Graviton Scattering in AdS at Two Loops

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Backgrounds

Scattering particles in AdS

- Help understand the gauge/gravity duality.
- Probe perturbative dynamics in curved spacetime.
- Most interesting to scatter gravitons.
- Described by boundary (conformal) correlators.

Direct computation is possible (Witten diagrams)
but HARD.

Viewing from the boundary

A new approach from the boundary side:
bootstrap the conformal correlator

(cf e.g. [Bissi, Sinha, Zhou, '22] and reference therein)

rough idea:

- Boundary theories with a weakly-coupled gravity dual is generally thought to have a large parameter (N).
- Expansion in N corresponds to loop expansion in the bulk.
- Large N expansion to the crossing equation induces recursive relations among CFT data at different orders.
- This resembles the unitarity methods where $2\Im T = T^\dagger T$.

The theory we discuss today

$\mathcal{N} = 4$ super Yang–Mill theory in 4d with $\mathrm{SU}(N)$

supergravity limit:

$$N \longrightarrow \infty, \qquad \lambda \longrightarrow \infty$$

type II sugra in $\mathrm{AdS}_5 \times \mathrm{S}^5$
graviton multiplet + Kaluza–Klein modes

Boundary description

- Chiral primary operators (CPO)

$$\mathcal{O}_p = y_{a_1} y_{a_2} \cdots y_{a_3} \mathrm{tr}(\phi^{a_1} \phi^{a_2} \cdots \phi^{a_p}) + \cdots ,$$

  y for R-symmetries,
  $\mathcal{O}_2 \Leftrightarrow$ graviton, $\mathcal{O}_{p>2} \Leftrightarrow$ Kaluza–Klein.

- CPO 4-point correlator

$$\langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle = (\text{some factor})\, \mathcal{G}_{p_1 p_2 p_3 p_4}(z, \bar{z}, \alpha, \bar{\alpha}),$$

  where the four variables are cross-ratios

$$\frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = u = z\bar{z}, \qquad \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = v = (1-z)(1-\bar{z}),$$

  (similar expressions for $\{\alpha, \bar{\alpha}\}$ related to y).

Large N expansion

Expand using the central charge $c \equiv \frac{N^2-1}{4}$

$$\mathcal{G}_{\{p\}} = \mathcal{G}^{(0)}_{\{p\}} + \frac{1}{c}\mathcal{G}^{(1)}_{\{p\}} + \frac{1}{c^2}\mathcal{G}^{(2)}_{\{p\}} + \frac{1}{c^3}\mathcal{G}^{(3)}_{\{p\}} + \cdots .$$

(For now ignore the $\lambda^{-1/2}$ corrections.)

Large N expansion

- $\mathcal{G}^{(0)}$: disconnected diagrams (no interaction);
  computable using mean field theory.
- $\mathcal{G}^{(1)}$: tree diagrams;
  arbitrary $\mathcal{G}^{(1)}_{p_1 p_2 p_3 p_4}$ known by now [Rastelli, Zhou, '14].
- $\mathcal{G}^{(2)}$: one loop;
  partial results are available recently ($\mathcal{G}^{(2)}_{22pp}, \mathcal{G}^{(2)}_{3333}, \ldots$), using a method to be discussed. [Alday, Caron-Huot, '17], [Aprile et al, '17], etc.
- $\mathcal{G}^{(3)}$: two loops;
  almost zero results before this work.

We target at $\mathcal{G}^{(3)}_{2222}$.

Spectrum

- (Super)conformal block expansion

$$\mathcal{G}_{2222} = \text{((protected) short)} + \underbrace{\sum_{t,l,m,n} \mathcal{A}_{t,l,m,n}}_{\text{(unprotected) long}} \,.$$

- Better organization (due to susy)

$$\mathcal{G}_{2222} = \mathcal{G}_{\text{free}} + \mathcal{I}(z,\bar{z},\alpha,\bar{\alpha})\mathcal{H}(z,\bar{z})$$

