration domains of the loop momenta are crucial for obtaining proper asymptotic expansions of the Feynman integrals. The method of regions is proposed to resolve this problem [54]. It leads to asymptotic expansions for general Feynman integrals in certain limits of external momenta and masses. The procedure can be summarized as follows: first divide the integration domain of the internal momenta into different regions $\mathcal{D}_i$ associated with a set of parameters $\mathcal{C}_i$ which are considered small in $\mathcal{D}_i$, and take the Taylor expansion for the integrand f in each region $\mathcal{D}_i$ with respect to the small parameters $\mathcal{C}_i$: $\mathcal{T}_{\mathcal{C}_i} f$; then integrate the new integrands $\mathcal{T}_{\mathcal{C}_i} f$ over the entire integration domain of the internal momenta. See [51] for detailed explanations of the method of regions.

The method of regions can be applied to general Feynman integrals, although a strict proof has not yet been achieved. However, for Feynman integrals in Euclidean space, the asymptotic expansion of the Feynman integrals can be formulated in a graph-theoretical language, namely the method of subgraphs [50–52]. The Feynman loop integrals for the CFT data are of the Euclidean type; therefore, their asymptotic expansions can be effectively studied using the method of subgraphs.

Let us go back to the one-loop integral $\mathcal{I}(1,1,1)$. As explained after (2.13), a naive Taylor expansion of the integrand with small q generates IR divergence. This problem can be cured by expanding the integrand with small k (2.15). Consider the remainder of above Taylor expansions up to the order n, denoted $\mathcal{T}^n_{q/k}$:

$$R^n\,\mathcal{I}(1,1,1) = \int \frac{d^D k}{(2\pi)^D}\frac{1}{k^2}\left((1-\mathcal{T}^n_k)\,\frac{1}{(p+k)^2}\right)\left((1-\mathcal{T}^n_q)\,\frac{1}{(k-q)^2}\right). \tag{2.17}$$

The second Taylor expansion $(1-\mathcal{T}^n_q)$ generates IR divergences for the integral, which is cured by the remainder from the first Taylor expansion $(1-\mathcal{T}^n_q)$, and vice versa for the integral in the UV limit. The operators in the remainder (2.17) can be interpreted graphically

$$R^n = (1-M^n_\gamma)(1-M^n_\Gamma), \tag{2.18}$$

in which M_Γ refers to the operation on the original Feynman diagram and only the momentum q_μ is considered small, while M_γ refers to the operation on the subgraph given by the propagator $\frac{1}{(k-q)^2}$ and both the momenta q_μ, k_μ are considered small. The original integral is given by

$$\begin{aligned}\mathcal{I}(1,1,1) &= (1-R^n)\mathcal{I}(1,1,1) + R^n\mathcal{I}(1,1,1) &(2.19)\\ &= (M^n_\Gamma + M^n_\gamma + M^n_\Gamma\cdot M^n_\gamma + R^n)\,\mathcal{I}(1,1,1). &(2.20)\end{aligned}$$

The first two terms consist of the one-loop integrals with a single scale, which can be evaluated using the formula (2.11). The third term generates a scaleless integral and vanishes (2.10). Taking the order of the Taylor expansions to infinity $n\to\infty$, the remainder term vanishes as well, and the original one-loop integral has an asymptotic expansion

$$\mathcal{I}(1,1,1) = M_\Gamma\mathcal{I}(1,1,1) + M_\gamma\mathcal{I}(1,1,1). \tag{2.21}$$

The operation M_Γ (M_γ) corresponds to the Taylor expansion with respect to small momentum q (momenta q,k) for the original Feynman diagram Γ (subgraph γ).

JHEP12(2025)066

This procedure can be generalized to higher-loop Feynman diagrams in Euclidean spacetime [50, 51]. Consider an L-loop Feynman diagram Γ in Euclidean space. Its external momenta are separated into two sets: the large and small external momenta $\{Q_1, Q_2, \ldots, Q_i\}$, $\{q_1, q_2, \ldots, q_j\}$. All the masses are tuned to zero to reach the IR fixed points. It can be shown that the remainder of the large Q or small q momenta expansion is given by the forest formula

$$R^n = \sum_S \prod_{\gamma \in S} \left(-M_\gamma^{n(\gamma)}\right), \tag{2.22}$$

where S denotes the set of non-overlapping subgraphs, i.e. the forests [62]. The rules to determine S will be explained later. The order of the Taylor expansion $n(\gamma)$ depends on the UV degree of divergence $\Lambda(\gamma)$ of the subgraph γ: $n(\gamma) = \Lambda(\gamma) + n$, so that the remainder R^n scales as $(q/Q)^n$ in the large Q (small q) limit [63]. The subgraphs γ in (2.22) are determined by two rules. First, the subgraphs $\gamma \subseteq \Gamma$ contain all the vertices with large external momenta Q. Besides, the new graph is 1PI after identifying the external vertices of γ with large momenta Q. Let $k_1, \ldots, k_{h(\gamma)}$ denote the loop momenta restricted in the subgraph γ, and $k_{h(\gamma)+1,\ldots,k_L}$ denote the loop momenta which also flow through the complementary part of the subgraph Γ/γ. When taking the Taylor expansion for the subgraph γ, the internal momenta are separated into two scales: the momenta $k_1, \ldots, k_{h(\gamma)}$ are considered large, while the momenta $k_{k(\gamma)+1,\ldots,k_L}$ are considered small. The operation $M_\gamma^{n(\gamma)}$ applies on the subgraph γ while commutes with its complementary part Γ/γ:

$$M_\gamma^{n(\gamma)} \mathcal{I}_\Gamma = \mathcal{I}_{\Gamma/\gamma} \circ M_\gamma^{n(\gamma)} \mathcal{I}_\gamma = \int \frac{dk_1}{(2\pi)^d} \cdots \frac{dk_L}{(2\pi)^d} \mathcal{P}_{\Gamma/\gamma} \mathcal{T}^{n(\gamma)}_{q_1,\cdots,q_j,k_{h(\gamma)+1},\ldots,k_L} \mathcal{P}_\gamma, \tag{2.23}$$

where $\mathcal{P}_\gamma$ represents the partial integrand from the subgraph γ and we use $\mathcal{I}_\gamma$ to denote its integral.

The remainder form (2.22) leads to an identity for the original Feynman integral $\mathcal{I}_\Gamma$:

$$\mathcal{I}_\Gamma = (1 - R^n)\, \mathcal{I}_\Gamma + R^n\, \mathcal{I}_\Gamma = \sum_\gamma \mathcal{I}_{\Gamma/\gamma} \circ M_\gamma^{n(\gamma)} \mathcal{I}_\gamma + R^n \mathcal{I}_\Gamma. \tag{2.24}$$

Here when expanding the subtraction $1 - R^n$, we have implicitly employed the fact that the multiple operations like $M_{\gamma_1} M_{\gamma_2} \mathcal{I}_\Gamma$ generate scaleless Feynman integrals which vanish automatically. Now take the infinity n limit, above formula gives the asymptotic expansion of the original Feynman integral

$$\mathcal{I}_\Gamma = \sum_\gamma \mathcal{I}_{\Gamma/\gamma} \circ M_\gamma \mathcal{I}_\gamma. \tag{2.25}$$

The subgraph expansion can be applied to general Feynman loop diagrams in Euclidean spacetime with external momenta separated into two scales.

Now we compute the subgraph expansion of the scalar 3-point correlator $\langle \mathcal{O}_1(p) \mathcal{O}_2(-p-q) \mathcal{O}_3(q) \rangle$ in the limit $q \ll p$. The correlator is proportional to the integral $\mathcal{I}(\nu_1, \nu_2, \nu_3)$

$$\mathcal{I}(\nu_1, \nu_2, \nu_3) = \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left((p+k)^2\right)^{\nu_3} \left(k^2\right)^{\nu_2} \left((k-q)^2\right)^{\nu_1}}. \tag{2.26}$$

This is a generalization of the integral (2.9) to general external scalars. The diagrammatic representation of the integral $\mathcal{I}(\nu_1, \nu_2, \nu_3)$ is shown in figure 1, in which the links should be considered as generalized scalar propagators: $1/(k^2)^{\nu_i}$. According to the rules for the

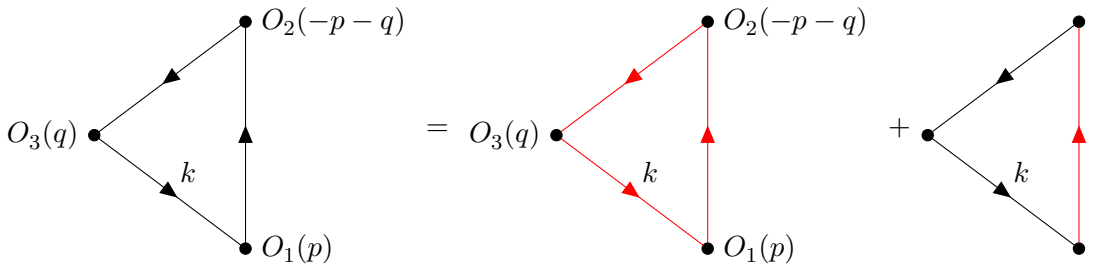

Diagram Γ Subgraph Γ Subgraph γ

Figure 1. Diagrammatic representation of $\mathcal{I}(\nu_1,\nu_2,\nu_3)$ and the two subgraphs in the expansion: Γ and γ marked in red color.

method of subgraphs (2.25), the subgraphs of the diagram $\mathcal{I}(\nu_1,\nu_2,\nu_3)$ consist of the 1PI subgraphs containing all the external vertices with large external momenta, i.e., $\mathcal{O}_1(p)$ and $\mathcal{O}_2(-p-q)$. So there are two subgraphs for the original Feynman diagram Γ: Γ itself and the link connecting $\mathcal{O}_1(p)$ with $\mathcal{O}_2(-p-q)$, see the graphs in red color in figure 1. We take Taylor expansions for the subgraphs with respect to the momenta of small scale. The external momentum q is consider small for all the subgraphs. The internal loop momentum k is considered large for the subgraph Γ; while for the subgraph γ, the loop momentum k also flows over Γ/γ so is considered small.

Note in figure 1 we have ignored another putative subgraph Γ/γ, i.e., the two links connecting operators $\{O_2,O_3\}$ and $\{O_3,O_1\}$. The reason is that the integral $\mathcal{I}_{\Gamma/\gamma}$ for this subgraph, after taking Taylor expansion with respect to k and q, becomes an integral with no scale and vanishes automatically[4]

$$\mathcal{I}_{\Gamma/\gamma}=\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left((k)^2\right)^{\nu_3}}\mathcal{T}_{k,q}\frac{1}{\left((k-p)^2\right)^{\nu_2}\left((k-p-q)^2\right)^{\nu_1}}=0. \tag{2.27}$$

We use $\mathcal{I}_{\gamma,n}$ to denote the contribution from the subgraph γ at the order n in the Taylor expansion. The leading terms of the two subgraphs are given by

$$\begin{aligned}\mathcal{I}_{\Gamma,0}&=\int\frac{d^Dk}{(2\pi)^D}\mathcal{T}_q^0\frac{1}{\left((p+k)^2\right)^{\nu_3}\left(k^2\right)^{\nu_2}\left((k-q)^2\right)^{\nu_1}}=\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left(k^2\right)^{\nu_1+\nu_2}\left((p+k)^2\right)^{\nu_3}}\\&=\frac{2^{-D}\pi^{-\frac{D}{2}}p^{-2D+\Delta_t}\Gamma\left(\frac{1}{2}\left(\Delta_1+\Delta_2-\Delta_3\right)\right)\Gamma\left(D-\frac{\Delta_t}{2}\right)\Gamma\left(\Delta_3-\frac{D}{2}\right)}{\Gamma\left(D-\Delta_3\right)\Gamma\left(\frac{1}{2}\left(D-\Delta_1-\Delta_2+\Delta_3\right)\right)\Gamma\left(\frac{1}{2}\left(-D+\Delta_t\right)\right)},\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}\mathcal{I}_{\gamma,0}&=\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left(k^2\right)^{\nu_2}\left((k-q)^2\right)^{\nu_1}}\mathcal{T}_{q,k}^0\frac{1}{\left((p+k)^2\right)^{\nu_3}}=\int\frac{d^Dk}{(2\pi)^D}\frac{1}{\left(p^2\right)^{\nu_3}\left(k^2\right)^{\nu_2}\left((k-q)^2\right)^{\nu_1}}\\&=\frac{2^{-D}\pi^{-\frac{D}{2}}p^{-D+\Delta_t-2\Delta_3}q^{2\Delta_3-D}\Gamma\left(\frac{1}{2}\left(\Delta_t-2\Delta_3\right)\right)\Gamma\left(\frac{1}{2}\left(\Delta_t-2\Delta_1\right)\right)\Gamma\left(\frac{D}{2}-\Delta_3\right)}{\Gamma\left(\Delta_3\right)\Gamma\left(\frac{1}{2}\left(D+\Delta_1-\Delta_2-\Delta_3\right)\right)\Gamma\left(\frac{1}{2}\left(D-\Delta_t+2\Delta_2\right)\right)}.\end{aligned} \tag{2.29}$$

[4]Here the internal momentum k has been shifted to $k\to k-p$ comparing with (2.26) so that the large external momentum p only flows inside the new subgraph Γ/γ.

JHEP12(2025)066

The two leading terms $\mathcal{I}_{\Gamma,0}$ and $\mathcal{I}_{\gamma,0}$ are consistent with the results (2.7) up to an overall normalization factor. Note the solution (2.7) is obtained using asymptotic behavior and integral properties of the Bessel functions. In contrast, the above results can be directly evaluated using the diagrammatic subgraph expansions. In the following sections of this work, we will use the method of subgraphs to compute the conformal 3-point correlators of conserved currents and higher loop Feynman integrals.

2.3 Comments on the subgraph expansion and operator product expansion

We provide physical interpretations for the subgraph expansions of the conformal 3-point correlator (2.28), (2.29). We show that the subgraph expansions automatically reproduce the singularities in the correlation functions in the OPE limit, and that the method of subgraphs provides a useful technique to study the OPE in momentum space.

In position space, the product of two operators can be expanded

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_i C_i(x, \partial_x)\mathcal{O}_i(x_2), \tag{2.30}$$

in which we define $x^\mu \equiv x_1^\mu - x_2^\mu$ for brevity. The above OPE is convergent in the regions where no other operators are inserted. In the short distance limit $x \to 0$, the $\mathcal{O}_3$-dependent part in the OPE is dominated by

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) \sim \frac{\lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}}\mathcal{O}_3(x_2) + \cdots . \tag{2.31}$$

Applying the above OPE in the conformal scalar 3-point correlator (2.1), it leads to

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle \sim \frac{\lambda_{\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}}\langle \mathcal{O}_3(x_2)\mathcal{O}_3(x_3)\rangle + \cdots . \tag{2.32}$$

The OPE in momentum space is given by the Fourier transformation of (2.30):

$$\mathcal{O}_1(p)\mathcal{O}_2(0) \equiv \int d^D x e^{ip\cdot x}\, \mathcal{O}(x)\mathcal{O}(0) \sim \sum_i \tilde{C}_i(p, \partial_p)\mathcal{O}_i(0). \tag{2.33}$$

The OPE limit in position space $x \to 0$ corresponds to the large momentum limit $p \to \infty$ in momentum space. One might naively expect that the correlation function in momentum space obtained from (2.33) should be identical to the Fourier transformation of the 3-point correlation function on the right-hand side of (2.32). However, this is not correct.

The Fourier transformation of the OPE limit in position space (2.32) gives

$$\langle \mathcal{O}_1(p)\mathcal{O}_2(-p-q)\mathcal{O}_3(q)\rangle \propto p^{\Delta_1+\Delta_2-\Delta_3} q^{2\Delta_3-D} + \cdots , \tag{2.34}$$

which is exactly the subgraph contribution $\mathcal{I}_\gamma$ (2.29). Recall that when evaluating $\mathcal{I}_\gamma$, the Taylor expansion $\mathcal{T}_{q,k}$ with respect to small q and k, or equivalently the large p limit has been applied. This is just the OPE limit $p \to \infty$ in momentum space. However, we have shown that in the subgraph expansion with large p, the whole conformal 3-point correlator also contains a q-regular term $\mathcal{I}_\Gamma$ contributed from the subgraph Γ. For the scalar $\mathcal{O}_3$ with $\Delta_3 > D/2$, the q-regular term is dominating and the naive OPE expansion (2.34) completely misses the asymptotic behavior of the correlator in this limit.

JHEP12(2025)066

The key is that the OPE in position space (2.30) is convergent only when no extra operator in between $\mathcal{O}_1(x)$ and $\mathcal{O}_2(0)$. However, for the OPE in momentum space (2.33), one integrates over the whole spacetime $x \in \mathbb{R}^D$, and when applying it in the 3-point correlator with an extra operator $\mathcal{O}(x_3)$, the integration domain covers the neighborhood of $\mathcal{O}(x_3)$: $x_3 \in \mathbb{U}_{x_3}$, where the operator $\mathcal{O}_1(x_1)$ is more close to the third operator $\mathcal{O}(x_3)$ and the OPE (2.30) for $\mathcal{O}_1(x_1) \times \mathcal{O}_2(x_2)$ is not convergent any more. The contribution from the integration domain $\mathbb{U}_{x_3}$ corresponds to the large loop momentum k which is comparable to the large external momentum $k \sim p$, and the integrand should be expanded with respect to large k and p, or equivalently small q denoted by $\mathcal{T}_q$ in (2.28), this is precisely the contribution $\mathcal{I}_\Gamma$ from the subgraph Γ! The correlation functions with coincident coordinates, e.g. $x_2 \to x_3$ are the so-called contact terms, and their Fourier transformations generate functions that are analytic in q [15]. The extra term $\mathcal{I}_\Gamma$ in the OPE limit plays an important role in the studies of the relation between scale and conformal invariance [41, 60] and also the conductivity at finite temperature [10, 11].

