Classical Yang-Baxter equation, Lagrangian multiforms and ultralocal integrable hierarchies

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Theoretical Physics Seminars

Based on **arXiv:2201.08286** *with M. Stoppato, B. Vicedo*

General context

Integrable classical field theories in $1+1$ dimensions

• Can be viewed as **Lagrangian systems** associated to an **action** with Lagrangian (density) $\mathscr{L}[u]$

$$S[u] = \int_\sigma \mathscr{L}[u] dx \wedge dt$$

NB: σ is a two-dimensional manifold and $\mathscr{L}[u] dx \wedge dt$ is a *volume* form.

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NB: σ is a two-dimensional manifold and $\mathscr{L}[u] dx \wedge dt$ is a *volume* form.

• Can also be viewed as (infinite dimensional) **Hamiltonian systems**.

$$H[u] = \int_\gamma \mathcal{H}[u]\, dx\,, \quad \gamma \subseteq \mathbb{R}$$

• (Liouville) integrability: e.g. countable number of charges in involution defining compatible flows on the fields of the theory.

$$\{H_i, H_j\} = 0\,, \quad \partial_{t_i} = \{\cdot, H_i\}\,, \quad [\partial_{t_i}, \partial_{t_j}] = 0$$

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$\rightarrow$ natural to think of an integrable systems as being part of an **integrable hierarchy**: The physical Hamiltonian is part of an infinite family $H_1, H_2, \dots$. The physical time is part of a hierarchy of times $t_1, t_2, \dots$.

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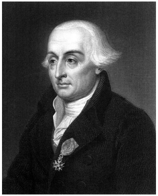
• Integrability, both classically and quantum mechanically, has been studied overwhelmingly from the Hamiltonian point of view (Liouville theorem, bi-Hamiltonian systems, Quantum Inverse Scattering method, etc.)

General context: Lagrange vs Hamilton?



Joseph-Louis Lagrange (1736-1813)

Which is more fundamental?

William Rowan Hamilton (1805-65)

• **Question**: how to capture/define (classical) integrability solely from the Lagrangian point of view? There is only one Lagrangian, as opposed to a hierarchy of Hamiltonians.

Plan

1. Variational criterion for integrability: Lagrangian multiforms

2. Lagrangian multiforms: key equations, properties, examples

3. How to construct a Lagrangian multiform?
a. Key example: Ablowitz-Kaup-Newell-Segur hierarchy
b. Important observations leading to generalisation

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

5. Conclusions, outlook, open questions

1. Variational criterion for integrability: Lagrangian multiforms

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1. Variational criterion for integrability: Lagrangian multiforms

Back to the question: how to define (classical) integrability from the Lagrangian point of view?

• Answer originally proposed in [Lobb, Nijhoff '09] (in the discrete setting). Presented here for field theories.

1. Replace the Lagrangian volume form (denote x, t by t_1, t_2)

$$\mathscr{L}[u] = \mathscr{L}_{12}[u] dt_1 \wedge dt_2$$

by a **Lagrangian multiform**

$$\mathscr{L}[u] = \sum_{i<j} \mathscr{L}_{ij}[u] dt_i \wedge dt_j$$

→ a two-form on a higher dimensional manifold $\mathcal{M}$ whose coordinates are the "times" t_i of the hierarchy.

2. Define an associated action

$$\mathcal{S}[u,\sigma] = \int_{\sigma} \sum_{i<j} \mathcal{L}_{ij}[u]\, dt_i \wedge dt_j \,.$$

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$$\mathcal{S}[u,\sigma] = \int_{\sigma} \sum_{i<j} \mathcal{L}_{ij}[u]\, dt_i \wedge dt_j \, .$$

and a **generalised variational principle**:

(i) A field u is critical for $\mathscr{L}[u]$ if it is a critical configuration of $\mathcal{S}[u,\sigma]$ **for "arbitrary" surface σ in $\mathcal{M}$**.

(ii) On critical configurations, the value of the action $\mathcal{S}[u,\sigma]$ is independent of σ: it is stationary with respect to local variations of the surface σ.

Intuition behind the proposed principle: The arbitrariness of σ implements variationally the idea of commuting Hamiltonian vectors fields in continuous setting.

