Spin matrix theory as non-relativistic limit of $\mathcal{N} = 4$ SYM

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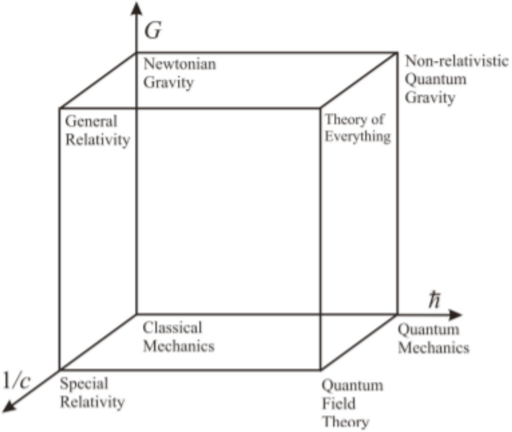
Work by Stefano Baiguera, Troels Harmark and Nico Wintergerst 2009.03799, 2012.08532, 2111.10149

The theory cube



AdS/CFT motivation

$\mathcal{N}=4$ SYM in adjoint of SU(N) group $\leftrightarrow$ Type IIB strings in AdS$_5 \times S^5$: Believed to be true for all couplings [Maldacena, 1997][Gubser et al, 1998][Witten, 1998]

1. Planar limit $N=\infty$ and the power of integrability [Minahan, Zarembo, 2002][Beisert et al, 2003]
2. Supersymmetric localization [Pestun, 2007]
3. Recent microstate counting of supersymmetric AdS$_5$ black hole [Kim, et al 2018] [Murthy et al, 2018] [Benini, Milan, 2018]

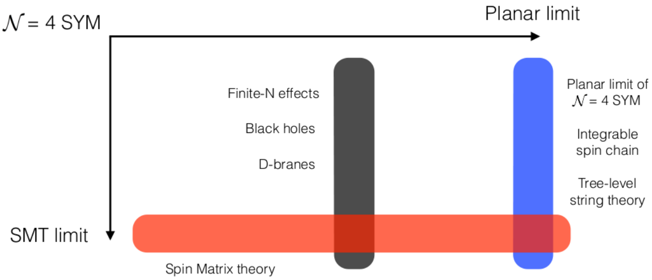
Problem

- Planar limit: gravity enters as $1/N$ perturbative corrections $\Rightarrow$ No access to black holes and D-branes dynamics
- Finite N but weak coupling: string theory is not geometrical

Spin Matrix Theory

Controlled finite N effects (strong coupled dynamics of gravity) and semiclassical geometry: Spin Matrix Theory limits [Harmark, Orselli, 2014].

- Decoupling limits of $\mathcal{N}=4$ SYM on $\mathbb{R}\times S^3 \Rightarrow$ the theory reduces to a subsector with only one-loop contributions of the dilatation operator [Harmark, Orselli, 2006][Harmark, Kristjansson, Orselli, 2006-07]
- Approach unitarity (BPS) bounds
- Understand how quantum gravity gets simplified in non-relativistic limit (as expansions of c^{-1})



Contents of $\mathcal{N} = 4$ SYM

The set of letters of $\mathcal{N} = 4$ SYM

- 6 indepedent gauge components $F_{\pm,0}, \bar{F}_{\pm,0}$
- 6 complex scalars $Z, W, X, \bar{Z}, \bar{W}, \bar{X}$
- 16 complex fermions $\chi_i, \bar{\chi}_i$, $i = 1, .., 8$
- 4 components of covariant derivatives $d_{1,2}$ and $\bar{d}_{1,2}$

The letters are specified by dimension D_0, $SO(4)$ spin (S_1, S_2) and R-charges (Q_1, Q_2, Q_3). The BPS letters are those satisfying

$$D_0 = S_1 + S_2 + Q_1 + Q_2 + Q_3$$

BPS letters

Letter	$SO(4)[S_1, S_2]$	Name in 0510251	$Q = \frac{1}{2}(Q_1 + Q_2)$	Q_3	D_0
Z	$[0, 0]$	Z	$\frac{1}{2}$	0	1
X	$[0, 0]$	X	$\frac{1}{2}$	0	1
W	$[0, 0]$	Y	0	1	1
$\bar{F}_+$	$[1, 1]$	F_{++}	0	0	2
χ_1	$[\frac{1}{2}, -\frac{1}{2}]$	$\psi_{0,+,+++}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
χ_2	$[-\frac{1}{2}, \frac{1}{2}]$	$\psi_{0,-,+++}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
$\bar{\chi}_3$	$[\frac{1}{2}, \frac{1}{2}]$	$\psi_{+,0,-++}$	0	$\frac{1}{2}$	$\frac{3}{2}$
$\bar{\chi}_5$	$[\frac{1}{2}, \frac{1}{2}]$	$\psi_{+,0,+-+}$	0	$\frac{1}{2}$	$\frac{3}{2}$
$\bar{\chi}_7$	$[\frac{1}{2}, \frac{1}{2}]$	$\psi_{+,0,++-}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$
d_1	$[1, 0]$	∂_{++}	0	0	1
d_2	$[0, 1]$	∂_{+-}	0	0	1

Dirac equation

$$d_1\chi_2 - d_2\chi_1 = 0$$

Decoupling

Assume $(\omega_1, \omega_2, \Omega_1, \Omega_2, \Omega_3) = (n_1\Omega, n_2\Omega, n_3\Omega, n_4\Omega, n_5\Omega)$ are the chemical potentials conjugate to the angular momenta and R-charges. The critical values are denoted as $(n_1, n_2, n_3, n_4, n_5)$, which is reached when $\Omega \to 1$. The critical overall charge is then $J \equiv n_1 S_1 + n_2 S_2 + n_3 Q_1 + n_4 Q_2 + n_5 Q_3$. We are interested in sectors $D_0 = J$. The partition function is

$$
\begin{aligned}
Z &= \mathrm{Tr}[e^{-\beta D + \beta\Omega J}] \\
&= \mathrm{Tr}[e^{-\beta(D_0 - J) + \beta(1-\Omega)J - \beta\lambda D_2 + \mathcal{O}(\lambda^{\frac{3}{2}})}]
\end{aligned}
$$

Take the following limit (decoupling limit)

$$
\beta \to \infty, \quad \Omega \to 1, \quad \lambda \to 0, \quad \tilde{\beta} = \beta(1-\Omega), \ \beta\lambda \text{ fixed}
$$

Then the effective partition function is just

$$
Z = \mathrm{Tr}_{D_0 = J}[e^{-\tilde{\beta}(D_0 + \tilde{\lambda} D_2)}]
$$

i.e. Only one-loop correction survives in the decoupling limit.

Non-relativistic

Recall that a known fact from special relativity is

$$E = \sqrt{m_0^2 c^4 + p^2 c^2} = m_0^2 c^2 + \frac{p^2}{2m_0} + \mathcal{O}(c^{-2})$$

The Newtonian mechanics is about the dynamics at the order c^0. A well-defined effective theory. The analogy in $\mathcal{N} = 4$ SYM is that we denote the classical conformal dimension by D_0. In the presence of weak interaction parametrized by 't Hooft coupling λ, the conformal dimension will receive the quantum correction

$$D = D_0 + \lambda D_2 + ...$$

where D_2 is the one loop correction. We can decouple the higher order of Feymann diagram corrections the same as we do for non-relativistic mechanics.

Spin Matrix theory [Harmark, Orselli, 2014]

Constructing D_2; as its letters carry both matrix indices from SU(N) and spin group indices from subgroup of PSU$(2, 2|4)$.