In short, the subgraph expansion of the conformal 3-point correlator provides a comprehensive description for the asymptotic behavior of the correlator in the OPE limit in momentum space, which is known to be a subtle problem [41, 60] as the integration domain in the Fourier transformation covers the neighborhood of the third operator where the OPE stops converging. In general such kind of problems can be carefully treated using the method of regions, and for the integrals in Euclidean space, the results are given by the subgraphs of the original integral. For the conformal correlators in Lorentzian space, the method of subgraphs does not apply and one needs to go back to the method of regions to evaluate contributions from different integration domains. We expect the method of subgraphs can play an important role for the conformal 4-point correlators and the related studies, in particular the conformal bootstrap in momentum space, for which we will provide more discussions at the end of this paper.

3 Subgraph expansions of conserved current 3-point correlators

We compute the conformal 3-point correlators of the conserved currents in momentum space using the subgraph expansion. It has been shown in [1, 2] that the conformal current 3-point correlators can be decomposed into those derived from free field theories, and the correlation functions have been solved in compact forms in position space. In momentum space, the conformal conserved current 3-point correlation functions can also be solved from the conformal Ward identities [39]. Nevertheless, they consist of complicated expressions that are not convenient for further study. The method of subgraphs, as explained previously, provides a simple graphical expansion for the conformal 3-point correlation functions, which is useful for related studies. Specifically, the results in this section provide the bases to compute the subleading order corrections in the $1/N$ expansion.

We study the three conformal 3-point correlators of conserved currents: $\langle JJJ\rangle$, $\langle TJJ\rangle$ and $\langle TTT\rangle$. The independent tensor structures in these correlators have been classified in [1, 2], which can be realized by free boson and free fermion theories. For the stress tensor 3-point correlator $\langle TTT\rangle$ in 4D, its tensor structures also contain those from the free vector field theory. In this work, we compute the conformal 3-point correlation functions in general

JHEP12(2025)066

dimensions. It has been known that the free vector field theory is not conformal in $D \neq 4$ [64]. Therefore, we focus on the tensor structures realized by the free boson and free fermion theories. The method in this work can be directly applied to the 4D free vector theory.

To compute the 3-point correlators in momentum space using Feynman diagrams, we need to specify the propagators and vertices of the external operators. Here we briefly introduce these ingredients for free scalars ϕ^i and free fermions ψ^i. A more detailed introduction to the interacting theories will be provided in section 4.1.

Consider the theories with N real scalars or N 4-component Dirac fermions. Their Lagrangians $\mathcal{L}_{s/f}$ are

$$\mathcal{L}_s = \frac{1}{2}\left(\partial\phi_0^i\right)^2, \quad \mathcal{L}_f = -\bar{\psi}_{0i}\partial\!\!\!/\psi_0^i, \tag{3.1}$$

and the propagators of the free scalars ϕ_0^i and free fermions ψ_0^i are given by

$$\langle\phi_0^i(p)\phi_0^j(-p)\rangle = \frac{\delta^{ij}}{p^2}, \quad \langle\psi_0^i(p)\bar{\psi}_{0j}(-p)\rangle = \frac{\delta^i_j p\!\!\!/}{p^2}. \tag{3.2}$$

The conserved spin 1 current J_μ^a and the stress tensor operator $T_{\mu\nu}$ read

$$J_\mu^a(p) = \frac{i}{2}\int\frac{d^Dk}{(2\pi)^D}\,(2k_\mu + p_\mu)\,\phi_0^i(-k)\,(t^a)^{ij}\,\phi_0^j(p+k), \tag{3.3}$$

$$T_{\mu\nu}(p) = \int\frac{d^Dk}{(2\pi)^D}\left(k_\mu k_\nu + \frac{k_\mu p_\nu + p_\mu k_\nu}{2} + \frac{D-2}{4(D-1)}p_\mu p_\nu\right)\phi_0^i(k+p)\phi_0^i(-k) - \text{trace}, \tag{3.4}$$

for the free scalar theory and

$$J_\mu^a(p) = -\int\frac{d^Dk}{(2\pi)^D}\bar{\psi}_{0i}(-k)\,(t^a)^i_j\,\gamma_\mu\psi_0^j(p+k), \tag{3.5}$$

$$T_{\mu\nu}(p) = -\frac{i}{2}\int\frac{d^Dk}{(2\pi)^D}\bar{\psi}_{0i}\gamma_\mu\left(k_\nu + \frac{1}{2}p_\nu\right)\psi_0^i(p+k) + (\mu\leftrightarrow\nu) - \text{trace}, \tag{3.6}$$

for the free fermion theory. In the perturbative computations, the expressions for J and T correspond to the vertices shown in figure 2. The momentum-dependent factors in the vertices are given by

$$U(p,k) = i(2k_\mu + p_\mu), \quad V(p,k) = 2k_\mu k_\nu + k_\mu p_\nu + p_\mu k_\nu + \frac{D-2}{2(D-1)}p_\mu p_\nu - \text{trace}, \tag{3.7}$$

for the free boson theory and

$$U(p,k) = -\gamma_\mu, \quad V(p,k) = -\frac{i}{4}\,(2k_\mu + p_\mu)\,\gamma_\nu + (\mu\leftrightarrow\nu) - \text{trace}, \tag{3.8}$$

for the free fermion theory.

In free field theories, it is straightforward to compute the conformal 3-point correlation functions of the conserved currents $\langle JJJ\rangle$, $\langle TJJ\rangle$ and $\langle TTT\rangle$ using Wick contractions. The final results are given by the one-loop Feynman integrals. In figure 3 we show the Feynman diagram for the correlator $\langle JJJ\rangle$, and the Feynman diagrams for $\langle TJJ\rangle$ and $\langle TTT\rangle$ have

JHEP12(2025)066

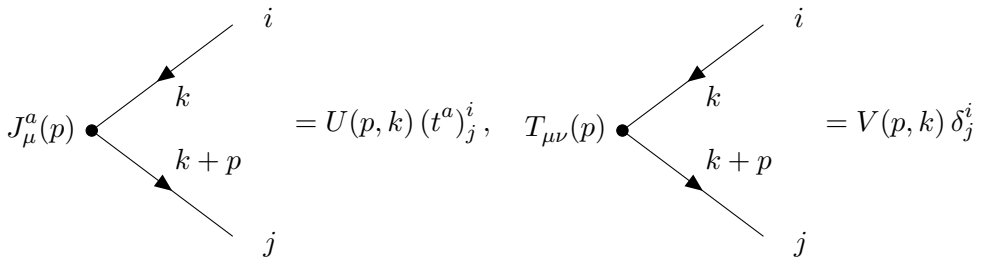

Figure 2. Vertices of the conserved currents $J^a_\mu(p)$ and $T_{\mu\nu}(p)$.

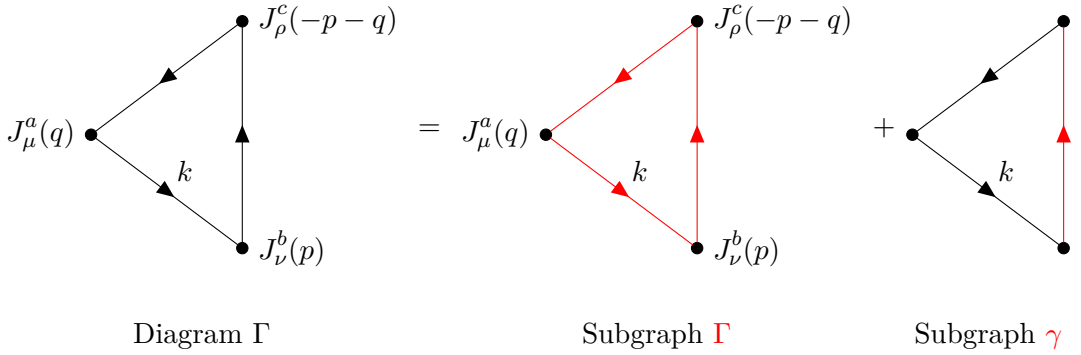

Figure 3. Feynman diagram for the current 3-point correlator $\langle J^a_\mu J^b_\nu J^c_\rho\rangle$ and its expansion with two subgraphs: Γ and γ marked in red color. The Feynman diagrams for the correlators $\langle TJJ\rangle$ and $\langle TTT\rangle$ are similar but with different vertices for the stress tensor.

the same geometry but different vertices. The subgraph expansions of the conserved current 3-point correlators are similar to those of the scalar 3-point correlator: there are two subgraphs Γ and γ, in which the integrands are expanded with respect to two sets of small momenta $\{q\}$ and $\{q,k\}$, respectively. Then the one-loop 2-scale Feynman integrals are reduced to one-loop 1-scale Feynman integrals, which can be evaluated analytically.

In principle, one can take the Taylor expansions of the integrands for each subgraph $\mathcal{T}_{q/q,k}$ to arbitrarily high orders. The resulting integrals are given by the series expansions of the small parameter q/p. For the Feynman diagrams in general QFTs, the expansions to high orders are needed to approximate the original integrals [56]. However, in CFTs the 3-point correlation functions are fixed by conformal symmetry up to a few free parameters, and the $1/N$ corrections are only applied to these free parameters. To extract these corrections from the Feynman loop integrals, we just need to keep the first few terms in the Taylor expansion of each subgraph.

In the following parts, we present the subgraph expansions for the correlators $\langle JJJ\rangle$, $\langle TJJ\rangle$ and $\langle TTT\rangle$ to the subleading order, which will be sufficient to extract the subleading corrections in the large N expansion. Higher order terms can be computed using the same method. We use $\mathcal{I}_{\gamma,i}$ to denote the i-th order contribution from the subgraph γ.

JHEP12(2025)066

3.1 Conserved current 3-point correlator $\langle JJJ\rangle$

In the free scalar theory, the spin-1 current 3-point correlation function is given by

$$\begin{aligned}\mathcal{I}_{\Gamma,0} = &-\frac{2^{1-D}\pi^{1-\frac{D}{2}}\left(p^2\right)^{\frac{D}{2}-3}\csc\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D}{2}\right)}{\Gamma(D)} \\ &\times\left(p_\mu\left((D-4)p_\nu p_\rho-(D-2)p^2\delta(\nu,\rho)\right)+p^2\left(p_\rho\delta(\mu,\nu)+p_\nu\delta(\mu,\rho)\right)\right),\end{aligned} \tag{3.9}$$

$$\begin{aligned}\mathcal{I}_{\Gamma,1} = &-\frac{2^{1-2D}\pi^{\frac{3}{2}-\frac{D}{2}}\left(p^2\right)^{\frac{D}{2}-4}\csc\left(\frac{\pi D}{2}\right)}{\Gamma\left(\frac{D+1}{2}\right)} \\ &\times\left(p^2\left((D-4)p_\mu p_\rho q_\nu+(D-4)p_\nu\left(p_\rho q_\mu+p_\mu q_\rho\right)\right)+(D-6)(D-4)p_\alpha p_\mu p_\nu p_\rho q_\alpha\right) \\ &+\delta-\text{dependent terms}\end{aligned} \tag{3.10}$$

$$\begin{aligned}\mathcal{I}_{\gamma,0} = &-\frac{2^{1-D}\pi^{1-\frac{D}{2}}\left(q^2\right)^{\frac{D}{2}-2}\csc\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D}{2}\right)}{\left(p^2\right)^2\Gamma(D)} \\ &\times\left(p^2p_\nu\left(q^2\delta(\mu,\rho)-q_\mu q_\rho\right)-p_\rho\left(p^2\left(q_\mu q_\nu-q^2\delta(\mu,\nu)\right)+q^2p_\mu p_\nu\right)+p_\alpha p_\nu p_\rho q_\alpha q_\mu\right).\end{aligned} \tag{3.11}$$

In the free fermion theory, the correlation function is

$$\begin{aligned}\mathcal{I}_{\Gamma,0} = &\frac{2^{1-2D}(D-2)\pi^{\frac{3}{2}-\frac{D}{2}}\left(p^2\right)^{\frac{D}{2}-3}\csc\left(\frac{\pi D}{2}\right)}{\Gamma\left(\frac{D+1}{2}\right)} \\ &\times p_\mu\left((D-4)p_\nu p_\rho-(D-2)p^2\delta(\nu,\rho)\right)+p^2\left(p_\rho\delta(\mu,\nu)+p_\nu\delta(\mu,\rho)\right),\end{aligned} \tag{3.12}$$

$$\begin{aligned}\mathcal{I}_{\Gamma,1} = &\frac{4^{-D}\pi^{\frac{3}{2}-\frac{D}{2}}\left(p^2\right)^{\frac{D}{2}-4}\csc\left(\frac{\pi D}{2}\right)}{\Gamma\left(\frac{D+1}{2}\right)}(D-4) \\ &\times\left(p_\nu\left((D-2)p_\rho\left((D-6)p_\mu p_\sigma q_\sigma+p^2q_\mu\right)+(2D-3)p^2p_\mu q_\rho\right)-p^2p_\mu p_\rho q_\nu\right) \\ &+\delta-\text{dependent terms}\end{aligned} \tag{3.13}$$

$$\begin{aligned}\mathcal{I}_{\gamma,0} = &\frac{2^{1-2D}(D-2)\pi^{\frac{3}{2}-\frac{D}{2}}\left(q^2\right)^{\frac{D}{2}-2}\csc\left(\frac{\pi D}{2}\right)}{p^2\Gamma\left(\frac{D+1}{2}\right)} \\ &\times\left(\delta(\nu,\rho)\left(q^2p_\mu-p_\alpha q_\alpha q_\mu\right)+p_\rho\left(q_\mu q_\nu-q^2\delta(\mu,\nu)\right)+p_\nu\left(q_\mu q_\rho-q^2\delta(\mu,\rho)\right)\right).\end{aligned} \tag{3.14}$$

3.2 Conserved current 3-point correlator $\langle TJJ\rangle$

The subgraph expansion of the conformal correlation function $\langle TJJ\rangle$ in the free scalar theory is given by

$$\begin{aligned}\mathcal{I}_{\Gamma,0} = &\frac{2^{1-2D}\pi^{\frac{3}{2}-\frac{D}{2}}\left(p^2\right)^{\frac{D}{2}-3}\csc\left(\frac{\pi D}{2}\right)}{D\Gamma\left(\frac{D+1}{2}\right)} \\ &\times\left(Dp_\mu p_\nu\left((D-4)p_\rho p_\sigma-(D-2)p^2\delta(\rho,\sigma)\right)-(D-4)p^2p_\rho p_\sigma\delta(\mu,\nu)\right. \\ &\left.+\left(p^2\right)^2\left(D\delta(\mu,\sigma)\delta(\nu,\rho)+D\delta(\mu,\rho)\delta(\nu,\sigma)+(D-4)\delta(\mu,\nu)\delta(\rho,\sigma)\right)\right),\end{aligned} \tag{3.15}$$

$$\mathcal{I}_{\Gamma,1} = \frac{4^{-D}\pi^{\frac{3}{2}-\frac{D}{2}}\left(p^2\right)^{\frac{D}{2}-4}\csc\left(\frac{\pi D}{2}\right)}{D\Gamma\left(\frac{D+1}{2}\right)}(D-4)Dp_\rho \tag{3.16}$$
$$\times\Big((D-6)p_\beta p_\mu p_\nu p_\sigma q_\beta + p^2\left(p_\mu p_\sigma q_\nu + p_\nu\left(p_\sigma q_\mu + 2p_\mu q_\sigma\right)\right)\Big) + \delta-\text{dependent terms},$$
$$\mathcal{I}_{\gamma,0} = \frac{4^{-D}\pi^{\frac{3}{2}-\frac{D}{2}}\left(q^2\right)^{\frac{D-4}{2}}\csc\left(\frac{\pi D}{2}\right)}{(D-1)(p^2)^2\Gamma\left(\frac{D+3}{2}\right)} \tag{3.17}$$
$$\times\Big(-\Big((D-1)q^2p_\mu q_\nu\left(p_\sigma q_\rho + p_\rho q_\sigma\right)\Big) + 2(D-2)p_\alpha q_\alpha q_\mu q_\nu\left(p_\sigma q_\rho + p_\rho q_\sigma\right)$$
$$+q_\mu\Big(-2q_\nu\Big((D-2)p^2q_\rho q_\sigma - 2q^2p_\rho p_\sigma\Big) - \Big((D-1)q^2p_\nu\left(p_\sigma q_\rho + p_\rho q_\sigma\right)\Big)\Big)\Big)$$
$$+\,\delta-\text{dependent terms}.$$

Note that the lowest non-vanishing term in $\mathcal{I}_\gamma$ appears at the order $O(q^D)$, in contrast to the order $O(q^{D-2})$ for the correlator $\langle JJJ\rangle$.

In the free fermion theory, the correlation function $\langle TJJ\rangle$ is

$$\mathcal{I}_{\Gamma,0} = -\frac{4^{-D}(D-2)\pi^{\frac{3}{2}-\frac{D}{2}}\left(p^2\right)^{\frac{D-6}{2}}\csc\left(\frac{\pi D}{2}\right)}{D\Gamma\left(\frac{D+1}{2}\right)} \tag{3.18}$$
$$\times\Big(p^2\Big(2(D-2)p^2\delta(\mu,\nu)\delta(\rho,\sigma)+Dp_\nu p_\sigma\delta(\mu,\rho)+p_\rho\left(Dp_\nu\delta(\mu,\sigma)-2(D-2)p_\sigma\delta(\mu,\nu)\right)\Big)$$
$$+Dp_\mu\Big(2p_\nu\Big((D-4)p_\rho p_\sigma-(D-2)p^2\delta(\rho,\sigma)\Big)+p^2\left(p_\sigma\delta(\nu,\rho)+p_\rho\delta(\nu,\sigma)\right)\Big)\Big),$$
$$\mathcal{I}_{\Gamma,1} = -\frac{2^{-D-1}(D-2)\pi^{1-\frac{D}{2}}z\left(p^2\right)^{\frac{D}{2}-4}\csc\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D}{2}\right)}{\Gamma(D+1)} \tag{3.19}$$
$$\times\Big((D-6)p_\alpha p_\mu p_\nu p_\sigma q_\alpha+p^2\left(p_\mu p_\sigma q_\nu+p_\nu\left(p_\sigma q_\mu+2p_\mu q_\sigma\right)\right)\Big)+\delta-\text{dependent terms},$$
$$\mathcal{I}_{\gamma,0} = O(q^{D+1}). \tag{3.20}$$

The subgraph γ vanishes at the order $O(q^D)$ in the free fermion theory.