$\mathcal{H}$ further decomposes into ordinary blocks

$$\mathcal{H} = \sum_{\tau,\ell} a_{\tau,\ell}\, g_{\tau+4,\ell}(z,\bar{z})$$

Spectrum

- Data not protected (i labels different long operators):

$$\tau_i = \tau_i^{(0)} + c^{-1}\gamma_i^{(1)} + c^{-2}\gamma_i^{(2)} + c^{-3}\gamma_i^{(3)} + \cdots ,$$
$$a_i = a_i^{(0)} + c^{-1}a_i^{(1)} + c^{-2}a_i^{(2)} + c^{-3}a_i^{(3)} + \cdots$$

  - $a_i^{(0)} \neq 0 \Rightarrow$ double-trace operators formed by CPOs, of the form $[\mathcal{O}\mathcal{O}]_{n,\ell} \equiv \mathcal{O}\Box^n\partial^\ell\mathcal{O}$.
  - $a_i^{(0)}, \tau_i^{(0)}$ are determined from mean field theory. In particular

$$\tau_i^{(0)} = 4 + 2n, \quad \text{for some } n,$$

- Plug this expansion into the block expansion of $\mathcal{H}$

$$\mathcal{H} = \sum_i \left(a_i^{(0)} + c^{-1}a_i^{(1)} + \cdots \right) g_{\tau_i^{(0)} + c^{-1}\gamma_i^{(1)} + \cdots , \ell}.$$

Structure of coefficients

- Note $g_{\tau,\ell}(z,\bar{z}) = (z\bar{z})^{\tau/2}(\cdots)$. The expansion in $1/c$ gives rise to powers of $\log(u)$ (s-channel).
- Max power is p at order c^{-p}, with the form

$$\mathcal{H}^{(p)} \subset \frac{1}{p!\,2^p} \log(u)^p \sum_{\tau^{(0)},\ell} \langle a^{(0)}(\gamma^{(1)})^p \rangle_{\tau^{(0)},\ell}\, g_{\tau^{(0)}+4,\ell}.$$

  $\langle \cdots \rangle$ due to operator degeneracy.
- The leading log coefficients above are completely determined by data of double-trace operators

$$\langle a^{(0)}(\gamma^{(1)})^p \rangle \sim \frac{\langle a^{(0)}\gamma^{(1)}\rangle^p}{\langle a^{(0)}\rangle^{p-1}}.$$

Structure of coefficients

- Coefficients in other terms take the generic form, e.g.,

	c^{-2}	c^{-3}
$\log(u)^0$	$\langle a^{(2)} \rangle$	$\langle a^{(3)} \rangle$
$\log(u)^1$	$\langle a^{(1)}\gamma^{(1)} + a^{(0)}\gamma^{(2)} \rangle$	$\langle a^{(2)}\gamma^{(1)} + a^{(1)}\gamma^{(2)} + a^{(0)}\gamma^{(3)} \rangle$
$\log(u)^2$	$\langle a^{(0)}(\gamma^{(1)})^2 \rangle$	$\langle a^{(1)}(\gamma^{(1)})^2 + 2a^{(0)}\gamma^{(1)}\gamma^{(2)} \rangle$
$\log(u)^3$		$\langle a^{(0)}(\gamma^{(1)})^3 \rangle$

- At each order c^{-p}
  - The new data $a^{(p)}$ and $\gamma^{(p)}$ only show up in the $\log(u)^0$ and $\log(u)^1$ coefficients.
  - $\log(u)^{p \geq 2}$ terms are in principle recursively determined by data at lower orders.

Now the question boils down to

How to work out $a^{(p)}$ and $\gamma^{(p)}$ at each c^{-p} order?

Luckily, $\log(u)^0$ and $\log(u)^1$ terms do NOT contribute to Lorentzian singularities of the correlator.

$$\mathrm{dDisc}\,\mathcal{H}(z,\bar{z}) = \mathcal{H}(z,\bar{z}) - \frac{1}{2}\mathcal{H}^{\circlearrowleft}(z,\bar{z}) - \frac{1}{2}\mathcal{H}^{\circlearrowright}(z,\bar{z}),$$

Some dispersive-like relations can be utilized.

