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Eigenvalue systems for integer orthogonal bases of multi-matrix invariants at finite N

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Abstract: Multi-matrix invariants, and in particular the scalar multi-trace operators of $\mathcal{N} = 4$ SYM with $\mathrm{U}(N)$ gauge symmetry, can be described using permutation centraliser algebras (PCA), which are generalisations of the symmetric group algebras and independent of N. Free-field two-point functions define an N-dependent inner product on the PCA, and bases of operators have been constructed which are orthogonal at finite N. Two such bases are well-known, the restricted Schur and covariant bases, and both definitions involve representation-theoretic quantities such as Young diagram labels, multiplicity labels, branching and Clebsch-Gordan coefficients for symmetric groups. The explicit computation of these coefficients grows rapidly in complexity as the operator length increases. We develop a new method for explicitly constructing all the operators with specified Young diagram labels, based on an N-independent integer eigensystem formulated in the PCA. The eigensystem construction naturally leads to orthogonal basis elements which are integer linear combinations of the multi-trace operators, and the N-dependence of their norms are simple known dimension factors. We provide examples and give computer codes in SageMath which efficiently implement the construction for operators of classical dimension up to 14. While the restricted Schur basis relies on the Artin-Wedderburn decomposition of symmetric group algebras, the covariant basis relies on a variant which we refer to as the Kronecker decomposition. Analogous decompositions exist for any finite group algebra and the eigenvalue construction of integer orthogonal bases extends to the group algebra of any finite group with rational characters.

Keywords: $1/N$ Expansion, AdS-CFT Correspondence, Gauge Symmetry, Matrix Models

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1 Introduction

An important problem in the AdS/CFT correspondence [1–3] is to understand the map between quantum states in the CFT and in the string theory side. The operator-state map in CFTs [4] means that the quantum states in CFT correspond to local operators. In any CFT with gauge symmetry, the local operators are constructed as gauge-invariant composites of the elementary fields. The study of the construction and correlators of general half-BPS operators in four-dimensional $\mathcal{N} = 4$ super Yang-Mills (SYM) with U(N) gauge group $\mathcal{N} = 4$ SYM [5] has allowed a detailed study of the physics of giant gravitons [6] and half-BPS geometries [7] in ten-dimensional space-time from the dual CFT point of view.

Half-BPS local operators in the U(N) theory are gauge-invariant holomorphic functions of a complex matrix Z of size $N \times N$. Since the half-BPS sector is protected, the inner product on the space of half-BPS operators defined by the free-field two-point functions is valid for any coupling. The operators of dimension L, which are polynomial gauge invariants of degree L, have an orthogonal basis labelled by Young diagrams with L boxes and no more than N rows. The elements of the basis are expressible as linear combinations of multi-trace operators with coefficients given by characters of the symmetric group S_L for the irreducible representation specified by the Young diagram, evaluated on a conjugacy class determined by the trace structure of the multi-trace operator. The norm of the operator in the free-field inner product is a polynomial in N which is related to dimensions of U(N) and S_L representations associated with the Young diagram [5].

The quarter-BPS and 1/8-BPS sectors contain gauge-invariant operators of more than one type of complex matrices, which we call multi-matrix operators or multi-matrix invariants. Multi-matrix operators are relevant to strings attached to giant gravitons and to brane-anti-brane systems. Finite N bases for multi-matrix operators were constructed in [8–12], which are orthogonal under the free-field inner product, and the norms of the basis elements were calculated. The number of elements in these finite N bases were shown ([13] and section 6 of [12]) to agree with finite N counting formulae for gauge invariant operators [14–17]. The basis in [10, 13] is known as the restricted Schur basis, and has its origins in studies of general (BPS or non-BPS) open string excitations of giant gravitons [18–20]. The simplest instance involves holomorphic gauge-invariant functions of two matrices. The construction of [8] was motivated by the brane-anti-brane interpretation of operators constructed from $Z, Z^\dagger$ and employed Walled Brauer algebras. The basis in [9] was studied with motivations coming from the quarter-BPS sector and involved holomorphic functions of two complex matrices, which are quarter-BPS at zero Yang-Mills coupling. This basis is known as the covariant basis for the two-matrix system, since the global U(2) rotating the two matrices into each other is manifest. The generalisation to multi-matrix systems for general global symmetry group was described in [12]. The existence of different orthogonal bases was explained in terms of Casimirs for enhanced symmetries in the zero coupling limit in [21]. The generalisation of the restricted Schur and covariant matrix bases to quiver gauge theories was given in [22]. Reviews of work on applications of these results on field theoretic multi-matrix combinatorics are available in [23–26].

It has been explained that the space of two-matrix gauge-invariant operators is closely related to the structure of an algebra $\mathcal{A}(\mu_1, \mu_2)$ which is an instance of a permutation centraliser algebra (PCA) [27]. $\mathcal{A}(\mu_1, \mu_2)$ is a sub-algebra of $\mathbb{C}[S_L]$, where $\mathbb{C}[S_L]$ is the group algebra of the symmetric group S_L of all permutations of $\{1, 2, \cdots, L\}$. It is the sub-algebra which commutes with the sub-group $S_{\mu_1} \times S_{\mu_2}$ of S_L, where $\mu_1 + \mu_2 = L$ and S_{μ_1} permutes $\{1, \cdots, \mu_1\}$ and S_{μ_2} permutes $\{\mu_1 + 1, \cdots, \mu_1 + \mu_2 = L\}$:

$$\mathcal{A}(\mu_1, \mu_2) = \Big\{\sigma \in \mathbb{C}[S_{\mu_1+\mu_2}] \,\Big|\, \gamma\sigma\gamma^{-1} = \sigma, \quad \forall \gamma \in S_{\mu_1} \times S_{\mu_2}\Big\}. \tag{1.1}$$

More generally, the gauge-invariant operators of M types of complex matrices are related to

$$\mathcal{A}(\mu) = \mathcal{A}(\mu_1, \mu_2, \ldots, \mu_M) = \Big\{\sigma \in \mathbb{C}[S_L] \,\Big|\, \gamma\sigma\gamma^{-1} = \sigma, \quad \forall \gamma \in S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_M} \subseteq S_L\Big\}, \tag{1.2}$$

$\mu_1 + \mu_2 + \cdots + \mu_M = L$. The structure constants of PCAs have been shown to arise in the correlators of Gaussian matrix models with background fields [28] and these algebras have been found to be useful in proving identities for Littlewood-Richardson coefficients which arise in quantum information theory [29].