Magnon example

The dispersion of a single magnon in $\mathcal{N} = 4$ SYM is [Beisert, 2005]

$$E - Q = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2 \frac{p}{2}} - 1$$

Take the small momenta limit, we have

$$E - Q \approx \sqrt{1 + \frac{\lambda p^2}{4\pi^2}} - 1$$

In the SMT decoupling limit, $\lambda \to 0$, this becomes

$$E - Q = \frac{\lambda p^2}{8\pi^2}$$

Zoo of decoupling limits

Without going to the details, it has been explored by [Harmark et al, 2007] to find all the possible decoupling limits. Except the trivial $U(1)$ decoupling limit, there are 12 nontrivial decoupling limits

- There are letters $D_0 = J$
- All the letters satisfy $D_0 \geq J$

The compact subsectors are [Harmark, Orselli, 2014; Baiguera, Harmark, YL, 2021]

- $SU(2)$ limit, $\vec{n} = (0, 0, 1, 1, 0)$. Letter: Z, X.
- $SU(1|1)$ limit ($XX_{\frac{1}{2}}$ Heisenberg spin chain), $\vec{n} = (\frac{2}{3}, 0, 1, \frac{2}{3}, \frac{2}{3})$. Letter: Z, χ_1
- $SU(1|2)$ limit ($t - J$ model), $\vec{n} = (\frac{1}{2}, 0, 1, 1, \frac{1}{2})$. Letter: Z, X, χ_1 [Beisert, Staudacher, 2005]
- $SU(2|3)$ limit, $\vec{n} = (0, 0, 1, 1, 1)$. Letter: Z, X, W, χ_1, χ_2

Zoo of decoupling limits

Non-compact SU$(1,1)$ kind subsectors are [Baiguera, Harmark, YL, Wintergerst, 2020, 2021]

- Bosonic $SU(1,1)$ limit ($XXX_{-\frac{1}{2}}$ Heisenberg model), $\vec{n} = (1,0,1,0,0)$. Letter: $d_1^n Z$.
- Fermionic $SU(1,1)$ limit, $\vec{n} = (1,0,\frac{2}{3},\frac{2}{3},\frac{2}{3})$. Letter: $d_1^n \chi_1$.
- SU$(1,1|1)$ limit, $\vec{n} = (1,0,1,\frac{1}{2},\frac{1}{2})$, Letters $d_1^n Z, d_1^n \chi_1$
- PSU$(1,1|2)$ limit, $\vec{n} = (1,0,1,1,0)$. Letters: $d_1^n Z, d_1^n X, d_1^n \chi_1, d_1^n \bar{\chi}_7$

Non-compact SU$(1,2)$ kind subsectors are [Baiguera, Harmark, YL, Wintergerst, 2020, 2022]

- SU$(1,2)$ limit, $\vec{n} = (1,1,0,0,0)$. Letter: $d_1^n d_2^k \bar{F}_+$
- SU$(1,2|1)$ limit, $\vec{n} = (1,1,\frac{1}{2},\frac{1}{2},0)$. Letter: $d_1^n d_2^k \bar{F}_+, d_1^n d_2^k \bar{\chi}_7$
- SU$(1,2|2)$ limit, $\vec{n} = (1,1,1,0,0)$. Letter: $d_1^n d_2^k \bar{F}_+,\ d_1^n d_2^k Z,\ d_1^n d_2^k \chi_1,\ d_1^n d_2^k \bar{\chi}_7$
- PSU$(1,2|3)$ limit, $\vec{n} = (1,1,1,1,1)$. Letter: $d_1^n d_2^k \bar{F}_+,\ d_1^n d_2^k Z,\ d_1^n d_2^k W,\ d_1^n d_2^k X,\ d_1^n d_2^k \chi_{1,2},\ d_1^n d_2^k \bar{\chi}_{3,5,7}$

The achievement so far

- Near $\frac{1}{16}$-BPS: PSU$(1,2|3)$ in progress
- Near $\frac{1}{8}$-BPS including other $1/4$-BPS etc
  1. SU$(1,2|2)$: [Baiguera, Harmark, YL, Wintergerst, 2020]
  2. SU$(2|3)$: [Baiguera, Harmark, YL, 2021]
  3. PSU$(1,1|2)$: [Baiguera, Harmark, YL, 2021]

$\mathcal{N} = 4$ SYM action

The action of $\mathcal{N} = 4$ SYM is

$$S = \int_{\mathbb{R}\times S^3} \sqrt{-\det g_{\mu\nu}}\, \mathrm{tr} \Big[-\frac{1}{4} F_{\mu\nu}^2 - |D_\mu \Phi_a|^2 - |\Phi_a|^2 - i\psi_a^\dagger \bar{\sigma}^\mu D_\mu \psi^A + g \sum_{A,B,a} C_{AB}^a \psi^A [\Phi_a, \psi^B] + g \sum_{A,B,a} \bar{C}^{aAB} \psi_A^\dagger [\Phi_a^\dagger, \psi_B^\dagger] - \frac{g^2}{2} \sum_{a,b} \left(|[\Phi_a, \Phi_b]|^2 + |[\Phi_a, \Phi_b^\dagger]|^2 \right) \Big]$$

We are expanding the fields in terms of S^3 harmonics such as [Ishiki, Takayama, Tsuchiya, 2006]

$$\Phi^a(t,\Omega) = \sum_{JM} \Phi^a_{JM}(t) Y_{JM}(\Omega), \quad A_i(t,\Omega) = \sum_{\rho=0,\pm1} \sum_{JM} A^\rho_{JM}(t) Y^\rho_{JMi}(\Omega)$$

$$\psi^A_\alpha(t,\Omega) = \sum_{\kappa=\pm1} \sum_{JM} \psi^A_{JM\kappa}(t) Y^\kappa_{JM\alpha}(\Omega)$$

S^3 harmonic functions

$$\nabla^2 \mathcal{Y}^{Ln}_{Jm,\tilde{J}\tilde{m}} = -(2J(J+1) + 2\tilde{J}(\tilde{J}+1) - L(L+1))\mathcal{Y}^{Ln}_{Jm,\tilde{J}\tilde{m}}$$

The function Y_{Jm} is the scalar harmonics

$$Y_{Jm} = \mathcal{Y}^{L=0,n=0}_{Jm,\tilde{J}\tilde{m}}$$

and triangle inequality fixes $\tilde{J} = J$. Vector harmonics are like

$$Y^{\rho=1}_{JMi} = i\mathcal{Y}^{1n,i}_{J+1,m;J\tilde{m}}, \quad Y^{\rho=-1}_{JMi} = -i\mathcal{Y}^{1n,i}_{Jm;J+1,\tilde{m}}, \quad Y^{\rho=0}_{JMi} = \mathcal{Y}^{1n,i}_{Jm;J\tilde{m}}$$

due to triangle inequality $|J - \tilde{J}| \leq L \leq J + \tilde{J}$. Note this means for $\rho = 1$, $|m| \leq J + 1$ while $|\tilde{m}| \leq J$. Identities

$$\int d\Omega (Y_{J_1 M_1})^* Y_{J_2 M_2} = \delta_{J_1 J_2}\delta_{M_1 M_2}$$

$$\int d\Omega (Y^{\rho_1}_{J_1 M_1 i}) * Y^{\rho_2}_{J_2 M_2 i} = \delta_{\rho_1\rho_2}\delta_{J_1 J_2}\delta_{M_1 M_2}$$

Vertex coefficients

We denote $M = (m, \tilde{m})$ as the $SO(4)$ spins. The $Y_{JM}, Y^i_{JM\rho}$ are all orthogonal among their kinds.

$$\mathcal{C}^{J_1 M_1}_{J_2 M_2; J_3 M_3} \equiv \int d\Omega \bar{Y}_{J_1 M_1} Y_{J_2 M_2} Y_{J_3 M_3}$$

$$= \sqrt{\frac{(2J_3+1)(2J_2+1)}{2J_1+1}} C^{J_1 m_1}_{J_2 m_2; J_3 m_3} C^{J_1 \tilde{m}_1}_{J_2 \tilde{m}_2; J_3 \tilde{m}_3}$$

$$\mathcal{D}^{J_1 M_1}_{J_2 M_2 \rho_2; J M \rho} \equiv \int d\Omega \bar{Y}_{J_1 M_1} Y^{\rho_2}_{J_2 M_2 i} Y^{\rho}_{JMi}$$

$$\mathcal{E}_{J_1 M_1 \rho_1; J_2 M_2 \rho_2; J M \rho} \equiv \int d\Omega \epsilon_{ijk} Y^{\rho_1}_{J_1 M_1 i} Y^{\rho_2}_{J_2 M_2 j} Y^{\rho}_{JMk}$$

$$\mathcal{F}^{J_1 M_1 \kappa_1}_{J_2 M_2 \kappa_2; JM} = \int d\Omega \bar{Y}^{\kappa_1}_{J_1 M_1 \alpha} Y^{\kappa_2}_{J_2 M_2 \alpha} Y_{JM}$$