Lorentzian inversion

- [Caron-Huot, '17]

$$c_{\tau,\ell} = \frac{1+(-1)^l}{4}\kappa_{\tau+2l+4} \times \int_0^1 \frac{\mathrm{d}z}{z^2}\frac{\mathrm{d}\bar{z}}{\bar{z}^2}\left(\frac{\bar{z}-z}{\bar{z}z}\right)^2 g_{\tau+l+1,2-\tau}(z,\bar{z})\,\mathrm{dDisc}(\mathcal{H}).$$

- This integral encodes the CFT data by

$$c_{\tau,\ell} = \sum_i \frac{a_i}{\tau-\tau_i}.$$

- Again, plug in the c^{-1} of the data.

Lorentzian inversion

$$
\begin{aligned}
c_{\tau,\ell} \supset \frac{1}{c^2} \sum_{\tau^{(0)}} &\left(\frac{\langle a^{(2)} \rangle}{\tau - \tau^{(0)}} + \frac{\langle a^{(1)}\gamma^{(1)} + a^{(0)}\gamma^{(2)} \rangle}{(\tau - \tau^{(0)})^2} + \frac{\langle a^{(0)}(\gamma^{(1)})^2 \rangle}{(\tau - \tau^{(0)})^3} \right) \\
&+ \frac{1}{c^3} \sum_{\tau^{(0)}} (\frac{\langle a^{(3)} \rangle}{\tau - \tau^{(0)}} + \frac{\langle a^{(2)}\gamma^{(1)} + a^{(1)}\gamma^{(2)} + a^{(0)}\gamma^{(3)} \rangle}{(\tau - \tau^{(0)})^2} \\
&\qquad + \frac{\langle a^{(1)}(\gamma^{(1)})^2 + 2a^{(0)}\gamma^{(1)}\gamma^{(2)} \rangle}{(\tau - \tau^{(0)})^3} + \frac{\langle a^{(0)}(\gamma^{(1)})^3 \rangle}{(\tau - \tau^{(0)})^4}).
\end{aligned}
$$

- Coefficients here are identical to those in the block expansion.
- Workflow at one loop [Alday, Caron-Huot, '17]:
  1. Compute $\langle a^{(0)}(\gamma^{(1)})^2 \rangle$ from lower-loop data $\langle a^{(0)} \rangle$ and $\langle a^{(0)}\gamma^{(1)} \rangle$ (need to solve operator degeneracy).
  2. Compute $c_{\tau,\ell}$ at order c^{-2} using Lorentzian inversion.
  3. Extract coefficients of the simple and double poles.

Tentative Computation

At two loops

1. Compute $\langle a^{(1)}(\gamma^{(1)})^2 + 2a^{(0)}\gamma^{(1)}\gamma^{(2)}\rangle$ and $\langle a^{(0)}(\gamma^{(1)})^3\rangle$ from lower-loop data.
   (If we accept the spectrum here still consists of double-trace operator only.)
2. Compute $c_{\tau,\ell}$ at order c^{-3} using Lorentzian inversion.
3. Extract coefficients of the simple and double poles.

Problem encountered...

Unfortunately, the $c_{\tau,\ell}$ computed in this way do not have fourth-order pole!

The problem lies in the entrance of triple-trace operators $[\mathcal{O}\mathcal{O}\mathcal{O}]$.

More Ingredients

Hidden symmetries

- Tree-level results suggests a hidden 10d conformal symmetry. [Caron-Huot, Trinh, '18]
- This dictates the leading log terms to be identical to

$$\mathcal{H}^{(p)}\Big|_{\log(u)^p} = \left[\Delta^{(8)}\right]^{p-1} \mathcal{F}^{(p)}(z,\bar{z}).$$

Here ($D_z = z^2\partial_z(1-z)\partial_z$)

$$\Delta^{(8)} = \frac{z\bar{z}}{\bar{z}-z} D_z(D_z-2)D_{\bar{z}}(D_{\bar{z}}-2)\frac{\bar{z}-z}{z\bar{z}},$$

$$\mathcal{F}^{(p)}(z,\bar{z}) = \sum_{|\vec{a}|=0}^{p} \frac{\rho_{\vec{a}}(z,\bar{z})}{(\bar{z}-z)^7} \underbrace{G(\vec{a};z)}_{\text{MPL}} + (z \leftrightarrow \bar{z}).$$

The components of vector $\vec{a}$ take values in $\{0,1\}$. $\rho_{\vec{a}}$ is a polynomial of weight 7 in each variable.