3.3 Conserved current 3-point correlator $\langle TTT\rangle$

The correlation function $\langle TTT\rangle$ has more complicated tensor structures. In the free scalar theory, it is given by

$$\mathcal{I}_{\Gamma,0} = \frac{2^{-2(D+1)}(D-2)\pi^{\frac{3}{2}-\frac{D}{2}}\left(p^2\right)^{D/2}\csc\left(\frac{\pi D}{2}\right)}{(D-1)p^6\Gamma\left(\frac{D+3}{2}\right)}\left(2(D-4)p_\alpha p_\beta p_\mu p_\nu p_\rho p_\sigma\right. \tag{3.21}$$
$$+\,p_\alpha\Big(-\Big((D-1)p^2p_\mu p_\nu\left(p_\sigma\delta(\beta,\rho)+p_\rho\delta(\beta,\sigma)\right)\Big)-2p^2p_\beta\delta(\mu,\nu)+p^2p_\mu p_\nu\delta(\rho,\sigma)\Big)$$
$$\left.+\,p^2p_\mu p_\nu\left(p_\rho\left(2p_\sigma\delta(\alpha,\beta)-(D-1)p_\beta\delta(\alpha,\sigma)\right)-(D-1)p_\beta p_\sigma\delta(\alpha,\rho)\right)\Big)+\cdots,$$

$$\mathcal{I}_{\Gamma,1} = \frac{4^{-D-1}(D-4)(D-2)\pi^{\frac{3}{2}-\frac{D}{2}}\left(p^2\right)^{\frac{D-8}{2}}\csc\left(\frac{\pi D}{2}\right)}{(D-1)\Gamma\left(\frac{D+3}{2}\right)}p_\rho p_\sigma \tag{3.22}$$
$$\times\left((D-6)p_\alpha p_\beta p_\mu p_\nu p_\tau q_\tau + p^2\left(p_\nu\left(2p_\mu\left(p_\beta q_\alpha + p_\alpha q_\beta\right) + p_\alpha p_\beta q_\mu\right) + p_\alpha p_\beta p_\mu q_\nu\right)\right)$$
$$+\,\delta-\text{dependent terms},$$
$$\mathcal{I}_{\gamma,0} = O(q^{D-1}). \tag{3.23}$$

For brevity, here we have ignored terms with more than one δ functions in $\mathcal{I}_{\Gamma,0}$, and all the δ-dependent terms in $\mathcal{I}_{\Gamma,1}$ and $\mathcal{I}_{\gamma,0}$.

The correlation function in the free fermion theory is

$$\mathcal{I}_{\Gamma,0} = -\frac{4^{-D}(D-4)(D-2)\pi^{\frac{3}{2}-\frac{D}{2}}\left(p^2\right)^{\frac{D}{2}-3}\csc\left(\frac{\pi D}{2}\right)}{\Gamma\left(\frac{D+3}{2}\right)}p_\alpha p_\beta p_\mu p_\nu p_\rho p_\sigma \tag{3.24}$$
$$+\,\delta-\text{dependent terms},$$
$$\mathcal{I}_{\Gamma,1} = \frac{p_\rho p_\sigma\left(2^{-2D-1}(D-4)(D-2)\pi^{\frac{3}{2}-\frac{D}{2}}\left(p^2\right)^{\frac{D}{2}-4}\csc\left(\frac{\pi D}{2}\right)\right)}{\Gamma\left(\frac{D+3}{2}\right)} \tag{3.25}$$
$$\times\left((D-6)p_\alpha p_\beta p_\gamma p_\mu p_\nu q_\gamma + p^2\left(p_\alpha p_\beta p_\nu q_\mu + p_\mu\left(2p_\nu\left(p_\beta q_\alpha + p_\alpha q_\beta\right) + p_\alpha p_\beta q_\nu\right)\right)\right)$$
$$+\,\delta-\text{dependent terms},$$
$$\mathcal{I}_{\gamma,0} = O(q^{D-1}). \tag{3.26}$$

The correlators with stress tensor $T_{\mu\nu}$ are found to have weaker singularity in q within the subgraph γ.

3.4 Compare the subgraph expansions with the 3D exact results

We verify the subgraph expansions of the conformal current 3-point correlators. It is straightforward to compare the above results with the solutions of the conformal Ward identities in momentum space [39]. This has been done for the scalar 3-point correlators in (2.28) and (2.29). Here we show another verification of the subgraph expansions based on the exact results in 3D, in which the Feynman diagrams generate conformal integrals and can be evaluated analytically for the conformal current 3-point correlators in free scalar and free fermion theories.

Let us take the correlator $\langle JJJ\rangle$ in the 3D free boson theory as an example. The correlation function is given by the Feynman diagram in figure 3 with vertices of the operator J, which corresponds to the integral

$$\langle J^a_\mu(q)J^b_\nu(p)J^c_\rho(-p-q)\rangle = \alpha_J\int\frac{d^3k}{(2\pi)^6}\frac{(2k_\mu - q_\mu)(p_\nu + 2k_\nu)(2k_\rho + p_\rho - q_\rho)}{k^2(k+p)^2(k-q)^2}, \tag{3.27}$$

where $\alpha_J = i^3\mathrm{tr}(t^a t^b t^c)$ is the symmetry factor and will be assumed implicitly. Using tensor reduction, above integral can be reduced to scalar conformal 3-point integrals or the 2-

JHEP12(2025)066

point integrals (2.11), both of which can be solved in compact forms. The exact integral in (3.27) is given by

$$\mathcal{I}_{JJJ,\,3\mathrm{D}} = \frac{p_\mu p_\nu p_\rho q \left(p^2 + pq + (2p+q) + 2q^2|p+q| + |p+q|^2\right) + \cdots}{8pq|p+q|(p+q+|p+q|)^3} + \frac{\delta(\mu,\nu)\left(p_\rho\left(p+|p+q|\right) - q_\rho\left(q+|p+q|\right)\right)}{8\left(p+q+|p+q|\right)^2} + \cdots \tag{3.28}$$

In the zero-momentum limit $q \to 0$, above correlation function admits a small q Taylor expansion, and indeed there are two parts which are respectively regular/singular in the small q limit. The results completely agree with the subgraph expansions (3.9)–(3.11). In particular, there is a q-singular sector in $\mathcal{I}_{JJJ,\,3\mathrm{D}}$ which contains a factor $1/q$:

$$-\frac{p_\nu p_\rho q_\mu}{8pq|p+q|} + \frac{p_\nu p_\rho q_\mu}{4q|p+q|\left(p+q+|p+q|\right)} + \cdots . \tag{3.29}$$

This sector corresponds to $\mathcal{I}_{\gamma,0}$ in the subgraph expansion (3.11).

We have verified the subgraph expansions of the correlators $\langle TJJ\rangle$ and $\langle TTT\rangle$ in both free scalar and free fermion theories using the 3D exact results, and found perfect agreements. Comparing with the conformal 3-point correlation functions solved from the conformal Ward identities, the subgraph expansions of the correlators provide diagrammatic representations for the complicated results, and are convenient for further studies of these correlators, e.g., the $1/N$ corrections for the conformal 3-point correlators which we will compute in the following two sections.

4 Method of subgraphs and the $1/N$ expansions: $\langle JJJ\rangle$

In this section, we compute the subleading order corrections to the conformal current 3-point correlators $\langle JJJ\rangle$ in the $1/N$ expansion. The correlation function $\langle JJJ\rangle$ is, in general, given by a linear combination of those from the free scalar and free fermion theories, which have already been computed in section 3. It is the coefficients in the linear combinations that receive $1/N$ corrections.

The higher order corrections to the conformal 3-point correlators involve 2-scale multi-loop Feynman diagrams, which are quite challenging to evaluate analytically. We use the method of subgraphs to simplify the loop computations. Here, we study the large N expansion of the correlator $\langle JJJ\rangle$ in the $O(N)$ vector model and GNY model. We aim to illustrate the role of the method of subgraphs in computing loop corrections to higher-point correlators. This method can be directly applied to general CFTs.

4.1 The large N diagrammatic techniques

The large N perturbative expansion has been widely applied to study critical phenomena, see [65] for a review. In this work, we follow the theoretical implementation of the large N expansion developed in [36, 66], which has been employed to compute the $1/N$ corrections to the conformal current 2-point correlators $\langle JJ\rangle$ and $\langle TT\rangle$.

JHEP12(2025)066

The critical $O(N)$ vector model can be realized as the IR fixed point of the theory

$$\mathcal{L} = \frac{1}{2}\left(\partial\phi_0^i\right)^2 + \lambda\left(\phi_0^i\phi_0^i\right)^2, \qquad i = 1, 2, \ldots, N. \tag{4.1}$$

Using the Hubbard-Stratonovich transformation, the Lagrangian can be rewritten as

$$\mathcal{L} = \frac{1}{2}\left(\partial\phi_0^i\right)^2 + \frac{1}{\sqrt{N}}\sigma_0\phi_0^i\phi_0^i - \frac{1}{4N\lambda}\sigma_0^2. \tag{4.2}$$

The propagator of the bare scalar field ϕ_0 has already been shown in (3.2). The auxiliary field σ_0 receives quantum loop corrections through the interaction $\sigma_0\phi_0^i\phi_0^i$ and becomes dynamical. According to the normalization in (4.2), its effective propagator is [36]

$$\langle\sigma_0(p)\sigma_0(-p)\rangle = \frac{C_\sigma^0}{\left(p^2\right)^{\frac{d}{2}-2+\epsilon}}, \quad C_\sigma^0 = 2^{D+1}(4\pi)^{\frac{D-3}{2}}\Gamma\left(\frac{D-1}{2}\right)\sin\left(\frac{\pi D}{2}\right), \tag{4.3}$$

in which the parameter ϵ denotes the shift of the scaling dimension of σ_0. It plays the role of a regulator in the loop computations and vanishes in the final results. The quadratic term σ_0^2 in (4.3) is irrelevant near the IR fixed point; therefore, the effective Lagrangian for the $O(N)$ critical point is

$$\mathcal{L}_{\text{IR}} = \frac{1}{2}\left(\partial\phi_0^i\right)^2 + \frac{1}{\sqrt{N}}\sigma_0\phi_0^i\phi_0^i. \tag{4.4}$$

Since we have introduced the regularization parameter ϵ for σ_0, its dimension is modified accordingly, which makes the vertex in $\mathcal{L}_{\text{IR}}$ dimensionful. To make the vertex dimensionless, an arbitrary renormalization scale μ is introduced in the vertex

$$\mathcal{L}_{\text{vertex}} = \mu^\epsilon\frac{1}{\sqrt{N}}\sigma_0\phi_0^i\phi_0^i. \tag{4.5}$$

The renormalization factor μ^ϵ will be implicitly assumed throughout this work.

It is straightforward to apply renormalization within the minimal subtraction scheme. The renormalization factors Z_ϕ and Z_σ are introduced for the bare operators ϕ_0 and σ_0. Then the physical fields ϕ and σ and given by

$$\phi^i = Z_\phi^{\frac{1}{2}}\phi_0^i, \quad \sigma = Z_\sigma^{\frac{1}{2}}\sigma_0, \tag{4.6}$$

which absorb the divergences in the loop computations

$$Z_\phi = 1 + \frac{1}{N}\frac{r_\phi}{\epsilon} + O(N^{-2}), \quad Z_\sigma = 1 + \frac{1}{N}\frac{r_\sigma}{\epsilon} + O(N^{-2}). \tag{4.7}$$

After renormalization, the 2-point correlators of the renormalized fields have the standard forms of the 2-point conformal correlation functions in momentum space

$$\langle\phi(p)^i\phi(-p)^j\rangle = \delta^{ij}\frac{C_\phi}{\left(p^2\right)^{\frac{D}{2}-\Delta_\phi}}, \quad \langle\sigma(p)\sigma(-p)\rangle = \frac{C_\sigma}{\left(p^2\right)^{\frac{D}{2}-\Delta_\sigma}}. \tag{4.8}$$

Both the scaling dimensions Δ_ϕ and the coefficient C_ϕ have been computed to the order $O(1/N)$ in [36]. We will compute the subleading order corrections to the coefficient C_σ and

JHEP12(2025)066

the anomalous dimension Δ_σ in section 5.3. We will also be interested in the conserved currents J and T, whose definitions have been given in (3.3) and (3.4). It has been found that at the order $O(1/N)$, the renormalization factor of the spin-1 conserved current J^a_μ is trivial, $Z_J = 1 + O(N^{-2})$, while the stress tensor operator $T_{\mu\nu}$ needs a special renormalization factor Z_T

$$T^p_{\mu\nu} = Z_T T_{\mu\nu}, \quad Z_T = 1 + \frac{1}{N}\left(\frac{r_T}{\epsilon} + s_T\right) + O(N^{-2}), \tag{4.9}$$

and the coefficients r_T, s_T are given by [36]

$$r_T = \frac{2\Gamma(D-2)\sin(\pi D/2)}{\pi\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(\frac{D}{2}+2\right)}, \qquad s_T = \frac{4\Gamma(D-2)\sin(\pi D/2)}{\pi\Gamma\left(\frac{D}{2}-1\right)\Gamma\left(\frac{D}{2}+2\right)}. \tag{4.10}$$

We now combine the large N diagrammatic approach and method of subgraphs to compute the $1/N$ corrections to the conformal current 3-point correlator $\langle JJJ\rangle$.

4.2 $\langle JJJ\rangle$ in the $O(N)$ vector model

Our strategy for compute the $1/N$ corrections to the conformal 3-point correlators can be briefly summarized as follows. We start with the effective Lagrangian (4.4), and write all the Feynman diagrams for the 3-point correlator $\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle$ up to the order $O(1/N)$. The Feynman diagrams are evaluated using the method of subgraphs, and the results contain divergences regularized by the parameter ϵ in (4.3). Then we introduce the renormalization factors $Z_{\mathcal{O}_i}$ to cancel the divergences. After renormalization, the Feynman loop integrals give the "physical" 3-point correlation function up to the order $O(1/N)$, which should be able to reproduce the correlation functions solved from the conformal Ward identities and also provide the subleading order corrections to the free parameters in the conformal 3-point correlators.

4.2.1 Subgraph expansions of the Feynman diagrams

The 3-point correlator $\langle J^a_\mu(q)J^b_\nu(p)J^c_\rho(-p-q)\rangle$ at the order $O(1/N)$ is given by the seven Feynman diagrams in figure 4

$$\langle J^a_\mu(q)J^b_\nu(p)J^c_\rho(-p-q)\rangle = D_0 + D_1 + \cdots + D_6 + O(1/N^2). \tag{4.11}$$

The exact results of the loop integrals in figure 4 are hard to evaluate analytically. Nevertheless, due to the constraints from the conformal symmetry, the conformal 3-point correlator $\langle J^a_\mu(q)J^b_\nu(p)J^c_\rho(-p-q)\rangle$ is fixed up to two free parameters C_J and λ_{JJJ}, or equivalently the coefficients n_s and n_f in the linear combination of free scalar and free fermion theories (1.3). The parameter C_J also appears in the 2-point correlator $\langle J^a_\mu(p)J^b_\nu(-p)\rangle$ and its subleading order correction has been computed in [36]. To evaluate the second parameter λ_{JJJ} to the subleading order, we take the zero-momentum limit $q \to 0$ for the correlator (4.11). The leading term of the correlation function in the zero-momentum limit degenerates to the 2-point correlation function and involves in the parameter C_J only. To fix the second parameter λ_{JJJ} it is necessary to expand the correlator $\langle J^a_\mu(q)J^b_\nu(p)J^c_\rho(-p-q)\rangle$ to the subleading order in the small q expansion. In other words, we evaluate the subgraphs of the Feynman diagrams to the subleading order in the small q expansion.

JHEP12(2025)066

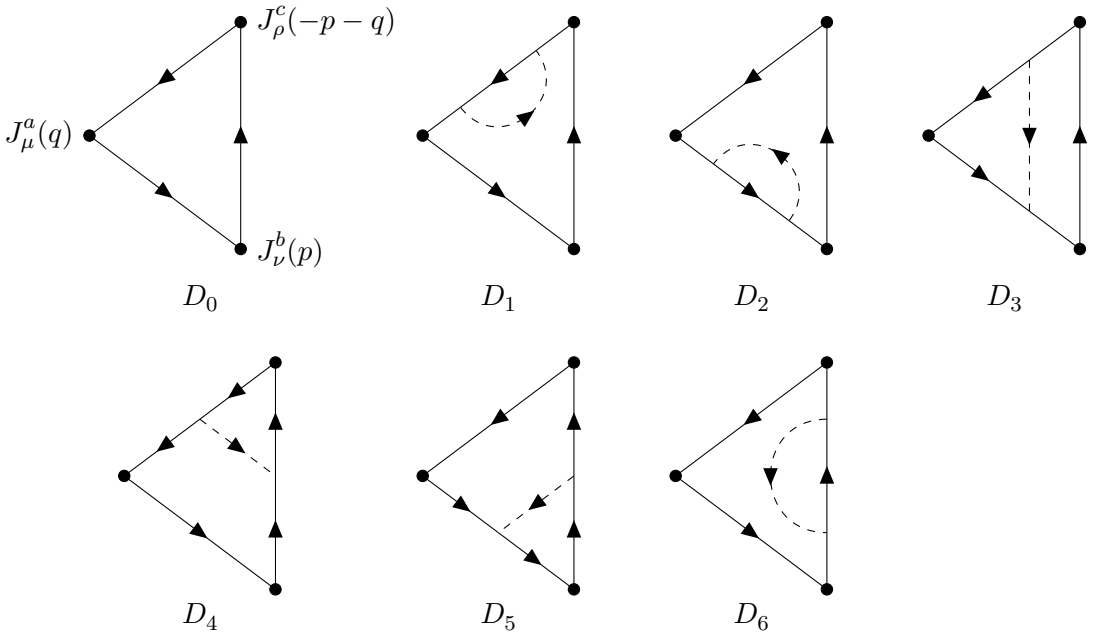

Figure 4. Feynman diagram for the conformal current 3-point correlator $\langle J^a_\mu J^b_\nu J^c_\rho \rangle$ up to the order $O(1/N)$.

