Tetrahedron Instantons

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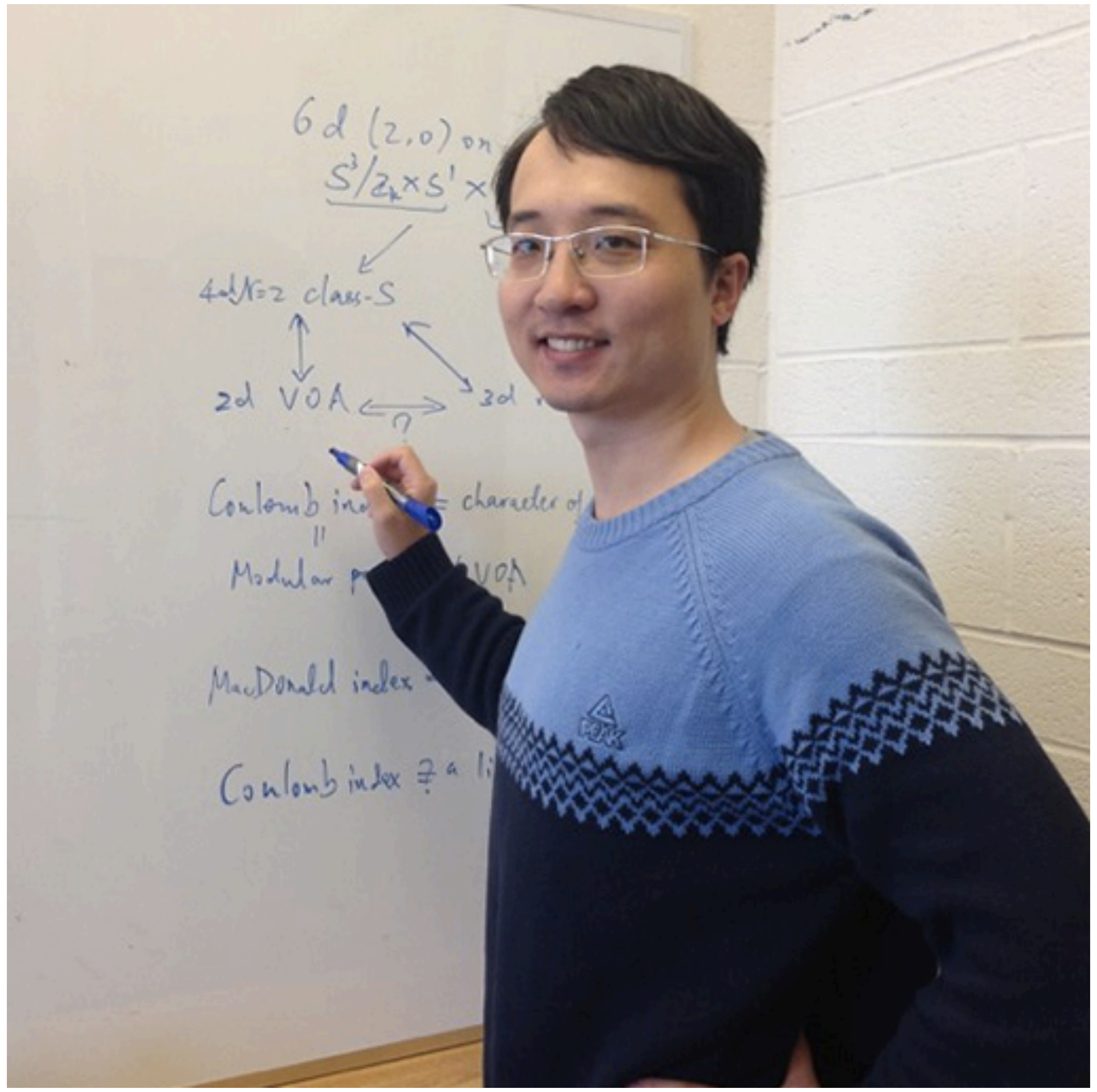
Based on work with



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and work in progress

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Plan for the talk

- Motivations:

① A generalization of Yang-Mills instantons

② A new class of generalized field theories

③ Nontrivial tests of M-theory/type IIA duality

- Properties of tetrahedron instantons:

1. Construction in string theory
2. Instanton moduli space
3. Instanton partition function

- The index of M-theory

Yang-Mills instantons

- The Euclidean action of 4d Yang-Mills theory is

$$S = -\frac{1}{2g^2}\int d^4x\,\mathrm{tr}F_{\mu\nu}^2, \quad F_{\mu\nu} = F^a_{\mu\nu}T_a, \quad [T_a, T_b] = f^c_{ab}T_c,$$

which can be written as

$$S = -\frac{1}{4g^2}\int d^4x\,\mathrm{tr}(F_{\mu\nu} \pm \tilde{F}_{\mu\nu})^2 \pm \frac{1}{2g^2}\int d^4x\,\mathrm{tr}(F_{\mu\nu}\tilde{F}_{\mu\nu}) \geq \frac{8\pi^2|k|}{g^2},$$

where $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$, and k is a topological invariant (instanton number).

- In the semi-classical approximation, the path integral is evaluated by expanding around all minima of the action. In addition to the perturbative vacua of the theory, there are other minima with finite action, the (anti-)instantons:

$$\tilde{F}_{\mu\nu} = \pm F_{\mu\nu}, \quad \frac{1}{16\pi^2}\int d^4x\,\mathrm{tr}F_{\mu\nu}\tilde{F}_{\mu\nu} \in \mathbb{Z}$$

Yang-Mills instantons

$$\tilde{F}_{\mu\nu} = \pm F_{\mu\nu}, \quad \frac{1}{16\pi^2}\int d^4x \, \mathrm{tr} F_{\mu\nu}\tilde{F}_{\mu\nu} \in \mathbb{Z}$$

- Example: BPST instanton ($\mathrm{SU}(2), k = 1$ in regular gauge): [Belavin, Polyakov, Schwarz, Tyupkin, 1975]

$$A_\mu(x) = \frac{2\sigma_{\mu\nu}(x-z)_\nu}{(x-z)^2 + \rho^2}.$$

- The instanton solution is characterized by eight free parameters: the position z_μ, the size ρ, and the global SU(2) gauge orientation.
- In general, the space of instanton solutions up to local gauge transformations is a smooth manifold, the moduli space of instantons $\mathcal{M}_{G,k}$.