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- Consequences of the generalised principle on simplest case:

$$\mathscr{L}[u] = \mathscr{L}_{12}[u] dt_1 \wedge dt_2 + \mathscr{L}_{13}[u] dt_1 \wedge dt_3 + \mathscr{L}_{23}[u] dt_2 \wedge dt_3$$

with

$$\mathscr{L}_{ij}[u] = \mathscr{L}_{ij}(u, u_{t_1}, u_{t_2}, u_{t_3}) \quad \text{(first order Lagrangians)}$$

2. Lagrangian multiforms: key equations and properties

If $\sigma = (t_1, t_2)$-plane then

$$S[u,\sigma] = \int_{\mathbb{R}^2} \mathscr{L}_{12}(u, u_{t_1}, u_{t_2}, u_{t_3}) dt_1 \wedge dt_2$$

and

$$\begin{aligned}
\delta_u S[u,\sigma] \quad = \quad & \int_{\mathbb{R}^2} \left(\frac{\partial \mathscr{L}_{12}}{\partial u} - \partial_{t_1} \frac{\partial \mathscr{L}_{12}}{\partial u_{t_1}} - \partial_{t_2} \frac{\partial \mathscr{L}_{12}}{\partial u_{t_2}} \right) \delta u \wedge dt_1 \wedge dt_2 \\
& + \int_{\mathbb{R}^2} \left(\partial_{t_1} \left(\frac{\partial \mathscr{L}_{12}}{\partial u_{t_1}} \delta u \right) + \partial_{t_2} \left(\frac{\partial \mathscr{L}_{12}}{\partial u_{t_2}} \delta u \right) \right) dt_1 \wedge dt_2 \\
& + \int_{\mathbb{R}^2} \left(\frac{\partial \mathscr{L}_{12}}{\partial u_{t_3}} \delta u_{t_3} \right) dt_1 \wedge dt_2
\end{aligned}$$

- Hence, one obtains:
  1. Euler-Lagrange equations for $\mathscr{L}_{12}$: $\frac{\delta \mathscr{L}_{12}}{\delta u} = 0$;
  2. boundary terms $\to 0$;
  3. New structural equation $\to \frac{\partial \mathscr{L}_{12}}{\partial u_{t_3}} = 0$.

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  2. boundary terms $\to 0$;
  3. New structural equation $\to \frac{\partial \mathscr{L}_{12}}{\partial u_{t_3}} = 0$.
- If $\sigma = \sigma_1 \cup \sigma_2$ (union of two half-planes) then

$$S[u, \sigma] = \int_{\sigma_1} \mathscr{L}_{12} dt_1 \wedge dt_2 + \int_{\sigma_2} \mathscr{L}_{13} dt_1 \wedge dt_3$$

- Similar derivation gives
  1. Euler-Lagrange equations for $\mathscr{L}_{12}$ and $\mathscr{L}_{13}$;
  2. $\frac{\partial \mathscr{L}_{12}}{\partial u_{t_3}} = 0$ as before and $\frac{\partial \mathscr{L}_{13}}{\partial u_{t_2}} = 0$;
  3. New structural equation

$$\frac{\partial \mathscr{L}_{12}}{\partial u_{t_2}} + \frac{\partial \mathscr{L}_{13}}{\partial u_{t_3}} = 0$$

2. Lagrangian multiforms: key equations and properties

Summary: generalised variational principle gives the **multi-time Euler-Lagrange equations** for the Lagrangian coefficients $\mathscr{L}_{ij}$ of $\mathscr{L}[u]$. [Suris, Vermeeren '15]

- **General structure:**

  1. Euler-Lagrange equations for each $\mathscr{L}_{ij}$;
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- Multi-time Euler-Lagrange equations rederived and generalised in several ways, e.g. [Sleigh, Nijhoff, Caudrelier '20].
- Outcome: compact formulation achieved using the variational bicomplex formalism

$$\delta d\mathscr{L}[u] = 0$$

Several advantages: coordinates independent formulation, valid for d-form $d = 1, 2, 3, \dots$, for higher order Lagrangians

Intuition behind the second requirement

• **On solutions**, the action is stationary with respect to local variations of the surface σ:

$$\mathcal{S}[u,\sigma] = \mathcal{S}[u,\sigma'] \Rightarrow \int_{\partial B} \mathcal{L}[u] = 0 \Rightarrow \int_{B} d\mathcal{L}[u] = 0$$

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With

$$\mathscr{L}[u]=\sum_{i<j}\mathscr{L}_{ij}[u]dt_i\wedge dt_j$$

$$d\mathscr{L}[u]=\sum_{i<j<k}\left(\partial_{t_k}\mathscr{L}_{ij}[u]+\partial_{t_j}\mathscr{L}_{ki}[u]+\partial_{t_i}\mathscr{L}_{jk}[u]\right)dt_i\wedge dt_j\wedge dt_k$$

so, in components,

$$\partial_{t_k}\mathscr{L}_{ij}[u]+\partial_{t_j}\mathscr{L}_{ki}[u]+\partial_{t_i}\mathscr{L}_{jk}[u]=0$$

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• $\delta d\mathscr{L}=0$ linked to commutativity of Hamiltonian flows and closure relation linked to known criterion $\{H_i, H_j\}=0$ (for certain Lagrangian 1-forms and 2-forms [Suris '13; Vermeeren '21])

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• The main topic today: link to classical r-matrix and classical Yang-Baxter equation.

3. How to construct a Lagrangian multiform?

- **Problem:** how to get the Lagrangians $\mathcal{L}_{ij}$ for all $i, j \geq 0$?