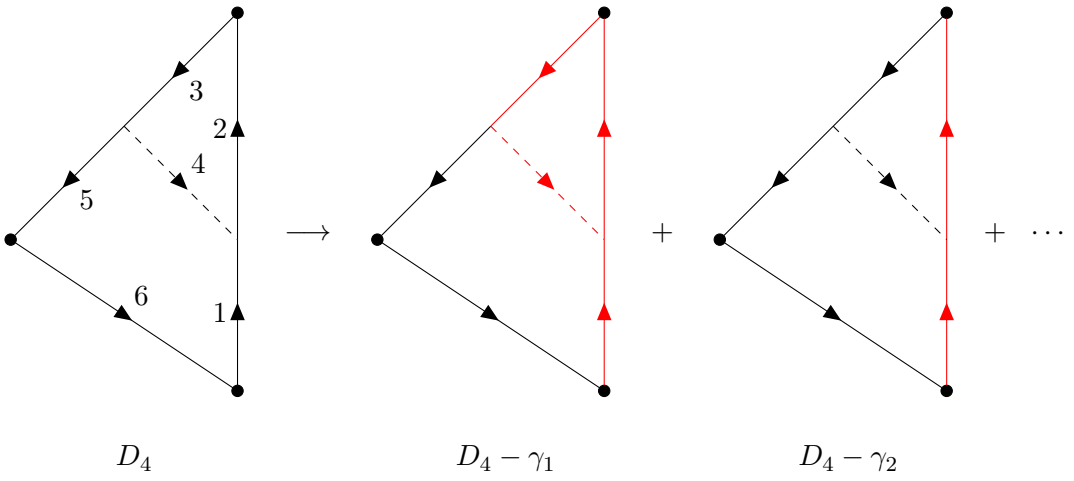

Figure 5. Subgraphs γ_1 (1234) and γ_2 (16) of the Feynman diagram D_4 in figure 4. The diagram D_4 itself constructs a trivial subgraph Γ. In addition, there are extra subgraphs of D_4 which relate to scaleless integrals.

As explained in section 2.2, the subgraph γ of a Feynman diagram is chosen based on the two rules: (a) γ contains all the vertices with large external momenta and (b) γ is 1PI after identifying all the vertices with large external momenta. We take the diagram D_4 as an example to show its expansion by subgraphs.

JHEP12(2025)066

The Feynman diagram D_4 is given by the 2-scale integral

$$D_4 \propto \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(2k_{1,\nu} + p_\nu)(2k_{1,\mu} - q_\mu)(2k_{2,\rho} + p_\rho - q_\rho)}{k_1^2 (k_1 + p)^2 (k_2 + p)^2 (k_1 - q)^2 (k_2 - q)^2 \left((k_1 - k_2)^2\right)^{\frac{d}{2}-2+\epsilon}}, \tag{4.12}$$

where we have omitted the overall normalization $\mathcal{N}_\sigma$ (4.3) for σ propagator and the algebraic factor $i^3 \mathrm{tr}(t^a t^b t^c)$ for brevity. All the subgraphs of D_4 contain the vertices $J^b_\nu(p)$ and $J^c_\rho(-p-q)$. There is a trivial subgraph of D_4 given the diagram itself $\Gamma = D_4$, and to evaluate the subgraph Γ, we simply take the Taylor expansion of the whole integrand in (4.12) with respect to the small momentum q

$$\mathcal{I}_\Gamma \propto \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \mathcal{T}_q \frac{(2k_{1,\nu} + p_\nu)(2k_{1,\mu} - q_\mu)(2k_{2,\rho} + p_\rho - q_\rho)}{k_1^2 (k_1 + p)^2 (k_2 + p)^2 (k_1 - q)^2 (k_2 - q)^2 \left((k_1 - k_2)^2\right)^{\frac{d}{2}-2+\epsilon}}. \tag{4.13}$$

After the Taylor expansion, the original 2-scale two-loop integral in (4.12) becomes a 1-scale two-loop integral with external momentum p. A typical example is given by

$$K(\Delta, p) \equiv \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{1}{k_1^2 (k_1 + p)^2 k_2^2 (k_2 + p)^2 \left((k_1 - k_2)^2\right)^{\Delta}}. \tag{4.14}$$

This is the so called "Kite" diagram, which can be analytically evaluated using the Gegenbauer polynomial technique [67, 68]. The price we pay is that by taking the Taylor expansion $\mathcal{T}_q$, the integrand has higher rank tensor structures. In this work, it suffices to take the Taylor expansion to the second order, which increases the rank of the tensor structures by 1. The Feynman integrals with tensor structures can be decomposed into scalar integrals using tensor reductions [69]. The final integrals in general dimension, even in the zero-momentum limit, are too cumbersome to show here. In 3D, the first few leading terms of the integral is given by

$$\begin{aligned} \mathcal{I}_\Gamma = & \frac{3 p_\alpha p_\mu p_\nu p_\rho q_\alpha - p^2 \left(p_\nu p_\rho q_\mu + p_\mu \left(p_\rho q_\nu + p_\nu (2p_\rho + q_\rho)\right)\right)}{24\pi^2 \left(p^2\right)^{5/2} \epsilon} \\ & + \frac{p^2 \left(15 p_\nu p_\rho q_\mu + p_\mu \left(3 p_\rho q_\nu + p_\nu (-6p_\rho - 13 q_\rho)\right)\right) + 39 p_\alpha p_\mu p_\nu p_\rho q_\alpha}{144\pi^2 \left(p^2\right)^{5/2}} \\ & + \frac{\delta-\text{dependent terms}}{\epsilon} + \delta-\text{dependent terms}. \end{aligned} \tag{4.15}$$

An important property of the subgraph Γ is that its leading order term is independent of the small momentum q. In contrast, the subgraphs γ will be shown to contain factors of q^{D-n}, which makes them radically different from $\mathcal{I}_\Gamma$.

Besides the subgraph Γ, the Feynman diagram D_4 also contains the subgraphs γ_1 and γ_2 depicted in red color in figure 5. For the subgraph γ_1, the internal momentum k_2 flows in the propagators (234) and is considered large; in contrast, the internal momentum k_1 flows

JHEP12(2025)066

in Γ/γ_1 so is considered small. Therefore, the integral for the subgraph γ_1 is

$$\mathcal{I}_{\gamma_1} \propto \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{1}{k_1^2 (k_1 - q)^2} \times \mathcal{T}_{k_1,q} \frac{(2k_{1,\nu} + p_\nu)(2k_{1,\mu} - q_\mu)(2k_{2,\rho} + p_\rho - q_\rho)}{(k_1 + p)^2 (k_2 + p)^2 (k_2 - q)^2 \left((k_1 - k_2)^2\right)^{\frac{d}{2}-2+\epsilon}}. \tag{4.16}$$

We take the above Taylor expansion $\mathcal{T}_{k_1,q}$ to second order, after which the new integration the two internal momenta k_1 and k_2 decouple. The integral $\mathcal{I}_{\gamma_1}$ consists of two one-loop integrals for k_1 and k_2 which can be evaluated directly.

$$\mathcal{I}_{\gamma_1} = \frac{2^{-D}(D-4)\pi^{-\frac{D}{2}} \left(q^2\right)^{\frac{D}{2}-2} \left(p_\alpha p_\nu p_\rho q_\alpha q_\mu - p_\rho \left(p^2 q_\mu q_\nu + q^2 p_\mu p_\nu\right)\right)}{(D-1)\Gamma\left(\frac{D}{2}+1\right)\left(p^2\right)^2 \epsilon} + \cdots. \tag{4.17}$$

The leading term contributes at the order $O(q^{D-2})$, and the terms with a pole $1/\epsilon$ will be canceled in the final results.

For the subgraph γ_2, the two internal momenta k_1 and k_2 are considered small, as both of them flow through the links in Γ/γ_2. The integral of the subgraph γ_2 is

$$\mathcal{I}_{\gamma_2} \propto \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{(2k_{1,\mu} - q_\mu)(2k_{1,\nu} + p_\nu)(2k_{2,\rho} + p_\rho - q_\rho)}{k_1^2 (k_1 - q)^2 (k_2 - q)^2 \left((k_1 - k_2)^2\right)^{\frac{d}{2}-2+\epsilon}} \times \mathcal{T}_{k_1,k_2,q} \frac{1}{(k_1 + p)^2 (k_2 + p)^2}. \tag{4.18}$$

Here, the Taylor expansion $\mathcal{T}_{k_1,k_2,q}$ is identical to the Taylor expansion with respect to the large external momentum p, through which the integral $\mathcal{I}_{\gamma_2}$ becomes a 1-scale (external momentum q) two-loop integral, reminiscent to the integral (4.13) but with a different scale parameter. At the leading order, the subgraph integral gives

$$\mathcal{I}_{\gamma_2} = \frac{2^{1-2D}(D-4)\pi^{1-D} \csc\left(\frac{\pi D}{2}\right)\left(q^2\right)^{\frac{D}{2}-1} p_\nu p_\rho q_\mu}{(D-2)\left(p^2\right)^2 \Gamma(D+1)} + O(q^D). \tag{4.19}$$

There is no singular term in ϵ, and the leading term contributes at the order $O(q^{D-1})$, higher than the leading term in $\mathcal{I}_{\gamma_1}$.

In figure 5, we have ignored more subgraphs of the diagram D_4. Actually, the links (143), (653), (14562), (14563) and (12356) also construct subgraphs of D_4. Nevertheless, they all correspond to vanishing integrals. For instance, the subgraph (145632) relates to the integral

$$\mathcal{I}_{145632} = \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} \frac{1}{(k_2 - q)^2} \times \mathcal{T}_{k_2,q} \frac{(2k_{1,\nu} + p_\nu)(2k_{1,\mu} - q_\mu)(2k_{2,\rho} + p_\rho - q_\rho)}{k_1^2 (k_1 + p)^2 (k_1 - q)^2 (k_2 + p)^2 \left((k_1 - k_2)^2\right)^{\frac{d}{2}-2+\epsilon}}, \tag{4.20}$$

in which the integration over k_2 becomes scaleless after taking the Taylor expansion

$$\mathcal{I}_{145632} = \int \frac{d^D k_2}{(2\pi)^D} \frac{1}{(k_2 - q)^2} \times (\text{polynomials of } k_2) = 0. \tag{4.21}$$

This explains why the subgraphs omitted in figure 5 have zero contributions to the original loop integral.

To summarize, besides the subgraph Γ constructed by the diagram D_4 itself, there are several subgraphs decomposed from D_4, and the leading contribution comes from the subgraph γ_1 in figure 5. The rest of the subgraphs either contributes at higher orders or correspond to scaleless integrals. We have done similar analysis for other Feynman diagrams in figure 4, and the dominating subgraphs γ are shown in figure 6. Note that the Feynman diagram D_3 has two subgraphs γ_1 and γ_2, both of which contribute at the order $O(q^{D-2})$:

$$\mathcal{I}_{D_3,\gamma_1} = \frac{2^{1-D}(4-D)\pi^{-\frac{D}{2}}\left(q^2\right)^{\frac{D}{2}-2}}{(D-1)\Gamma\left(\frac{D}{2}+1\right)p^4\epsilon}\left(p_\rho\left(p^2 q_\mu q_\nu + q^2 p_\mu p_\nu\right) + p^2 p_\nu q_\mu q_\rho - p_\alpha p_\nu p_\rho q_\alpha q_\mu\right) + \cdots,$$

$$\mathcal{I}_{D_3,\gamma_2} = -\frac{2^{-D}(4-D)\pi^{-\frac{D}{2}}\left(q^2\right)^{\frac{D}{2}-2}}{(D-1)\Gamma\left(\frac{D}{2}+1\right)p^4\epsilon}\left(p_\rho\left(p^2 q_\mu q_\nu + q^2 p_\mu p_\nu\right) + p^2 p_\nu q_\mu q_\rho - p_\alpha p_\nu p_\rho q_\alpha q_\mu\right) + \cdots.$$

The two subgraphs have the same tensor structure in the ϵ-divergent terms, but their ϵ-regular terms are different.

Using the subgraph expansion, the 3-point correlator (4.11) can be expanded as

$$\langle J^a_\mu(q) J^b_\nu(p) J^c_\rho(-p-q)\rangle = \mathcal{S}_\Gamma + \mathcal{S}_\gamma, \tag{4.22}$$

with

$$\mathcal{S}_\Gamma = \sum_{i=0}^{6} \mathcal{I}_{D_i,\Gamma}, \quad \mathcal{S}_\gamma = \sum_{i=0}^{6} \mathcal{I}_{D_i,\gamma_1} + \mathcal{I}_{D_3,\gamma_2}, \tag{4.23}$$

where $\mathcal{I}_{D_i,X}$ denotes the integral of the subgraph X of the Feynman diagram D_i. In the small momentum limit $q \to 0$, the two sectors $\mathcal{S}_\Gamma$ and $\mathcal{S}_\gamma$ scale as:

$$\mathcal{S}_\Gamma \sim q^0, \quad \mathcal{S}_\gamma \sim q^{D-2}. \tag{4.24}$$

As a result, in general dimensions the two sectors do not overlap, and the subleading order corrections to the free parameters C_J, λ_{JJJ} can be extracted from either $\mathcal{S}_\Gamma$ or $\mathcal{S}_\gamma$. We have verified the results from the two different sectors are identical, as they should be.

4.2.2 Renormalization of the 3-point correlator

In general, the 3-point correlators obtained from the Feynman diagrams are divergent, and one needs to introduce a renormalization factor Z_J to absorb the infinities

$$\langle J^{\text{phy},a}_\mu J^{\text{phy},b}_\nu J^{\text{phy},c}_\rho\rangle = Z_J^3 \, \langle J^a_\mu J^b_\nu J^c_\rho\rangle. \tag{4.25}$$

Here, for the spin-1 conserved current J^a_μ, its renormalization factor Z_J has been evaluated in [36], and the result is $Z_J = 1 + O(1/N^2)$. Therefore, at order $O(1/N)$, no renormalization

JHEP12(2025)066

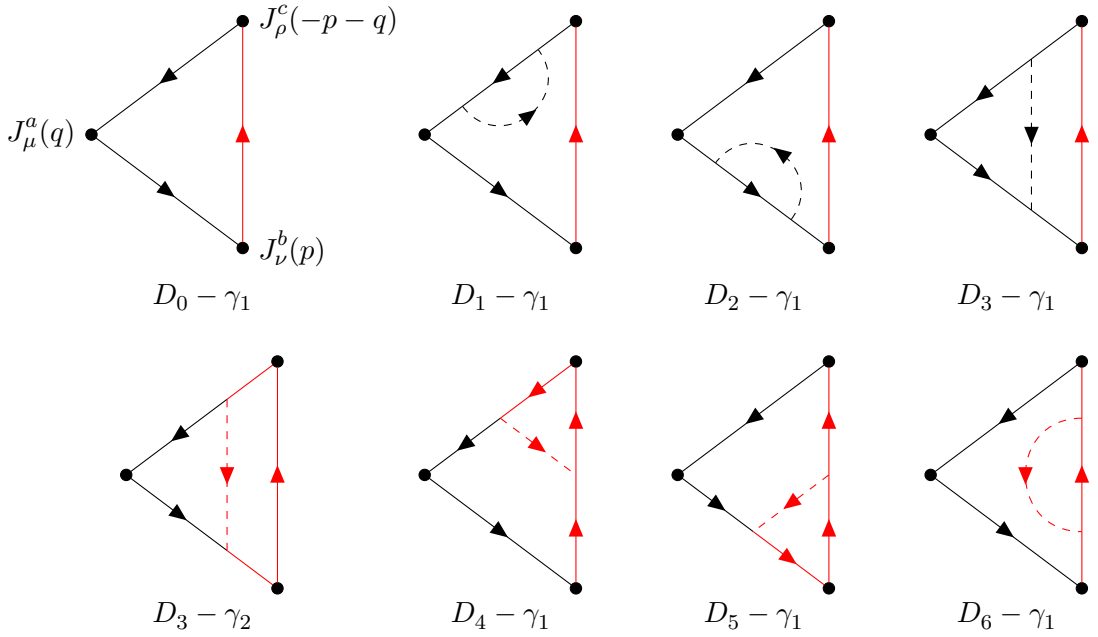

Figure 6. The subgraphs γ marked in red color give the dominating subgraphs (besides the original diagram Γ) of the Feynman diagrams D_i up to the order $O(1/N)$.

for the correlator $\langle JJJ\rangle$ is needed. The net results $\mathcal{S}_\Gamma$ and $\mathcal{S}_\gamma$ from the Feynman loop integrals should be finite. This has been confirmed in our loop computations. We have evaluated the Feynman integrals for the subgraphs γ in figure 5 in general dimensions. After combining all the contributions of $\mathcal{S}_\gamma$ in (4.23), the poles in ϵ and the logarithmic terms $\log p$, $\log q$ arising from the ϵ-regularization of the σ propagator have been cancelled. At the leading order $O(N^0)$, the result is given by the free boson theory (3.11). At the next-to-leading order, $O(1/N)$ the result $\mathcal{S}_{\gamma,1/N}$ is

$$\begin{aligned}\mathcal{S}_{\gamma,1/N} =& \frac{2^{5-D}(3D-4)\pi^{-\frac{D}{2}}\left(q^2\right)^{\frac{D}{2}-2} p_\nu p_\rho \left(q^2 p_\mu - p_\alpha q_\alpha q_\mu\right)}{(D-2)^2(D-1)D^2\Gamma\left(\frac{D}{2}-2\right)p^4 N} \\ &+ \frac{2^{3-D}(D-4)\pi^{-\frac{D}{2}}\left(q^2\right)^{\frac{D}{2}-2}}{(D-2)D\Gamma\left(\frac{D}{2}+1\right)p^2 N}\left(p_\rho q_\mu q_\nu + p_\nu q_\mu q_\rho\right) + \delta-\text{dependent terms.}\end{aligned} \tag{4.26}$$

The integrals of the subgraphs $\mathcal{I}_{D_i,\Gamma}$ are rather cumbersome in general dimensions; therefore, we evaluate the integrals $\mathcal{I}_{D_i,\Gamma}$ within fixed dimension $D=3$. At order $O(N^0)$, it is given by the free boson theory (3.9) and (3.10). At the subleading order $O(1/N)$, the result is

$$\begin{aligned}\mathcal{S}_{\Gamma,1/N}|_{D=3} =& \frac{p^2\left(2p_\nu p_\rho q_\mu + p_\mu\left(3p_\rho q_\nu + p_\nu\left(4p_\rho + q_\rho\right)\right)\right) - 6p_\alpha p_\mu p_\nu p_\rho q_\alpha}{9\pi^2\left(p^2\right)^{5/2} N} \\ &+ \delta-\text{dependent terms.}\end{aligned} \tag{4.27}$$

Like the subgraphs γ, the poles and logarithmic terms are canceled after collecting all the contributions from D_i, and the final result $\mathcal{S}_\Gamma|_{D=3}$ is finite without renormalization.