Yang-Mills instantons: ADHM construction

- All SU(n) instantons with instanton number k can be constructed from the ADHM data:

1. Two $k \times k$ complex matrices B_1, B_2, one $k \times n$ complex matrix I, and one $n \times k$ complex matrix J
2. Moment maps $\mu^{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J, \quad \mu^{\mathbb{C}} = [B_1, B_2] + IJ$
3. U(k) symmetry: $\left(B_a, I, J\right) \sim \left(gB_a g^{-1}, gI, Jg^{-1}\right), \quad g \in \mathrm{U}(k)$

[Atiyah, Drinfeld, Hitchin, Manin, 1978]

- The moduli space $\mathscr{M}_{n,k} \cong \left\{ \left(B_1, B_2, I, J\right) \middle| \mu^{\mathbb{R}} = \mu^{\mathbb{C}} = 0 \right\} \Big/ \mathrm{U}(k)$

- To avoid the non-compactness of $\mathscr{M}_{n,k}$ due to small instantons, Nakajima introduced a smooth manifold $\widetilde{\mathscr{M}}_{n,k}$, which can be obtained from the Uhlenbeck compactification of $\mathscr{M}_{n,k}$ by resolving the singularities

[Nakajima, 1994]

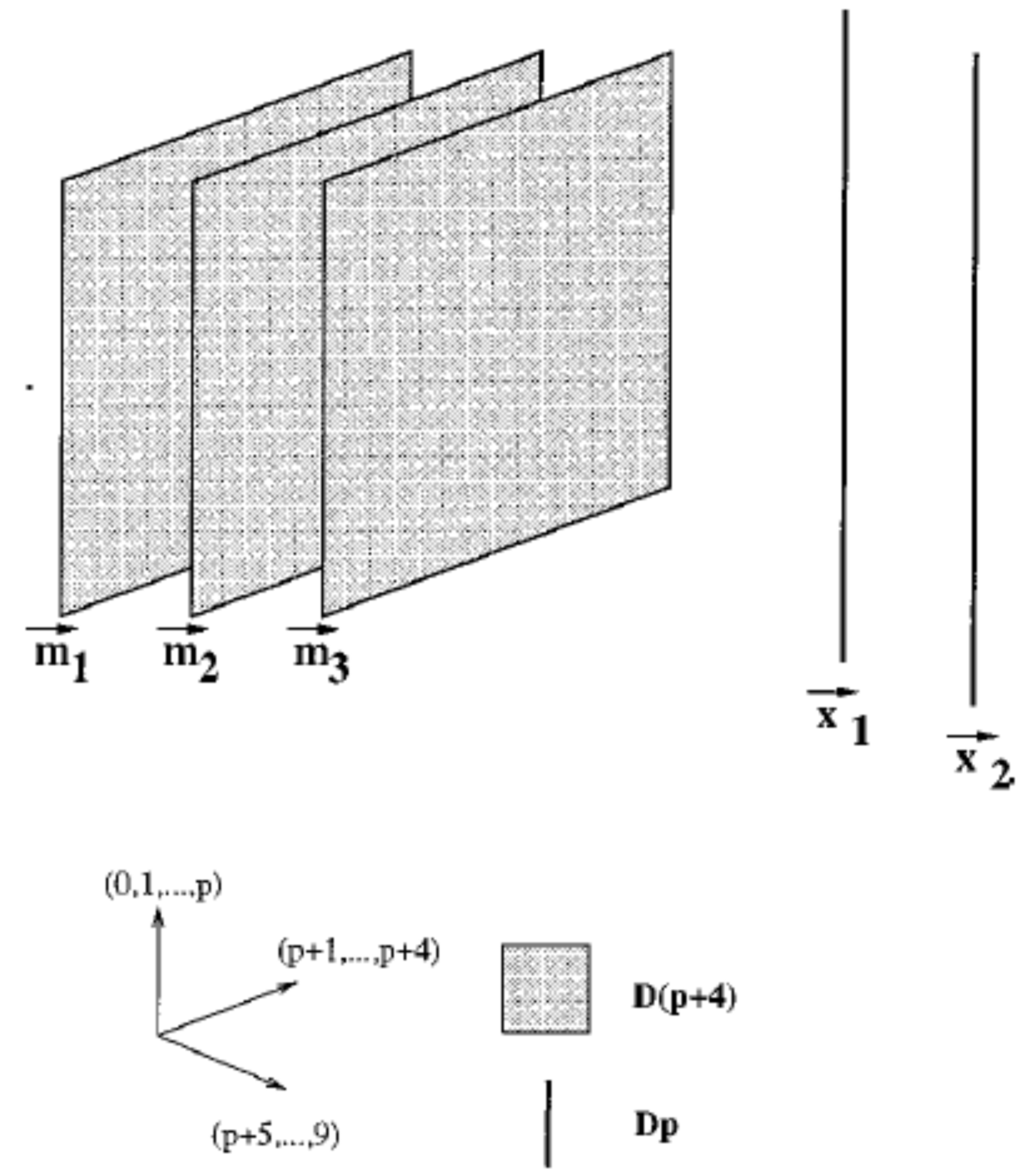
$$\widetilde{\mathscr{M}}_{n,k} \cong \left\{ \left(B_1, B_2, I, J\right) \middle| \mu^{\mathbb{R}} - r \cdot \mathbb{I}_k = \mu^{\mathbb{C}} = 0 \right\} \Big/ \mathrm{U}(k), \quad r > 0$$

Yang-Mills instantons: String theory

- k Dp-branes probing a stack of n coincident D(p+4)-branes in type II string theory ⇒ SU(n) instantons with instanton number k [Witten, 1994; Douglas 1995]

$\mathcal{M}_{n,k} \cong$ Higgs branch of supersymmetric gauge theory on Dp-branes.

- Nekrasov and Schwarz interpreted $\widetilde{\mathcal{M}}_{n,k}$ as the moduli space of U(n) instantons on $\mathbb{C}^2_\Theta$. [Nekrasov, Schwarz, 1998]
- $\widetilde{\mathcal{M}}_{n,k}$ in string theory: turn on a nonzero constant background B-field. [Seiberg, Witten, 1999]



Instanton partition function

Nekrasov introduced the instanton partition function [Nekrasov, 2002]

$$\mathscr{Z} = \sum_{k \geq 0} \mathsf{q}^k \mathscr{Z}_k, \quad \mathscr{Z}_k = \int_{\widetilde{\mathscr{M}}_{n,k}, \mathbf{T}} \cdots$$

The equivariant group $\mathbf{T}$: a maximal torus of $\mathrm{U}(1)^2 \times \mathrm{U}(n)$, which rotate the spacetime $\mathbb{C}^2$ and the gauge orientation at infinity.

$\mathscr{Z}$ is the non-perturbative part of the supersymmetric partition function of a 4d $\mathscr{N} = 2$ theory (or its higher-dimensional lift) in the Omega background.

$\mathscr{Z}$ can be evaluated exactly using localization techniques. The result can be expressed as a statistical sum over a collection of random partitions.

Instanton partition function

$\mathcal{Z}$

Virasoro/W-algebra conformal blocks
[Alday, Gaiotto, Tachikawa, 2009]

Seiberg-Witten theory
[Nekrasov, Okounkov, 2003]

Refined topological strings on CY3
[Iqbal, Kozcaz, Vafa, 2007]

Quantum integrable systems
[Nekrasov, Shatashvili, 2009]

Topological strings on Riemann surfaces
[Nekrasov, 2009]

Dijkgraaf-Vafa matrix models
[Dijkgraaf, Vafa, 2002]

...

Generalized field theories

A generalized field theory is constructed by merging several ordinary field theories across defects. Its spacetime X contains several intersecting components, $X = \cup_A X_A$. The fields and the gauge groups $G_A = G|_A$ on different components can be different, and the matter fields living on the intersection $X_A \cap X_B$ transform in the bifundamental representation of the product group $G_A \times G_B$.

Example: Spiked instantons

The instanton partition function of spiked instantons provides a unified treatment of instanton partition functions of 4d $\mathcal{N} = 2$ theories, with local or surface defects.