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• Technical and difficult problem: several methods (brute force, variational symmetries, discrete to continuum). Results essentially for a finite number of levels in the hierarchy [Suris, Vermeeren '16; Sleigh, Nijhoff, Caudrelier '19; Vermeeren '19; Petrera, Vermeeren '19]

3. How to construct a Lagrangian multiform?

Example: Nonlinear Schrödinger and modified KdV levels in Ablowitz-Kaup-Newell-Segur hierarchy

$$q_2 - \frac{i}{2}q_{11} + iq^2r = 0, \qquad r_2 + \frac{i}{2}r_{11} - iqr^2 = 0,$$

$$q_3 + \frac{1}{4}q_{111} - \frac{3}{2}qrq_1 = 0, \qquad r_3 + \frac{1}{4}r_{111} - \frac{3}{2}qrr_1 = 0\,.$$

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Lagrangians

$$\mathscr{L}_{12} = \frac{1}{2}(rq_2 - qr_2) + \frac{i}{2}q_1r_1 + \frac{i}{2}q^2r^2$$

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$$\begin{aligned}\mathscr{L}_{23} &= \frac{1}{4}(q_2r_{11}-r_2q_{11})-\frac{i}{2}(q_3r_1+r_3q_1)+\frac{1}{8}(q_1r_{12}-r_1q_{12})\\ &+ \frac{3qr}{8}(qr_2-rq_2)-\frac{i}{8}q_{11}r_{11}+\frac{i}{4}qr(qr_{11}+rq_{11})\\ &- \frac{i}{8}(qr_1-rq_1)^2-\frac{i}{2}q^3r^3.\end{aligned}$$

Our method: Confluence of several ideas

1) Generating formalism and "compounding the hierarchy" idea advocated e.g. in [Nijhoff '83] in the Lagrangian formalism.

2) Zakharov-Mikhailov insighful result on Lagrangian formulation of zero curvature equations for rational Lax pairs. [Zakharov, Mikhailov '80]

3) Flaschka-Newell-Ratiu (FNR) construction of the Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy [Flaschka, Newell, Ratiu '83] and the comparison of their generating function for conservation laws with our known first few covariant Hamiltonians $\mathcal{H}_{ij}$ for AKNS [Caudrelier, Stoppato '20].

3. How to construct a Lagrangian multiform?

Idea 1): generating Lagrangian multiform

- Assemble the Lagrangian coefficients $\mathscr{L}_{ij}$ into a formal series

$$\mathscr{L}(\lambda, \mu) = \sum_{i,j=0}^{\infty} \frac{\mathscr{L}_{ij}}{\lambda^{i+1}\mu^{j+1}}$$

- Propose a formula for $\mathscr{L}(\lambda, \mu)$.

3. How to construct a Lagrangian multiform?

Idea 2) and 3): form of $\mathscr{L}(\lambda,\mu)$

- $\mathscr{L}(\lambda,\mu) = K(\lambda,\mu) - V(\lambda,\mu)$ with

$$K(\lambda,\mu) = \mathrm{Tr}\big(\phi(\mu)^{-1}\partial_\lambda\phi(\mu)Q_0 - \phi(\lambda)^{-1}\partial_\mu\phi(\lambda)Q_0\big)\,,$$

$$V(\lambda,\mu) = -\frac{1}{2}\mathrm{Tr}\frac{(Q(\lambda)-Q(\mu))^2}{\lambda-\mu}\,.$$

where $Q_0 = -i\sigma_3$, $\partial_\mu \equiv \sum_{k=0}^{\infty}\frac{1}{\mu^{k+1}}\partial_{t_k}$, and

$$\phi(\lambda) = \mathbb{1} + \sum_{j=1}^{\infty}\frac{\phi_j}{\lambda^j}\,,\quad Q(\lambda) = \phi(\lambda)Q_0\phi^{-1}(\lambda)$$

$\rightarrow$ formal dressing in sl_2 loop algebra.

3. How to construct a Lagrangian multiform?

Why am I claiming that we have a Lagrangian multiform for the AKNS hierarchy?

- [Flaschka, Newell, Ratiu '83] showed that, with

$$Q(\lambda) = \sum_{j=0}^{\infty} Q_j \lambda^{-j}\,, \quad Q_j = \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \in \mathfrak{sl}_2\,, \quad Q_0 = -i\sigma_3\,,$$

all (positive) AKNS flows can be be written as

$$\partial_{t_k} Q(\lambda) = [V^{(k)}(\lambda), Q(\lambda)]\,, \quad k \geq 0$$

where

$$V^{(k)}(\lambda) = P_+(\lambda^k Q(\lambda)) = \sum_{j=0}^{k} Q_j \lambda^{k-j} \quad (\textbf{Lax matrix} \text{ for } t_k \text{ flow})$$