In this paper we develop efficient algorithms to compute orthogonal bases for the permutation centraliser algebras $\mathcal{A}(\mu_1, \mu_2)$. These lead to orthogonal bases for the two-matrix system, which are closely related to the restricted Schur and covariant bases.

The restricted Schur basis for gauge-invariant composites of matrices Z, W of size $N \times N$, of degree μ_1 in Z and degree μ_2 in W takes the form

$$\mathcal{O}^{R,(r_1,r_2)}_{\nu_-,\nu_+}[Z, W] \tag{1.3}$$

where r_1, r_2, R are Young diagrams with $\mu_1, \mu_2, \mu_1 + \mu_2 \equiv L$ boxes respectively, with their number of rows should be less or equal to N. These Young diagrams label irreducible representations of symmetric groups, denoted by $V^{S_{\mu_1}}_{r_1}, V^{S_{\mu_2}}_{r_2}, V^{S_{\mu_1+\mu_2}}_{R}$ respectively. The index $\nu_\mp$ is an integer ranging over the Littlewood-Richardson coefficient $g(r_1, r_2; R)$, which has an interpretation as the multiplicity in the reduction of the irreducible representation $V^{S_{\mu_1+\mu_2}}_{R}$ to the representation $V^{S_{\mu_1}}_{r_1} \otimes V^{S_{\mu_2}}_{r_2}$. The restricted Schur basis is closely related to the Artin-Wedderburn decomposition of $\mathcal{A}(\mu)$. The explicit formula for the operators (1.3) involves branching coefficients for this reduction (for the details see [10, 11, 21]). The explicit construction of these branching coefficients has been discussed in the mathematical physics literature [30–37] and become very complex when μ_1, μ_2 increase up to say 4, 5. Nevertheless the Young diagram structure of the restricted Schur basis has been useful in identifying new large N integrable sectors in the two-matrix system (see [38–40] and references therein).

The covariant basis elements [9, 12] for gauge-invariant composites of Z, W of size $N \times N$ of total degree L in the two matrices take the form

$$\mathcal{O}^{R,\Lambda,M_\Lambda,\tau}[Z, W] \tag{1.4}$$

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where R, Λ are Young diagrams with L boxes whose length (number of non-vanishing rows) obey $\ell(R) \leq N, \ell(\Lambda) \leq 2$. Λ is associated with a representation $V_\Lambda^{S_L}$ of S_L and also by Schur-Weyl duality with a representation $V_\Lambda^{\mathrm{U}(2)}$ of U(2). M_Λ denotes a state in the representation $V_\Lambda^{\mathrm{U}(2)}$ of U(2). The index τ runs over the multiplicity of $V_\Lambda^{S_L}$ in the tensor product decomposition of $V_R^{S_L} \otimes V_R^{S_L}$. This multiplicity is equal to the Kronecker coefficient $C(R, R, \Lambda)$ for the triple of representations (R, R, Λ) of S_L and is also equal to the number of times the trivial representation of S_L appears in this tensor product $V_R^{S_L} \otimes V_R^{S_L} \otimes V_\Lambda^{S_L}$. In this two-matrix case, the state label M_Λ can further be decomposed into a pair of non-negative integers (μ_1, μ_2) with $\mu_1 + \mu_2 = L$, along with a branching multiplicity β. The integers μ_1 and μ_2 are the numbers of Z and W respectively in the composite operator, and the index β ranges over the Kotska number $K_{\Lambda(\mu_1,\mu_2)}$, which represents how many times the trivial representation of $S_{\mu_1} \times S_{\mu_2}$ appears when the representation $V_\Lambda^{S_L}$ is restricted from S_L to the subgroup $S_{\mu_1} \times S_{\mu_2}$. The explicit formula for the basis elements involves Clebsch-Gordan coefficients for the overlaps of the states in $V_R^{S_L} \otimes V_R^{S_L}$ with the states in the subspace $V_\Lambda^{S_L}$, as well as the branching coefficients for the trivial representation of $S_{\mu_1} \times S_{\mu_2}$ in $V_\Lambda^{S_L}$. Explicit examples at small μ_1, μ_2 in [9, 12] were given using standard group theory constructions of symmetric group Clebsch-Gordan coefficients [41], but these constructions become rapidly very complex as L increases.

We will explain in section 2.2 that the covariant construction is closely related to a variant of Artin-Wedderburn decomposition of $\mathbb{C}[S_L]$ which we call Kronecker decomposition.

We will describe an eigenvalue system for matrices derived from a selection of central elements in $\mathcal{A}(\mu_1, \mu_2)$, which allows the construction of multi-matrix invariants of restricted Schur type in the vector space of dimension $g(r_1, r_2; R)^2$

$$\bigoplus_{\nu_-,\nu_+=1}^{g(r_1,r_2;R)} \text{Span} \left(\mathcal{O}^{R,(r_1,r_2)}_{\nu_-,\nu_+}[Z, W]\right) \tag{1.5}$$

The matrices in this eigenvalue system have non-negative integer entries. The eigenvalues are also integers, as they are expressible in terms of characters of symmetric group elements in the irreducible representations r_1, r_2, R. This integrality of the matrices and their eigenvalues means that the eigenvectors can be constructed as integer linear combinations of multi-trace functions of Z, W by standard algorithms for null vectors of integer matrices based on Hermite normal forms. The output of the null vector algorithms can be efficiently orthogonalised, in the planar limit of the free-field inner product by a Gram-Schmidt procedure. This naturally produces rational linear combinations of multi-traces but can easily be adapted to produce integer combinations. Interestingly, the orthogonal basis for the planar inner product thus produced is automatically orthogonal under the finite N inner product.