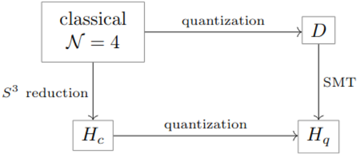
$$\mathcal{G}^{J_1 M_1 \kappa_1}_{J_2 M_2 \kappa_2; J M \rho} = \int d\Omega \sigma^i_{\alpha\beta} \bar{Y}^{\kappa_1}_{J_1 M_1 \alpha} Y^{\kappa_2}_{J_2 M_2 \beta} Y^{\rho}_{JMi}$$

$$
\begin{aligned}
H_{\text{int}} = \sum_{J_i, M_i, \kappa_i, \rho_i} \text{tr} \Big\{ & ig\mathcal{C}^{J_2M_2}_{J_1M_1;JM}\, \chi_{JM} \left([(Z^\dagger_a)_{J_2M_2}, (\Pi^{(\Phi)\dagger}_a)_{J_1M_1}] + [Z^a_{J_1M_1}, \Pi^{a(\Phi)}_{J_2M_2}] \right) \\
& -4g\sqrt{J_1(J_1+1)}\mathcal{D}^{J_2M_2}_{J_1M_10;JM\rho}\, A^{JM}_{(\rho)}[Z^a_{J_1M_1}, (Z^\dagger_a)_{J_2M_2}] \\
& +g\mathcal{F}^{J_1M_1\kappa_1}_{J_2M_2\kappa_2;JM}\, \chi_{JM}\{(\Psi^\dagger_A)_{J_1M_1\kappa_1}, \Psi^A_{J_2M_2\kappa_2}\} \\
& +g\mathcal{G}^{J_1M_1\kappa_1}_{J_2M_2\kappa_2;JM\rho}\, A^{JM}_{(\rho)}\{(\Psi^\dagger_A)_{J_1M_1\kappa_1}, \Psi^A_{J_2M_2\kappa_2}\} \\
& +\frac{g^2}{2}\mathcal{C}^{J_2M_2}_{J_1M_1;JM}\mathcal{C}^{J_3M_3}_{J_4M_4;JM}[Z^a_{J_1M_1}, (Z^\dagger_a)_{J_2M_2}][Z^b_{J_3M_3}, (Z^\dagger_b)_{J_4M_4}] \\
& -\sqrt{2}ig(-1)^{-m_1+\tilde{m}_1+\frac{\kappa_1}{2}}\mathcal{F}^{J_1,-M_1,\kappa_1}_{J_2M_2\kappa_2;JM}\psi^4_{J_2M_2\kappa_2}[(Z_a)^{JM}, \Psi^a_{J_1M_1\kappa_1}] \\
& +\sqrt{2}ig(-1)^{-m_1+\tilde{m}_1+\frac{\kappa_1}{2}}\mathcal{F}^{J_1,-M_1,\kappa_1}_{J_2M_2\kappa_2;JM}\, \epsilon_{abc}\Psi^a_{J_1M_1\kappa_1}[(Z^\dagger_b)^{JM}, \Psi^c_{J_2M_2\kappa_2}] \\
& +\sqrt{2}ig(-1)^{m_2-\tilde{m}_2+\frac{\kappa_2}{2}}\mathcal{F}^{J_1M_1\kappa_1}_{J_2,-M_2,\kappa_2;JM}(\Psi^\dagger_4)_{J_2M_2\kappa_2}[(Z^\dagger_a)^{JM}, (\Psi^\dagger_a)_{J_1M_1\kappa_1}] \\
& -\sqrt{2}ig(-1)^{m_2-\tilde{m}_2+\frac{\kappa_2}{2}}\mathcal{F}^{J_1M_1\kappa_1}_{J_2,-M_2,\kappa_2;JM}\, \epsilon_{abc}(\Psi^\dagger_a)_{J_1M_1\kappa_1}[(Z_b)^{JM}, (\Psi^\dagger_c)_{J_2M_2\kappa_2}] \\
& +ig\mathcal{D}^{JM}_{J_1M_1\rho_1;J_2M_2\rho_2}\, \chi_{JM}[\Pi^{J_1M_1}_{(\rho_1)}, A^{J_2M_2}_{(\rho_2)}] \\
& +g^2\mathcal{C}^{JM}_{J_2M_2;J_4,-M_4}\mathcal{D}_{JM;J_1M_1\rho_1;J_3M_3\rho_3}[A^{J_1M_1}_{(\rho_1)}, Z^a_{J_2M_2}][A^{J_3M_3}_{(\rho_3)}, (Z^\dagger_a)_{J_4M_4}] \\
& +2ig\rho_1(J_1+1)\mathcal{E}_{J_1M_1\rho_1;J_2M_2\rho_2;J_3M_3\rho_3}A^{J_1M_1}_{(\rho_1)}[A^{J_2M_2}_{(\rho_2)}, A^{J_3M_3}_{(\rho_3)}] \\
& -\frac{g^2}{2}\mathcal{D}^{JM}_{J_1M_1\rho_1;J_3M_3\rho_3}\mathcal{D}_{JM;J_2M_2\rho_2;J_4M_4\rho_4}[A^{J_1M_1}_{(\rho_1)}, A^{J_2M_2}_{(\rho_2)}][A^{J_3M_3}_{(\rho_3)}, A^{J_4M_4}_{(\rho_4)}] \\
& -2g\sqrt{J_1(J_1+1)}\mathcal{D}_{J_2M_2;J_1M_10;JM\rho}\, \chi_{J_1M_1}[\chi_{J_2M_2}, A^{JM}_{(\rho)}] \\
& +\frac{g^2}{2}\mathcal{C}^{JM}_{J_1M_1;J_3M_3}\mathcal{D}_{JM;J_2M_2\rho_2;J_4M_4\rho_4}[\chi_{J_1M_1}, A^{J_2M_2}_{(\rho_2)}][\chi_{J_3M_3}, A^{J_4M_4}_{(\rho_4)}] \\
& +g^2\mathcal{C}^{JM}_{J_1M_1;J_2M_2}\mathcal{C}_{JM;J_3M_3;J_4M_4}[\chi_{J_1M_1}, Z^a_{J_2M_2}][\chi_{J_3M_3}, (Z^\dagger_a)_{J_4M_4}] \Big\} \, .
\end{aligned}
\tag{B.40}
$$

Effect of spherical reduction

- Most of the massive modes on S^3 decouple and only very few dynamical modes survive
- Non-dynamical modes can still contribute to the interactions of dynamical modes

Scheme of the procedure [Harmark, Wintergerst, 2019]:



Sectors with effective (1+1)-dimensional theories

Focus on BPS bounds

$$H \geq S_1 + \sum_{i=1}^{3} \omega_i Q_i$$

- S_1, S_2 Cartan generators for rotations on S^3
- Q_i Cartan generators of $SU(4)$ R-symmetry group
- ω_i chemical potentials characterizing the bound

Spin Matrix Theory limit

$$\lambda \to 0\,, \qquad H_2 = \frac{H - S_1 - \sum_{i=1}^{3} \omega_i Q_i}{\lambda} \text{ finite }, \qquad N \text{ fixed}$$

Sectors	Combination of $SU(4)$ Cartan charges $\sum_{i=1}^{3} \omega_i Q_i$
SU$(1,1)$ bosonic	Q_1
SU$(1,1)$ fermionic	$\frac{2}{3}(Q_1 + Q_2 + Q_3)$
SU$(1,1\|1)$	$Q_1 + \frac{1}{2}(Q_2 + Q_3)$
PSU$(1,1\|2)$	$Q_1 + Q_2$

Hamiltonian analysis

Focus on gauge kinetic term + minimal coupling

$$S = \int \sqrt{-g}\,\mathrm{tr}(-\frac{1}{4}F_{\mu\nu}^2 + A^\mu j_\mu)$$

Canonical momenta:

$$\Pi_0 = \frac{\delta S}{\delta \dot{A}_0} = 0\,, \qquad \Pi_i = \frac{\delta S}{\delta \dot{A}_i} = F_{0i}$$

We work in Coulomb gauge, imposed via a Lagrange multiplier η. Hamiltonian:

$$H = \int \sqrt{-g}\,\mathrm{tr}(\frac{1}{2}\Pi_i^2 + \frac{1}{4}F_{ij}^2 - A_0(\nabla_i \Pi^i - j_0)) - A^i j_i + \eta \nabla_i A^i\,,$$

with constraints

$$\nabla_i \Pi^i - j_0 = 0\,, \quad \nabla_i A^i = 0$$

- We treat A_0 as a Lagrange multiplier enforcing Gauss' law: it is no longer a dynamical variable.
- They are two second-class constraints, corresponding to the two unphysical degrees of freedom.