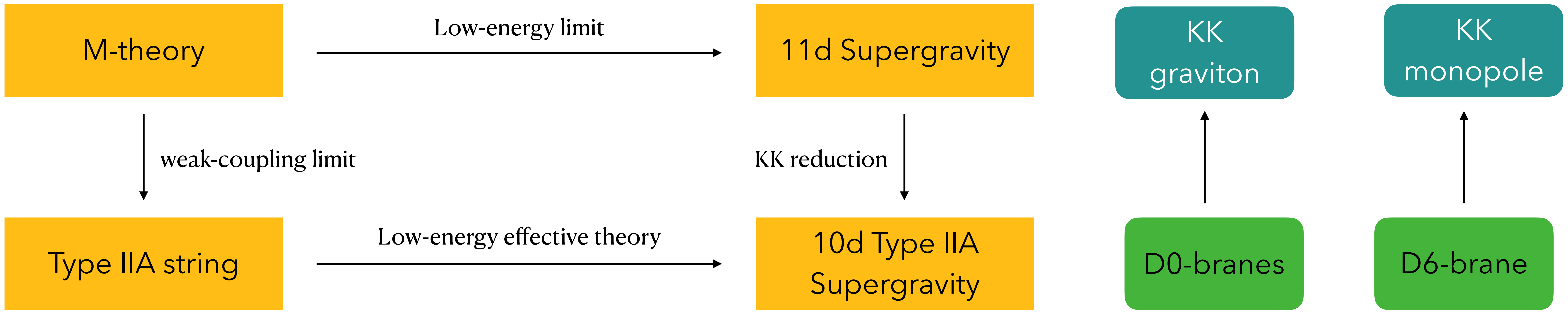
[Nekrasov, 2015; Nekrasov, Prabhakar, 2016]

The D1–D5 system for spiked instantons.

$\mathbf{R}^{1,9}$	1	2	3	4	5	6	7	8	9	0
$\mathbf{C}^4 \times \mathbf{R}^{1,1}$	z^1		z^2		z^3		z^4		x	t
D1									×	×
$D5_{(12)}$	×	×	×	×					×	×
$D5_{(13)}$	×	×			×	×			×	×
$D5_{(14)}$	×	×					×	×	×	×
$D5_{(23)}$			×	×	×	×			×	×
$D5_{(24)}$			×	×			×	×	×	×
$D5_{(34)}$					×	×	×	×	×	×

$$B = \sum_{a=1}^{4} b_a dx^{2a-1} \wedge dx^{2a}$$

M-theory/Type IIA duality



A bound state of a D6-brane and k D0-branes on $\mathbb{S}^1$ can be lifted to an 11d bound state of k KK gravitons on $\mathbb{S}^1 \times \mathbb{C}^3 \times \mathbb{TN}$.

$$\mathscr{Z}^{\mathrm{M}}(\vec{g})\left[\mathbb{S}^1 \times \mathbb{C}^3 \times \mathbb{TN}\right] = \sum_{k=0}^{\infty} Z_{D6-k\,D0}(\vec{\mu})$$

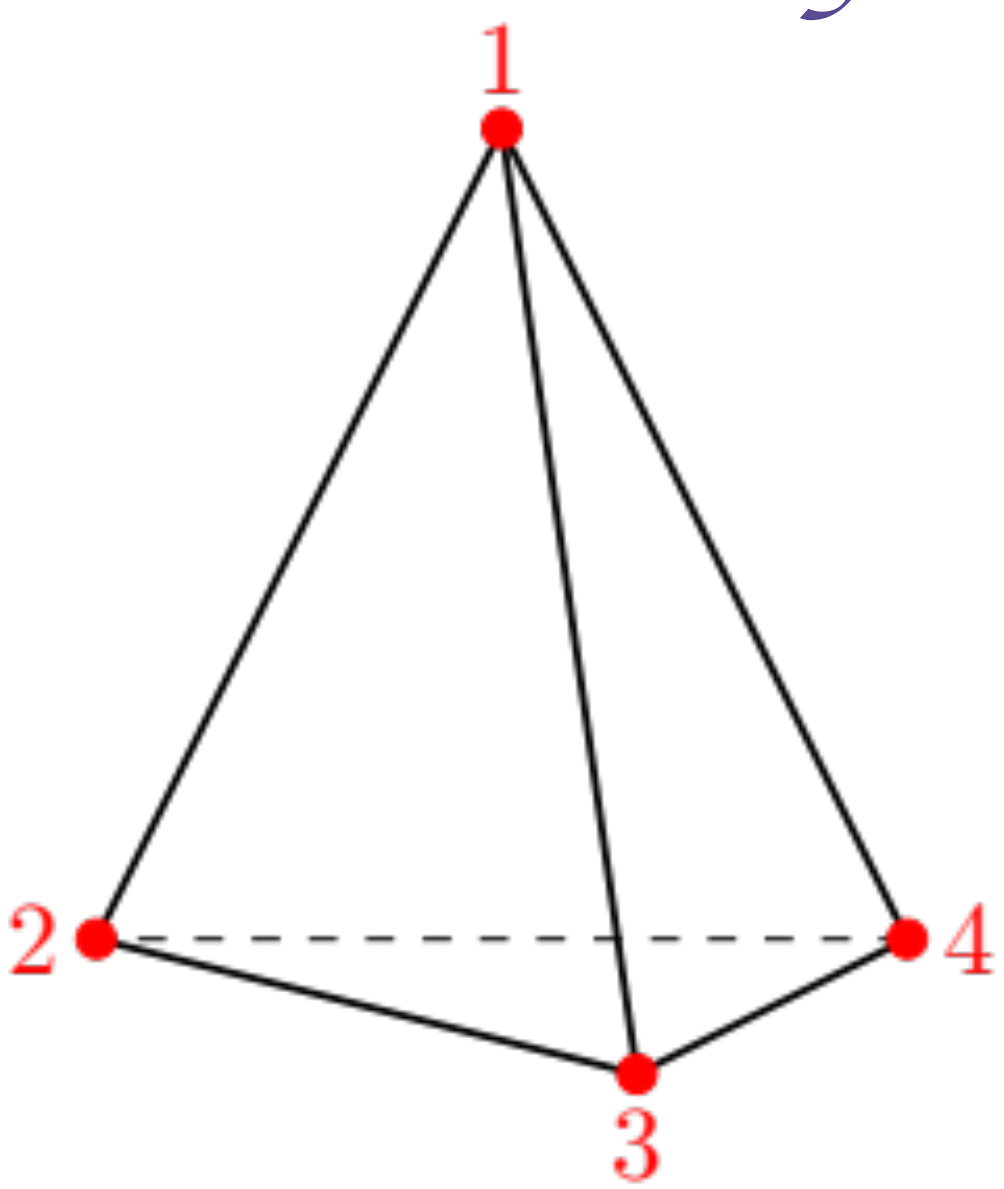
Geometric parameters

Fugacities of global symmetry

[Nekrasov, 2009;
Nekrasov, Okounkov, 2014;
Benini, Bonelli, Poggi, Tanzini, 2018]

Tetrahedron instantons

Our aim: Study D0-branes probing a configuration of intersecting D6-branes.