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- **Zero curvature equations** for the whole hierarchy hold

$$\partial_{t_j}V^{(k)}(\lambda)-\partial_{t_k}V^{(j)}(\lambda)+[V^{(k)}(\lambda),V^{(j)}(\lambda)]=0\quad j,k\geq 0$$

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- $(V^{(1)}(\lambda),V^{(2)}(\lambda))$= Lax pair for NLS, $(V^{(1)}(\lambda),V^{(3)}(\lambda))$= Lax

3. How to construct a Lagrangian multiform?

- Now, introduce formal series

$$\partial_\mu \equiv \sum_{k=0}^{\infty} \frac{1}{\mu^{k+1}} \partial_{t_k}\,, \quad \frac{1}{\mu-\lambda} \equiv \sum_{k=0}^{\infty} \frac{\lambda^k}{\mu^{k+1}}$$

to get

$$\partial_{t_k} Q(\lambda) = [V^{(k)}(\lambda), Q(\lambda)] \quad k \geq 0 \Leftrightarrow \partial_\mu Q(\lambda) = \left[\frac{Q(\mu)}{\mu-\lambda}, Q(\lambda)\right].$$

→ **Generating Lax equation for integrable hierarchy.**

Now we have

Theorem

$\mathscr{L}(\lambda,\mu)$ is a Lagrangian multiform for the AKNS hierarchy equations i.e.

$$\delta d\mathscr{L}=0 \Leftrightarrow \partial_\mu Q(\lambda)=\left[\frac{Q(\mu)}{\mu-\lambda},Q(\lambda)\right],$$

and $d\mathscr{L}=0$ on these equations (closure relation). In generating form, the latter is equivalent to

$$\partial_\nu\mathscr{L}(\lambda,\mu)+\partial_\lambda\mathscr{L}(\mu,\nu)+\partial_\mu\mathscr{L}(\nu,\lambda)=0.$$

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Theorem

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Corollary: The generating Lax equation for the AKNS hierarchy is variational! FNR had shown the flows were Hamiltonian but no Lagrangian interpretation was known.

3. How to construct a Lagrangian multiform?

- Euler-Lagrange eqs for $\mathscr{L}_{ij}$ are equivalent to the corresponding zero curvature equation

$$\partial_{t_i} V^{(j)}(\lambda) - \partial_{t_j} V^{(i)}(\lambda) + [V^{(j)}(\lambda), V^{(i)}(\lambda)] = 0$$

- Explicit calculation reproduces

$$\mathscr{L}_{12} = \frac{1}{2}(rq_2 - qr_2) + \frac{i}{2}q_1 r_1 + \frac{i}{2}q^2 r^2$$

$$\mathscr{L}_{13} = \frac{1}{2}(rq_3 - qr_3) - \frac{1}{8}(r_1 q_{11} - q_1 r_{11}) - \frac{3qr}{8}(rq_1 - qr_1)$$

and gives other systematically.

Reinterpretation with classical r-matrix

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•The kernel $1/(\mu-\lambda)$ is typical of the **rational r-matrix**

$$r_{12}(\lambda,\mu)=\frac{P_{12}}{(\mu-\lambda)}$$

$$P_{12}=\begin{pmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{pmatrix}\ \text{permutation operator on } \mathbb{C}^2\otimes\mathbb{C}^2$$

Reinterpretation with classical r-matrix

- Then

$$\partial_\mu Q(\lambda) = \left[\frac{Q(\mu)}{\mu - \lambda}, Q(\lambda)\right] \Leftrightarrow \partial_\mu Q_1(\lambda) = \left[\mathrm{Tr}_2 r_{12}(\lambda,\mu) Q_2(\mu), Q_1(\lambda)\right] .$$

Reinterpretation with classical r-matrix

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- Generating function for the Lax matrices

$$V(\lambda,\mu) = \mathrm{Tr}_2 r_{12}(\lambda,\mu) Q_2(\mu) = \sum_{k=0}^{\infty} \frac{1}{\mu^{k+1}} V^{(k)}(\lambda)$$

Reinterpretation with classical r-matrix

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- Generating function for the Lax matrices

$$V(\lambda,\mu) = \text{Tr}_2 r_{12}(\lambda,\mu) Q_2(\mu) = \sum_{k=0}^{\infty} \frac{1}{\mu^{k+1}} V^{(k)}(\lambda)$$

- It can be shown [Avan, Caudrelier '16] that each $V^{(k)}(\lambda)$ satisfies a Sklyanin (Lie-Poisson) bracket

$$\{V_1^{(k)}(\lambda), V_2^{(k)}(\mu)\}_k = [r_{12}(\lambda,\mu), V_1^{(k)}(\lambda) + V_2^{(k)}(\mu)]$$

Reinterpretation with classical r-matrix

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$$\{V_1^{(k)}(\lambda), V_2^{(k)}(\mu)\}_k = [r_{12}(\lambda,\mu), V_1^{(k)}(\lambda) + V_2^{(k)}(\mu)]$$