Decomposing into spherical harmonics on S^3 [Ishiki, Takayama, Tsuchiya, 2006] the constraints become

$$2i\sqrt{J(J+1)}\Pi_{(0)}^{Jm\tilde{m}} + j_0^{\dagger\, Jm\tilde{m}} = 0\,, \quad A_{(0)}^{Jm\tilde{m}} = 0\,.$$

We can directly solve the constraints for $A_{(0)}^{Jm\tilde{m}}$ and its simplectic partner $\Pi_{(0)}^{Jm\tilde{m}}$ and plug the solution into the Hamiltonian without changing the Poisson brackets:

$$H = \mathrm{tr}\sum_{J,m,\tilde{m}}\left\{\sum_{\rho=\pm1}\left(\frac{1}{2}|\Pi_{(\rho)}^{Jm\tilde{m}}|^2 + \frac{1}{2}(2J+2)^2|A_{(\rho)}^{Jm\tilde{m}}|^2 + A_{(\rho)}^{Jm\tilde{m}} j_{(\rho)}^{\dagger\, Jm\tilde{m}}\right) + \frac{1}{8J(J+1)}|j_0^{Jm\tilde{m}}|^2\right\}.$$

Charges

Keep in mind charges and angular momentum can also be written in terms of modes. For example

$$Q_a^{\Phi} = i\sum_{J,M}\mathrm{tr}(\Phi_a^{JM}\Pi_a^{JM} - \Phi_a^{\dagger JM}\Pi_a^{\dagger JM})$$

Bosonic SU$(1,1)$ example

As we have known, the decoupling condition for bosonic SU$(1,1)$ is $H_0 = Q_1 + S_1$.

$$H_0 - S_1 - Q_1 = \sum_{J,m,\tilde{m}} \mathrm{tr}\Big\{ |\Pi_a^{Jm\tilde{m}} + i(\delta_a^1 + \tilde{m} - m)\Phi_a^{\dagger Jm\tilde{m}}|^2 + (\omega_J^2 - (\delta_a^1 + \tilde{m} - m)^2)|\Phi_a^{Jm\tilde{m}}|^2$$

$$+ \sum_{\kappa=\pm 1} \left(\sum_{A=1,4} \left(\omega_J^\psi + m - \tilde{m} - \frac{\kappa}{2}\right) \psi^\dagger_{JM,\kappa,A} \psi^A_{JM,\kappa} + \sum_{A=2,3} \left(\omega_J^\psi + m - \tilde{m} + \frac{\kappa}{2}\right) \psi^\dagger_{JM,\kappa,A} \psi^A_{JM,\kappa}$$

$$+ \sum_{\rho=-1,1} \frac{1}{2} \left(|\Pi_{(\rho)}^{Jm\tilde{m}} - i(m-\tilde{m}) A_{(\rho)}^{\dagger\, Jm\tilde{m}}|^2 + (\omega_{A,J}^2 - (m-\tilde{m})^2)|A_{(\rho)}^{Jm\tilde{m}}|^2 \right) \Big\},$$

with $\omega_J = 2J+1$, $\omega_J^\psi = 2J + \frac{3}{2}$ and $\omega_{A,J} \equiv 2J+2$. We thus obtain the set of constraints

$$A_{(\rho)}^{Jm\tilde{m}} = -\frac{1}{\omega_{A,J}^2 - (m-\tilde{m})^2} j_{(\rho)}^{Jm\tilde{m}}, \quad \Pi_{(\rho)}^{Jm\tilde{m}} = 0,$$

$$\Phi_{a=2,3}^{Jm\tilde{m}} = 0, \quad \Pi_{a=2,3}^{Jm\tilde{m}} = 0,$$

$$\Phi_1^{Jm\tilde{m}} = 0, \quad \Pi_1^{Jm\tilde{m}} = 0 \quad (\text{except when } \ J = -m = \tilde{m}),$$

$$\Pi_1^{J,-J,J} + i\omega_J \Phi_1^{\dagger\, J,-J,J} = 0.$$

Solutions

The only allowed dynamics comes from the momenta saturated modes! Then the free part of Hamiltonian H_0 is written as

$$H_0 = S_1 + Q_1 = \sum_{n=0}^{\infty} (n+1) \mathrm{tr} |\Phi_n|^2$$

where we define $\Phi_{2J} = \sqrt{2\omega_J} \Phi_1^{J,-J,J}$ which is subject to Poisson bracket

$$\{\Phi_{J'}, \Phi_J^\dagger\}_D = i\delta_{JJ'}$$

Note matrix indices are suppressed

The interaction Hamiltonian should be

$$\sum_{J,m,\tilde{m}} \mathrm{tr}\left(\frac{1}{8J(J+1)} |j_0^{Jm\tilde{m}}|^2 - \sum_{\rho=\pm 1} \frac{1}{2(\omega_{A,J}^2 - (m-\tilde{m})^2)} |j_{(\rho)}^{Jm\tilde{m}}|^2 \right).$$

The relevant interaction terms involving scalars in the Hamiltonian H of $\mathcal{N}=4$ SYM (Notation $\mathcal{J}=(J,-J,J)$) and the convention is $A^0 = \sum_{JM} \chi_{JM} Y_{JM}$

$$\begin{aligned} \sum_{J,m,\tilde{m}} \mathrm{tr}\Big\{ & \frac{g^2}{2} \mathcal{C}^{\mathcal{J}_2}_{\mathcal{J}_1,JM} \mathcal{C}^{\mathcal{J}_3}_{\mathcal{J}_4,JM} [\Phi_1^{\mathcal{J}_1}, \Phi_1^{\mathcal{J}_2\,\dagger}][\Phi_1^{\mathcal{J}_3}, \Phi_1^{\mathcal{J}_4\,\dagger}] \\ & + ig \mathcal{C}^{\mathcal{J}_2}_{\mathcal{J}_1,JM} \chi^{JM} \left([\Phi_1^{\mathcal{J}_2\,\dagger}, \Pi_1^{\mathcal{J}_1\,\dagger}] + [\Phi_1^{\mathcal{J}_1}, \Pi_1^{\mathcal{J}_2}] \right) \\ & -4g\sqrt{J_1(J_1+1)} \mathcal{D}^{\mathcal{J}_2}_{\mathcal{J}_1,JM\rho} A^{JM}_{(\rho)} [\Phi_1^{\mathcal{J}_1}, \Phi_1^{\mathcal{J}_2\,\dagger}] \Big\}. \end{aligned}$$

where one can read

$$\begin{aligned} j_0^{\dagger\, Jm\tilde{m}} &= ig \mathcal{C}^{\mathcal{J}_2}_{\mathcal{J}_1,JM} \left([\Phi_1^{\mathcal{J}_2\,\dagger}, \Pi_1^{\mathcal{J}_1\,\dagger}] + [\Phi_1^{\mathcal{J}_1}, \Pi_1^{\mathcal{J}_2}] \right) \\ j_{(\rho)}^{\dagger\, Jm\tilde{m}} &= -4g\sqrt{J_1(J_1+1)} \mathcal{D}^{\mathcal{J}_2}_{\mathcal{J}_1,JM\rho} [\Phi_1^{\mathcal{J}_1}, \Phi_1^{\mathcal{J}_2\,\dagger}] \end{aligned}$$

The hamiltonian is then

$$H_{\mathrm{int}}=\frac{1}{8N}\sum_{J,m}\sum_{J_1,J_2,J_3,J_4}\left(\prod_{i=1}^{4}\frac{1}{\sqrt{\omega_{J_i}}}\right)\mathrm{tr}\Bigg[\left(\frac{(1+J_1+J_2)(1+J_3+J_4)}{J(J+1)}+1\right)\mathcal{C}^{\mathcal{J}_2}_{\mathcal{J}_1,JM}\mathcal{C}^{\mathcal{J}_3}_{\mathcal{J}_4,JM}$$
$$-\sum_{\rho=\pm1}\frac{4}{(J+1)^2-m^2}\sqrt{J_1(J_1+1)}\sqrt{J_4(J_4+1)}\mathcal{D}^{\mathcal{J}_2}_{\mathcal{J}_1,JM\rho}\bar{\mathcal{D}}^{\mathcal{J}_3}_{\mathcal{J}_4,JM\rho}\Bigg][\Phi_{2J_1},\Phi^{\dagger}_{2J_2}][\Phi_{2J_3},\Phi^{\dagger}_{2J_4}]\,.$$

The sum is actually telescopic!