Correlator at one loop

- The leading log

$$\mathcal{H}^{(2)}\big|_{\log(u)^2} = \Delta^{(8)}\mathcal{F}^{(2)}(z,\bar{z}).$$

- This structure extends to the entire correlator at one loop, with a slight modification [Aprile et al, '19]

$$\mathcal{H}^{(2)} = \Delta^{(8)}\mathcal{L}^{(2)} + \frac{1}{4}\mathcal{H}^{(1)},$$

$$\mathcal{L}^{(2)} = \sum_{|\vec{a}|+|\vec{a}'|=0}^{4} \frac{\rho_{\vec{a},\vec{a}'}(z,\bar{z})}{(\bar{z}-z)^7} G(\vec{a};z)G(\vec{a}';\bar{z}).$$

  - $\rho_{\vec{a},\vec{a}'}$ is again some polynomial of degree 7 in each variable.
  - The total weight of each term including that of the coefficient does not exceed 4.

Ansatz

Main ansatz

$$\mathcal{H}^{(3)} = \left[\Delta^{(8)}\right]^2 \mathcal{L}^{(3)} + a_2 \mathcal{H}^{(2)} + a_1 \mathcal{H}^{(1)}.$$

Functions that can appear

- Up to one loop the correlator as a function belongs to multiple polylogarithms, with maximal weight $2p$.
- The correlator is single-valued on the Euclidean slice $\bar{z} = z^*$.
- Crossing symmetries of $\mathcal{G}_{2222}$ forms an S_3 group acting on the correlator. It dictates that

$$\frac{(z-\bar{z})^4}{(z\bar{z})^4}\mathcal{H}$$

  has to be fully crossing invariant.
- By definition $\mathcal{H}_{2222}$ is invariant under exchanging $z \leftrightarrow \bar{z}$, which forms a $\mathbb{Z}_2$ group.
- These together makes an $\mathrm{S}_3 \times \mathbb{Z}_2$ group, whose representations fall into six types $\{\mathbf{1}^\pm, \bar{\mathbf{1}}^\pm, \mathbf{2}^\pm\}$.
- The above combination transforms in $\mathbf{1}^+$ (singlet).

Basis

- We want a complete set of independent single-valued MPLs up to weight 6 that transforms according to one of the above representations.
- This is possible when one restricts to a finite set of symbol alphabets.
  - A symbol of an MPL is (roughly) an algebraic structure capturing info of its singularities, e.g.

$$
\begin{aligned}
\mathcal{S}\log(x) &= x, \\
\mathcal{S}\log(x)\log(y) &= x \otimes y + y \otimes x, \\
\mathcal{S}\mathrm{Li}_2(x) &= -(1-x) \otimes x.
\end{aligned}
$$

  - An alphabet is the collection of all symbol entries.
  - Up to one loop, the alphabet is restricted to

$$
\{z, \bar{z}, 1-z, 1-\bar{z}\}.
$$

Basis

- Three loops calls for an extra $z - \bar{z}$ in the alphabets.
- We want a complete set of independent single-valued MPLs up to weight 6 transforming in one of $\{\mathbf{1}^{\pm}, \bar{\mathbf{1}}^{\pm}, \mathbf{2}^{\pm}\}$, with symbol alphabets $\{z, \bar{z}, 1 - z, 1 - \bar{z}, z - \bar{z}\}$.
- Denote these functions as

$$G^{\mathrm{SV}}_{w,r,i}(z, \bar{z}), \tag{1}$$

  w: weight, r: representation, i: extra degeneracy.
- There is a well-defined algorithm for recursively working out these functions (choice is not unique when degeneracy occurs). [Chavez, Duhr, '12]

Basis

- Counting of functions without $z - \bar{z}$

$w \backslash r$	$\mathbf{1}^+$	$\mathbf{1}^-$	$\bar{\mathbf{1}}^+$	$\bar{\mathbf{1}}^-$	$\mathbf{2}^+$	$\mathbf{2}^-$
0	1	0	0	0	0	0
1	0	0	0	0	1	0
2	2	1	0	0	1	0
3	2	0	1	0	3	1
4	5	3	1	0	5	2
5	7	3	4	2	11	5
6	15	10	6	3	20	12

- Functions with $z - \bar{z}$ starts to appear at weight 3, where there is a unique one. It turns out functions at higher weights are not needed.