- Jacobi identity ensured by the Classical Yang-Baxter equation

$$\left[r_{12}(\lambda,\mu), r_{13}(\lambda,\nu)\right] + \left[r_{12}(\lambda,\mu), r_{23}(\mu,\nu)\right] - \left[r_{13}(\lambda,\nu), r_{32}(\nu,\mu)\right] = 0.$$

Liouville integrability from classical r-matrix formalism

- From

$$\{V_1^{(k)}(\lambda), V_2^{(k)}(\mu)\}_k = [r_{12}(\lambda,\mu), V_1^{(k)}(\lambda) + V_2^{(k)}(\mu)]$$

the **monodromy matrix** $T^{(k)}(\lambda)$ associated to $V^{(k)}(\lambda)$ satisfies Sklyanin quadratic Poisson bracket

$$\{T_1^{(k)}(\lambda), T_2^{(k)}(\mu)\}_k = [r_{12}(\lambda,\mu), T_1^{(k)}(\lambda)T_2^{(k)}(\mu)]$$

Liouville integrability from classical r-matrix formalism

- From

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- Consequence

$$\{\mathrm{Tr}\, T^{(k)}(\lambda), \mathrm{Tr}\, T^{(k)}(\mu)\}_k = 0 \Rightarrow \{H_i, H_j\}_k = 0$$

Key observations for our AKNS generating Lagrangian multiform : beyond a single hierarchy.

1. The potential term in $\mathscr{L}(\lambda,\mu)$ has a characteristic form

$$\text{Tr}_{12}\left(r_{12}(\lambda,\mu)Q_1(\lambda)Q_2(\mu)\right)$$

where $r_{12}(\lambda,\mu)=\frac{P_{12}}{\mu-\lambda}$ is the rational r-matrix.

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

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where $r_{12}(\lambda, \mu) = \frac{P_{12}}{\mu - \lambda}$ is the rational r-matrix.

$\rightarrow$ How about replacing this particular r-matrix with another (skew-symmetric) r-matrix?

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

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4. A generating Lagrangian multiform for ultralocal field theories and CYBE

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3. The Pauli matrix in $Q_0 = -i\sigma_3$ is a special choice of constant element in the underlying loop algebra of sl_2 from which the phase space is built as a (co)adjoint orbit.

$\rightarrow$ How about considering other elements in the loop algebra to construct different phase spaces and even considering other Lie algebras than sl_2?

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

• Careful implementation of these natural observations involves using the **Lie algebra of $\mathfrak{g}$-valued adèles** associated with a Lie algebra $\mathfrak{g}$ instead of the loop algebra of sl_2 [Semenov-Tian-Shansky '08].

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

- Careful implementation of these natural observations involves using the **Lie algebra of $\mathfrak{g}$-valued adèles** associated with a Lie algebra $\mathfrak{g}$ instead of the loop algebra of sl_2 [Semenov-Tian-Shansky '08].

- In a nutshell, with $\lambda_a = \lambda - a$ for $a \in \mathbb{C}$ and $\lambda_\infty = \frac{1}{\lambda}$,

$$\boldsymbol{\mathcal{A}}_{\boldsymbol{\lambda}}(\mathfrak{g}) := \coprod_{a \in \mathbb{C}P^1} \mathfrak{g} \otimes \mathbb{C}(\!(\lambda_a)\!)\,,$$

An element $\boldsymbol{X}(\boldsymbol{\lambda}) = (X^a(\lambda_a))_{a \in \mathbb{C}P^1}$ of this algebra consist of tuples with all but finitely many of the formal Laurent series $X^a(\lambda_a) \in \mathfrak{g} \otimes \mathbb{C}(\!(\lambda_a)\!)$ being Taylor series in λ_a, *i.e.* there exists a finite subset $S \subset \mathbb{C}P^1$ such that $X^a(\lambda_a) \in \mathfrak{g} \otimes \mathbb{C}[\![\lambda_a]\!]$ for every $a \in \mathbb{C} \setminus S$.

Schematic implementation of the generalisation

$$\mathfrak{sl}_2 \qquad \rightarrow \qquad \mathfrak{g}$$

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

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Q_0 = -i\sigma_3 & \to & (\boldsymbol{\iota}_{\boldsymbol{\lambda}} F(\lambda))_- \text{ collection of principal parts} \\
 & & \text{of } \mathfrak{g}\text{-valued rational function } F(\lambda) \\
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$Q_0 = -i\sigma_3 \quad \to \quad (\boldsymbol{\iota_\lambda} F(\lambda))_-$ collection of principal parts of $\mathfrak{g}$-valued rational function $F(\lambda)$ with poles in finite set S

$\frac{P_{12}}{\mu-\lambda} \quad \to$ any skew-symmetric r-matrix $r_{12}(\lambda, \mu)$