$$\sum_{J=\Delta J} f(J+1)-f(J)$$

This means the sum cancel each other except the boundary term $J=\Delta J=J_2-J_1=J_3-J_4$. Then it simplifies to

$$\begin{aligned}H_{\mathrm{int}}^{(J_1\neq J_2)}&=\frac{1}{4N}\mathrm{tr}\sum_{J_1,J_4\geq 0}\sum_{\Delta J>0}\frac{1}{\Delta J}[\Phi_{2J_1},\Phi^{\dagger}_{2J_1+2\Delta J}][\Phi_{2J_4+2\Delta J},\Phi^{\dagger}_{2J_4}]\\&=\frac{1}{2N}\sum_{l=1}^{\infty}\frac{1}{l}\mathrm{tr}\left(q_l^{\dagger}q_l\right)\end{aligned}$$

where we defined scalar block as

$$q_l=\sum_{n=0}^{\infty}[\Phi^{\dagger}_n,\Phi_{n+l}]$$

General procedure:

- Isolate the propagating modes in a given near-BPS limit from the quadratic classical Hamiltonian
- Derive the form of the current which couple to the gauge field from the $\mathcal{N} = 4$ SYM action – of order λ
- Integrate out additional non-dynamical modes giving rise to effective interactions in a given near-BPS limit
- Derive the interacting Hamiltonian from

$$H_{\text{int}} = \lim_{g \to 0} \frac{H - S_1 - \sum_{i=1}^{3} \omega_i Q_i}{g^2 N}$$

Summary of results

The full Hamiltonian of SU$(1,1)$ bosonic sector is

$$H = L_0 + \frac{\tilde{g}^2}{2N} \sum_{l=0}^{\infty} \frac{1}{l} \mathrm{tr} \left(q_l^{\dagger} q_l \right)$$

For SU$(1,1|1)$ subsector including bosons and fermions,

$$H_{\mathrm{int}} = \frac{1}{2N} \sum_{l=0}^{\infty} \frac{1}{l} \mathrm{tr} \left(\hat{q}_l^{\dagger} \hat{q}_l \right) + \frac{1}{2N} \sum_{l=0}^{\infty} \mathrm{tr} \left(F_l^{\dagger} F_l \right)$$

where

$$\hat{q}_l = q_l + \tilde{q}_l$$

$$\tilde{q}_l = \sum_{n=0}^{\infty} \sqrt{\frac{n+1}{n+l+1}} \{ \psi_n^{\dagger}, \psi_{n+l} \}$$

$$F_l = \sum_{n=0}^{\infty} \frac{[\psi_{n+l}, \Phi_m^{\dagger}]}{\sqrt{n+l+1}}$$

Dynamical gauge fields in SU(1, 2)

[Baiguera, Harmark, Lei, 2020; Baiguera, Harmark, Lei, 2021]

New features:

- As

$$S_1^{(A)} = \sum_{J,m,\tilde{m}} \sum_{\rho=-1,1} \frac{i}{2}(\tilde{m} - m)\text{tr}\left(A_{(\rho)}^{Jm\tilde{m}}\Pi_{(\rho)}^{Jm\tilde{m}} - A_{(\rho)}^{\dagger\, Jm\tilde{m}}\Pi_{(\rho)}^{\dagger\, Jm\tilde{m}}\right)$$

$$S_2^{(A)} = \sum_{J,m,\tilde{m}} \sum_{\rho=-1,1} \frac{i}{2}(m + \tilde{m})\text{tr}\left(A_{(\rho)}^{Jm\tilde{m}}\Pi_{(\rho)}^{Jm\tilde{m}} - A_{(\rho)}^{\dagger\, Jm\tilde{m}}\Pi_{(\rho)}^{\dagger\, Jm\tilde{m}}\right)$$

  The decoupling condition $H_0 = S_1 + S_2$ will only constrain $\tilde{m}$ to be momenta saturated.

- The letters are of the form $d_1^n d_2^k \bar{F}_+$. The (n, k) are related to (J, m) by

$$n = J - m, \quad k = J + m$$

$\mathrm{SU}(1,2)$ subsector

Since the gauge fields are neutral under the $\mathrm{SU}(4)$ R-symmetry, all the limits contain at quadratic order the combination

$$H_0 - S_1 - S_2 = \sum_{J,M} \sum_{\rho=-1,1} \frac{1}{2} \left(|\Pi^{JM}_{(\rho)} - 2i\tilde{m} A^{\dagger\, JM}_{(\rho)}|^2 + (\omega^2_{A,J} - 4\tilde{m}^2)|A^{JM}_{(\rho)}|^2 \right)$$

which must vanish. This implies the constraint for all the non-dynamical modes of the gauge field

$$\Pi^{JM}_{(\rho)} - 2i\tilde{m} A^{\dagger\, JM}_{(\rho)} = 0$$

consitency of the constraints with the time evolution implies that

$$\{H, \Pi^{JM}_{(\rho)} - 2i\tilde{m} A^{\dagger\, JM}_{(\rho)}\} = (\omega^2_{A,J} - 4\tilde{m}^2) A^{\dagger\, Jm\tilde{m}}_{(\rho)} + j^{\dagger\, Jm\tilde{m}}_{(\rho)} = 0\,,$$

Then the interaction Hamiltonian will be

$$H - S_1 - S_2 = \mathrm{tr} \left(\sum_{J,m,\tilde{m}} \frac{1}{8J(J+1)} |j_0^{Jm\tilde{m}}|^2 - \sum_{\rho=\pm1} \sum_{J,m,\tilde{m}} \frac{1}{2(\omega^2_{A,J} - 4\tilde{m}^2)} |j^{Jm\tilde{m}}_{(\rho)}|^2 \right)$$

SU(1, 2) Hamiltonian

Defining $\Delta J \equiv J_1 - J_2 = J_4 - J_3$, we find that when $\Delta J \neq 0$ the only contribution arises from the boundary of summation

$$\tfrac{1}{4N} \sum_{J_i m_i} \delta^{J_1 - J_2}_{J_4 - J_3} \tfrac{1}{|\Delta J|} \sqrt{\tfrac{(1+J_1)(1+J_4)}{(1+J_2)(1+J_3)}} \sqrt{\tfrac{(1+2J_1)(1+2J_4)}{(1+2J_2)(1+2J_3)}} C^{J_2 m_2}_{|\Delta J| \Delta m, J_1 m_1} C^{J_3 m_3}_{|\Delta J| \Delta m, J_4 m_1} \times \mathrm{tr}\Big([A_{J_1 m_1}, A^\dagger_{J_2 m_2}][A_{J_3 m_3}, A^\dagger_{J_4 m_4}] \Big)$$

The block is

$$\mathfrak{q}_{l,\Delta\mu} \equiv \sum_{\mu_1,\mu_2} \sum_{s_2=0}^{\infty} C^{\frac{s_2+l}{2}, \frac{\mu_2+\Delta\mu}{2}}_{\frac{s_2}{2}, \frac{\mu_2}{2}; \frac{l}{2}, \frac{\Delta\mu}{2}} \sqrt{\frac{(s_2+1)(s_2+2)}{(s_2+l+1)(s_2+l+2)}} [A^\dagger_{s_2\mu_2}, A_{s_2+l,\mu_2+\Delta\mu}]$$

The final Hamiltonian is then

$$H_{\text{int}} = \sum_{l=1}^{\infty} \sum_{\Delta\mu=-l}^{l} \frac{1}{l} \mathrm{tr}(\mathfrak{q}^\dagger_{l,\Delta\mu} \mathfrak{q}_{l,\Delta\mu})$$

SU$(1,2|2)$ subsector

The decoupling condition $H_0 - S_1 - S_2 - Q_1 = 0$.

There is an extra scalar Φ and two fermions $\psi_{2,3} = \bar{\chi}_{5,7}$ in the SU$(1,2|2)$ subsector. Evaluate the BPS bound $H - S_1 - S_2 - Q_1 \geq 0$ at quadratic order We need to consider a few terms:

- Terms mediated by the non-dynamical modes of the gauge field via the currents the Hamiltonian. Quartic interactions involving only scalar fields, only gauge fields and the mixed combination of two scalars with two gauge fields.
- Cubic Yukawa terms between one scalar field and two fermions.
- Terms containing dynamical gauge fields mediated by non-dynamical scalars or fermions.