	$\mathbb{S}^1_t$	$\mathbb{C}_1$		$\mathbb{C}_2$		$\mathbb{C}_3$		$\mathbb{C}_4$		
	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
k D0	–	•	•	•	•	•	•	•	•	•
n_{123} D6_{123}	–	–	–	–	–	–	–	•	•	•
n_{124} D6_{124}	–	–	–	–	–	•	•	–	–	•
n_{134} D6_{134}	–	–	–	•	•	–	–	–	–	•
n_{234} D6_{234}	–	•	•	–	–	–	–	–	–	•

Vertex $a \in \underline{4} = \{1,2,3,4\} \leftrightarrow \mathbb{C}_a$

Face $A \in \underline{4}^\vee = \{(123),(124),(134),(234)\}$

$$\leftrightarrow \mathbb{C}^3_A = \prod_{a\in A} \mathbb{C}_a \subset \mathbb{C}^4$$

$\check{A} = \underline{4}\backslash A$

$$B = \sum_{a\in\underline{4}} b_a dx^{2a-1} \wedge dx^{2a}$$

$$e^{2\pi i v_a} = \frac{1+ib_a}{1-ib_a}, \quad -\frac{1}{2} < v_a < \frac{1}{2}$$

BPS bound states

- Supersymmetry is completely broken for generic v_a.
- When $v_1 = v_2 = v_3 = v_4 = \frac{1}{6} + \frac{r}{3}$, the scalar potential is given by

$$V = \mathrm{Tr}\left(\sum_{a\in \underline{4}} [B_a, B_a^\dagger] + \sum_{A\in \underline{4}^\vee} I_A I_A^\dagger - r\right)^2 + \sum_{A\in \underline{4}^\vee} \mathrm{Tr}\left|B_{\check{A}} I_A\right|^2 + \sum_{a<b\in \underline{4}} \mathrm{Tr}\left|[B_a, B_b]\right|^2 .$$

Here B_a and I_A are from D0-D0 strings and D0-D6$_A$ strings, respectively.

- In order to obtain an analogue of $\widetilde{\mathscr{M}}_{n,k}$, we need to take $r > 0$. In this case, supersymmetry is broken in the original string theory vacuum, but is restored after tachyon condensation. The low-energy theory on k D0-branes can be described by a supersymmetric gauged matrix model with gauge group $\mathrm{U}(k)$, and it preserves two supercharges $Q_+, \bar{Q}_+$.

Instanton moduli space

- The moduli space of tetrahedron instantons: the space of solutions to $V = 0$ modulo the gauge symmetry $\mathrm{U}(k)$,

$$\mathfrak{M}_{\vec{n},k} \cong \left\{ \left(B_a \in \mathrm{End}\left(\mathbb{C}^k\right), I_A \in \mathrm{Hom}\left(\mathbb{C}^{n_A}, \mathbb{C}^k\right) \right) \middle| \mu^{\mathbb{R}} - r \cdot \mathbb{I}_k = \mu^{\mathbb{C}} = \sigma = 0 \right\} \Big/ \mathrm{U}(k)$$

$$\mu^{\mathbb{R}} = \sum_{a \in \underline{4}} [B_a, B_a^\dagger] + \sum_{A \in \underline{4}^\vee} I_A I_A^\dagger, \ \mu^{\mathbb{C}} = \left(\mu^{\mathbb{C}}_{ab} = [B_a, B_b] \right)_{a,b \in \underline{4}}, \ \sigma = \left(\sigma_A = B_{\check{A}} I_A \right)_{A \in \underline{4}^\vee}$$

$$\left(B_a, I_A\right) \sim \left(g B_a g^{-1}, g I_A\right), \quad g \in \mathrm{U}(k)$$

- If $\vec{n} = \left(n_{123} = 1,0,0,0\right)$, $\mathfrak{M}_{\vec{n},k}$ reduces to the moduli space of Donaldson-Thomas invariants on $\mathbb{C}^3$. [Cirafici, Sinkovics, Szabo, 2008]

[Nekrasov, 2017; Nekrasov, Piazzalunga, 2018]

- If we drop σ-equations, $\mathfrak{M}_{\vec{n},k}$ becomes the moduli space of magnificent four model.
- The virtual dimension (# components of matrices - # constraints - # gauge) of $\mathfrak{M}_{\vec{n},k}$ is 0.

Instanton moduli space with k=1

- $\vec{n} = \left(n_{123} = n, 0, 0, 0\right)$: B_1, B_2, B_3 are unconstrained complex numbers, $I_{124} = I_{134} = I_{234} = 0$, and

$$B_4 I_{123} = 0, \quad \sum_{\alpha=1}^{n} \left| I_{123,\alpha} \right|^2 = r, \quad I_{123} \sim e^{i\theta} I_{123}.$$

Therefore, $\mathfrak{M}_{(n,0,0,0),1} \cong \mathbb{C}^3 \times \mathbb{CP}^{n-1}$.

- $\vec{n} = \left(n_{123} = n, n_{124} = m, 0, 0\right)$: B_1, B_2 are unconstrained complex numbers, $I_{134} = I_{234} = 0$, and

$$B_3 I_{124} = B_4 I_{123} = 0, \quad \sum_{\alpha=1}^{n} \left| I_{123,\alpha} \right|^2 + \sum_{\alpha=1}^{m} \left| I_{124,\alpha} \right|^2 = r, \quad I_A \sim e^{i\theta} I_A$$

Therefore, $\mathfrak{M}_{(n,m,0,0),1} \cong \mathbb{C}^2 \times \left(\mathbb{C}^* \times \mathbb{CP}^{n-1} \cup \mathbb{C}^* \times \mathbb{CP}^{m-1} \cup \mathbb{CP}^{n+m-1} \right)$.

In general, $\mathfrak{M}_{\vec{n},k}$ consists of several smooth manifolds with different actual dimensions.

Instanton partition function

We define the (K-theoretical) instanton partition function as

$$Z = \sum_{k=0}^{\infty} (-p)^k Z_k = \sum_{k=0}^{\infty} (-p)^k \hat{A}_{\mathbf{T}} \left(\mathfrak{M}_{\vec{n},k} \right) = \sum_{k=0}^{\infty} (-p)^k \mathrm{Tr}_{\mathscr{H}_k} \left[(-1)^F \prod_{a \in \underline{4}} q_a^{\mathscr{J}_a} \prod_{A \in \underline{4}^\vee} \prod_{\alpha=1}^{n_A} t_{A,\alpha}^{T_{A,\alpha}} \right]_{\prod_{a\in\underline{4}} q_a = 1}$$

$\mathbf{T}$: maximal torus of $\mathrm{U}(1)^3 \times \prod_{A \in \underline{4}^\vee} \mathrm{U}(n_A)$

$\mathscr{H}_k$: the Hilbert space of the worldvolume theory with k D0-branes

$\mathscr{J}_a$: the generator of the $\mathrm{U}(1)_a$ rotation, satisfying $[\mathscr{J}_a, Q_+] = -Q_+$, $[\mathscr{J}_a, \bar{Q}_+] = \bar{Q}_+$

$T_{(A,\alpha)}$: the Cartan generators of the symmetry group $\mathrm{U}(n_A)$

.