Extending these considerations to the covariant basis (1.4) for each R, Λ, M_Λ, we will describe an eigenvalue system allowing the construction of multi-matrix invariants in the vector space of dimension $C(R, R, \Lambda)$

$$\bigoplus_{\tau=1}^{C(R,R,\Lambda)} \text{Span} \left(\mathcal{O}^{R,\Lambda,M_\Lambda,\tau}[Z, W]\right). \tag{1.6}$$

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In this case the eigenvalue system is derived by considering the matrix elements of the appropriate actions of central elements of $\mathbb{C}[S_L]$. The left or right action selects R and the adjoint action selects Λ. The integrality of the eigenvalue system leads to a basis for (1.6) which consists of integer linear combinations of multi-traces. The states of different M_Λ are related by the generators of U(2), expressed as operators on

$$\bigoplus_{\substack{\mu_1,\mu_2\\ \mu_1+\mu_2=L}} \mathcal{A}(\mu_1,\mu_2). \tag{1.7}$$

Similar eigenvalue methods have been developed in the context of representation-theoretic orthogonal bases arising in tensor models with $U(N)$ symmetries [42] as well as matrix [43] and tensor [44] models with S_N symmetries. In a close similarity to (1.3) and (1.4) the basis labels in these cases include representation labels as well as appropriate multiplicity labels. These applications, as well as the present one, exploit properties of centres of symmetric group algebras whereby the characters of a small set conjugacy classes of S_L, which multiplicatively generate the centres, suffice to identify an irreducible representation of S_L [45, 46].

The paper is organised as follows. In section 2 we review the connection between multi-matrix invariants, symmetric group algebras, permutation centraliser algebras and the previously mentioned representation bases. An important result in section 3 is that the representation bases of $\mathcal{A}(\mu)$ is characterised as eigenstates of an integer eigensystem constructed from certain linear combinations in $\mathbb{C}[S_L]$ acting on $\mathcal{A}(\mu)$. We explain how to construct integer eigenvector solutions to the eigensystem by using the Hermite normal form from the theory of integer matrices. A similar construction is applied to the Artin-Wedderburn and the Kronecker decompositions of $\mathbb{C}[S_L]$ in section 4. Section 5 extends this dicussion to general finite group algebras, specialises to rational groups where analogous integrality properties hold. Appendix A details the notation used for group and representation theoretic quantities. Details on the algorithm used to compute integer bases for $\mathcal{A}(\mu_1,\mu_2)$ are given in B. This paper is accompanied by a computer code which easily produces the results up to $L=10$ by laptop, and in principle works up to $L=14$. We give a complete set of the restricted and covariant basis elements for $\mathcal{A}(\mu_1,\mu_2)$ up to $L=5$, and give an interesting example at $L=6$ in appendix C. Following section 4, we give a complete set of bases elements for $\mathbb{C}[S_L]$ up to $L=4$ in appendix D.

2 Symmetric group algebras and multi-matrix invariants

Physically, we are interested in constructing an orthogonal basis of all multi-trace operators of $\mathcal{N}=4$ SYM. The $\mathcal{N}=4$ SYM contains six real scalar fields in the adjoint representation of the gauge group $U(N)$, denoted by Φ^I_{ij} with $I=1,2,\ldots,6$ and $i,j=1,2,\ldots,N$. The $\mathfrak{su}(2)$ sector of $\mathcal{N}=4$ SYM is made of two complex scalars, $Z=\Phi^5+\sqrt{-1}\,\Phi^6$ and $W=\Phi^3+\sqrt{-1}\,\Phi^4$. The general multi-trace operator in the $\mathfrak{su}(2)$ sector of length $L=\mu_1+\mu_2$ can be written as

$$\mathcal{O}_g[Z,W]=\sum_{i_1,i_2,\ldots,i_{\mu_1+\mu_2}=1}^{N}\prod_{a=1}^{\mu_1} Z_{i_a i_{g(a)}} \prod_{b=\mu_1+1}^{\mu_1+\mu_2} W_{i_b i_{g(b)}} \tag{2.1}$$

with g in the symmetric group $S_{\mu_1+\mu_2}$. It is straightforward to generalize (2.1) to all multi-trace operators made of M complex scalars as

$$\mathcal{O}_g(\vec{a}) = \sum_{i_1,i_2,\ldots,i_L=1}^{N} X^{a_1}_{i_1 i_{g(1)}} X^{a_2}_{i_2 i_{g(2)}} \ldots X^{a_L}_{i_L i_{g(L)}} \tag{2.2}$$

where $a_i \in \{1,2,\ldots,M\}$. We call $\mathcal{O}_g(\vec{a})$ multi-matrix invariants, or gauge-invariant operators. As will be shown below, the multi-matrix invariants possess a conjugation symmetry

$$\mathcal{O}_g(a_1,a_2\ldots a_L) = \mathcal{O}_{\gamma g \gamma^{-1}}(a_{\gamma(1)}\, a_{\gamma(2)}\, \ldots a_{\gamma(L)}), \qquad \forall \gamma \in S_L\,. \tag{2.3}$$

An extra sign is needed in this equation when X^a are fermionic fields. One example of such situations is the $\mathfrak{su}(2|3)$ sector of $\mathcal{N}=4$ SYM consisting of three complex scalars and two fermions [47, 48].

2.1 Notation

The symmetric group algebra $\mathbb{C}[S_L]$ helps organise contractions of tensor indices of multi-matrix $\mathrm{U}(N)$ invariants, and Young diagrams are useful for describing the irreducible representations of $\mathbb{C}[S_L]$. We begin by fixing our notation about $\mathbb{C}[S_L]$ and Young diagrams.