JHEP12(2025)066

4.2.3 Compare with the correlation functions from conformal symmetry

Now we are ready to extract the $1/N$ corrections to the free parameters C_J and λ_{JJJ} — or equivalently, the n_s and n_f from the Feynman integrals $\mathcal{S}_\Gamma$ and $\mathcal{S}_\gamma$. Since $\mathcal{S}_\Gamma$ and $\mathcal{S}_\gamma$ represent two distinct sectors of the correlation functions, both of them contain the same information on n_s and n_f at the order $O(1/N)$.

We start with the sector $\mathcal{S}_\gamma$, which has been evaluated in general dimensions. As shown in the seminal work [1], the conformal 3-point correlator $\langle JJJ\rangle$ is strongly restricted by the conformal symmetry. The general conformal 3-point correlator $\langle JJJ\rangle$ consists of functions generated by the free scalar and free fermion theories. The two free parameters in the correlation function can be defined through

$$\langle JJJ\rangle_{\text{crit. } O(N)} = n_s\langle JJJ\rangle_{\text{free scalar}} + n_f\langle JJJ\rangle_{\text{free fermion}}. \tag{4.28}$$

At order $O(N^0)$, we have $n_s = 1$ and $n_f = 0$. At order $O(1/N)$, the correlation function $\mathcal{S}_{\gamma,1/N}$ is a combination of $\langle JJJ\rangle_{\text{free scalar}}$ and $\langle JJJ\rangle_{\text{free fermion}}$. The parameters n_s and n_f can be simply fixed by comparing $\mathcal{S}_{\gamma,1/N}$ in (4.26) with $\langle JJJ\rangle_{\text{free scalar}}$ in (3.11) and $\langle JJJ\rangle_{\text{free fermion}}$ in (3.14), which are given by

$$n_s = 1 - \frac{2^{D+2}(3D-4)\sin\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{\pi^{3/2}(D-2)^2D^2\Gamma\left(\frac{D}{2}-2\right)N}, \tag{4.29}$$

$$n_f = -\frac{2^{D+3}\sin\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{\pi^{3/2}(D-2)^2D^2\Gamma\left(\frac{D}{2}-2\right)N}. \tag{4.30}$$

The above results can also be obtained from the subgraph sector $\mathcal{S}_\Gamma$ (4.27). We compare the result $\mathcal{S}_{\Gamma,1/N}|_{D=3}$ with correlation functions from the free scalar and free fermion theories in section 3.1. The 3D results are

$$n_s|_{D=3} = -\frac{80}{9\pi^2N}, \qquad n_f|_{D=3} = -\frac{32}{9\pi^2N}, \tag{4.31}$$

which agree with the solutions of the subgraph sector $\mathcal{S}_\gamma$ in (4.29) and (4.30). This provides a consistency check for the method of subgraphs.

4.3 $\langle JJJ\rangle$ in the GNY model

We use the same method to compute the $1/N$ corrections to the free parameters in $\langle JJJ\rangle$ in the GNY model.[5] The effective Lagrangian for the IR fixed point reads

$$\mathcal{L}_{\text{IR},f} = -\bar{\psi}_{0i}\partial\!\!\!/\psi_0^i + \frac{1}{\sqrt{N}}\sigma_0\bar{\psi}_{0i}\psi_0^i. \tag{4.32}$$

The propagator of the bare fermion operator is provided in (3.2). The critical scalar σ_0 has an effective propagator:

$$\langle\sigma_0(p)\sigma_0(-p)\rangle = \frac{\mathcal{M}_\sigma}{\left(p^2\right)^{\frac{d}{2}-1+\epsilon}}, \quad \mathcal{M}_\sigma = -2^{D+1}(4\pi)^{\frac{D-3}{2}}\Gamma\left(\frac{D-1}{2}\right)\sin\left(\frac{\pi D}{2}\right), \tag{4.33}$$

[5] See the recent works [70, 71] for bootstrap studies of the GNY model.

where the parameter ϵ is introduced to regularize the divergences in the loop integrals, like we have done in the critical $O(N)$ vector model. The vertices of the spin-1 conserved current J^a_μ and stress tensor $T_{\mu\nu}$ have been presented in (3.5) and (3.6). The normalization of the bare operators ψ^i_0, σ_0, the spin-1 conserved current J^a_μ, and the stress tensor $T_{\mu\nu}$ in GNY model have been studied in [36]. An interesting property is that the renormalization factor for the operator J^a_μ is $Z_J = 1 + O(1/N^2)$; therefore, the renormalization for the 3-point correlator $\langle JJJ\rangle$ is trivial at the order $O(1/N)$.

We compute the 3-point correlator $\langle J^a_\mu(q)J^b_\nu(p)J^c_\rho(-p-q)\rangle$ in GNY model using the Feynman diagrams to the order $O(1/N)$. Actually, the Feynman diagrams for this correlator have the same geometry as those in figure 4 for the critical $O(N)$ vector model. The reason is that the propagators and vertices defined by the Lagrangian (4.32) have the same structures as the critical $O(N)$ vector model. Differences appear in the vertices of J^a; however, they only affect the elements in the numerators of the integrands.

We computed the Feynman diagrams using subgraph expansions, and the leading subgraphs γ for the Feynman diagrams are also given by those in figure 6, but with propagators and vertices replaced by those of the GNY model. We focus on the subgraph sector $\mathcal{S}_\gamma$ for simplicity. The leading term at the order $O(N^0)$ is given by the free fermion theory (3.14). The subleading terms at the order $O(1/N)$ are given by the subgraphs in figure 6. After collecting all the contributions from each subgraph, the net result is

$$\mathcal{S}_\gamma = \frac{2^{4-D}\pi^{-\frac{D}{2}}\left(q^2\right)^{\frac{D}{2}-2}\left(q_\mu\left((D-1)p^2\left(p_\rho q_\nu + p_\nu q_\rho\right) + 2p_\alpha p_\nu p_\rho q_\alpha\right) - 2q^2 p_\mu p_\nu p_\rho\right)}{(D-1)D^2\Gamma\left(\frac{D}{2}-1\right)p^4 N} + \delta-\text{dependent terms.} \tag{4.34}$$

We decompose the correlation function into free scalar and free fermion parts:

$$\langle JJJ\rangle_{\text{GNY}} = n_s\langle JJJ\rangle_{\text{free scalar}} + n_f\langle JJJ\rangle_{\text{free fermion}}. \tag{4.35}$$

By matching the subgraph integrals on both sides, we obtain the free parameters

$$n_s = \frac{2^{D+2}\sin\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{\pi^{3/2}D^2\Gamma\left(\frac{D}{2}-1\right)N}, \tag{4.36}$$

$$n_f = 1 + \frac{2(D+1)\sin\left(\frac{\pi D}{2}\right)\Gamma(D-1)}{\pi\Gamma\left(\frac{D}{2}+1\right)^2 N}. \tag{4.37}$$

The two parameters can also be evaluated from the subgraph sector $\mathcal{S}_\Gamma$. For simplicity, we compute the integrations of the subgraph Γ in 3D. The result is

$$\mathcal{S}_{\Gamma,1/N}|_{D=3} = \frac{3p_\alpha p_\mu p_\nu p_\rho q_\alpha - p^2\left(p_\nu p_\rho q_\mu + p_\mu\left(p_\nu\left(2p_\rho + 5q_\rho\right) - 3p_\rho q_\nu\right)\right)}{9\pi^2 p^5 N} + \delta-\text{dependent terms.} \tag{4.38}$$

Comparing with the correlation functions from the free scalar and free fermion theories, we obtain

$$n_s|_{D=3} = -\frac{32}{9\pi^2 N}, \qquad n_f|_{D=3} = -\frac{128}{9\pi^2 N}, \tag{4.39}$$

which are well consistent with the results (4.36) and (4.37) obtained from $\mathcal{S}_\gamma$.

JHEP12(2025)066

5 $1/N$ expansions of $\langle JJ\mathcal{O}\rangle$ and conductivity at finite temperature

Near the IR fixed point, some fundamental physical parameters can be computed using conformal perturbation theory. In particular, the conductivity near the quantum critical point can be estimated using the CFT data related to the spin-1 conserved current operators. In this section, we briefly review the conformal perturbation studies of the conductivity near the IR fixed point [8–11], which motivates our perturbative study of the 3-point correlator $\langle JJ\mathcal{O}\rangle$. Then we apply the method of subgraphs to compute the $1/N$ corrections to the correlators $\langle JJ\mathcal{O}\rangle$ in the critical $O(N)$ vector model.

5.1 Conformal correlator $\langle JJ\mathcal{O}\rangle$ and conductivity at finite temperature

Consider a general CFT in $2+1$ dimensions with coordinates $x_\mu = (t, \mathbf{x})$. The imaginary frequency conductivity, $\omega(ik)$, is defined by the 2-point correlator of the spin-1 conserved current J_μ in the space-like direction, e.g., $\mu = 2$:

$$\frac{\omega(ik)}{\omega_Q} = -\frac{1}{k}\langle J_{\mu=2}(\mathbf{k})J_{\nu=2}(-\mathbf{k})\rangle_T + \text{contact terms}, \tag{5.1}$$

where $\omega_Q = e^2/\hbar$ is the quantum unit of conductance, and the external momentum $\mathbf{k}$ is taken to be in the time direction, $\mathbf{k} = (k, 0, 0)$, where k is the analytic continuation of the Matsubara frequencies $k_n = 2\pi nT$. The 2-point correlator is evaluated at a finite temperature T. In the large momentum limit $k \gg T$, the thermal 2-point correlator $\langle J_2(\mathbf{k})J_2(-\mathbf{k})\rangle_T$ can be evaluated using the OPE

$$\lim_{k\gg|\mathbf{q}|} J_{\mu=2}(\mathbf{k})J_{\nu=2}(-\mathbf{k}+\mathbf{q}) = -c_1|k|\delta^3(\mathbf{q}) - \frac{c_2}{|k|^{\Delta-1}}\mathcal{O}(\mathbf{q}) + \cdots, \tag{5.2}$$

where c_1 and c_2 are the OPE coefficients in momentum space, and we have ignored the contributions from operators with higher scaling dimensions — in particular, the stress tensor $T_{\mu\nu}$. Here we have implicitly assumed that the lowest scalar operator $\mathcal{O}$ is relevant; therefore, in the OPE limit, it provides the subleading order contribution to the thermal 2-point function. Applying the above OPE in the thermal 2-point correlator in (5.1), it gives

$$\frac{\omega(ik)}{\omega_Q} = \frac{C_J}{32} + \lambda_{JJ\mathcal{O}}\frac{C_J}{4\pi}\frac{\Gamma(\Delta_\mathcal{O}+1)\sin\left(\frac{\pi}{2}\Delta_\mathcal{O}\right)}{2-\Delta_\mathcal{O}}f(T,k) + \cdots \tag{5.3}$$

in which the factor $f(T,k)$ is related to the thermal one-point function of the scalar $\langle\mathcal{O}\rangle_T$; see [8] for more details. The parameters C_J and $\lambda_{JJ\mathcal{O}}$ are the usual OPE coefficients defined in position space; they relate to the parameters c_1 and c_2 in (5.2) up to the constants from Fourier transformations.

The conductivity at finite temperature (5.1) can be generalized to a CFT deformed by a relevant operator $\mathcal{O}$

$$S_h = S_{\text{CFT}} - h\int d^D x\mathcal{O}. \tag{5.4}$$

The conductivity in the theory S_h with $h \ll k$ can also be estimated using the conformal perturbation theory [11]. Again, the leading two terms are related to the correlator $\langle J_2(k)J_2(-k)\rangle$ and the 3-point correlator $\langle\mathcal{O}(q)J_2(k)J_2(-k-q)\rangle$.

JHEP12(2025)066

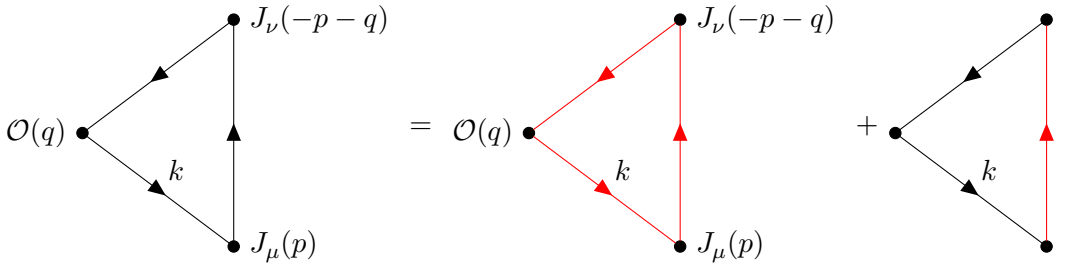

Diagram Γ Subgraph Γ Subgraph γ

Figure 7. Diagrammatic representation of $\mathcal{I}(\nu_1, \nu_2, \nu_3)$ and the two subgraphs in the expansion: Γ and γ marked in red color.

We compute the conformal 3-point correlator $\langle \mathcal{O}(q) J_\mu(p) J_\nu(-p-q) \rangle$ in momentum space. We follow the normalization of the 3-point correlator in the bootstrap work [57], which has obtained a nonperturbative estimation of the OPE coefficient λ_{JJO} for the critical $O(2)$ vector model. Assume that the scalar $\mathcal{O}$ with scaling dimension Δ is normalized as

$$\langle \mathcal{O}(0)\mathcal{O}(x) \rangle = \frac{C_\mathcal{O}}{\left(x^2\right)^\Delta}. \tag{5.5}$$

The correlator $\langle \mathcal{O} J_\mu J_\nu \rangle$ in position space is given by [57, 72, 73]

$$\langle \mathcal{O}(x_1) J_\mu(x_2) J_\nu(x_3) \rangle = \frac{C_J C_\mathcal{O}^{1/2}}{(4\pi)^2} \lambda_{JJ\mathcal{O}} \frac{(\Delta - D + 1) H_{\mu\nu} + \Delta V_{1,\mu} V_{2,\nu}}{\Delta\, x_{12}^\Delta x_{13}^\Delta x_{23}^{2D-2-\Delta}}, \tag{5.6}$$

where we have ignored the global symmetry indices. The correlation function contains only one independent parameter, $\lambda_{JJ\mathcal{O}}$, while the tensor structures in (5.6) are defined as follows

$$H_{\mu\nu} = \delta(\mu,\nu) - 2\frac{x_{23,\mu} x_{23,\nu}}{x_{23}^2}, \tag{5.7}$$

$$V_{1,\mu} = \frac{x_{23}}{x_{12} x_{13}} x_{12,\mu} + \frac{x_{12}}{x_{23} x_{13}} x_{23,\mu}, \tag{5.8}$$

$$V_{2,\nu} = \frac{x_{23}}{x_{12} x_{13}} x_{13,\nu} - \frac{x_{13}}{x_{23} x_{12}} x_{23,\nu}, \tag{5.9}$$

where $x_{ij,\mu} = x_{i,\mu} - x_{j,\mu}$ and $x_{ij} = \sqrt{x_{ij}^2}$.

The correlator $\langle \mathcal{O}(q) J_\mu(p) J_\nu(-p-q) \rangle$ in momentum space can be obtained by Fourier transformation of the above formula. There could be extra contributions from contact terms in position space, whose Fourier transformations only give terms analytical in the external momenta. Like the scalar 3-point correlator, the Fourier transformation of (5.6) leads to one-loop integrals (2.26) associated with tensor structures for the spinning operators. We

JHEP12(2025)066

take subgraph expansions in the small q limit for these integrals. The results consist of two terms which are regular/singular in q

$$\langle \mathcal{O}(q) J_\mu(p) J_\nu(-p-q)\rangle = \frac{C_J C_{\mathcal{O}}^{1/2}}{(4\pi)^2}\lambda_{JJ\mathcal{O}} \tag{5.10}$$
$$\times\left(\frac{2^{3-D}\pi^{D+\frac{1}{2}}\Gamma\left(2-\frac{\Delta}{2}\right)\left(p^2\right)^{\frac{\Delta}{2}-2}\Gamma\left(\frac{1}{2}(D-\Delta+2)\right)\Gamma\left(\Delta-\frac{D}{2}\right)\left(p^2\delta(\mu,\nu)-p_\mu p_\nu\right)}{\Gamma\left(\frac{\Delta}{2}+1\right)^2\Gamma\left(\frac{1}{2}(D-\Delta+1)\right)\Gamma\left(\frac{1}{2}(D+\Delta-2)\right)}\right.$$
$$\left.+\frac{2^{2-\Delta}\pi^D(-D+\Delta+1)\Gamma\left(\frac{D}{2}-\Delta\right)\Gamma\left(\frac{1}{2}(-D+\Delta+2)\right)\left(p^2\right)^{\frac{1}{2}(D-\Delta-2)}\left(q^2\right)^{\Delta-\frac{D}{2}}\delta(\mu,\nu)}{\Gamma(\Delta+1)\Gamma\left(D-\frac{\Delta}{2}-1\right)}+\cdots\right),$$

where we only keep the leading terms in the subgraph expansions. The q-singular (q-regular) term is from the Subgraph γ (Γ). For the scalars with scaling dimension $\Delta < d/2$, the q-singular term dominates in the small q limit.