Expectation value of codimension-two defects

Up to now, we treat all D6-branes on equal footing, but we can choose the physical spacetime to be $\mathbb{S}^1_t \times \mathbb{C}^3_{123}$, so that the bound states of D0- and D6$_{123}$-branes give rise to instantons on $\mathbb{C}^3_{123}$, while the remaining D6-branes will produce codimension-two defects.

$$Z = \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} \int \prod_{i=1}^{k} d\phi_i \left[\left(Z_k^{0-0} Z_k^{0-6_{123}} \right) \left(\prod_{A \in \underline{4}^\vee \setminus \{(123)\}} Z_k^{0-6_A} \right) \right] = \left\langle \prod_{A \in \underline{4}^\vee \setminus \{(123)\}} \mathcal{O}_A \right\rangle_{\mathrm{DT}},$$

where $Z_{\mathrm{DT}} = \sum_{k=0}^{\infty} \frac{(-p)^k}{k!} \int \prod_{i=1}^{k} d\phi_i Z_k^{0-0} Z_k^{0-6_{123}}$.

Instanton partition function

Applying the supersymmetric localization techniques, we can express Z_k as

$$Z_k = q_4^{k^2} \left(\frac{(1-q_1q_2)(1-q_1q_3)(1-q_2q_3)}{\prod_{a\in\underline{4}}(1-q_a)} \prod_{A\in\underline{4}^\vee} q_A^{n_A/2} \right)^k \times$$

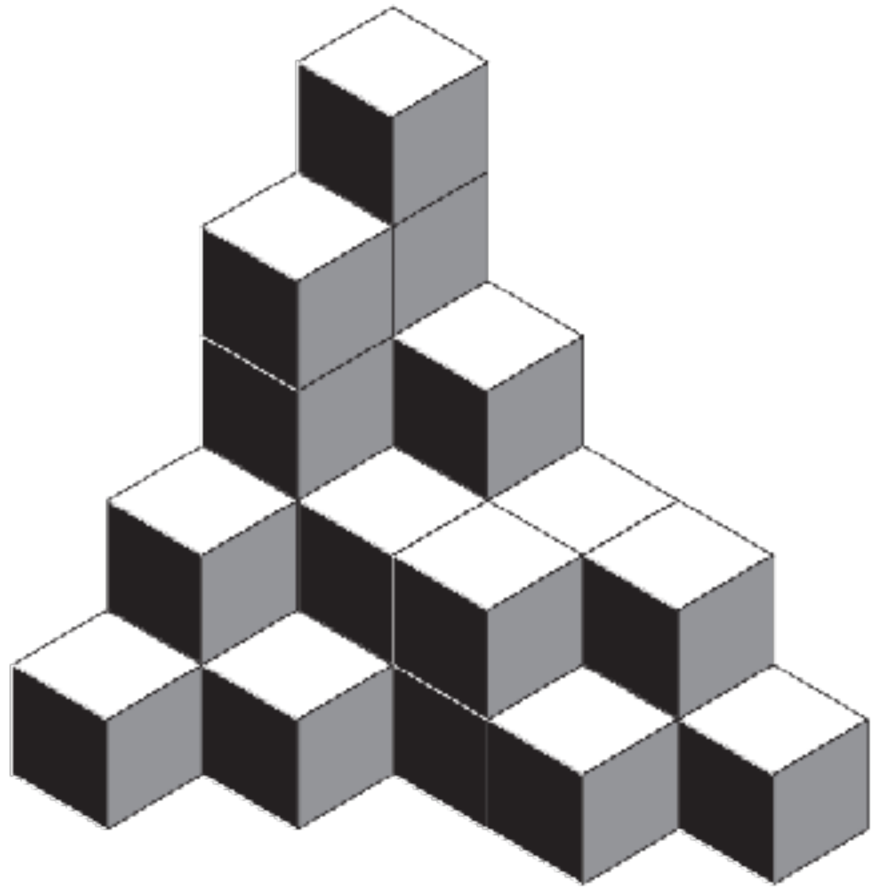
$$\times \frac{1}{k!} \int \prod_{i=1}^{k} \frac{dx_i}{x_i} \prod_{\substack{i,j=1 \\ i\neq j}}^{k} \frac{(x_j-x_i)(x_j-q_1q_2x_i)(x_j-q_1q_3x_i)(x_j-q_2q_3x_i)}{\prod_{a\in\underline{4}}(x_j-q_ax_i)} \times \prod_{i=1}^{k} \prod_{A\in\underline{4}^\vee} \prod_{\alpha=1}^{n_A} \frac{(x_i - q_A^{-1}t_{A,\alpha})}{(x_i - t_{A,\alpha})}.$$

The contour integral is evaluated using the *Jeffrey-Kirwan residue prescription*, and the poles are labeled by a collection of plane partitions $\vec{\pi} = \left\{\pi^{(A,\alpha)}\right\}$. [Jeffrey, Kirwan, 1993]

The poles:

$$\left\{x_i\right\} = \left\{t_{A,\alpha} q_a^{1-s_X} q_b^{1-s_Y} q_c^{1-s_Z}, \left(s_X, s_Y, s_Z\right) \in \pi^{(A,\alpha)}, A = \left(abc\right) \in \underline{4}^\vee, \alpha = 1,\cdots, n_A\right\}$$

Each plane partition $\pi = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \cdots \\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} & \cdots \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$, $\quad \pi_{x,y} \geq \pi_{x+1,y}, \pi_{x,y+1} \geq 0.$



Plethystic exponential form

Remarkably, the instanton partition function allows a plethystic expression

$$Z = \sum_{k=0}^{\infty} (-p)^k Z_k = \mathrm{PE}_{\vec{q},p} \left\{ \frac{[q_1q_2]\,[q_1q_3]\,[q_2q_3]}{\prod_{a\in\underline{4}} [q_a]} \frac{[Q]}{[Q^{\frac{1}{2}}p]\,[Q^{\frac{1}{2}}p^{-1}]} \right\}, \quad Q = \prod_{A\in\underline{4}^\vee} q_{\check{A}}^{n_A}$$

$= \mathbb{F}_{\vec{n}}(\vec{q},p)$: single-particle seed

$[x] = x^{\frac{1}{2}} - x^{-\frac{1}{2}}$. Plethystic exponent $\mathrm{PE}_{\vec{x}}\{f(x_1,\cdots,x_m)\} = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f(x_1^n,\cdots,x_m^n)\right]$.

This is a generalization of the plethystic expression for Donaldson-Thomas invariants.

Singular when $p \to Q^{\pm\frac{1}{2}}$

[Nekrasov, 2009; Nekrasov, Okounkov, 2014]

Magnificent four and Tachyon condensation

Magnificent four model: a system of D0-branes probing a D8-brane and an anti-D8-brane, with a strong background B-field.

[Nekrasov, 2017]

Its instanton partition function looks very similar

$$Z^{MF} = \mathrm{PE}_{\vec{q},p,\mu}\left\{ \frac{[q_1q_2]\,[q_1q_3]\,[q_2q_3]}{\prod_{a\in\underline{4}}[q_a]} \frac{[\mu]}{\left[\mu^{\frac{1}{2}}p\right]\left[\mu^{\frac{1}{2}}p^{-1}\right]} \right\}$$

where μ encodes the relative position of the D8-brane and the anti-D8-brane.

The instanton partition function $Z^{MF} = Z$ when

$$\mu = Q = \prod_{A\in\underline{4}^\vee} q_{\check{A}}^{n_A}$$

indicating that the annihilation of the D8-brane and the anti-D8-brane leaves behind a system of D6-branes.

Decomposition property

$\mathbb{F}_{\vec{n}}\left(\vec{q},p\right)$ is independent of $t_{A,\alpha}$. We can take all D6-branes to be widely separated.