2.1.1 Group algebra actions

Let $g_1\,, g_2$ be elements in $S_L\,$. The composition of two permutations acts on $i \in \{1,2,\ldots,L\}$ as

$$(g_1\, g_2)\,(i) = g_2(g_1(i)). \tag{2.4}$$

Let $\sigma \in \mathbb{C}[S_L]$ be given by

$$\sigma = \sum_{g\in S_L} c_g(\sigma)\, g, \qquad c_g(\sigma) \in \mathbb{C}. \tag{2.5}$$

We define the δ-function by

$$\delta(g) = \begin{cases} 1 & g = \mathbf{1} \\ 0 & \text{otherwise} \end{cases} \tag{2.6}$$

and extend it linearly to the group algebra $\mathbb{C}[S_L]$ by

$$\delta(\sigma) = \sum_{g\in S_L} c_g(\sigma)\, \delta(g) = c_{\mathbf{1}}(\sigma). \tag{2.7}$$

The left and right actions of $\mathbb{C}[S_L]$ are denoted by the following linear operators

$$m^{\mathfrak{L}}[\sigma](\tau) = \sigma\tau, \qquad m^{\mathfrak{R}}[\sigma](\tau) = \tau\sigma, \qquad \sigma,\tau \in \mathbb{C}[S_L]. \tag{2.8}$$

The left and right actions always commute,

$$m^{\mathfrak{L}}[\sigma_1]\Big(m^{\mathfrak{R}}[\sigma_2](h)\Big) = m^{\mathfrak{R}}[\sigma_2]\Big(m^{\mathfrak{L}}[\sigma_1](h)\Big) = \sigma_1 h\, \sigma_2. \tag{2.9}$$

Expanding the left or right action of $\mathbb{C}[S_L]$ in a basis of S_L , we obtain their matrix elements as

$$\sigma h = \sum_{g\in S_L} m^{\mathfrak{L}}_{gh}[\sigma]\, g, \qquad h\sigma = \sum_{g\in S_L} m^{\mathfrak{R}}_{gh}[\sigma]\, g, \tag{2.10}$$

which can be written as

$$m^{\mathfrak{L}}_{gh}[\sigma] = \delta\Big(g^{-1}\sigma h\Big), \qquad m^{\mathfrak{R}}_{gh}[\sigma] = \delta\Big(g^{-1}h\sigma\Big). \tag{2.11}$$

We also introduce the involution

$$\mathcal{S}(\sigma) = \sum_{g\in S_L} c_g(\sigma)\, g^{-1} \tag{2.12}$$

which is known as the antipode in the context of group Hopf algebras. The adjoint action of $\mathbb{C}[S_L]$, often called conjugation in the literature, is denoted by

$$m^{\mathfrak{ad}}[\sigma](h) = \sum_{g\in S_L} c_g(\sigma)\, ghg^{-1}, \qquad \text{for} \quad \sigma = \sum_{g\in S_L} c_g(\sigma)\, g, \quad \sigma, h \in \mathbb{C}[S_L]. \tag{2.13}$$

The matrix elements of the adjoint action is denoted by

$$m^{\mathfrak{ad}}[\sigma](h) = \sum_{g\in S_L} m^{\mathfrak{ad}}_{gh}[\sigma]\, g. \tag{2.14}$$

2.1.2 Centre of $\mathbb{C}[S_L]$

The centre of $\mathbb{C}[S_L]$, denoted $\mathcal{Z}(\mathbb{C}[S_L])$, is the subalgebra defined by

$$\mathcal{Z}(\mathbb{C}[S_L]) = \Big\{ z \in \mathbb{C}[S_L] \,\Big|\, zg = gz \text{ for all } g \in S_L \Big\}. \tag{2.15}$$

The centre has two important bases. The first basis is labelled by conjugacy classes of S_L. Let $\rho \vdash L$ (here $\vdash L$ means an integer partition of L, see equation (2.35)) and C_ρ the corresponding conjugacy class of S_L, we define

$$z_\rho \equiv \frac{1}{|\mathrm{Stab}(g_\rho)|} \sum_{\gamma\in S_L} \gamma\, g_\rho\, \gamma^{-1} \tag{2.16}$$

for any $g_\rho \in C_\rho$, where $\mathrm{Stab}(g_\rho)$ is the stabiliser subgroup of S_L, which leaves g_p invariant. That z_ρ is central is proven as follows

$$hz_\rho = \frac{1}{|\mathrm{Stab}(g_\rho)|} \sum_{\gamma\in S_L} h\gamma\, g_\rho\, \gamma^{-1} = \frac{1}{|\mathrm{Stab}(g_\rho)|} \sum_{\gamma'\in S_L} \gamma'\, g_\rho\, \gamma'^{-1} h = z_\rho h \tag{2.17}$$

where we defined $\gamma' = h\gamma$. The second basis is a set of the projection operators to the irreducible representation $R \vdash L$,

$$P^R = \frac{d_R}{L!} \sum_{g\in S_L} \chi^R(g)\, g^{-1} \tag{2.18}$$

where $\chi^R(g)$ is a character of S_L. That P^R is central follows from the fact that $\chi^R(g)$ is a class function. The two bases are related by an eigensystem

$$m^{\mathfrak{L}}[z_\rho](P^R) = z_\rho P^R = \frac{\chi^R(z_\rho)}{d_R} P^R \tag{2.19}$$

and the eigenvalues are called normalised characters. The last equality follows from Schur's lemma. As we will see in subsequent sections, generalisations of this eigenvalue equation exist for permutation centraliser algebras and play an important role in the eigenvalue method used in this paper.

There exists a subset of the above equations that uniquely distinguish all P^R. It was proven in [45, section 3.4] (see [46, section 3.1] for the general group algebra case) that the following two statements are equivalent

- A set of central elements $z_1, z_2, \ldots, z_k \in \mathcal{Z}(\mathbb{C}[S_L])$ multiplicatively generate the centre.
- The ordered list of normalised characters $(\frac{\chi^R(z_1)}{d_R}, \ldots, \frac{\chi^R(z_k)}{d_R})$ uniquely determine $R \vdash L$.

The following facts, which were observed in [45], contribute to the efficiency of our algorithms. Let $p \in \{1, \ldots, L\}$ and define

$$T_p = z_{(p,1^{L-p})}, \tag{2.20}$$

which is a sum of all the elements of S_L with a single cycle of length p and $L-p$ cycles of length 1. It was then observed that T_2 generates $\mathcal{Z}(\mathbb{C}[S_L])$ for $L = 2, 3, 4, 5, 7$, and T_2 together with T_3 generates the centre for $L = 6, 8, 9, \ldots, 14$. Therefore, for $L \leq 14$ it is sufficient to solve the eigensystem

$$m^{\mathfrak{L}}[T_2](P^R) = \frac{\chi^R(T_2)}{d_R} P^R, \quad m^{\mathfrak{L}}[T_3](P^R) = \frac{\chi^R(T_3)}{d_R} P^R, \tag{2.21}$$

to find the projectors P^R. More generally, there exists a $k_* < L$ such that $T_2, \ldots, T_{k^*}$ generate the centre but $T_2, \ldots, T_{k^*-1}$ do not. An important observation is that k_* is typically much smaller than L.