In this work, we consider the conductivity near the critical $O(N)$ vector model. The leading order result of $\omega(ik)$ in the large k limit is given by the spin-1 conserved current central charge C_J. The parameter C_J in the critical $O(N)$ vector model has been computed to the order $O(1/N)$ in [33–36]. The subleading order contribution to $\omega(ik)$ depends on the OPE coefficient $\lambda_{JJ\sigma}$. The leading order result of $\lambda_{JJ\sigma}$ has been computed in [8]. We will compute its next-to-leading order correction using the method of subgraphs.

5.2 $\langle JJ\sigma_T\rangle$ in the $O(N)$ vector model

In the critical $O(N)$ vector model, there are two relevant scalars in the $J\times J$ OPE: the $O(N)$ traceless symmetric scalar $\sigma_T = \phi_i\phi_j - \frac{1}{N}\delta_{ij}\phi_k\phi_k$ and the $O(N)$ singlet $\sigma = \phi_i\phi_i$.[6] It is the coefficient of the correlator $\langle JJ\sigma\rangle$ that appears in the conductivity formula (5.1). The coefficient $\langle JJ\sigma_T\rangle$ is expected to play a significant role in bootstrap studies involving conserved currents. It turns out that subgraph expansions of the two 3-point correlators show interesting differences.

We first study the 3-point correlator with a non-singlet scalar $\langle JJ\sigma_T\rangle$. The scalar operator σ_T is associated with a renormalization factor Z_{σ_T}:

$$\sigma_T^{\text{phy}} = Z_{\sigma_T}\sigma_T, \qquad Z_{\sigma_T} = 1+\frac{1}{N}\frac{r_{\sigma_T}}{\epsilon}. \tag{5.11}$$

The renormalization factor Z_{σ_T} is fixed by the cancellation of the divergences in the propagator $\langle\sigma_T(p)\sigma_T(-p)\rangle$, after which the correlation function $\langle\sigma_T(p)\sigma_T(-p)\rangle$ takes the standard form fixed by conformal symmetry

$$\langle\sigma_T^{\text{phy}}(p)\sigma_T^{\text{phy}}(-p)\rangle = Z_{\sigma_T}^2\langle\sigma_T(p)\sigma_T(-p)\rangle = \frac{C_{\sigma_T}}{\left(p^2\right)^{\frac{D}{2}-\Delta_{\sigma_T}}}. \tag{5.12}$$

[6] The expressions $\phi_i\phi_j - \frac{1}{N}\delta_{ij}\phi_k\phi_k$ and $\phi_i\phi_i$ give the physical operators only for the free scalar theory or in the large N limit. At finite N, these operators are modified by the renormalization factors.

JHEP12(2025)066

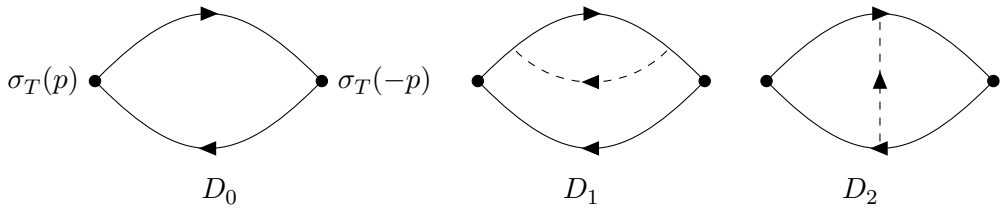

Figure 8. Feynman diagrams for the $\langle\sigma_T(p)\sigma_T(-p)\rangle$ propagator.

At the order $O(1/N)$, the propagator is given by the Feynman diagrams in figure 8, from which the factors $r_{\sigma_T}, C_{\sigma_T}$ and Δ_{σ_T} can be solved to the order $O(1/N)$. The results read

$$r_{\sigma_T} = -\frac{2^D \sin\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{\pi^{3/2} D\Gamma\left(\frac{D}{2}\right)}, \quad \Delta_{\sigma_T} = D - 2 - \frac{2^D \sin\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{\pi^{3/2}\Gamma\left(\frac{D}{2}+1\right) N}. \tag{5.13}$$

The scaling dimension Δ_{σ_T} agrees with previous results [27, 32]. The coefficients C_{σ_T} is given by

$$C_{\sigma_T} = C^0_{\sigma_T}\left(1 + \frac{C^1_{\sigma_T}}{N} + O\left(\frac{1}{N^2}\right)\right), \tag{5.14}$$

in which

$$C^0_{\sigma_T} = -\frac{2^{3-2D}\pi^{\frac{3}{2}-\frac{D}{2}}\csc\left(\frac{\pi D}{2}\right)}{\Gamma\left(\frac{D-1}{2}\right)}, \tag{5.15}$$

$$C^1_{\sigma_T} = \left(\frac{8\mathcal{C}_{\mathrm{O(N)}}}{D-4} + \frac{128 - 12(D-6)(D-4)D}{(D-4)^2(D-2)D}\right)\frac{2\sin\left(\frac{\pi D}{2}\right)\Gamma(D-2)}{\pi\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(\frac{D}{2}+1\right)N}, \tag{5.16}$$

where $\mathcal{C}_{\mathrm{O(N)}} = \psi\left(3-\frac{D}{2}\right) + \psi(D-1) - \psi\left(\frac{D}{2}\right) + \psi(1)$, and $\psi(x) = \Gamma'(x)/\Gamma(x)$.

The Feynman diagrams for the 3-point correlator $\langle\sigma_T(q)J^a_\mu(p)J^a_\nu(-p-q)\rangle$ have the same geometry as the seven Feynman diagrams in figure 4, but with the first vertex $J^a_\mu(q)$ replaced by the scalar vertex $\sigma_T(q)$. We use the subgraph expansions to evaluate the integrals — in particular the subgraphs Γ of the original diagrams, which are regular in q in the small q expansion. Since this 3-point correlator has one independent tensor structure, it suffices to keep the leading order in q in the small q expansion. The integrals of these subgraphs are too cumbersome to be shown here or in appendices; instead, we present the integrals in a *Mathematica* file attached to this submission.

Combining the renormalization factor of σ_T and the loop integrals to the order $O(1/N)$, we extract the coefficient $\lambda_{JJ\sigma_T}$ of the 3-point correlator $\langle\sigma_T(q)J^a_\mu(p)J^a_\nu(-p-q)\rangle$. Using the normalization (5.10), the coefficient reads

$$\lambda_{JJ\sigma_T} = D - 2 + \frac{((D-12)D+16)(\cos(\pi D)-1)\Gamma\left(2-\frac{D}{2}\right)\Gamma(D-2)}{\pi^2 D\Gamma\left(\frac{D}{2}+1\right)N} + O(1/N^2). \tag{5.17}$$

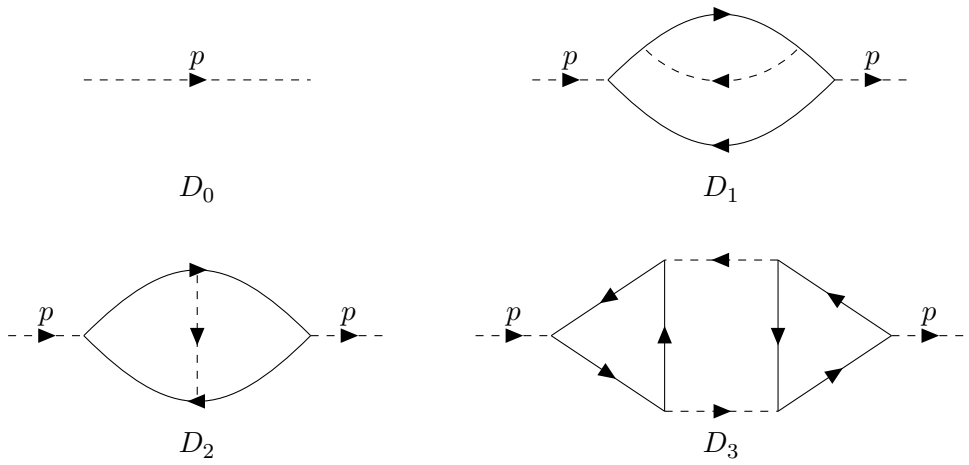

Figure 9. Feynman diagrams for the propagator of the singlet scalar $\langle\sigma(p)\sigma(-p)\rangle$ up to the order $O(1/N)$.

We emphasize that for the above 3-point correlator, it is the subgraphs Γ which produce the q-regular term in (5.10). The integrals of the subgraphs γ are not mentioned here, but they generate the q-singular term in $\langle JJ\sigma_T\rangle$. In contrast, the roles of the subgraphs Γ and γ are switched for the 3-point correlator $\langle\sigma_T(q)J^a_\mu(p)J^a_\nu(-p-q)\rangle$.

5.3 $\langle JJ\sigma\rangle$ in the $O(N)$ vector model

The critical scalar σ has a propagator (4.8), and its renormalization factor Z_σ is defined in (4.7). Up to the order $O(1/N)$, the 2-point correlator $\langle\sigma(p)\sigma(-p)\rangle$ is given by the four Feynman diagrams in figure 9. The Feynman diagrams can be evaluated using the formulas shown before, except for the three-loop diagram D_3, which is of Aslamazov-Larkin type [74]. Details of these integrals are presented in an attached *Mathematica* file.

Based on the Feynman integrals in figure 9, the renormalization factor of the field σ can be solved to the subleading order

$$Z_\sigma = 1 + \frac{r_\sigma}{\Delta}\frac{1}{N} + O(1/N^2), \quad r_\sigma = \frac{2^D \sin\left(\frac{\pi D}{2}\right)\Gamma\left(\frac{D-1}{2}\right)}{\pi^{3/2}\Gamma\left(\frac{D}{2}+1\right)}. \tag{5.18}$$

The scaling dimension Δ_σ and the coefficient C_σ are given by

$$\Delta_\sigma = 2 + \frac{4(D-2)(D-1)\eta_{\rm ON}}{(D-4)}\frac{1}{N} + O(1/N^2), \tag{5.19}$$

and

$$C_\sigma = C^0_\sigma\left(1 + \frac{C^1_\sigma}{N} + O(1/N^2)\right), \tag{5.20}$$

$$C^1_\sigma = \frac{\eta_{\rm ON}}{N}\left(\frac{2\mathcal{C}_{\rm ON}((D-3)D+4)}{4-D} - \frac{2\left((D-4)D\left(2(D-4)D^2+D+36\right)+64\right)}{(D-4)^2(D-2)D}\right), \tag{5.21}$$

JHEP12(2025)066

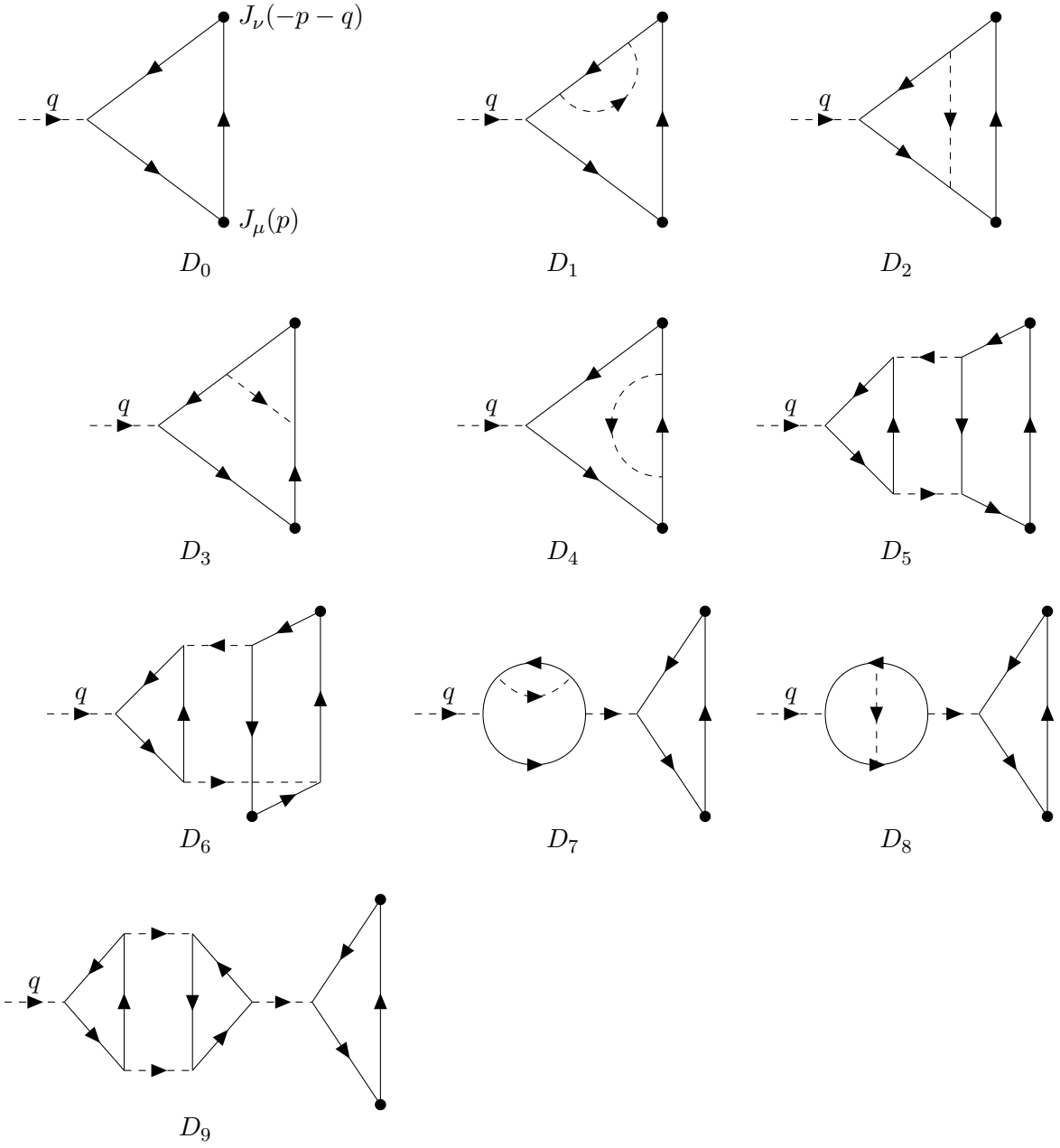

Figure 10. Feynman diagrams for the 3-point correlator $\langle\sigma(q)J_\mu(p)J_\nu(-p-q)\rangle$ up to the order $O(1/N)$. There are extra Feynman diagrams similar to D_1 and D_3 but with the σ propagator connecting with different solid lines. However, since we only keep the leading terms in the small q expansion, these diagrams are identical to D_1 or D_3.

where

$$\eta_{\rm ON} = \frac{2\sin\left(\frac{\pi D}{2}\right)\Gamma(D-2)}{\pi\Gamma\left(\frac{D}{2}-2\right)\Gamma\left(\frac{D}{2}+1\right)}. \tag{5.22}$$

The anomalous dimension of σ agrees with previous result [28, 30].

JHEP12(2025)066

To the subleading order, the 3-point correlator $\langle\sigma(q)J_\mu(p)J_\nu(-p-q)\rangle$ can be expanded by the Feynman diagrams in figure 10. The Feynman diagrams $D_i,\ i=0,1,\ldots,4$ are the same as the Feynman diagrams in figure 4, attached with an extra σ propagator. The Feynman diagrams D_7, D_8, and D_9 are the combinations of the 1PI loop corrections of the σ propagator and the one-loop diagram of the correlator $\langle JJ\phi^2\rangle$. The three-loop integrals D_5 and D_6 are the generalizations of the Aslamazov-Larkin diagram to three external momenta, which are hard to evaluate analytically. We use the method of subgraphs to compute the two integrals in the small q limit. Since the 3-point correlator contains only one independent coefficient $\lambda_{JJ\sigma}$, to fix its $1/N$ correction we only need to keep the leading order term in the small q expansion of each subgraph.

We focus on the subgraphs of the Feynman diagrams which generate the q-regular terms in the small q expansion (5.10) of the correlator $\langle JJ\sigma\rangle$. We separate the Feynman integrals $\mathcal{I}_{\langle\sigma JJ\rangle}$ in figure 10 into two components:

$$\mathcal{I}_{\langle\sigma JJ\rangle}=\mathcal{P}_\sigma\cdot\mathcal{I}_{\langle JJ\phi^2\rangle}\propto\left(q^2\right)^{2-\frac{D}{2}}\cdot\mathcal{I}_{\langle JJ\phi^2\rangle}, \tag{5.23}$$

where $\mathcal{P}_\sigma$ gives the σ propagator or its 1PI loop corrections, while $\mathcal{I}_{\langle JJ\phi^2\rangle}$ denotes the contributions from the amputated diagrams in figure 10 without the σ propagators. The subgraph expansion will be taken with respect to these amputated diagrams. The subgraphs consist of two sectors Γ and γ, which depend on the external momenta as follows

$$\mathcal{I}_{\langle JJ\phi^2\rangle,\Gamma}\propto\left(p^2\right)^{\frac{D}{2}-3}p_\mu p_\nu+\cdots, \tag{5.24}$$

$$\mathcal{I}_{\langle JJ\phi^2\rangle,\gamma}\propto\left(p^2\right)^{-1}\left(q^2\right)^{\frac{D}{2}-2}p_\mu p_\nu+\cdots. \tag{5.25}$$

Combining with the σ propagator factor $\mathcal{P}_\sigma$ in (5.23), it is the subgraphs γ of the amputated diagram which generate the q-regular term in the 3-point correlator $\langle\sigma JJ\rangle$. This is opposite to the 3-point correlator $\langle JJ\sigma_T\rangle$.