$\Rightarrow$ decomposition property:

$$\mathbb{F}_{(n,0,0,0)}\left(\vec{q},p\right) = \sum_{a=1}^{n} \mathbb{F}_{(1,0,0,0)}\left(\vec{q},q_4^{a-\frac{n+1}{2}}p\right),$$

$$\mathbb{F}_{(n,m,0,0)}\left(\vec{q},p\right) = \sum_{a=1}^{n} \mathbb{F}_{(1,0,0,0)}\left(\vec{q},q_3^{\frac{m}{2}}q_4^{a-\frac{n+1}{2}}p\right) + \sum_{b=1}^{m} \mathbb{F}_{(0,1,0,0)}\left(\vec{q},q_3^{b-\frac{m+1}{2}}q_4^{-\frac{n}{2}}p\right),$$

$$\cdots\cdots$$

Cohomological limit

We introduce $q_a = e^{\beta\varepsilon_a}$. Taking the limit $\beta \to 0$ while keeping ε_a and p fixed, we have

$$\mathbb{F}_{\vec{n}}\left(\vec{q}, p\right) \to \frac{p}{(1-p)^2} \sum_{A \in \underline{4}^\vee} r_A n_A, \quad r_A = -\frac{\prod_{a<b\in A}\left(\varepsilon_a + \varepsilon_b\right)}{\prod_{a\in A}\varepsilon_a}$$

The cohomological instanton partition function (D-instanton probing intersecting D5-branes):

$$\mathcal{Z}^\flat\left(\vec{\varepsilon}, p\right) = \lim_{\beta\to 0} Z\left(\vec{q}, p\right) = \prod_{A\in\underline{4}^\vee} \mathcal{M}_3(p)^{r_A n_A},$$

where $\mathcal{M}_3(p)$ is the MacMahon function

$$\mathcal{M}_3(p) = \sum_{k=0}^{\infty} \mathrm{PL}(k) p^k = \prod_{m=1}^{\infty} \frac{1}{\left(1-p^m\right)^m} = \mathrm{PE}\left[\frac{p}{(1-p)^2}\right]$$

It is interesting that the all-genus A-model topological string partition function of $\mathbb{C}^3_{123}$ is

$$Z^{\mathrm{top}}\left(\varepsilon_1, \varepsilon_2, \varepsilon_3, \hbar\right) = \exp \sum_{g=0}^{\infty} \hbar^{2g-2} \mathcal{F}_g\left(\varepsilon_1, \varepsilon_2, \varepsilon_3\right) = \left(\mathcal{M}_3\left(-e^{\mathrm{i}\hbar}\right)\right)^{\frac{(\varepsilon_1+\varepsilon_2)(\varepsilon_1+\varepsilon_3)(\varepsilon_2+\varepsilon_3)}{\varepsilon_1\varepsilon_2\varepsilon_3}}$$

M-theory Index

A system of intersecting D6-branes ⇒ superposition of KK monopoles, described by a non-compact Calabi-Yau fivefold $\mathscr{X}$.

Compute the twisted index of M-theory

$$\mathscr{Z}^{\mathrm{M}}\left[\mathbb{S}^1_t \rtimes_g \mathscr{X}\right]\left(v_1, \cdots, v_5\right) = \mathrm{PE}_{\overrightarrow{v}}\left\{\mathscr{F}^{\mathrm{M}}\left(v_1, \cdots, v_5\right)\right\}$$

SU(5) **isometry of** $\mathscr{X}$, $\prod_{i=1}^{5} v_i = 1$

Mapping $\left(v_1, \cdots, v_5\right) \to \left(\overrightarrow{q}, p\right)$, we find $\mathscr{F}^{\mathrm{M}}\left(v_1, \cdots, v_5\right) = \mathbb{F}\left(\overrightarrow{q}, p\right) + \mathscr{P}\left(\overrightarrow{q}\right)$.

$\mathscr{P}\left(\overrightarrow{q}\right)$: the contribution without D0-branes.

$$\mathscr{Z}^{\mathrm{M}}\left[\mathbb{S}^1_t \rtimes_g \mathscr{X}\right]\left(v_1, \cdots, v_5\right) = Z^{\mathrm{pert}}\left(\overrightarrow{q}\right) Z\left(\overrightarrow{q}, p\right)$$

M-theory Index

Basic example: $\vec{n} = (n_{123} = 1, n_{124} = n_{134} = n_{234} = 0)$:

M-theory side: $\mathscr{F}^{\mathrm{M}}\left(v_1, \cdots, v_5\right)\left[\mathbb{S}^1_t \rtimes_g \mathbb{C}^5\right] = -\frac{\sum_{i=1}^{5}[v_i^2]}{\prod_{i=1}^{5}[v_i]}$

S_5 symmetry

Type IIA string theory side: $\mathbb{F}_{(1,0,0,0)}\left(\vec{q}, p\right) = \frac{[q_1q_2][q_1q_3][q_2q_3]}{\prod_{a\in\underline{4}}[q_a]}\frac{[q_4]}{\left[q_4^{\frac{1}{2}}p\right]\left[q_4^{\frac{1}{2}}p^{-1}\right]}$

$S_3 \times S_2$ symmetry

We can find the dictionary:

$$\mathscr{F}^{\mathrm{M}}\left(v_1 = q_1, v_2 = q_2, v_3 = q_3, v_4 = q_4^{\frac{1}{2}}p, v_5 = q_4^{\frac{1}{2}}p^{-1}\right)\left[\mathbb{S}^1_t \rtimes_g \mathbb{C}^5\right] = \mathbb{F}_{(1,0,0,0)}\left(\vec{q}, p\right) + \frac{[q_4]}{[q_1][q_2][q_3]}$$

Using the decomposition property, we can also find the correspondence for general $\vec{n}$.

Summary

- We introduced the tetrahedron instantons, which can be realized in string theory by D0-branes probing a configuration of intersecting D6-branes with a suitable background B-field.
- We studied the moduli space of tetrahedron instantons.
- The instanton partition function can be computed exactly, and allows a plethystic expression.
- Lifting the type IIA configuration to M-theory, the instanton partition function can be reproduced from the M-theory index.

Outlook

Our discussion can be extension in several directions:

① Other spacetime $\mathbb{S}^1_t \times \mathrm{CY}_4 \times \mathbb{R}$ in IIA string theory.

② Adding D2- and D4-branes. The M-theory index will also receive contributions from M-branes.

Thank you!