Overall Hamiltonian in SU$(1,2|2)$

Summing all the interactions, we find

$$
\begin{aligned}
H_{\rm int} &= \frac{1}{2N}\sum_{l=1}^{\infty}\sum_{\Delta\mu=-l}^{l}\frac{1}{l}\mathrm{tr}\left(\mathbf{Q}^{\dagger}_{l,\Delta\mu}\mathbf{Q}_{l,\Delta\mu}\right) \\
&+ \frac{1}{2N}\sum_{a=2,3}\sum_{l=0}^{\infty}\sum_{\Delta\mu=-l}^{l}\mathrm{tr}\left((F^{\dagger}_{a}+K^{\dagger}_{a})_{l,\Delta\mu}(F^{a}+K^{a})_{l,\Delta\mu}\right) \\
&+ \frac{1}{2N}\sum_{l=0}^{\infty}\sum_{\Delta\mu=-l}^{l}\mathrm{tr}\left(W^{\dagger}_{l,\Delta\mu}W_{l,\Delta\mu}\right)
\end{aligned}
$$

where

$$
\mathbf{Q}_{l,\Delta\mu} \equiv q_{l,\Delta\mu} + \tilde{q}_{l,\Delta\mu} + \mathfrak{q}_{l,\Delta\mu}
$$

Blocks

$$q_{l,\Delta\mu} \equiv \sum_{s_2=0}^{\infty} \sum_{\mu_2=-s_2}^{s_2} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{s_2}{2},\frac{\mu_2}{2};\frac{l}{2},\frac{\Delta\mu}{2}} [\Phi^{\dagger}_{s_2\mu_2}, \Phi_{s_2+l,\mu_2+\Delta\mu}]$$

$$\tilde{q}_{l,\Delta\mu} \equiv \sum_{a=1,2} \sum_{s_2=0}^{\infty} \sum_{\mu_2=-s_2}^{s_2} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{s_2}{2},\frac{\mu_2}{2};\frac{l}{2},\frac{\Delta\mu}{2}} \sqrt{\frac{s_2+1}{s_2+l+1}} \{(\zeta^{\dagger}_a)_{s_2\mu_2}, (\zeta^a)_{s_2+l,\mu_2+\Delta\mu}\}$$

$$\mathfrak{q}_{l,\Delta\mu} \equiv \sum_{\mu_1,\mu_2} \sum_{s_2=0}^{\infty} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{s_2}{2},\frac{\mu_2}{2};\frac{l}{2},\frac{\Delta\mu}{2}} \sqrt{\frac{(s_2+1)(s_2+2)}{(s_2+l+1)(s_2+l+2)}} [A^{\dagger}_{s_2\mu_2}, A_{s_2+l,\mu_2+\Delta\mu}]$$

$$(F^a)_{l,\Delta\mu} \equiv \sum_{s_2=0}^{\infty} \sum_{\mu_2=-s_2}^{s_2} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{l}{2},\frac{\Delta\mu}{2};\frac{s_2}{2},\frac{\mu_2}{2}} \epsilon^{ab} \frac{[(\zeta_b)_{s_2+l,\mu_2+\Delta\mu}, \Phi^{\dagger}_{s_2,\mu_2}]}{\sqrt{s_2+l+1}}$$

$$(K^a)_{l,\Delta\mu} \equiv \sum_{s_2=0}^{\infty} \sum_{\mu_2=-s_2}^{s_2} \sqrt{\frac{s_2+1}{(s_2+l+1)(s_2+l+2)}} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{l}{2},\frac{\Delta\mu}{2};\frac{s_2}{2},\frac{\mu_2}{2}} [(\zeta^{\dagger}_a)_{s_2,\mu_2}, A_{s_2+l,\mu_2+\Delta\mu}]$$

$$W_{l,\Delta\mu} = \sum_{s_2=0}^{\infty} \sum_{\mu_2=-s_2}^{s_2} \sqrt{\frac{l+1}{(s_2+l+1)(s_2+l+2)}} C^{\frac{s_2+l}{2},\frac{\mu_2+\Delta\mu}{2}}_{\frac{s_2}{2},\frac{\mu_2}{2};\frac{l}{2},\frac{\Delta\mu}{2}} [\Phi^{\dagger}_{s_2\mu_2}, A_{s_2+l,\mu_2+\Delta\mu}]$$

$$
\begin{aligned}
H_{\text{int}} = \sum_{J_i,M_i,\kappa_i,\rho_i} \text{tr}\Big\{ & ig\mathcal{C}^{J_2M_2}_{J_1M_1;JM}\,\chi_{JM}\left([(Z^\dagger_a)_{J_2M_2},(\Pi^{(\Phi)\dagger}_a)_{J_1M_1}] + [Z^a_{J_1M_1},\Pi^{a(\Phi)}_{J_2M_2}]\right) \\
& -4g\sqrt{J_1(J_1+1)}\mathcal{D}^{J_2M_2}_{J_1M_10;JM\rho}\,A^{JM}_{(\rho)}[Z^a_{J_1M_1},(Z^\dagger_a)_{J_2M_2}] \\
& +g\mathcal{F}^{J_1M_1\kappa_1}_{J_2M_2\kappa_2;JM}\,\chi_{JM}\{(\Psi^\dagger_A)_{J_1M_1\kappa_1},\Psi^A_{J_2M_2\kappa_2}\} \\
& +g\mathcal{G}^{J_1M_1\kappa_1}_{J_2M_2\kappa_2;JM\rho}\,A^{JM}_{(\rho)}\{(\Psi^\dagger_A)_{J_1M_1\kappa_1},\Psi^A_{J_2M_2\kappa_2}\} \\
& +\frac{g^2}{2}\mathcal{C}^{J_2M_2}_{J_1M_1;JM}\mathcal{C}^{J_3M_3}_{J_4M_4;JM}[Z^a_{J_1M_1},(Z^\dagger_a)_{J_2M_2}][Z^b_{J_3M_3},(Z^\dagger_b)_{J_4M_4}] \\
& -\sqrt{2}ig(-1)^{-m_1+\tilde{m}_1+\frac{\kappa_1}{2}}\mathcal{F}^{J_1,-M_1,\kappa_1}_{J_2M_2\kappa_2;JM}\psi^4_{J_2M_2\kappa_2}[(Z_a)^{JM},\Psi^a_{J_1M_1\kappa_1}] \\
& +\sqrt{2}ig(-1)^{-m_1+\tilde{m}_1+\frac{\kappa_1}{2}}\mathcal{F}^{J_1,-M_1,\kappa_1}_{J_2M_2\kappa_2;JM}\,\epsilon_{abc}\Psi^a_{J_1M_1\kappa_1}[(Z^\dagger_b)^{JM},\Psi^c_{J_2M_2\kappa_2}] \\
& +\sqrt{2}ig(-1)^{m_2-\tilde{m}_2+\frac{\kappa_2}{2}}\mathcal{F}^{J_1M_1\kappa_1}_{J_2,-M_2,\kappa_2;JM}(\Psi^\dagger_4)_{J_2M_2\kappa_2}[(Z^\dagger_a)^{JM},(\Psi^\dagger_a)_{J_1M_1\kappa_1}] \\
& -\sqrt{2}ig(-1)^{m_2-\tilde{m}_2+\frac{\kappa_2}{2}}\mathcal{F}^{J_1M_1\kappa_1}_{J_2,-M_2,\kappa_2;JM}\,\epsilon_{abc}(\Psi^\dagger_a)_{J_1M_1\kappa_1}[(Z_b)^{JM},(\Psi^\dagger_c)_{J_2M_2\kappa_2}] \\
& +ig\mathcal{D}^{JM}_{J_1M_1\rho_1;J_2M_2\rho_2}\,\chi_{JM}[\Pi^{J_1M_1}_{(\rho_1)},A^{J_2M_2}_{(\rho_2)}] \\
& +g^2\mathcal{C}^{JM}_{J_2M_2;J_4,-M_4}\mathcal{D}_{JM;J_1M_1\rho_1;J_3M_3\rho_3}[A^{J_1M_1}_{(\rho_1)},Z^a_{J_2M_2}][A^{J_3M_3}_{(\rho_3)},(Z^\dagger_a)_{J_4M_4}] \\
& +2ig\rho_1(J_1+1)\mathcal{E}_{J_1M_1\rho_1;J_2M_2\rho_2;J_3M_3\rho_3}A^{J_1M_1}_{(\rho_1)}[A^{J_2M_2}_{(\rho_2)},A^{J_3M_3}_{(\rho_3)}] \\
& -\frac{g^2}{2}\mathcal{D}^{JM}_{J_1M_1\rho_1;J_3M_3\rho_3}\mathcal{D}_{JM;J_2M_2\rho_2;J_4M_4\rho_4}[A^{J_1M_1}_{(\rho_1)},A^{J_2M_2}_{(\rho_2)}][A^{J_3M_3}_{(\rho_3)},A^{J_4M_4}_{(\rho_4)}] \\
& -2g\sqrt{J_1(J_1+1)}\mathcal{D}_{J_2M_2;J_1M_10;JM\rho}\,\chi_{J_1M_1}[\chi_{J_2M_2},A^{JM}_{(\rho)}] \\
& +\frac{g^2}{2}\mathcal{C}^{JM}_{J_1M_1;J_3M_3}\mathcal{D}_{JM;J_2M_2\rho_2;J_4M_4\rho_4}[\chi_{J_1M_1},A^{J_2M_2}_{(\rho_2)}][\chi_{J_3M_3},A^{J_4M_4}_{(\rho_4)}] \\
& +g^2\mathcal{C}^{JM}_{J_1M_1;J_2M_2}\mathcal{C}_{JM;J_3M_3;J_4M_4}[\chi_{J_1M_1},Z^a_{J_2M_2}][\chi_{J_3M_3},(Z^\dagger_a)_{J_4M_4}]\Big\}\,.
\end{aligned}
\tag{B.40}
$$