Ansatz

$$\mathcal{H}^{(3)} = \left[\Delta^{(8)}\right]^2 \mathcal{L}^{(3)} + a_2 \mathcal{H}^{(2)} + a_1 \mathcal{H}^{(1)},$$

$$\mathcal{L}^{(3)} = \sum_{w=0}^{6} \sum_{r,i} \frac{\omega_i^{w,r}(z,\bar{z})}{(\bar{z}-z)^7} G^{\mathrm{SV}}_{w,r,i}(z,\bar{z}),$$

$$\omega_i^{w,r} = \sum_{j,k=0}^{7} c_{i,j,k}^{w,r} z^j \bar{z}^k, \quad c_{i,j,k}^{w,r\pm} = \mp c_{i,k,j}^{w,r\pm}.$$

Bootstrap

Constraints on $\mathcal{L}^{(3)}$: general

1. In Euclidean region $\mathcal{L}^{(3)}$ should be finite at $z = \bar{z}$.
2. As a sum of s-channel blocks with identical external operators, exchanging operator 1 and 2 should leave $\mathcal{L}^{(3)}$ unchanged

$$\mathcal{L}^{(3)}(z, \bar{z}) = \mathcal{L}^{(3)}\left(\frac{\bar{z}}{\bar{z}-1}, \frac{z}{z-1}\right).$$

Constraints on $\mathcal{L}^{(3)}$: SYM

A When expanding $\mathcal{L}^{(3)}$ in the s-channel, all the $\log(u)^p$ terms with $p > 3$ have to vanish.

B The $\log(u)^3$ terms of $\mathcal{L}^{(3)}$ should match known data

$$\mathcal{L}^{(3)}(z,\bar{z})\big|_{\log(u)^3} = \mathcal{F}^{(3)}(z,\bar{z}),$$

This is because the leading log terms are determined solely by double-trace operators, and the recursive data $\langle a^{(0)}(\gamma^{(1)})^3\rangle$ can be trusted.

Constraints on $\mathcal{H}^{(3)}$: general

3 $\mathcal{H}^{(3)}$ should respect the full crossing symmetries. With the help of the $\mathrm{S}_3 \times \mathbb{Z}_2$ SVMPL basis, this is equivalent to requiring that in

$$\frac{(z-\bar{z})^4}{z^4\bar{z}^4}\mathcal{H}^{(3)} \equiv \sum_{w,r,i} \Omega_i^{w,r}(z,\bar{z})\, G^{\mathrm{SV}}_{w,r,i}(z,\bar{z}),$$

the rational coefficient functions $\Omega_i^{w,r}(z,\bar{z})$ transform in a way such that each term on RHS is an S_3 invariant.

Constraints on $\mathcal{H}^{(3)}$: SYM

C Tree-level $\mathcal{H}^{(1)}$ has poles at $z = 1$ and $\bar{z} = 1$, which are not expected to be present in $\mathcal{H}^{(3)}$. So there should be cancellation between $\mathcal{H}^{(1)}$ and $\left[\Delta^{(8)}\right]^2 \mathcal{L}^{(3)}$. This fixes

$$a_1 = -\frac{1}{16}.$$

D Recursive data for the subleading log $\langle a^{(1)}(\gamma^{(1)})^2 + 2a^{(0)}\gamma^{(1)}\gamma^{(2)}\rangle$ can be trusted at twist 4, which consists of a unique double-trace operator

$$\langle a^{(1)}(\gamma^{(1)})^2 + 2a^{(0)}\gamma^{(1)}\gamma^{(2)}\rangle_{4,\ell} = \frac{2\langle a^{(0)}\gamma^{(1)}\rangle_{4,\ell}\langle a^{(1)}\gamma^{(1)} + a^{(0)}\gamma^{(2)}\rangle_{4,\ell}}{\langle a^{(0)}\rangle_{4,\ell}} - \frac{\langle a^{(0)}\gamma^{(1)}\rangle_{4,\ell}^2\langle a^{(1)}\rangle_{4,\ell}}{\langle a^{(0)}\rangle_{4,\ell}^2}.$$