Research Overview

My research interests span the areas of quantum field theory, supersymmetry, string theory and mathematical physics. In addition to today's topic, I have been working on

① $\mathcal{N} = 2$ supersymmetric gauge theories

- Derivation of Seiberg-Witten geometry via instanton counting
- Alday-Gaiotto-Tachikawa correspondence via non-perturbative Dyson-Schwinger equations [with Saebyeok Jeong]
- First-principle calculation of effective gravitational couplings [with Jan Manschot and Gregory Moore]
- Correspondence with 2d topological strings

② Donaldson invariants of four-manifolds

- K-theoretical/elliptic Donaldson invariants [with Heeyeon Kim, Jan Manschot, Gregory Moore and Runkai Tao]

③ $\mathcal{N} = 1$ theories of class S_k [with Thomas Bourton and Elli Pomoni]

④ Supersymmetric localization computations in field/supergravity theories [with Jun Nian]

⑤ Generalization of the notion of symmetries beyond group theory [with Enrico Andriolo, Hanno Bertle, Elli Pomoni and Konstantinos Zoubos]

- Hidden quantum $\mathcal{N} = 4$ superconformal symmetry in $\mathcal{N} = 2$ superconformal theories

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
k D1	–	•	•	•	•	•	•	•	•	–
n_{123} D7$_{123}$	–	–	–	–	–	–	–	•	•	–
n_{124} D7$_{124}$	–	–	–	–	–	•	•	–	–	–
n_{134} D7$_{134}$	–	–	–	•	•	–	–	–	–	–
n_{234} D7$_{234}$	–	•	•	–	–	–	–	–	–	–

- Two quartets of matrices

$$\vec{B} = \left(B_a\right)_{a\in\underline{4}}, B_a \in \mathrm{End}\left(\mathbb{C}^k\right),$$

$$\vec{I} = \left(I_A\right)_{A\in\underline{4}^\vee}, I_A \in \mathrm{Hom}\left(\mathbb{C}^{n_A}, \mathbb{C}^k\right)$$

Strings	$\mathcal{N}=(2,2)$	$\mathcal{N}=(0,2)$	$\left(\mathrm{U}(k), \mathrm{U}(n_A)\right)$
D1-D1	Vector	Vector Υ	$(\mathrm{Adj}, 1)$
		Chiral $\Phi_{\breve{A}} = B_{\breve{A}} + \cdots$	
	Chiral $(a \in A)$	Chiral $\Phi_a = B_a + \cdots$	
		Fermi $\Psi_{a,-} = \psi_{a,-} + \cdots$	
D1-D7$_A$	Chiral	Chiral $\Phi_A = I_A + \cdots$	$(k, \overline{n_A})$
		Fermi $\Psi_{A,-} = \psi_{A,-} + \cdots$	

$$\mathbb{R}^{1,9} \cong \mathbb{S}^1_t \times \prod_{a\in\underline{4}} \mathbb{C}_a \times \mathbb{R}_9$$

$$Z = \sum_{k=0}^{\infty} \mathfrak{q}^k \chi_{\mathbf{T}}\left(\mathfrak{M}_{\vec{n},k}\right) = \sum_{k=0}^{\infty} \mathfrak{q}^k \mathrm{Tr}_{\mathcal{H}_k}\left[(-1)^F q^{H_L} \bar{q}^{H_R} \prod_{a\in\underline{4}} e^{2\pi \mathrm{i} \varepsilon_a \mathcal{J}_a} \prod_{A\in\underline{4}^\vee} \prod_{\alpha=1}^{n_A} e^{2\pi \mathrm{i} \mathfrak{a}_{A,\alpha} T_{A,\alpha}}\right]_{\sum_{a\in\underline{4}} \varepsilon_a = 0}$$

Free field representation

- Free massless multi-component scalar field φ on $\mathbb{T}^2$:

$$\left\langle \varphi_i(z,\bar{z})\,\varphi_j(0,0) \right\rangle_{\mathbb{T}^2} = -\log\left| \frac{\theta_1(z|\,\tau)}{2\pi\eta(\tau)^3} \exp\left(-\frac{\pi\,(\mathrm{Im}z)^2}{\mathrm{Im}\tau} \right) \right|^2 \delta_{i,j}.$$

- Introduce a vertex operator

$$\mathscr{V}_{\alpha,\rho}(z,\bar{z}) =: \exp\left[\mathrm{i}\sum_{i=1}^{7} \alpha_i \varphi_i\left(z+\rho_i, \bar{z}+\rho_i\right)\right] :: \exp\left[-\mathrm{i}\sum_{i=1}^{7} \alpha_i \varphi_i\left(z-\rho_i, \bar{z}-\rho_i\right)\right] :,$$

where $\alpha = \left(\mathrm{i},\mathrm{i},\mathrm{i},\mathrm{i},1,1,1\right)$ and $\rho = \frac{1}{2}\left(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_{12},\varepsilon_{13},\varepsilon_{23}\right)$.

- Introduce a linear source operator $\Upsilon = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma dz \sum_{A\in\underline{4}^\vee} \varpi_A(z)\partial_z\varphi_{\check{A}}(z)$, where Γ is a loop arount $z=0$ encircling all $\pm\rho_i$, and $\varpi_A(z) = \sum_{\alpha=1}^{n_A} \log\theta_1\left(z - \mathrm{a}_{A,\alpha} - \frac{1}{2}\varepsilon_A \,\middle|\, \tau \right)$.

- The instanton partition function admits a free field representation: $Z = \left\langle e^{\Upsilon} e^{\mathfrak{q}\oint_{\mathscr{C}}\mathscr{V}_{\alpha,\rho}(z)dz} \right\rangle^{\mathrm{hol}}_{\mathbb{T}^2}$.

[Seiberg, Witten, 1999]

Noncommutative Instantons

- Open strings connecting D-branes in the presence of a strong background B-field can usually be described by noncommutative field theory.
- The spacetime becomes $\mathbb{R}^{1,1} \times \mathbb{C}^4_\Theta$, with
  $\left[z_a, z_b\right] = \left[\bar{z}_a, \bar{z}_b\right] = 0, \quad \left[z_a, \bar{z}_b\right] = -2\Theta\delta_{ab}$
-

Instanton partition function

Applying the supersymmetric localization techniques, $Z_k = \frac{1}{k!}\int \prod_{i=1}^{k} d\phi_i \left(Z_k^{1-1} \prod_{A\in \underline{4}^\vee} Z_k^{1-7_A} \right)$, where

$$Z_k^{1-1} = \left[\frac{2\pi\eta(\tau)^3\theta_1(\varepsilon_{12}|\tau)\theta_1(\varepsilon_{13}|\tau)\theta_1(\varepsilon_{23}|\tau)}{\theta_1(\varepsilon_1|\tau)\theta_1(\varepsilon_2|\tau)\theta_1(\varepsilon_3|\tau)\theta_1(\varepsilon_4|\tau)}\right]^k \times$$