The amputated diagrams for the Feynman diagrams $D_i,\ i=0,1,2,3,4,7,8,9$ have the same geometry as those in figure 4, and their subgraph expansions are given by those in figure 6. The three-loop Feynman diagrams D_5 and D_6 contain more complicated subgraphs. In figure 11, we show the subgraphs for the Feynman diagram D_5 and D_6 in figure 10. There are more subgraphs for the diagram D_5, but they lead to scaleless integrals and are ignored in figure 11. The integrals for the subgraphs γ are significantly simplified compared with the original Feynman integrals. In particular, the original 2-scale three-loop Feynman diagrams D_5 and D_6 reduce to the Aslamazov-Larkin diagrams in the subgraphs D_{5,γ_1} and D_{6,γ_1}, and to the Kite diagrams in the subgraphs D_{5,γ_2}, D_{6,γ_2} and D_{6,γ_3}. To evaluate the subgraph D_{6,γ_3}, we have employed the following identity between two Kite diagrams in 3D:

$$K(D-3+2\epsilon,p)p^2+2K(D-4+2\epsilon,p)\Big|_{D=3}=-\frac{3}{16\pi^2}+O(\epsilon), \tag{5.26}$$

which we have verified numerically. Its generalization to general D is not known yet.

The integrals of the subgraphs are presented in the attached *Mathematica* file. For simplicity, we focus on the leading order results in 3D, and ignore the terms with δ-functions.

JHEP12(2025)066

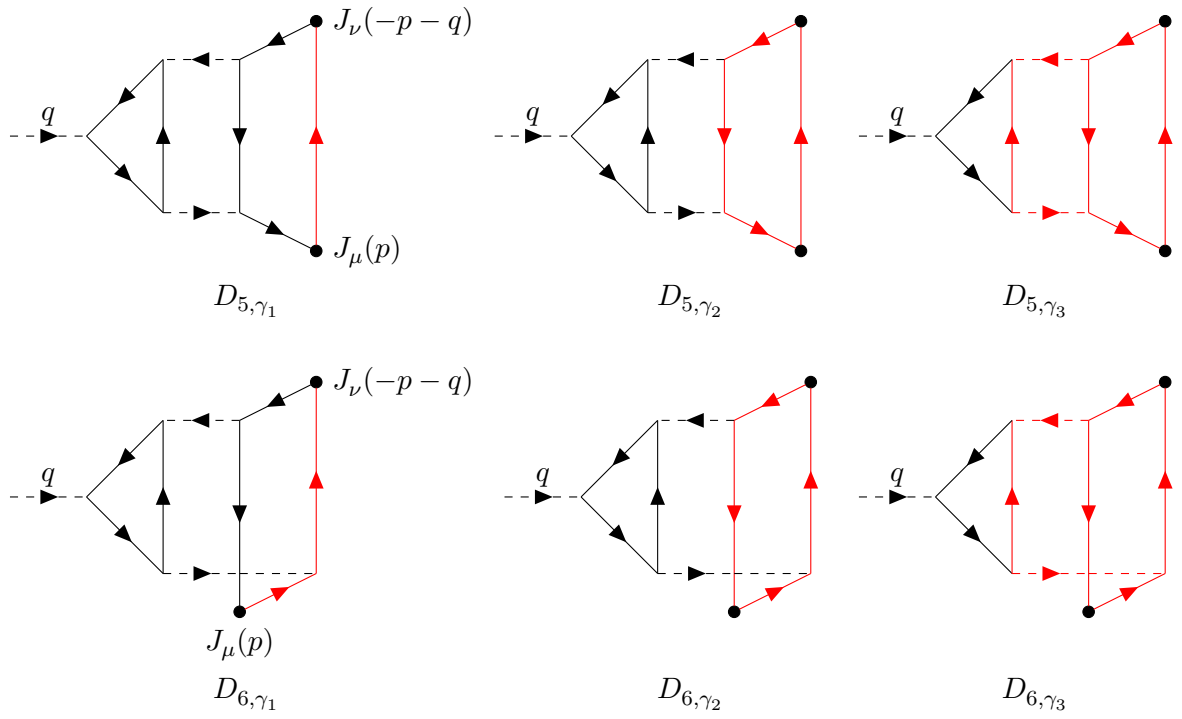

Figure 11. Subgraphs (in red color) of the Feynman diagrams D_5 and D_6 in figure 10.

Collecting all subgraph contributions, the final result is given by

$$\langle\sigma(q)J_\mu(p)J_\nu(-p-q)\rangle|_{D=3} = \frac{2p_\rho p_\sigma}{p^2} - \frac{32\left(3\log\left(p^2\right)-2\right)p_\rho p_\sigma}{9\pi^2 p^2 N} + \cdots . \tag{5.27}$$

Comparing with the same conformal 3-point correlation function (5.10) solved from conformal symmetry, we obtain

$$\lambda_{JJ\sigma}|_{D=3} = \frac{4}{\pi\sqrt{N}} - \frac{80}{9\pi^3\sqrt{N}N} + O(1/N^{5/2}). \tag{5.28}$$

For the $O(2)$ vector model, the leading order result is $\lambda_{JJ\sigma} \simeq 0.900$ [57], and the subleading order correction leads to $\lambda_{JJ\sigma} \simeq 0.799$, while the nonperturbative bootstrap result gives $\lambda_{JJ\sigma} = 0.645(4)$ [57]. There is a notable difference between the large N expansion and the nonperturbative result, which is expected for small $N = 2$.

The $1/N$ correction to the parameter $\lambda_{JJ\sigma}$ in (5.28) provides necessary ingredient to compute the conductivity near the $O(N)$ critical points at finite temperature. The next-to-next-to-leading-order term of the conductivity $\omega(ik)$ arises from the stress tensor, whose contribution depends on the 3-point correlator $\langle TJJ\rangle$, and we leave its $1/N$ correction for future work.

JHEP12(2025)066

6 Conclusion and discussion

We have developed a new approach to compute the subleading order corrections to the conformal current 3-point correlators in the large N expansion. The main technical challenge is that both the conformal 3-point correlators in momentum space and the Feynman loop integrals with three external momenta are hard to evaluate analytically. We introduced the method of subgraphs to resolve this problem, which can generate graph-theoretical expansions for the 3-point loop integrals. Specifically, we diagrammatically expanded the original 2-scale loop integrals in terms of the 1-scale loop integrals, which are much easier to evaluate analytically. Such diagrammatic expansions of the Feynman integrals are subtle in Lorentzian spacetime, while the Feynman loop integrals for CFT data are given in Euclidean spacetime and can be expanded in terms of their subgraphs. This provides a powerful approach to compute the multi-point correlation functions in general CFTs. We have applied the method of subgraphs for the following problems:

- The small momentum expansions of the conformal conserved current 3-point correlators $\langle JJJ\rangle$, $\langle TJJ\rangle$, and $\langle TTT\rangle$.

  The conformal 3-point correlation functions can be solved from the conformal Ward identities in momentum space, while the solutions in general dimensions are given by complicated integrals and are hard to apply for the $1/N$ perturbative computations. Using the method of subgraphs, these correlators can be expanded in a graph-theoretical way. The subgraph expansions provide diagrammatic representations for the behaviors of the correlation functions in the zero-momentum limit. In particular, the method of subgraphs captures the singularities of the correlator in the OPE limit in momentum space.

- The $1/N$ corrections to the conformal 3-point correlators $\langle JJJ\rangle$ in the critical $O(N)$ vector model and GNY model.

  The conformal conserved current 3-point correlators play important roles in various directions, while they are only known at the leading order. We use the correlator $\langle JJJ\rangle$ as an example to demonstrate that the method of subgraphs provides a powerful approach to compute the $1/N$ corrections. Different from the subgraph expansions in general QFTs, for CFTs, one needs to keep only the first few terms in the subgraph expansions to obtain the $1/N$ corrections.

- The $1/N$ corrections to the conformal 3-point correlators $\langle JJ\mathcal{O}\rangle$ in the critical $O(N)$ vector model.

  This problem is motivated by the fundamental role of the correlator $\langle JJ\sigma\rangle$ in evaluating the conductivity of quantum critical systems at finite temperature. We computed the coefficients for both the lowest $O(N)$ traceless symmetric scalar σ_T and the singlet scalar σ, which can be used to compute the subleading term of the conductivity. An interesting observation is that, due to the attachment of the σ propagator in the Feynman diagrams for $\langle JJ\sigma\rangle$, the subgraphs Γ and γ switch their roles in reproducing the asymptotic behaviors of the correlators $\langle JJ\mathcal{O}\rangle$ in the subgraph expansion.

JHEP12(2025)066

We expect this work to pave the way to perturbatively study the conformal current 3-point correlators in general CFTs and their applications in various directions. We elaborate on part of the problems below.

It is straightforward to apply the method of subgraphs to compute the subleading order corrections to the conformal current 3-point correlators $\langle JJ\mathcal{O}\rangle$ and $\langle JJT\rangle$ in the conformal gauge theories, e.g., the IR fixed points of the QED_3 coupled with massless fermions [75, 76]. The theories are strongly coupled in the IR, and they provide ideal candidates for the modern conformal bootstrap studies [77, 78]. The conserved currents are important elements in the bootstrap equations, and the large N perturbative estimations on the conserved current 3-point correlators can provide necessary input data for the bootstrap computations.

We have computed the conserved current 3-point correlator $\langle JJ\mathcal{O}\rangle$ in the critical $O(N)$ vector model, motivated by its application in the conductivity at finite temperature [5, 7–11]. The next dominant contribution comes from the stress tensor $T_{\mu\nu}$. For the CFTs without relevant singlet scalars, e.g., the conformal QED_3 coupled with many-flavor massless fermions, the stress tensor provides the dominating contribution to the thermal conductivity in addition to the vacuum state. Therefore, it is important to evaluate the conformal current 3-point correlators $\langle JJT\rangle$ in these theories to provide better estimations of the conductivities near the IR fixed points. The leading order results of the correlator $\langle JJT\rangle$ are given by the free fermion and free scalar theories, which have been computed in [5]. The Feynman diagrams for the correlator $\langle JJT\rangle$ to order $O(1/N)$ are similar to those for $\langle JJ\sigma\rangle$ in figure 10, but with a new vertex for the stress tensor $T_{\mu\nu}$. We expect the subleading order correction to $\langle JJT\rangle$ can be solved using the method developed in this work, and it would be interesting to compare the perturbative results with the Monte Carlo simulations [7, 8, 10, 79, 80].

It is tempting to generalize the studies of conformal current 3-point correlators to the parity-breaking CFTs. A prominent family of such theories is the Chern-Simons matter field theories in 3D, which exhibit interesting conductivity properties at finite temperature; see [81] for a recent study. Prior to the Feynman loop computations, it is necessary to know the conformal symmetry constraints on the parity-odd 3-point correlation functions [82–85]. To compute the $1/N$ corrections, it would be helpful to get the diagrammatic representations of the parity-odd correlation functions. A remarkable property of the 3D Chern-Simons matter field theories is the so-called bosonization, namely the duality between the Chern-Simons theories coupled with fermions and scalars. A classical example of the 3D bosonization is provided by the $\text{U}(N)_k$ Chern-Simons theory coupled to a fundamental boson or fermion [86–88]. In the planar limit $N \to \infty$ with fixed 't Hooft coupling $\lambda = N/k$, all the current 3-point correlators can be solved through a bootstrap approach [89, 90], and the results confirm the bosonization duality at large N. The duality is conjectured to be true even at finite N, with more details on the relations between the gauge groups and matter fields [91, 92]. It would be interesting to study this duality at finite N by computing the $1/N$ corrections of the conformal current 3-point correlators. The 3D bosonization can also be realized by the Abelian Chern-Simons theories coupled to matter fields [93–96], in which different theories are connected through the $\text{SL}(2, Z)$ actions and form a duality web. The duality web has a close relation to the deconfined quantum critical points [97]. The current 3-point correlators with $1/N$ corrections can help to decode the underlying physics of these theories.

JHEP12(2025)066

We have shown that the subgraph expansion is closely related to the operator product expansion in momentum space. Specifically, it provides a diagrammatic classification of the contributions from different regions in the integration domain. This approach can be straightforwardly generalized to the conformal 4-point correlators. The conformal 4-point correlator and its OPE are the key ingredients for the bootstrap studies, which are usually formulated in position space [77, 78]. There have been attempts to implement bootstrap methods in momentum space [98–101]. The challenge is that the OPE for conformal 4-point correlator is quite subtle in momentum space, as the external operators have complicated coincident limits, which requires careful classification of the integration domains. We expect the method of subgraphs can help to resolve this problem in a graph-theoretical approach. In the conformal 4-point correlator, there are three independent external momenta and the correlation functions consist of 3-scale loop integrals, for which it needs to take two subgraph expansions sequentially to realize the OPE of the 4-point correlator. It would be interesting to understand how the associativity of the OPEs and also the crossing symmetry are realized in the final results. Moreover, there are constraints from the consistency conditions, including unitarity and locality [102]. These constraints have been explored in Mellin space and lead to remarkable results for the CFT data [103, 104]. It would be intriguing to explore these constraints further in momentum space. We leave this problem for future study.

Acknowledgments

The author would like to thank Rajeev Erramilli and Ke Ren for the helpful discussions and collaborations on related projects. The author is in debt to Christoph Dlapa and Zhengwen Liu for the instructive discussions on the method of regions and method of subgraphs. The author is grateful to Matthew Mitchell, David Poland, Kostas Skenderis, Yuan Xin and Gang Yang for valuable discussions. The author would like to thank David Poland and the Yale Physics Department for the support during early stage of this work. The author thanks the organizers of the ICBS 2025 and Bootstrap 2025 for the support during the course of this work. This research was supported by the Startup Funding 4007022314 of the Southeast University, the Southeast University Global Engagement of Excellence Fund No. 3360682401E, and the National Natural Science Foundation of China funding No. 12375061.

Data Availability Statement. This article has data included as electronic supplementary material.

Code Availability Statement. This article has no associated code or the code will not be deposited.

Supplementary material and data. The following files are provided as supplementary material/data and are accessible through the article's webpage.

- `Integrals.nb` (Feynman integrals in this paper). The Feynman integrals appear in the $1/N$ expansion of the current three-point correlators. The results are too cumbersome to be presented in the manuscript. Details of these integrals are provided in this Mathematica file for reference.

JHEP12(2025)066

References

[1] H. Osborn and A.C. Petkou, *Implications of conformal invariance in field theories for general dimensions*, *Annals Phys.* **231** (1994) 311 [hep-th/9307010] [INSPIRE].

[2] J. Erdmenger and H. Osborn, *Conserved currents and the energy momentum tensor in conformally invariant theories for general dimensions*, *Nucl. Phys. B* **483** (1997) 431 [hep-th/9605009] [INSPIRE].

[3] D.M. Hofman and J. Maldacena, *Conformal collider physics: Energy and charge correlations*, *JHEP* **05** (2008) 012 [arXiv:0803.1467] [INSPIRE].

[4] A. Buchel et al., *Holographic GB gravity in arbitrary dimensions*, *JHEP* **03** (2010) 111 [arXiv:0911.4257] [INSPIRE].

[5] D. Chowdhury et al., *Multipoint correlators of conformal field theories: implications for quantum critical transport*, *Phys. Rev. B* **87** (2013) 085138 [arXiv:1210.5247] [INSPIRE].

[6] D.M. Hofman et al., *A Proof of the Conformal Collider Bounds*, *JHEP* **06** (2016) 111 [arXiv:1603.03771] [INSPIRE].

[7] W. Witczak-Krempa, E. Sørensen and S. Sachdev, *The dynamics of quantum criticality via Quantum Monte Carlo and holography*, *Nature Phys.* **10** (2014) 361 [arXiv:1309.2941] [INSPIRE].

[8] E. Katz, S. Sachdev, E.S. Sørensen and W. Witczak-Krempa, *Conformal field theories at nonzero temperature: Operator product expansions, Monte Carlo, and holography*, *Phys. Rev. B* **90** (2014) 245109 [arXiv:1409.3841] [INSPIRE].

[9] W. Witczak-Krempa, *Constraining Quantum Critical Dynamics: (2+1)D Ising Model and Beyond*, *Phys. Rev. Lett.* **114** (2015) 177201 [arXiv:1501.03495] [INSPIRE].

[10] A. Lucas, S. Gazit, D. Podolsky and W. Witczak-Krempa, *Dynamical response near quantum critical points*, *Phys. Rev. Lett.* **118** (2017) 056601 [arXiv:1608.02586] [INSPIRE].

[11] A. Lucas, T. Sierens and W. Witczak-Krempa, *Quantum critical response: from conformal perturbation theory to holography*, *JHEP* **07** (2017) 149 [arXiv:1704.05461] [INSPIRE].

[12] X.O. Camanho, J.D. Edelstein, J. Maldacena and A. Zhiboedov, *Causality Constraints on Corrections to the Graviton Three-Point Coupling*, *JHEP* **02** (2016) 020 [arXiv:1407.5597] [INSPIRE].

[13] N. Afkhami-Jeddi, T. Hartman, S. Kundu and A. Tajdini, *Einstein gravity 3-point functions from conformal field theory*, *JHEP* **12** (2017) 049 [arXiv:1610.09378] [INSPIRE].

[14] J.M. Maldacena, *Non-Gaussian features of primordial fluctuations in single field inflationary models*, *JHEP* **05** (2003) 013 [astro-ph/0210603] [INSPIRE].

[15] J.M. Maldacena and G.L. Pimentel, *On graviton non-Gaussianities during inflation*, *JHEP* **09** (2011) 045 [arXiv:1104.2846] [INSPIRE].

[16] I. Antoniadis, P.O. Mazur and E. Mottola, *Conformal Invariance, Dark Energy, and CMB Non-Gaussianity*, *JCAP* **09** (2012) 024 [arXiv:1103.4164] [INSPIRE].

JHEP12(2025)066

[17] A. Bzowski, P. McFadden and K. Skenderis, *Holography for inflation using conformal perturbation theory*, *JHEP* **04** (2013) 047 [arXiv:1211.4550] [INSPIRE].

[18] N. Arkani-Hamed, D. Baumann, H. Lee and G.L. Pimentel, *The Cosmological Bootstrap: Inflationary Correlators from Symmetries and Singularities*, *JHEP* **04** (2020) 105 [arXiv:1811.00024] [INSPIRE].

[19] D. Baumann et al., *The Cosmological Bootstrap: Spinning Correlators from Symmetries and Factorization*, *SciPost Phys.* **11** (2021) 071 [arXiv:2005.04234] [INSPIRE].

[20] C.-H. Chang et al., *Bootstrapping the 3d Ising stress tensor*, *JHEP* **03** (2025) 136 [arXiv:2411.15300] [INSPIRE].