$\mathscr{L}(\lambda, \mu) \quad \to \quad \mathscr{L}(\boldsymbol{\lambda}, \boldsymbol{\mu})$ = collection of $\mathscr{L}^{a,b}(\lambda_a, \mu_b)$

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

- **Elementary Lagrangians** computed as

$$\mathscr{L}^{a,b}_{m,n} \coloneqq \text{res}^{\lambda}_{a}\, \text{res}^{\mu}_{b}\, \mathscr{L}^{a,b}(\lambda_a, \mu_b)\lambda^{-m-1}d\lambda\, \mu^{-n-1}d\mu$$

- **Elementary Lax matrices** $V^a_m(\lambda)$ similarly computed from

$$\boldsymbol{V}(\lambda; \boldsymbol{\mu}) \coloneqq \text{Tr}_2\left(\boldsymbol{\iota}_{\boldsymbol{\mu}} r_{12}(\lambda, \mu)\boldsymbol{Q}_2(\boldsymbol{\mu})\right)$$

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- Key message: Euler-Lagrange eqs for $\mathscr{L}^{a,b}_{m,n}$ equivalent to zero curvature equation for times t^a_m, t^b_n

$$\partial_{t^b_n} V^a_m(\lambda) - \partial_{t^a_m} V^b_n(\lambda) + \left[V^a_m(\lambda), V^b_n(\lambda)\right] = 0$$

$\rightarrow$ **Full integrable hierarchy in variational form.**

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

Main results

Theorem

The generating Lax equation

$$\mathcal{D}_{\boldsymbol{\mu}}\boldsymbol{Q}_1(\boldsymbol{\lambda}) = \left[\, \mathrm{Tr}_2 \left(\boldsymbol{\iota}_{\boldsymbol{\lambda}}\boldsymbol{\iota}_{\boldsymbol{\mu}} r_{12}(\lambda,\mu)\boldsymbol{Q}_2(\boldsymbol{\mu})\right), \boldsymbol{Q}_1(\boldsymbol{\lambda})\right]. \tag{1}$$

is variational: it derives from the multiform EL eqs for $\mathscr{L}(\boldsymbol{\lambda},\boldsymbol{\mu})$. The closure relation in generating form

$$\mathcal{D}_{\boldsymbol{\nu}}\mathscr{L}(\boldsymbol{\lambda},\boldsymbol{\mu}) + \mathcal{D}_{\boldsymbol{\mu}}\mathscr{L}(\boldsymbol{\nu},\boldsymbol{\lambda}) + \mathcal{D}_{\boldsymbol{\lambda}}\mathscr{L}(\boldsymbol{\mu},\boldsymbol{\nu}) = 0$$

holds as a consequence of the CYBE equation.

Theorem

The flows (1) on the Lie algebra of $\mathfrak{g}$-valued adèles commute as a consequence of the CYBE

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

Theorem

The CYBE also ensures that the generating zero curvature equation holds

$$\mathcal{D}_{\boldsymbol{\nu}} \boldsymbol{V}(\lambda;\boldsymbol{\mu}) - \mathcal{D}_{\boldsymbol{\mu}} \boldsymbol{V}(\lambda;\boldsymbol{\nu}) + \left[\boldsymbol{V}(\lambda;\boldsymbol{\mu}), \boldsymbol{V}(\lambda;\boldsymbol{\nu})\right] = 0,$$

where

$$\boldsymbol{V}(\lambda;\boldsymbol{\mu}) \coloneqq \mathrm{Tr}_2\left(\boldsymbol{\iota}_{\boldsymbol{\mu}} r_{12}(\lambda,\mu) \boldsymbol{Q}_2(\boldsymbol{\mu})\right)$$

generates the local Lax matrices as

$$V^b(\lambda;\mu_b) = \sum_{n=-N_b}^{\infty} V_n^b(\lambda)\mu_b^n\,, \quad b \in \mathbb{C}\,,$$

$$V^\infty(\lambda;\mu_\infty) = \sum_{n=-N_\infty}^{\infty} V_n^\infty(\lambda)\mu_\infty^{n+k+1}\,.$$

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

Procedure to get examples.

Choose:

(i) a skew-symmetric r-matrix (rational or trig for us),

(ii) an effective divisor $\mathcal{D} := \sum_{a\in S} N_a a$, with support given by a finite subset $S \subset \mathbb{C}P^1$,

(ii) a Lie algebra $\mathfrak{g}$ which for simplicity we take to be either gl_N or sl_N,

(iv) a $\mathfrak{g}$-valued rational function $F(\lambda) \in R_\lambda(\mathfrak{g})$ with poles divisor $(F)_\infty = \mathcal{D}$, *i.e.* with a pole of order N_a at each point $a \in S$.

Recovering the original AKNS example.