2.1.3 Basis for multi-matrix U(N) invariants

Denote an N-dimensional vector space by $V_N = \text{Span}_{\mathbb{C}}\,(e_i \,|\, i \in \{1, 2, \ldots, N\})$, and its dual vector space by $\overline{V}_N = \text{Span}_{\mathbb{C}}\,(\check{e}_j \,|\, j \in \{1, 2, \ldots, N\})$. Consider a set of M linear operators

$$X^a : V_N \to V_N, \qquad a = 1, 2, \ldots, M \tag{2.22}$$

which defines a set of $N \times N$ matrices as $X^a e_i = \sum_{j=1}^N X^a_{ji}\, e_j$. For a given tuple $\vec{a} = (a_1, a_2, \ldots, a_L) \in \{1, 2, \ldots, M\}^{\times L}$, we define the tensor product of linear operators

$$\mathcal{O}(\vec{a}) = X^{a_1} \otimes X^{a_2} \otimes \cdots \otimes X^{a_L}. \tag{2.23}$$

The operator $\mathcal{O}(\vec{a})$ acts on

$$V_N^{\otimes L} = \text{Span}_{\mathbb{C}}\,(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_L} \,|\, i_1, \ldots, i_L = 1, \ldots, N) \tag{2.24}$$

as

$$\mathcal{O}(\vec{a})\, e_{i_1} \otimes \cdots \otimes e_{i_L} = X^{a_1} e_{i_1} \otimes \cdots \otimes X^{a_L} e_{i_L} = \prod_{k=1}^{L} X^{a_k}_{j_k i_k}\; e_{j_1} \otimes \cdots \otimes e_{j_L}. \tag{2.25}$$

We define an operator $\mathcal{L} : S_L \to \mathrm{End}(V_N^{\otimes L})$ as the permutation of factors of the tensor product

$$\mathcal{L}(g)\,|e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_L}\rangle = |e_{i_{g(1)}} \otimes e_{i_{g(2)}} \otimes \cdots \otimes e_{i_{g(L)}}\rangle \tag{2.26}$$

$$\langle \check{e}_{j_1} \otimes \check{e}_{j_2} \otimes \cdots \otimes \check{e}_{j_L}|\,\mathcal{L}(g^{-1}) = \langle \check{e}_{j_{g(1)}} \otimes \check{e}_{j_{g(2)}} \otimes \cdots \otimes \check{e}_{j_{g(L)}}|\,. \tag{2.27}$$

This permutation operator obeys the composition rule $\mathcal{L}(g_1)\mathcal{L}(g_2) = \mathcal{L}(g_1 g_2)$ under the convention (2.4). It acts on the operator $\mathcal{O}(\vec{a})$ as

$$\mathcal{L}(g)\,\mathcal{O}(\vec{a})\,\mathcal{L}(g^{-1}) = \mathcal{O}(g\cdot\vec{a}) \equiv X^{a_{g(1)}} \otimes X^{a_{g(2)}} \otimes \cdots \otimes X^{a_{g(L)}}, \tag{2.28}$$

where

$$g\cdot\vec{a} = (a_{g(1)}, a_{g(2)}, \ldots, a_{g(L)}). \tag{2.29}$$

By combining (2.26)–(2.28), we find

$$\prod_{\ell=1}^{L} X^{a_{g(\ell)}}_{j_{g(\ell)} i_{g(\ell)}} = \prod_{k=1}^{L} X^{a_k}_{j_k i_k} \tag{2.30}$$

which means that the product of matrix elements are invariant under the relabelling.

By taking the trace over the tensor product space $V_N^{\otimes L}$, we obtain the multi-matrix invariant (2.2)

$$\mathcal{O}_g(\vec{a}) \equiv \mathrm{tr}_{V_N^{\otimes L}}(\mathcal{L}(g)\mathcal{O}(\vec{a})) = \sum_{j_1,j_2,\ldots,j_L=1}^{N} X^{a_1}_{j_1 j_{g(1)}} X^{a_2}_{j_2 j_{g(2)}} \ldots X^{a_L}_{j_L j_{g(L)}}\,. \tag{2.31}$$

It defines a map between the pairs $(g, \vec{a})$ and the space of multi-traces of multiple matrices. For a fixed $\vec{a}$ this map can be extended linearly to elements of the group algebra $\mathbb{C}[S_L]$. Let $\sigma \in \mathbb{C}[S_L]$ be given by (2.5). The corresponding sum of multi-matrix invariants is given by

$$\mathcal{O}_\sigma(\vec{a}) = \sum_{g\in S_L} c_g(\sigma)\,\mathcal{O}_g(\vec{a}). \tag{2.32}$$

From the cyclic property of the trace, we have

$$\sum_{i_1,i_2,\ldots,i_L} \langle \check{e}_{i_1} \otimes \ldots|\,\mathcal{L}(\sigma)\mathcal{O}(\vec{a})\,|e_{i_1} \otimes \ldots\rangle = \sum_{i_1,i_2,\ldots,i_L} \langle \check{e}_{i_1} \otimes \ldots|\,\mathcal{L}(\gamma)\mathcal{L}(\sigma)\mathcal{O}(\vec{a})\mathcal{L}(\gamma^{-1})\,|e_{i_1} \otimes \ldots\rangle \tag{2.33}$$

for any $\gamma \in S_L$. By evaluating this equation with the help of (2.26)–(2.28), we can derive the identity for the multi-matrix invariants (2.3),

$$\mathcal{O}_\sigma(\vec{a}) = \mathcal{O}_{\gamma\sigma\gamma^{-1}}(\gamma\cdot\vec{a}), \qquad \forall\gamma \in S_L\,. \tag{2.34}$$