[21] Z. Li and D. Poland, *Searching for gauge theories with the conformal bootstrap*, *JHEP* **03** (2021) 172 [arXiv:2005.01721] [INSPIRE].

[22] S. Albayrak et al., *Bootstrapping N_f=4 conformal QED_3*, *Phys. Rev. D* **105** (2022) 085008 [arXiv:2112.02106] [INSPIRE].

[23] S.S. Pufu, *Anomalous dimensions of monopole operators in three-dimensional quantum electrodynamics*, *Phys. Rev. D* **89** (2014) 065016 [arXiv:1303.6125] [INSPIRE].

[24] S.M. Chester and S.S. Pufu, *Anomalous dimensions of scalar operators in QED_3*, *JHEP* **08** (2016) 069 [arXiv:1603.05582] [INSPIRE].

[25] S.M. Chester, L.V. Iliesiu, M. Mezei and S.S. Pufu, *Monopole Operators in U(1) Chern-Simons-Matter Theories*, *JHEP* **05** (2018) 157 [arXiv:1710.00654] [INSPIRE].

[26] J. Henriksson, *The critical O(N) CFT: Methods and conformal data*, *Phys. Rept.* **1002** (2023) 1 [arXiv:2201.09520] [INSPIRE].

[27] S.-K. Ma, *Scaling Variables and Dimensions*, *Phys. Rev. A* **10** (1974) 1818 [INSPIRE].

[28] Y. Okabe and M. Oku, *$1/n$ Expansion Up to Order $1/n^2$. 3. Critical Exponents γ and ν for $d = 3$*, *Prog. Theor. Phys.* **60** (1978) 1287 [INSPIRE].

[29] D.I. Kazakov, O.V. Tarasov and A.A. Vladimirov, *Calculation of Critical Exponents by Quantum Field Theory Methods*, *Sov. Phys. JETP* **50** (1979) 521 [INSPIRE].

[30] A.N. Vasiliev, Y.M. Pismak and Y.R. Khonkonen, *$1/n$ expansion: Calculation of the exponents η and ν in the order $1/n^2$ for arbitrary number of dimensions*, *Theor. Math. Phys.* **47** (1981) 465 [INSPIRE].

[31] S.E. Derkachov and A.N. Manashov, *The simple scheme for the calculation of the anomalous dimensions of composite operators in the 1/N expansion*, *Nucl. Phys. B* **522** (1998) 301 [hep-th/9710015] [INSPIRE].

[32] J.A. Gracey, *Crossover exponent in O(N) phi**4 theory at O(1 / N**2)*, *Phys. Rev. E* **66** (2002) 027102 [cond-mat/0206098] [INSPIRE].

[33] A.C. Petkou, *C(T) and C(J) up to next-to-leading order in 1/N in the conformally invariant 0(N) vector model for $2 < d < 4$*, *Phys. Lett. B* **359** (1995) 101 [hep-th/9506116] [INSPIRE].

[34] Y. Huh, P. Strack and S. Sachdev, *Conserved current correlators of conformal field theories in 2+1 dimensions*, *Phys. Rev. B* **88** (2013) 155109 [*Erratum ibid.* **90** (2014) 199902] [arXiv:1307.6863] [INSPIRE].

[35] Y. Huh and P. Strack, *Stress tensor and current correlators of interacting conformal field theories in 2+1 dimensions: Fermionic Dirac matter coupled to U(1) gauge field*, *JHEP* **01** (2015) 147 [*Erratum ibid.* **03** (2016) 054] [arXiv:1410.1902] [INSPIRE].

JHEP12(2025)066

[36] K. Diab et al., *On C_J and C_T in the Gross-Neveu and O(N) models*, *J. Phys. A* **49** (2016) 405402 [arXiv:1601.07198] [INSPIRE].

[37] M. Goykhman and M. Smolkin, *Vector model in various dimensions*, *Phys. Rev. D* **102** (2020) 025003 [arXiv:1911.08298] [INSPIRE].

[38] N. Chai, M. Goykhman and R. Sinha, *Conformal correlators in the critical O(N) vector model*, *Phys. Rev. D* **105** (2022) 086026 [arXiv:2103.10168] [INSPIRE].

[39] A. Bzowski, P. McFadden and K. Skenderis, *Implications of conformal invariance in momentum space*, *JHEP* **03** (2014) 111 [arXiv:1304.7760] [INSPIRE].

[40] C. Corianò, L. Delle Rose, E. Mottola and M. Serino, *Solving the Conformal Constraints for Scalar Operators in Momentum Space and the Evaluation of Feynman's Master Integrals*, *JHEP* **07** (2013) 011 [arXiv:1304.6944] [INSPIRE].

[41] A. Bzowski and K. Skenderis, *Comments on scale and conformal invariance*, *JHEP* **08** (2014) 027 [arXiv:1402.3208] [INSPIRE].

[42] A. Bzowski, P. McFadden and K. Skenderis, *Scalar 3-point functions in CFT: renormalisation, beta functions and anomalies*, *JHEP* **03** (2016) 066 [arXiv:1510.08442] [INSPIRE].

[43] A. Bzowski, P. McFadden and K. Skenderis, *Renormalised 3-point functions of stress tensors and conserved currents in CFT*, *JHEP* **11** (2018) 153 [arXiv:1711.09105] [INSPIRE].

[44] M. Gillioz, *Conformal 3-point functions and the Lorentzian OPE in momentum space*, *Commun. Math. Phys.* **379** (2020) 227 [arXiv:1909.00878] [INSPIRE].

[45] A. Bzowski, P. McFadden and K. Skenderis, *Conformal n-point functions in momentum space*, *Phys. Rev. Lett.* **124** (2020) 131602 [arXiv:1910.10162] [INSPIRE].

[46] A. Bzowski, P. McFadden and K. Skenderis, *Conformal correlators as simplex integrals in momentum space*, *JHEP* **01** (2021) 192 [arXiv:2008.07543] [INSPIRE].

[47] C. Corianò and M.M. Maglio, *Conformal field theory in momentum space and anomaly actions in gravity: The analysis of three- and four-point function*, *Phys. Rept.* **952** (2022) 1 [arXiv:2005.06873] [INSPIRE].

[48] M. Gillioz, *From Schwinger to Wightman: all conformal 3-point functions in momentum space*, arXiv:2109.15140 [INSPIRE].

[49] R. Marotta, K. Skenderis and M. Verma, *Momentum space CFT correlators of non-conserved spinning operators*, *JHEP* **03** (2023) 196 [arXiv:2212.13135] [INSPIRE].

[50] V.A. Smirnov, *Asymptotic expansions in momenta and masses and calculation of Feynman diagrams*, *Mod. Phys. Lett. A* **10** (1995) 1485 [hep-th/9412063] [INSPIRE].

[51] V.A. Smirnov, *Applied asymptotic expansions in momenta and masses*, *Springer Tracts Mod. Phys.* **177** (2002) 1 [INSPIRE].

[52] V.A. Smirnov, *Analytic tools for Feynman integrals*, Springer (2012) [DOI:10.1007/978-3-642-34886-0] [INSPIRE].

[53] V.A. Smirnov, *Expansion by Regions: An Overview*, in *Anti-Differentiation and the Calculation of Feynman Amplitudes*, Springer (2021) [DOI:10.1007/978-3-030-80219-6_18] [arXiv:2406.11475] [INSPIRE].

[54] M. Beneke and V.A. Smirnov, *Asymptotic expansion of Feynman integrals near threshold*, *Nucl. Phys. B* **522** (1998) 321 [hep-ph/9711391] [INSPIRE].

JHEP12(2025)066

[55] E. Gardi et al., *The on-shell expansion: from Landau equations to the Newton polytope*, *JHEP* **07** (2023) 197 [arXiv:2211.14845] [INSPIRE].

[56] V.A. Smirnov, *Simplifying the large-mass expansion of Feynman integrals*, *Theor. Math. Phys.* **219** (2024) 986 [arXiv:2307.00387] [INSPIRE].

[57] M. Reehorst, E. Trevisani and A. Vichi, *Mixed Scalar-Current bootstrap in three dimensions*, *JHEP* **12** (2020) 156 [arXiv:1911.05747] [INSPIRE].

[58] A.I. Davydychev, *Recursive algorithm for evaluating vertex-type Feynman integrals*, *J. Phys. A* **25** (1992) 5587 [INSPIRE].

[59] A. Prudnikov, Y. Brychkov and O. Marichev, *Integrals and Series. Vol. 2. Special Functions*, vol. 2, Gordon and Breach Science Publishers (1992).

[60] A. Dymarsky et al., *Scale Invariance, Conformality, and Generalized Free Fields*, *JHEP* **02** (2016) 099 [arXiv:1402.6322] [INSPIRE].

[61] S. Weinberg, *High-energy behavior in quantum field theory*, *Phys. Rev.* **118** (1960) 838 [INSPIRE].

[62] W. Zimmermann, *Convergence of Bogolyubov's method of renormalization in momentum space*, *Commun. Math. Phys.* **15** (1969) 208 [INSPIRE].

[63] V.A. Smirnov, *Asymptotic expansions in limits of large momenta and masses*, *Commun. Math. Phys.* **134** (1990) 109 [INSPIRE].

[64] S. El-Showk, Y. Nakayama and S. Rychkov, *What Maxwell Theory in D<>4 teaches us about scale and conformal invariance*, *Nucl. Phys. B* **848** (2011) 578 [arXiv:1101.5385] [INSPIRE].

[65] M. Moshe and J. Zinn-Justin, *Quantum field theory in the large N limit: A Review*, *Phys. Rept.* **385** (2003) 69 [hep-th/0306133] [INSPIRE].

[66] S. Giombi, G. Tarnopolsky and I.R. Klebanov, *On C_J and C_T in Conformal QED*, *JHEP* **08** (2016) 156 [arXiv:1602.01076] [INSPIRE].

[67] K.G. Chetyrkin, A.L. Kataev and F.V. Tkachov, *New Approach to Evaluation of Multiloop Feynman Integrals: The Gegenbauer Polynomial x Space Technique*, *Nucl. Phys. B* **174** (1980) 345 [INSPIRE].

[68] A.V. Kotikov, *The Gegenbauer polynomial technique: The evaluation of a class of Feynman diagrams*, *Phys. Lett. B* **375** (1996) 240 [hep-ph/9512270] [INSPIRE].

[69] A.I. Davydychev, *A simple formula for reducing Feynman diagrams to scalar integrals*, *Phys. Lett. B* **263** (1991) 107 [INSPIRE].

[70] R.S. Erramilli et al., *The Gross-Neveu-Yukawa archipelago*, *JHEP* **02** (2023) 036 [arXiv:2210.02492] [INSPIRE].

[71] M.S. Mitchell and D. Poland, *Bounding irrelevant operators in the 3d Gross-Neveu-Yukawa CFTs*, *JHEP* **09** (2024) 134 [arXiv:2406.12974] [INSPIRE].

[72] A. Dymarsky, J. Penedones, E. Trevisani and A. Vichi, *Charting the space of 3D CFTs with a continuous global symmetry*, *JHEP* **05** (2019) 098 [arXiv:1705.04278] [INSPIRE].

[73] J.-F. Fortin, W.-J. Ma, V. Prilepina and W. Skiba, *Conformal conserved currents in embedding space*, *JHEP* **01** (2022) 185 [arXiv:2012.05227] [INSPIRE].

[74] L.G. Aslamasov and A.I. Larkin, *The influence of fluctuation pairing of electrons on the conductivity of normal metal*, *Phys. Lett. A* **26** (1968) 238.

JHEP12(2025)066

[75] T.W. Appelquist, M.J. Bowick, D. Karabali and L.C.R. Wijewardhana, *Spontaneous Chiral Symmetry Breaking in Three-Dimensional QED*, *Phys. Rev. D* **33** (1986) 3704 [INSPIRE].

[76] T. Appelquist, D. Nash and L.C.R. Wijewardhana, *Critical Behavior in (2+1)-Dimensional QED*, *Phys. Rev. Lett.* **60** (1988) 2575 [INSPIRE].

[77] R. Rattazzi, V.S. Rychkov, E. Tonni and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **12** (2008) 031 [arXiv:0807.0004] [INSPIRE].

[78] D. Poland, S. Rychkov and A. Vichi, *The Conformal Bootstrap: Theory, Numerical Techniques, and Applications*, *Rev. Mod. Phys.* **91** (2019) 015002 [arXiv:1805.04405] [INSPIRE].

[79] K. Chen et al., *Universal Conductivity in a Two-Dimensional Superfluid-to-Insulator Quantum Critical System*, *Phys. Rev. Lett.* **112** (2014) 030402 [arXiv:1309.5635] [INSPIRE].

[80] S. Gazit, D. Podolsky and A. Auerbach, *Critical Capacitance and Charge-Vortex Duality Near the Superfluid-to-Insulator Transition*, *Phys. Rev. Lett.* **113** (2014) 240601.

[81] A. Maity, H. Guo, S. Sachdev and V. Tripathi, *Thermal Hall response of an Abelian chiral spin liquid at finite temperatures*, *Phys. Rev. B* **111** (2025) 205119 [arXiv:2411.15304] [INSPIRE].

[82] S. Jain et al., *Momentum space parity-odd CFT 3-point functions*, *JHEP* **08** (2021) 089 [arXiv:2101.11635] [INSPIRE].

[83] C. Corianò, S. Lionetti and M.M. Maglio, *Parity-odd 3-point functions from CFT in momentum space and the chiral anomaly*, *Eur. Phys. J. C* **83** (2023) 502 [arXiv:2303.10710] [INSPIRE].

[84] C. Corianò, S. Lionetti and M.M. Maglio, *Parity-violating CFT and the gravitational chiral anomaly*, *Phys. Rev. D* **109** (2024) 045004 [arXiv:2309.05374] [INSPIRE].

[85] C. Corianò, S. Lionetti and M.M. Maglio, *CFT correlators and CP-violating trace anomalies*, *Eur. Phys. J. C* **83** (2023) 839 [arXiv:2307.03038] [INSPIRE].

[86] S. Giombi et al., *Chern-Simons Theory with Vector Fermion Matter*, *Eur. Phys. J. C* **72** (2012) 2112 [arXiv:1110.4386] [INSPIRE].

[87] O. Aharony, G. Gur-Ari and R. Yacoby, $d = 3$ *Bosonic Vector Models Coupled to Chern-Simons Gauge Theories*, *JHEP* **03** (2012) 037 [arXiv:1110.4382] [INSPIRE].

[88] O. Aharony, G. Gur-Ari and R. Yacoby, *Correlation Functions of Large N Chern-Simons-Matter Theories and Bosonization in Three Dimensions*, *JHEP* **12** (2012) 028 [arXiv:1207.4593] [INSPIRE].

[89] J. Maldacena and A. Zhiboedov, *Constraining Conformal Field Theories with A Higher Spin Symmetry*, *J. Phys. A* **46** (2013) 214011 [arXiv:1112.1016] [INSPIRE].

[90] J. Maldacena and A. Zhiboedov, *Constraining conformal field theories with a slightly broken higher spin symmetry*, *Class. Quant. Grav.* **30** (2013) 104003 [arXiv:1204.3882] [INSPIRE].

[91] O. Aharony, *Baryons, monopoles and dualities in Chern-Simons-matter theories*, *JHEP* **02** (2016) 093 [arXiv:1512.00161] [INSPIRE].

[92] O. Aharony, F. Benini, P.-S. Hsin and N. Seiberg, *Chern-Simons-matter dualities with SO and USp gauge groups*, *JHEP* **02** (2017) 072 [arXiv:1611.07874] [INSPIRE].

[93] N. Seiberg, T. Senthil, C. Wang and E. Witten, *A Duality Web in 2+1 Dimensions and Condensed Matter Physics*, *Annals Phys.* **374** (2016) 395 [arXiv:1606.01989] [INSPIRE].

[94] A. Karch and D. Tong, *Particle-Vortex Duality from 3d Bosonization*, *Phys. Rev. X* **6** (2016) 031043 [arXiv:1606.01893] [INSPIRE].

JHEP12(2025)066

[95] S. Kachru, M. Mulligan, G. Torroba and H. Wang, *Bosonization and Mirror Symmetry*, *Phys. Rev. D* **94** (2016) 085009 [arXiv:1608.05077] [INSPIRE].

[96] C. Wang et al., *Deconfined quantum critical points: symmetries and dualities*, *Phys. Rev. X* **7** (2017) 031051 [arXiv:1703.02426] [INSPIRE].

[97] T. Senthil et al., *Deconfined Quantum Critical Points*, *Science* **303** (2004) 1490 [cond-mat/0311326] [INSPIRE].

[98] H. Isono, T. Noumi and G. Shiu, *Momentum space approach to crossing symmetric CFT correlators*, *JHEP* **07** (2018) 136 [arXiv:1805.11107] [INSPIRE].

[99] H. Isono, T. Noumi and G. Shiu, *Momentum space approach to crossing symmetric CFT correlators. Part II. General spacetime dimension*, *JHEP* **10** (2019) 183 [arXiv:1908.04572] [INSPIRE].

[100] M. Gillioz, X. Lu, M.A. Luty and G. Mikaberidze, *Convergent Momentum-Space OPE and Bootstrap Equations in Conformal Field Theory*, *JHEP* **03** (2020) 102 [arXiv:1912.05550] [INSPIRE].

[101] M. Gillioz, *The momentum-space conformal bootstrap in 2d*, arXiv:2502.21227 [INSPIRE].

[102] A.M. Polyakov, *Non-Hamiltonian approach to conformal quantum field theory*, *Zh. Eksp. Teor. Fiz.* **66** (1974) 23 [INSPIRE].

[103] R. Gopakumar, A. Kaviraj, K. Sen and A. Sinha, *Conformal Bootstrap in Mellin Space*, *Phys. Rev. Lett.* **118** (2017) 081601 [arXiv:1609.00572] [INSPIRE].

[104] R. Gopakumar, A. Kaviraj, K. Sen and A. Sinha, *A Mellin space approach to the conformal bootstrap*, *JHEP* **05** (2017) 027 [arXiv:1611.08407] [INSPIRE].

JHEP12(2025)066