Fix the data as

$$S = \{\infty\}\,, \quad N_\infty = 0\,, \quad \mathfrak{g} = \mathrm{sl}_2\,, \quad F(\lambda) = -i\sigma_3\,,$$

and choose the rational r-matrix $r_{12}(\lambda,\mu) = \frac{P_{12}}{\mu-\lambda}$.

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

Sine-Gordon hierarchy

For the hierarchy of the sine-Gordon equation (in light-cone coords)

$$u_{xy} + \sin u = 0\,,$$

we fix $S = \{0, \infty\}$, $N_0 = 1 = N_\infty$, $\mathfrak{g} = \mathrm{sl}_2$,

$$F(\lambda) = \frac{i}{2}\left(\frac{1}{\lambda}\sigma_+ + \sigma_- - \sigma_+ - \lambda\sigma_-\right)$$

and we choose the trigonometric r-matrix

$$r_{12}^{\mathrm{trig}}(\lambda, \mu) = \frac{1}{2}\left(P_{12}^{+} - P_{12}^{-} + \frac{\mu + \lambda}{\mu - \lambda}P_{12}\right)$$

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

We can derive all elementary Lagrangians. We find

$$\mathscr{L}_{\mathrm{sG}} \equiv \mathscr{L}_{00}^{0,\infty} = -\frac{1}{4}u_x u_y - \frac{1}{2}\cos u$$

$$\mathscr{L}_{\mathrm{mKdV}} \equiv \mathscr{L}_{01}^{\infty,\infty} = \frac{1}{4}u_x u_z + \frac{1}{16}u_x^4 - \frac{1}{4}u_{xx}^2 - \frac{i}{4}\partial_x\left(\frac{1}{6}u_x^3 + i u_x u_{xx}\right)$$

$$\begin{aligned}\mathscr{L}_{\mathrm{mixed}} \equiv \mathscr{L}_{01}^{0,\infty} \quad = \quad & -\frac{1}{4}u_y u_z - \frac{1}{2}u_{xx}(u_{xy} + \sin u) + \frac{1}{4}u_x^2 \cos u \\ & -\frac{i}{4}\partial_y\left(\frac{1}{6}u_x^3 + i u_x u_{xx}\right)\end{aligned}$$

Recover the results of [Suris '16].

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

Hierarchies of Zakharov-Mikhailov type

Correspond to Lax matrices of Zakharov-Shabat type: rational Lax matrices with prescribed pole structures.

- In our setup, choose the following data

$$S = \{a_1, \dots, a_P\} \subset \mathbb{C}\,, \quad P > 0\,, \quad \mathfrak{g} = \mathrm{gl}_N\,,$$

$$F(\lambda) = -\sum_{i=1}^{P} \sum_{r=0}^{n_i} \frac{A_{ir}}{(\lambda - a_i)^{r+1}}\,.$$

- Each $A_{ir} \in \mathrm{gl}_N$ is a non-dynamical constant matrix.

- r-matrix can be the rational (original Zakharov-Mikhailov case) or trigonometric (new models). Even in rational case, obtain full hierarchy, not just a single model/level.

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

Most famous example: Faddeev-Reshetikhin version of Principal chiral model

- 2 simple poles $a, b = -a$ in S,

$$F(\lambda) = -\frac{A}{(\lambda - a)} - \frac{B}{(\lambda + a)}\,.$$

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- Lowest elementary Lax matrices for times $t^{a}_{-1} \equiv \xi$, $t^{-a}_{-1} \equiv \eta$

$$V^{a}_{-1}(\lambda) = \frac{\phi A \phi^{-1}}{\lambda - a} \equiv \frac{J_0}{\lambda - a}\,, \quad V^{b}_{-1}(\lambda) = \frac{\psi B \psi^{-1}}{\lambda + a} \equiv \frac{J_1}{\lambda + a}$$

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- Zero curvature equations

$$\partial_\eta J_0 + \frac{1}{2a}[J_0, J_1] = 0\,, \quad \partial_\xi J_1 + \frac{1}{2a}[J_0, J_1] = 0\,.$$

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

- We get the lowest elementary Lagrangian as

$$\mathscr{L}^{ab}_{-1-1} = \mathrm{Tr}\left(\phi^{-1}\partial_\eta\phi A - \psi^{-1}\partial_\xi\psi B - \frac{\phi A\phi^{-1}\,\psi B\psi^{-1}}{2a}\right)\,.$$

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

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• Procedure produces new models/hierarchies that are automatically integrable. NB: not the same as taking combination of flows within a hierarchy.

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• Example: couple nonlinear Schrödinger to Faddeev-Reshetikhin.

$$S = \{a, -a, \infty\}\,, \quad a \in \mathbb{C}^\times\,, N_a = N_b = 1\,, \quad N_\infty = 0\,, \quad \mathfrak{g} = \mathrm{sl}_2\,,$$

$$F(\lambda) = -i\alpha\sigma_3 + \frac{A}{\lambda - a} + \frac{B}{\lambda + a} \equiv \alpha F^{AKNS}(\lambda) + F^{FR}(\lambda)\,,$$

where A, B are constant sl_2 matrices.