This fits

$$a_2 = \frac{5}{4}.$$

Constraints on $\mathcal{H}^{(3)}$: SYM

E Bulk-point limit should reduce to the flat-space four-graviton scattering amplitude in 10d. Specifically here, the leading divergence of $\mathrm{dDisc}\,\mathcal{H}(z^{\circlearrowleft},\bar{z})$ at the bulk point limit should match the discontinuity of the scattering amplitude $\mathcal{A}$ across the t-channel cut.

$$
\begin{aligned}
&(\bar{z}-z)^{23}\mathrm{dDisc}\,\mathcal{H}^{(3)}(z^{\circlearrowleft},\bar{z})\big|_{z\to\bar{z}}\\
=&\frac{\Gamma(22)(1-\bar{z})^{11}\bar{z}^{24}}{8\pi^8}\mathrm{Disc}_{x>1}\mathcal{A}^{(2)}(x)\big|_{x=1/\bar{z}},
\end{aligned}
\tag{2}
$$

The two-loop supergravity amplitude $\mathcal{A}^{(2)}(x)$ has been computed in [Bissi et al, '20].

Constraints on $\mathcal{H}^{(3)}$: SYM

F By construction the ansatz does not necessarily produces OPE data with sufficient analyticity in spin. At two loops analyticity should hold down to $\ell = 6$. For this purpose we require that the ansatz be self-consistent under Lorentzian inversion.

- Insert the ansatz into the inversion integral to generate tentative CFT data at two loops.
- Feed these data into conformal block expansions in the t-channel and re-summed over all spins and over twists up to a cutoff τ_{cutoff}.
- Compare the resulting expression with the ansatz itself in the expansion near $\bar{z} = 1$ (which is valid up to certain order related to τ_{cutoff}).

At $\log(v)^2$ additional freedom exits in the $\ell = 0$ coefficients. At $\log(v)^1$ and $\log(v)^0$ in $\ell = 2, 4$ as well.

Result

- After all the above computations

$$\mathcal{H}^{(3)} = \left[\Delta^{(8)}\right]^2 \mathcal{L}^{(3)} + \frac{5}{4}\mathcal{H}^{(2)} - \frac{1}{16}\mathcal{H}^{(1)} + (\text{counterterms}).$$

- We are left with a few remaining degrees of freedom.
- All of them can be identified as coefficients of counterterms (which is beyond the scope of this work), except for ONE!
- This unique freedom looks a somewhat misterious.
  By far we are not aware of additional concrete PHYSICAL constraints that ultimately fixes it.

However...

There are two observations which seem to strongly suggests a unique value for this dof.

- The number of indepedent functions in the ansatz basis significantly reduces after the bootstrap. If one asks to further reduce the number, the ONLY possibility is to fix this dof.
- In principle $\mathcal{L}^{(3)}$ does not have to respect the full crossing symmetry. If one insists on doing this, then it is necessary to fix this dof to the SAME value.

Resulting space of functions

$w \backslash r$	$\mathbf{1}^+$	$\mathbf{1}^-$	$\bar{\mathbf{1}}^+$	$\bar{\mathbf{1}}^-$	$\mathbf{2}^+$	$\mathbf{2}^-$
0	1	0	0	0	0	0
1	0	0	0	0	1	0
2	$2 \to 1$	1	0	0	1	0
3	$2 \to 1$	0	1	0	$3 \to 2$	1
4	$5 \to 2$	$3 \to 2$	1	0	$5 \to 1$	2
5	$7 \to 1$	$3 \to 1$	$4 \to 1$	$2 \to 1$	$11 \to 3$	$5 \to 3$
6	$15 \to 0$	$10 \to 2$	$6 \to 0$	$3 \to 1$	$20 \to 0$	$12 \to 2$

Outlook

- Determine the string correction.
- Data for triple-trace operators.
- Other two-loop correlators.
- Structure of $\mathcal{H}_{2222}$ at even higher loops.
- Understand the hidden conformal symmetries at loop level.

Thank you very much!

Questions & comments are welcome.