2.1.4 Representations and Young diagrams

The irreducible representations of S_L are labelled by integer partitions of L, or equivalently Young diagrams with L boxes. Young diagrams with L boxes are also denoted by

$$R = [R_1, R_2, \ldots, R_\ell] \vdash L, \qquad R_1 \geq R_2 \geq \cdots \geq R_\ell, \qquad \sum_{i=1}^{\ell} R_i = L, \qquad \ell \equiv \ell(R). \tag{2.35}$$

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The dimension of the irreducible representation $V_R^{S_L}$ for $R \vdash L$ can be computed combinatorially using Young diagrams. Given a Young diagram R, we label the boxes using coordinates (i,j) where for example $(1,1)$ labels the top-left box. Then the dimension $d_R = \dim V_R^{S_L}$ is

$$d_R = \frac{L!}{\text{hook}_R}\,, \qquad \text{hook}_R = \prod_{(i,j)\in R} \Big(\text{hook length at } (i,j)\Big). \tag{2.36}$$

From Schur-Weyl duality, R also labels an irreducible representation of $\mathrm{U}(N)$. The dimension of the irreducible representation $V_R^{\mathrm{U}(N)}$ labelled by R is

$$\mathrm{Dim}_N(R) = \frac{d_R}{L!}\,\mathrm{Wt}_N(R)\,, \qquad \mathrm{Wt}_N(R) = \prod_{(i,j)\in R} (N+i-j). \tag{2.37}$$

We use the symbol e_I^R as the I-th basis element of $V_R^{S_L}$ with $I = 1,2,\dots,d_R$. We introduce the inner product of S_L by (e_I^R, e_J^S). The representation R is called unitary if the group action respects the inner product

$$(g\, e_I^R, g\, e_J^R) = (e_I^R, e_J^R), \qquad \forall g \in S_L\,. \tag{2.38}$$

When the inner product is diagonal, $(e_I^R, e_J^S) = \delta^{RS}\,\delta_{IJ}$, we use Dirac's bracket notation

$$e_J^S = \left|\begin{matrix} S \\ J \end{matrix}\right\rangle, \qquad (e_I^R, \cdot) = \left\langle\begin{matrix} R \\ I \end{matrix}\right|, \qquad \left\langle\begin{matrix} R \\ I \end{matrix}\middle|\begin{matrix} S \\ J \end{matrix}\right\rangle = \delta^{RS}\,\delta_{IJ}\,. \tag{2.39}$$

The group element $g \in S_L$ acts on this basis as

$$g\left|\begin{matrix} R \\ I \end{matrix}\right\rangle = \sum_{J=1}^{d_R} D_{JI}^R(g)\left|\begin{matrix} R \\ J \end{matrix}\right\rangle \tag{2.40}$$

where $D_{JI}^R(g)$ is the matrix element of the irreducible representation R. The matrix element for $g_1 g_2$ is

$$D_{JI}^R(g_1 g_2) = \sum_{K=1}^{d_R} D_{JK}^R(g_1) D_{KI}^R(g_2). \tag{2.41}$$

It follows that

$$\left\langle\begin{matrix} R \\ I \end{matrix}\middle|\,\sigma\,\middle|\begin{matrix} S \\ J \end{matrix}\right\rangle = \delta^{RS}\, D_{IJ}^R(\sigma). \tag{2.42}$$

The unitary representation R can also be defined by using the matrix elements as

$$D_{IJ}^R(g^{-1}) = D_{JI}^{\overline{R}}(g) \tag{2.43}$$

where $\overline{R}$ is the complex conjugate representation of R. The character of $V_R^{S_L}$ is defined by

$$\chi^R(\sigma) = \sum_{I=1}^{d_R} D_{II}^R(\sigma). \tag{2.44}$$

Since (2.39) is a complete basis, the projection operator to $V_R^{S_L}$ can also be written as

$$P^R = \sum_{I=1}^{d_R} \left| \begin{matrix} R \\ I \end{matrix} \right\rangle \left\langle \begin{matrix} R \\ I \end{matrix} \right|. \tag{2.45}$$

The dimension d_R is equal to the number of standard Young tableaux of shape R. For example, when $R = [3,1] = \ydiagram{3,1}$, we find $d_R = 3$ and

$$\left\{ \left| \begin{matrix} R \\ 1 \end{matrix} \right\rangle, \left| \begin{matrix} R \\ 2 \end{matrix} \right\rangle, \left| \begin{matrix} R \\ 3 \end{matrix} \right\rangle \right\} = \left\{ \young(123,4)\,, \young(124,3)\,, \young(134,2) \right\}. \tag{2.46}$$

2.2 Two decompositions of $\mathbb{C}[S_L]$

We will discuss two interesting decompositions of $\mathbb{C}[S_L]$. In the first decomposition, $\mathbb{C}[S_L]$ decomposes into a direct sum of matrix sub-algebras, which transform as $V_R^{S_L} \otimes V_R^{S_L}$ under the simultaneous left and right action of $\mathbb{C}[S_L]$ on itself. We refer to this as the Artin-Wedderburn decomposition since it is an example of the Artin-Wedderburn decomposition of general semi-simple associative algebras into simple algebras. It is also known, in the context of finite groups, as the Maschke's decomposition, since Maschke's decomposition establishes the semi-simplicity of the group algebras of finite groups [49, 50]. In the second decomposition, the simultaneous left and right actions of $\mathbb{C}[S_L]$ on $\mathbb{C}[S_L]$ are recognised as an action of $\mathbb{C}[S_L] \otimes \mathbb{C}[S_L]$ and we decompose the matrix blocks $V_R^{S_L} \otimes V_R^{S_L}$ under the diagonal embedding $\mathbb{C}[S_L] \hookrightarrow \mathbb{C}[S_L] \otimes \mathbb{C}[S_L]$. We call it Kronecker decomposition, as it involves a multiplicity space whose dimension is equal to the Kronecker coefficient $C(R, R, \Lambda)$. These Kronecker coefficient multiplicities have appeared in the construction of covariant bases of matrix operators [9, 12].