Representations of SU$(1, 2)$

Like the global symmetry of $\mathcal{N} = 4$ SYM, the algebra can be represented by oscillators

$$[\mathbf{a}_\alpha, \mathbf{a}^\dagger_\beta] = \delta_{\alpha\beta}, \quad [\mathbf{b}_{\dot\alpha}, \mathbf{b}^\dagger_{\dot\beta}] = \delta_{\dot\alpha\dot\beta}, \quad \{\mathbf{c}^\dagger_a, \mathbf{c}_b\} = \delta_{ab}$$

Such that

$$\begin{aligned}
L_0 &= \frac{1}{2}(1 + \mathbf{a}^\dagger_1\mathbf{a}_1 + \mathbf{b}^\dagger_1\mathbf{b}_1), \quad L_1 = \mathbf{a}^\dagger_1\mathbf{b}^\dagger_1, \quad L_{-1} = \mathbf{a}_1\mathbf{b}_1 \\
\tilde{L}_0 &= \frac{1}{2}(1 + \mathbf{a}^\dagger_2\mathbf{a}_2 + \mathbf{b}^\dagger_1\mathbf{b}_1), \quad \tilde{L}_1 = \mathbf{a}^\dagger_2\mathbf{b}^\dagger_1, \quad \tilde{L}_{-1} = \mathbf{a}_2\mathbf{b}_1 \\
J_+ &= \mathbf{a}^\dagger_1\mathbf{a}_2, \qquad J_- = \mathbf{a}^\dagger_2\mathbf{a}_1
\end{aligned}$$

There are a few merits using this

- $L_1, \tilde{L}_1$ are d_1, d_2 operations respectively. Two spatial directions are treated equally.
- If $\tilde{L}_{0,\pm}$ is turned off, we can reacquire the algebra SU$(1, 1)$ of subsectors. They are ghost like
- The descendants are labelled by (n, k) symmetrically.

Classification of (p,q) representations

Like SU$(1,1)$ shown in the notes, we need to understand how the generators act on a given state and preserve the unitarity. Recall SU(N) has $N-1$ independent Casimir operators. Thus we need

$$C_2 = -1 - x_1x_2 - x_2x_3 - x_3x_1 = p + q + \frac{1}{3}(p^2 + pq + q^2)$$

$$C_3 = x_1x_2x_3 = \frac{1}{27}(p-q)(p+2q+3)(q+2p+3)$$

Remind the Casimir of SU$(1,1)$ is

$$C = -j(j-1)$$

- Principle representation: p, q can be complex (analogous to continuous)
- p-series: p is integer while q is not
- q-series: q is integer while p is not
- $(p+q)$-series: Neither of p, q is integer but $p+q$ is
- Integer series: $p, q \in \mathbb{Z}$
- Supplementary series

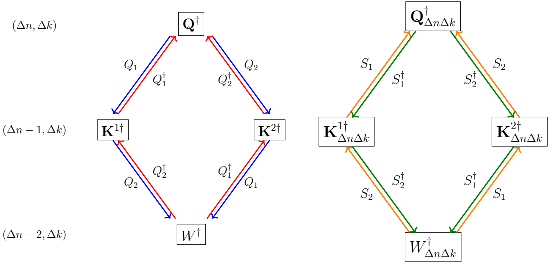
Representations

- **Gauge field**. The gauge field $\bar{F}_+$ is parametrized by $(p,q)=(0,0)$ representation.
- **Fermion**: The fermions $\bar{\chi}_{5,7}$ are parametrized by $(p,q)=(0,-1)$ representation.
- **Scalar**: The scalar Z is parametrized by $(p,q)=(0,-2)$ representation.

Symmetry actions

What about blocks?

- The block $\mathbf{Q}^\dagger_{nk}$ is parametrized by $(p,q)=(0,-3)$ representation. This is a representation of fermion!
- The fermionic block $\mathbf{K}^{a\dagger}_{nk}=(K^{a\dagger}_{nk}+F^{a\dagger}_{nk})$ are parametrized by $(p,q)=(0,-2)$ representation.
- The scalar block $W^\dagger_{nk}$ is parametrized by $(p,q)=(0,-1)$ representation.



The blocks and letters in SU$(1,2|2)$ are forming $\mathcal{N}=2$ vector multiplets.

$\mathrm{SU}(2|3)$ subsector

The decoupling condition: $H_0 = Q_1 + Q_2 + Q_3$

There are three scalars $\Phi_{1,2,3}$ and two chiral fermions $\chi_{1,2}$ in this sector. The full SMT Hamiltonian by spherical reduction is

$$\begin{aligned}H_{\mathrm{int}} =& \frac{1}{4N}\mathrm{tr}\left([\Phi_b^\dagger, \Phi_a^\dagger][\Phi_a, \Phi_b]\right) + \frac{1}{4N}\mathrm{tr}\left(\{\chi_\beta^\dagger, \chi_\alpha^\dagger\}\{\chi_\alpha, \chi_\beta\}\right)\\ &+ \frac{1}{2N}\mathrm{tr}\left([\Phi_a^\dagger, \chi_\beta^\dagger][\chi_\beta, \Phi_a]\right)\end{aligned}$$

D/F term

D-term means

$$[W^\dagger, W][W^\dagger, W]$$

while F-term means

$$[W, W][W^\dagger, W^\dagger]$$

The Hamiltonian in this subsector are made by F-terms.

The nature of SU$(2|3)$ Hamiltonian

Let's focus on scalar part.

- $\Phi^\dagger_{1,2,3}$ transform in fundamental representation $\mathbf{3} = (1, 0)$ of SU(3).
- Bilinear blocks $[\Phi^\dagger_a, \Phi^\dagger_b]$ transforms in $\bar{\mathbf{3}}$ as $\epsilon_{abc}\Phi_c$, because

$$\mathbf{3} \times \mathbf{3} = \mathbf{6} + \bar{\mathbf{3}}$$

  The F-term interaction is just the singlet in

$$\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{8} + \mathbf{1}$$

- We can use Jacobi identity to write tr $\left([\Phi^\dagger_b, \Phi^\dagger_a][\Phi_a, \Phi_b]\right)$ as D-term. The basic eight D-term blocks

$$[\Phi^\dagger_a, \Phi_b], \quad [\Phi^\dagger_a, \Phi_a] - [\Phi^\dagger_b, \Phi_b]$$

  transform in $(1, 1) \equiv \mathbf{8}$ which is the adjoint representation of SU(3). The Hamiltonian in D-term is just the *quadratic Casimir* of SU(3).

PSU(1, 1|2) subsector

The quantum version of the Hamiltonian was obtained in [Bellucci, Casteill, 06]. We have decoupling limit $H_0 = S_1 + Q_1 + Q_2$.