• α couples the two theories: $\alpha = 0$ gives a pure FR theory while sending α to infinity produces a pure AKNS hierarchy.

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

- Extract model at lowest level. Can compute Lax matrices and zero curvature equations and Lagrangian that produces those equations.

$$\partial_\eta J_0 + \frac{1}{2a}\,[J_0, J_1] + \alpha\,[J_0, V_{NLS}(a)] = 0\,,$$

$$\partial_\xi J_1 + \frac{1}{2a}\,[J_0, J_1] - \alpha\,[U_{NLS}(-a), J_1] = 0\,,$$

$$\alpha\partial_\xi Q_1 + i\alpha^2[\sigma_3, Q_2] + i\alpha[J_0, \sigma_3] = 0\,,$$

$$\alpha\partial_\eta Q_1 - \alpha\partial_\xi Q_2 + \alpha^2[Q_1, Q_2] - ia\alpha[J_0, \sigma_3] - i\alpha[\sigma_3, J_1] + \alpha[J_0, Q_1] = 0\,.$$

- If needed, can compute all higher levels in hierarchy.

5. Conclusions

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1. We brought together the theory of Lagrangian multiforms and of the classical r-matrix and CYBE for the first time.
$\rightarrow$ Conceptual by-products: (i) CYBE acquires a variational interpretation for first time; (ii) closure relation established as fundamental criterion for "Lagrangian integrability".

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$\rightarrow$ Conceptual by-products: (i) CYBE acquires a variational interpretation for first time; (ii) closure relation established as fundamental criterion for "Lagrangian integrability".

2. **Constructive** approach to **derive** (not guess), from minimal (algebraic) input, Lax pairs and Lagrangians for corresponding zero curvature equations. Applicable to large variety of integrable hierarchies, old and new.

Many open questions.

Classical level

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• Connection between our results and $4d$ Chern-Simons construction of integrable field theories. Extend results from [Caudrelier, Stoppato, Vicedo '21] to entire hierarchies encoded in our generating Lagrangian multiform?

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• What about non ultralocal integrable theories? How to relate with other important construction of non ultralocal integrable theories via affine Gaudin models?

Quantum level

Many open questions.

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Quantum level

• Covariant quantization of integrable field theories? Relation to quantum R matrix and quantum YBE?

THANK YOU!

4. A generating Lagrangian multiform for ultralocal field theories and CYBE

In formulas,

$$\mathscr{L}(\boldsymbol{\lambda},\boldsymbol{\mu}) := \boldsymbol{K}(\boldsymbol{\lambda},\boldsymbol{\mu}) - \boldsymbol{U}(\boldsymbol{\lambda},\boldsymbol{\mu}) = (\mathscr{L}^{a,b}(\lambda_a,\mu_b))_{a,b\in\mathbb{C}P^1}$$

Kinetic and potential terms

$$\begin{aligned}\boldsymbol{K}(\boldsymbol{\lambda},\boldsymbol{\mu}) &= \operatorname{Tr}\left(\boldsymbol{\phi}(\boldsymbol{\lambda})^{-1}\mathcal{D}_{\boldsymbol{\mu}}\boldsymbol{\phi}(\boldsymbol{\lambda})(\boldsymbol{\iota_\lambda}F(\lambda))_-\right)\\ &\qquad - \operatorname{Tr}\left(\boldsymbol{\phi}(\boldsymbol{\mu})^{-1}\mathcal{D}_{\boldsymbol{\lambda}}\boldsymbol{\phi}(\boldsymbol{\mu})(\boldsymbol{\iota_\mu}F(\mu))_-\right),\end{aligned}$$

$$\boldsymbol{U}(\boldsymbol{\lambda},\boldsymbol{\mu}) = \frac{1}{2}\operatorname{Tr}_{12}\left((\boldsymbol{\iota_\lambda}\boldsymbol{\iota_\mu} + \boldsymbol{\iota_\mu}\boldsymbol{\iota_\lambda})r_{12}(\lambda,\mu)\boldsymbol{Q}_1(\boldsymbol{\lambda})\boldsymbol{Q}_2(\boldsymbol{\mu})\right).$$

Generating Lax equation reads

$$\mathcal{D}_{\boldsymbol{\mu}}\boldsymbol{Q}_1(\boldsymbol{\lambda}) = \left[\operatorname{Tr}_2\left(\boldsymbol{\iota_\lambda}\boldsymbol{\iota_\mu}r_{12}(\lambda,\mu)\boldsymbol{Q}_2(\boldsymbol{\mu})\right),\boldsymbol{Q}_1(\boldsymbol{\lambda})\right]. \qquad (2)$$

derives from multiform EL eqs.