2.2.1 Artin-Wedderburn decomposition

Any irreducible representations of S_L satisfy the orthogonality relation

$$\sum_{g \in S_L} D^R_{IJ}(g^{-1})\, D^S_{KL}(g) = \frac{|S_L|}{d_R}\, \delta^{RS}\, \delta_{IL}\, \delta_{JK}\,. \tag{2.47}$$

All irreducible representations of S_L can be chosen real and unitary, so that $D^R_{IJ}(g^{-1}) = D^R_{JI}(g)$. With this assumption, we define the following elements of $\mathbb{C}[S_L]$,

$$Q^R_{IJ} = \frac{d_R}{|S_L|} \sum_{g \in S_L} D^R_{JI}(g)\, g^{-1}. \tag{2.48}$$

Using (2.47), one can prove that they satisfy the relation [51, Proposition 11, page 49],

$$Q^R_{IJ}\, Q^S_{KL} = \delta^{RS}\, \delta_{JK}\, Q^R_{IL}\,. \tag{2.49}$$

These elements are called matrix units (borrowing the language in [52, Theorem 3.7]). The matrix units form a complete basis of $\mathbb{C}[S_L]$ making the following group theoretical identity manifest

$$|S_L| = L! = \sum_{R \vdash L} d_R^2. \tag{2.50}$$

The existence of a complete basis of matrix units is guaranteed by the Artin-Wedderburn theorem. The Artin-Wedderburn theorem, applied to $\mathbb{C}[S_L]$, states that $\mathbb{C}[S_L]$ decomposes into the direct sum of simple subalgebras, and each simple subalgebra is a matrix algebra over an irreducible representation of S_L [51, section 6.2]. The Artin-Wedderburn decomposition of $\mathbb{C}[S_L]$ is written as

$$\mathbb{C}[S_L] \cong \bigoplus_{R \vdash L} \text{Mat}\,(V_R^{S_L}) \tag{2.51}$$

where $\text{Mat}(V_R^{S_L})$ is the matrix algebra over $V_R^{S_L}$, consisting of $d_R \times d_R$ matrices. The matrix units Q_{IJ}^R in (2.49) can be identified as a complete basis of $\text{Mat}\,(V_R^{S_L})$.

The matrix algebra over V_R can be written as a tensor product

$$\text{Mat}(V_R^{S_L}) \cong V_R^{S_L} \otimes V_R^{S_L} \,. \tag{2.52}$$

To see this, consider group actions on the matrix units. From the definition in (2.48), we have

$$\begin{aligned} h\, Q_{IJ}^R &= \frac{d_R}{|S_L|} \sum_{g \in S_L} D_{JI}^R(g)\, hg^{-1} \\ &= \frac{d_R}{|S_L|} \sum_{k \in S_L} \sum_{K=1}^{d_R} D_{JK}^R(k) D_{KI}^R(h)\, k^{-1} = \sum_{K=1}^{d_R} D_{KI}^R(h)\, Q_{KJ}^R. \end{aligned} \tag{2.53}$$

where in the second equality we defined $k^{-1} = hg^{-1}$. Following similar steps we also have

$$Q_{IJ}^R\, h^{-1} = \sum_{K=1}^{d_R} D_{JK}^R(h^{-1})\, Q_{IK}^R = \sum_{K=1}^{d_R} D_{KJ}^R(h)\, Q_{IK}^R \,, \tag{2.54}$$

where we used the fact that the representation matrices $D_{IJ}^R(h)$ are real and unitary. This argument shows that under the following group action

$$(h, h') \cdot Q_{IJ}^R \equiv h\, Q_{IJ}^R\, (h')^{-1} \,, \qquad (h, h') \in S_L \times S_L \tag{2.55}$$

the subscripts of Q_{IJ}^R separately form a basis for irreducible representations $V_R^{S_L} \otimes V_R^{S_L}$ of $S_L \times S_L$, which is (2.52).

The centre $\mathcal{Z}(\mathbb{C}[S_L])$ acts on the matrix units as the multiplication of a constant known as a normalised character,

$$m^{\mathfrak{L}}[z](Q_{IJ}^R) = m^{\mathfrak{R}}[z](Q_{IJ}^R) = \frac{\chi^R(z)}{d_R}\, Q_{IJ}^R \tag{2.56}$$

where we used (2.53), (2.54) and Schur's lemma which implies that $D_{IJ}^R(z) = \chi^R(z)\, \delta_{IJ}/d_R$ for $z \in \mathcal{Z}(\mathbb{C}[S_L])$.

2.2.2 Kronecker decomposition

Recall that the matrix algebra over the irreducible representation $V_R^{S_L}$ can be regarded as a tensor product of a pair of $V_R^{S_L}$'s (2.52). We decompose this tensor product into a sum of irreducible representations,

$$\mathbb{C}[S_L] \;\cong\; \bigoplus_{R \vdash L} V_R^{S_L} \otimes V_R^{S_L} \;\cong\; \bigoplus_{R \vdash L} \bigoplus_{\Lambda \vdash L} V_\Lambda^{S_L} \otimes V_{R,R,\Lambda} \tag{2.57}$$

where the multiplicity space $V_{R,R,\Lambda}$ has the dimension equal to the Kronecker coefficient,

$$\dim V_{R,R,\Lambda} = C(R,R,\Lambda) = \frac{1}{|S_L|} \sum_{g\in S_L} \chi^R(g)\chi^R(g)\chi^\Lambda(g). \tag{2.58}$$

We take an explicit basis of $V_R^{S_L}$ as in section 2.1.4 and rewrite the irreducible decomposition (2.57) as

$$\left|\begin{matrix} R \\ I \end{matrix}\right\rangle \otimes \left|\begin{matrix} R \\ J \end{matrix}\right\rangle = \sum_{\Lambda\vdash L} \sum_{\tau=1}^{C(R,R,\Lambda)} \sum_{K=1}^{d_\Lambda} S^{\tau}{}^{\Lambda}_{K}{}^{R}_{I}{}^{R}_{J} \left|\begin{matrix} \Lambda \\ K \end{matrix}\ \tau\right\rangle \tag{2.59}$$

where $S^{\tau}{}^{\Lambda}_{K}{}^{R}_{I}{}^{R}_{J}$ is called the Clebsch-Gordan coefficient and τ is a label for an orthogonal basis in the multiplicity space. You can read more about these in appendix A.2.