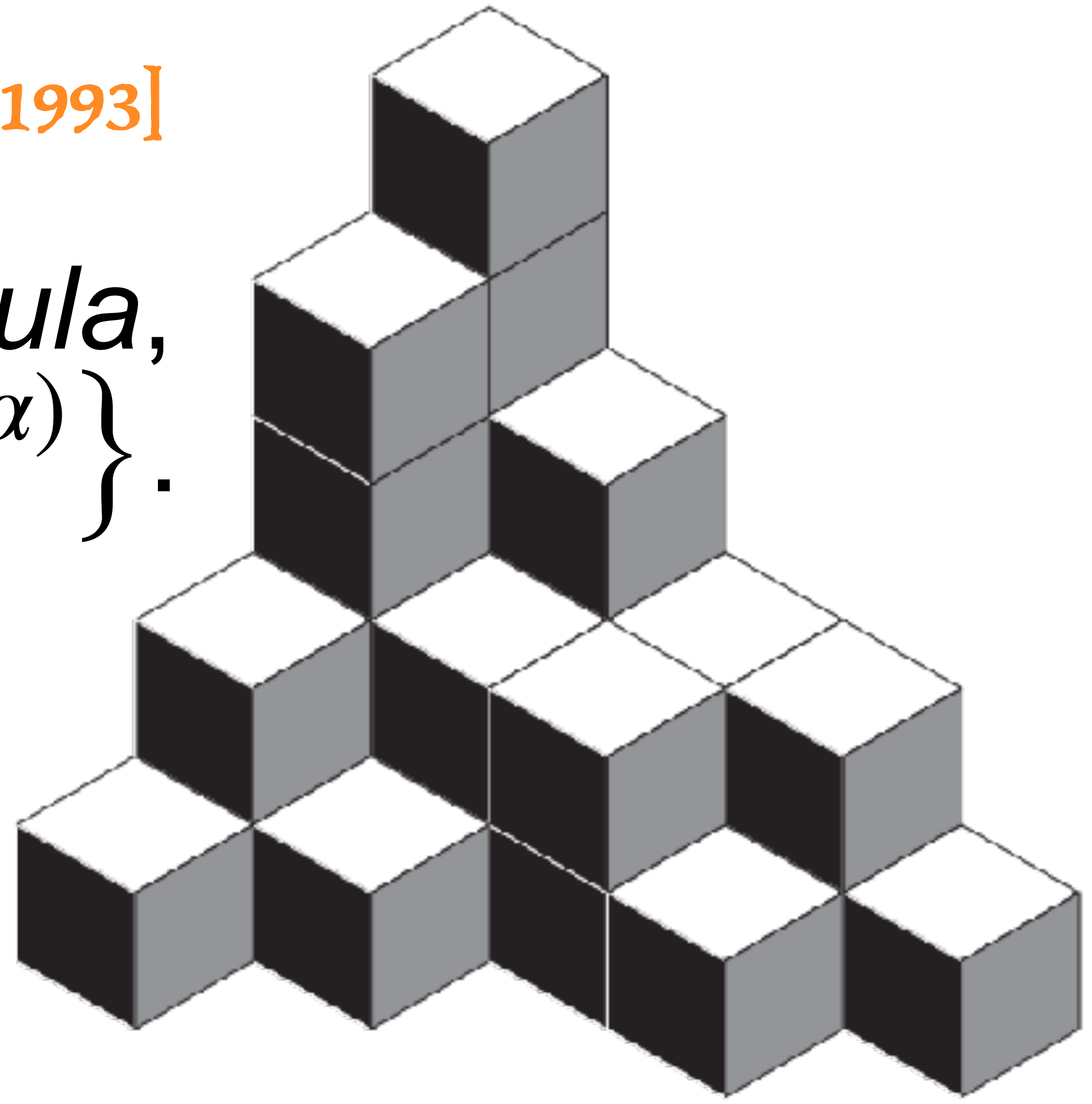
$$\times \prod_{\substack{i,j=1 \\ i\neq j}}^{k} \frac{\theta_1(\phi_{ij}|\tau)\theta_1(\phi_{ij}+\varepsilon_{12}|\tau)\theta_1(\phi_{ij}+\varepsilon_{13}|\tau)\theta_1(\phi_{ij}+\varepsilon_{23}|\tau)}{\theta_1(\phi_{ij}+\varepsilon_1|\tau)\theta_1(\phi_{ij}+\varepsilon_2|\tau)\theta_1(\phi_{ij}+\varepsilon_3|\tau)\theta_1(\phi_{ij}+\varepsilon_4|\tau)},$$

$$Z_k^{1-7_A} = \prod_{i=1}^{k}\prod_{\alpha=1}^{n_A} \frac{\theta_1(\phi_i - \mathsf{a}_{A,\alpha} - \varepsilon_A|\tau)}{\theta_1(\phi_i - \mathsf{a}_{A,\alpha}|\tau)},$$

[Jeffrey, Kirwan, 1993]

The contour integral is evaluated using the *Jeffrey-Kirwan residue formula*, and the poles are labeled by a collection of plane partitions $\vec{\pi} = \left\{\pi^{(A,\alpha)}\right\}$.

Each plane partition $\pi = \begin{pmatrix} \pi_{1,1} & \pi_{1,2} & \pi_{1,3} & \cdots \\ \pi_{2,1} & \pi_{2,2} & \pi_{2,3} & \cdots \\ \pi_{3,1} & \pi_{3,2} & \pi_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$, $\quad \pi_{x,y} \geq \pi_{x+1,y}, \pi_{x,y+1} \geq 0.$



Dimensional reduction

- Performing a T-duality, we get a D0-D6 system in type IIA superstring theory.
- The K-theoretical instanton partition function is computed by the generalized Witten indices.

$$Z_k = \int_{(\mathfrak{M}_{\vec{n},k})} \hat{A}_{\mathbf{T}}\left(\mathfrak{M}_{\vec{n},k}\right) = \mathrm{Tr}_{\mathscr{H}_k}\left[(-1)^F \prod_{a\in\underline{4}} q_a^{\mathscr{J}_a} \prod_{A\in\underline{4}^\vee} \prod_{\alpha=1}^{n_A} t_{A,\alpha}^{T_{A,\alpha}}\right]_{\prod_{a\in\underline{4}} q_a=1}$$

$$q_a = e^{2\pi \mathrm{i}\varepsilon_a}$$

$$t_{A,\alpha} = e^{2\pi \mathrm{i} a_{A,\alpha}}$$

$$\begin{aligned} Z_k = \; & q_4^{k^2}\left(\frac{(1-q_1q_2)(1-q_1q_3)(1-q_2q_3)}{\prod_{a\in\underline{4}}(1-q_a)} \prod_{A\in\underline{4}^\vee} q_A^{n_A/2}\right)^k \times \\ & \times \frac{1}{k!}\int \prod_{i=1}^{k} \frac{dx_i}{x_i} \prod_{\substack{i,j=1 \\ i\neq j}}^{k} \frac{(x_j-x_i)(x_j-q_1q_2x_i)(x_j-q_1q_3x_i)(x_j-q_2q_3x_i)}{\prod_{a\in\underline{4}}(x_j-q_ax_i)} \times \\ & \times \prod_{i=1}^{k} \prod_{A\in\underline{4}^\vee} \prod_{\alpha=1}^{n_A} \frac{(x_i - q_A^{-1} t_{A,\alpha})}{(x_i - t_{A,\alpha})} \end{aligned}$$

- Similar reduction to D(-1)-D5 system will give the rational instanton partition function.

Low-energy worldvolume theory

$v_1 = -v_2 = v_3 = -v_4$
for spiked instantons

Share a common (0,2) susy

When $v_1 = v_2 = v_3 = v_4 = \frac{1}{6} + \frac{r}{3}$, the scalar potential is given by

$$V = \mathrm{Tr}\left(\sum_{a \in \underline{4}} [B_a, B_a^\dagger] + \sum_{A \in \underline{4}^\vee} I_A I_A^\dagger - r \right)^2 + \sum_{A \in \underline{4}^\vee} \mathrm{Tr} \left| B_{\check{A}} I_A \right|^2 + \sum_{a < b \in \underline{4}} \mathrm{Tr} \left| [B_a, B_b] \right|^2 .$$

When $r > 0$, susy is broken in the original string theory vacuum, but is restored after transitioning to a nearby vacuum via tachyon condensation.

The moduli space of tetrahedron instantons: the space of solutions to $V = 0$ modulo the gauge symmetry $\mathrm{U}(k)$