The letters are two scalars $\Phi_{1,2}$, a chiral fermion $\chi_1 = \psi_1$ and antichiral fermion $\bar{\chi}_7 = \psi_2$, including their descendants generated by d_1.

$$
\begin{aligned}
H_{\text{int}} = {} & H_B + \frac{1}{N}\sum_{l=1}^{\infty}\frac{1}{l} : \mathrm{tr}\left(\mathbf{Q}_l^{\dagger}\,\mathbf{Q}_l\right) : + \frac{1}{N}\sum_{l=0}^{\infty} : \mathrm{tr}\Big((F_{ab})_l^{\dagger}(F_{ab})_l\Big) : \\
& - \frac{1}{N}\sum_{l=0}^{\infty}\sum_{m,n=0}^{\infty}\frac{1}{m+n+l+1} : \mathrm{tr}\left(\epsilon^{ac}\epsilon^{bd}[(\Phi_a^{\dagger})_m,(\Phi_b)_{m+l}][(\Phi_c^{\dagger})_{n+l},(\Phi_d)_n]\right) : \\
& + \frac{1}{N}\sum_{l=0}^{\infty}\sum_{m,n=0}^{\infty}\frac{\sqrt{(m+1)(n+1)}}{\sqrt{(m+l+1)(n+l+1)}}\frac{: \mathrm{tr}\left(\epsilon^{ac}\epsilon^{bd}\{(\psi_a^{\dagger})_m,(\psi_b)_{m+l}\}\{(\psi_c^{\dagger})_{n+l},(\psi_d)_n\}\right) :}{m+n+l+2} \\
& + \frac{1}{N}\sum_{l=0}^{\infty}\sum_{m,n=0}^{\infty}\sqrt{\frac{m+1}{n+l+1}}\frac{\epsilon^{ac}\epsilon^{bd}}{m+n+l+2} : \mathrm{tr}\left([(\psi_a^{\dagger})_m,(\Phi_b)_{m+l+1}][(\psi_c^{\dagger})_{n+l},(\Phi_d)_n]\right) : \\
& - \frac{1}{N}\sum_{l=0}^{\infty}\sum_{m,n=0}^{\infty}\sqrt{\frac{m+1}{n+l+1}}\frac{\epsilon^{ac}\epsilon^{bd}}{m+n+l+2} : \mathrm{tr}\left([(\Phi_a^{\dagger})_{m+l+1},(\psi_b)_m][(\Phi_c^{\dagger})_n,(\psi_d)_{n+l}]\right) : ,
\end{aligned}
$$

Positiveness

Although the Hamiltonian is tedious, we can show

$$\hat{\mathcal{Q}}^\dagger=\sum_{m,n=0}^{\infty}\Big[\tfrac{1}{\sqrt{n+1}}\mathrm{tr}\Big([(\Phi_a^\dagger)_{m+n+1},(\Phi_a)_m](\psi_2)_n\Big)+\sqrt{\tfrac{m+1}{(n+1)(m+n+2)}}\mathrm{tr}\Big(\{(\psi_1^\dagger)_{m+n+1},(\psi_1)_m\}(\psi_2)_n\Big)$$
$$+\tfrac{1}{2}\sqrt{\tfrac{m+n+2}{(m+1)(n+1)}}\mathrm{tr}\Big(\{(\psi_2^\dagger)_{m+n+1},(\psi_2)_m\}(\psi_2)_n\Big)-\tfrac{1}{2\sqrt{m+n+1}}\,\epsilon^{ab}\mathrm{tr}\Big((\psi_1^\dagger)_{m+n}[(\Phi_a)_m,(\Phi_b)_n]\Big)\Big]$$

We can then show

$$\{\hat{\mathcal{Q}},\hat{\mathcal{Q}}^\dagger\}=H_{\mathrm{int}}$$

The cubic supercharges are from the extra PSU$(1|1)^2$ symmetry of this subsector [Beisert, Zwiebel, 2007]

Representation

How can the positiveness be manifest as square of blocks?

F-term problem

We define

$$J_L = \sum_{n=0}^{L} [\Phi^1_{L-n}, \Phi^2_n]$$

and $L = m + n + l$ We can show

$$-\frac{1}{2N}\sum_{l=0}^{\infty}\sum_{m,n=0}^{\infty}\frac{1}{m+n+l+1}\text{tr}\left(\epsilon^{ac}\epsilon^{bd}[(\Phi^\dagger_a)_m,(\Phi_b)_{m+l}][(\Phi^\dagger_c)_{n+l},(\Phi_d)_n]\right)$$

$$= \sum_{L=0}^{\infty}\frac{1}{L+1}\text{tr}(J^\dagger_L J_L)$$

To derive this we need to use the Jacobi identity

$$\text{tr}([\Phi_a,\Phi_b][\Phi^\dagger_b,\Phi^\dagger_a]) = \text{tr}([\Phi_b,\Phi^\dagger_a][\Phi_a,\Phi^\dagger_b]) - \text{tr}([\Phi_a,\Phi^\dagger_a][\Phi_b,\Phi^\dagger_b])$$

Then PSU$(1,1|2)$ symmetry generator action:

$$(L_+)_D J^\dagger_L = (L+1)J^\dagger_{L+1}\,, \quad (L_+)_D J_L = -(L+1)J_{L-1}$$

F-term problem

The fermionic term contains non-trivial coefficients. Therefore, Jacobi identity

$$\mathrm{tr}\{\psi_1, \psi_2\}\{\psi_3, \psi_4\} = -\mathrm{tr}\{\psi_1, \psi_3\}\{\psi_2, \psi_4\} - \mathrm{tr}\{\psi_1, \psi_4\}\{\psi_2, \psi_3\}$$

or

$$f_{ABE} f_{CDE} - f_{ACE} f_{BDE} + f_{ADE} f_{BCE} = 0$$

is not enough. We would also need general d-type coupling

$$f_{ABE} f_{CDE} = \frac{2}{N}(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}) + d_{ACE} d_{BDE} - d_{BCE} d_{ADE}$$

which are from

$$T^A T^B = \frac{1}{2N}\delta_{AB}\mathbb{1} + \frac{1}{2}(d_{ABC} + i f_{ABC}) T^C$$

as $\{\psi_1, \psi_2\} = i\psi_1^A \psi_2^B f_{ABC} T_C$.

In SU(3),

$$f_{ABE}f_{CDE} - f_{ADE}f_{BCE} = 3d_{ACE}d_{BDE} - \delta_{AB}\delta_{CD} - \delta_{AD}\delta_{BC} + \delta_{AC}\delta_{BD}$$

we can then

$$\begin{aligned}
&\mathrm{tr}\{\psi_1, \psi_3\}\{\psi_2, \psi_4\} - \mathrm{tr}\{\psi_1, \psi_4\}\{\psi_2, \psi_3\} \\
=&\frac{1}{2}\Big(f_{ABE}f_{CDE} + f_{ADE}f_{CBE}\Big)\psi_1^A\psi_2^C\psi_3^B\psi_4^D \\
=&\frac{1}{2}\Big(3d_{ACE}d_{BDE} - \delta_{AB}\delta_{CD} - \delta_{AD}\delta_{BC} + \delta_{AC}\delta_{BD}\Big)\psi_1^A\psi_2^C\psi_3^B\psi_4^D
\end{aligned}$$

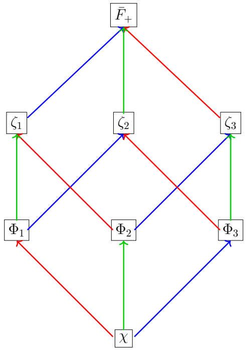
These are neither commutator nor the anticommutator of fermionic letters. Simply means the blocks are not transforming in the adjoint representations of color group SU(N).

Summary: New physics in each subsector

- SU$(2|3)$: $\chi_{1,2}, Z, W, X$, the fermionic doublet block $\{\chi_1, \chi_2\}$ and scalar block, pure F-term
- SU$(1,1|1)$, telescopic sum, pure D-term tr$(q_l^\dagger q_l)$, infinite modes
- PSU$(1,1|2) \times SU(2)_F$: $\chi_1, \bar{\chi}_7, Z, X, d_1$, infinite modes$+$ F-term, fermionic doublet block? SU$(2)_F$ automorphism; Not clear block form. Nontrivial interaction between color group and spin group
- SU$(1,2|2)$: $F, \Phi, \bar{\chi}_{5,7}, d_{1,2}$, new $[A, \Phi]$ blocks
- PSU$(1,2|3)$: $F, Z, W, X, \chi_{1,2}, \bar{\chi}_{3,5,7}, d_{1,2}$,
  1. Dirac equation
  2. Fermion triplet v.s. emergent doublet
  3. Block form possible?
  4. Summation technical problem

$\mathcal{N} = 3$ vector multiplet?

But we know the Lagrangian of $\mathcal{N} = 3$ are all respecting $\mathcal{N} = 4$ SUSY.



Some future work

- Finish PSU$(1,2|3)$ subsector [Baiguera, Harmark, Lei, 2022]
- Local field theory in SU$(1,2)$ subsector [Baiguera, Harmark, Lei, 2022] and relation to chiral algebra?
- Relation to the work SU$(1,D)$ field theory: [Lambert, Mouland, Orchard, 2022]
- $\frac{1}{16}$−BPS black hole interpretation
- Understanding how Kerr/CFT appears holographically based on AdS$_5$/CFT$_4$ (Based on [Goldstein, Jejjala, Lei, Leuven, Li, 2019]).

$$\text{PSU}(1,2|3) \to \text{PSU}(1,1|2)$$

- Factorization of partition function with finite N?
- Relation to strings in TNC gravity [Harmark, Hartong, Obers, Yan, 2018, 2021]