Now consider the following transformation of the matrix units,

$$\mathcal{Q}^{R,\Lambda,\tau}_{K} = \sum_{I,J=1}^{d_R} S^{\tau}{}^{\Lambda}_{K}{}^{R}_{I}{}^{R}_{J}\, Q^R_{IJ}\,. \tag{2.60}$$

Since this is a unitary transformation, the elements $\{\mathcal{Q}^{R,\Lambda,\tau}_{K}\}$ also form a complete basis of $\mathbb{C}[S_L]$,

$$\mathbb{C}[S_L] \cong \mathrm{Span}_{\mathbb{C}} \left(\mathcal{Q}^{R,\Lambda,\tau}_{K} \,\middle|\, R,\Lambda \vdash L, K\in\{1,\ldots,d_\Lambda\},\ \tau\in\{1,\ldots,C(R,R,\Lambda)\} \right). \tag{2.61}$$

We call (2.60) Kronecker basis and (2.61) Kronecker decomposition. This equation is consistent with the identity

$$|S_L| = L! = \sum_{R,\Lambda\vdash L} C(R,R,\Lambda)\, d_\Lambda\,. \tag{2.62}$$

It turns out that the subscript K of the Kronecker basis forms an irreducible representation $V_\Lambda^{S_L}$. In particular, if we consider the adjoint action of $\gamma\in S_L$ defined in (2.13), we will find

$$m^{\mathfrak{ad}}[\gamma](\mathcal{Q}^{R,\Lambda,\tau}_{K}) = \sum_{K'} \mathcal{Q}^{R,\Lambda,\tau}_{K'}\, D^\Lambda_{K'K}(\gamma). \tag{2.63}$$

To see this, we combine (2.48) and (2.60) as

$$\begin{aligned} m^{\mathfrak{ad}}[\gamma](\mathcal{Q}^{R,\Lambda,\tau}_{K}) &= \frac{d_R}{|S_L|} \sum_{g\in S_L} \sum_{I,J} S^{\tau}{}^{\Lambda}_{K}{}^{R}_{I}{}^{R}_{J}\, D^R_{JI}(g)\, \gamma g^{-1}\gamma^{-1} \\ &= \frac{d_R}{|S_L|} \sum_{g\in S_L} \sum_{I,J,I',J'} S^{\tau}{}^{\Lambda}_{K}{}^{R}_{I}{}^{R}_{J}\, D^R_{JJ'}(\gamma^{-1}) D^R_{J'I'}(g) D^R_{I'I}(\gamma)\, g^{-1}\,. \end{aligned} \tag{2.64}$$

Since R is a real and unitary representation, we have $D^R_{I'I}(\gamma) = D^R_{II'}(\gamma^{-1})$. Using the equivariance property of Clebsch-Gordan coefficients (A.19), the equation (2.64) simplifies as

$$m^{\mathfrak{ad}}[\gamma](\mathcal{Q}^{R,\Lambda,\tau}_{K}) = \frac{d_R}{|S_L|} \sum_{g\in S_L} \sum_{I',J',K'} D^\Lambda_{KK'}(\gamma^{-1})\, S^{\tau}{}^{\Lambda}_{K'}{}^{R}_{I'}{}^{R}_{J'}\, D^R_{J'I'}(g)\, g^{-1} = \sum_{K'} \mathcal{Q}^{R,\Lambda,\tau}_{K'}\, D^\Lambda_{K'K}(\gamma) \tag{2.65}$$

which is (2.63).

Let us compute the inner product of the Kronecker basis using the δ-function defined by (2.6). We write

$$\mathcal{S}(\mathcal{Q}^{R,\Lambda,\tau}_{K}) = \frac{d_R}{|S_L|} \sum_{\sigma\in S_L} \sum_{I,J} S^{\tau}{}^{\Lambda}_{K}{}^{R}_{I}{}^{R}_{J}\, D^R_{JI}(\sigma)\, \sigma \tag{2.66}$$

$$\mathcal{Q}^{R',\Lambda',\tau'}_{K'} = \frac{d_{R'}}{|S_L|} \sum_{\sigma'\in S_L} \sum_{I',J'} S^{\tau'}{}^{\Lambda'}_{K'}{}^{R'}_{I'}{}^{R'}_{J'}\, D^{R'}_{J'I'}(\sigma')\, \sigma'^{-1} \tag{2.67}$$

and use the relation

$$\sum_{\sigma,\sigma'\in S_L} \delta\Big(D^R_{JI}(\sigma)\, D^{R'}_{J'I'}(\sigma')\, \sigma\, \sigma'^{-1}\Big) = \sum_{\sigma} D^R_{JI}(\sigma)\, D^{R'}_{J'I'}(\sigma) = \frac{|S_L|}{d_R}\, \delta^{RR'} \delta_{JJ'}\, \delta_{II'} \tag{2.68}$$

together with the orthogonality of the Clebsch-Gordan coefficients (A.18). Then we find

$$\delta\Big(\mathcal{S}(\mathcal{Q}^{R,\Lambda,\tau}_{K})\, \mathcal{Q}^{R',\Lambda',\tau'}_{K'}\Big) = \frac{d_R}{|S_L|}\, \delta^{RR'}\, \delta^{\Lambda\Lambda'}\, \delta^{\tau\tau'}\, \delta_{KK'}\,. \tag{2.69}$$

This δ-function inner product is the same as the planar two-point function of $\mathcal{N}=4$ SYM at zero coupling, as will be discussed in section 3.5.

Let us consider the adjoint action of the centre on $\mathbb{C}[S_L]$ and in particular on the basis $\mathcal{Q}^{R,\Lambda,\tau}_{K}$. Following the notation in section 2.1.1, we write

$$m^{\mathfrak{ad}}[z](\sigma) = \sum_{g\in S_L} c_g(z)\, g^{-1}\sigma g, \qquad \text{for} \quad z = \sum_{g\in S_L} c_g(z)\, g \;\in\; \mathcal{Z}(\mathbb{C}[S_L]). \tag{2.70}$$

The left and right action of the center $\mathcal{Z}(\mathbb{C}[S_L])$ on $\mathbb{C}[S_L]$ are identical. This left/right action commutes with the adjoint action of $\mathcal{Z}(\mathbb{C}[S_L])$ on $\mathbb{C}[S_L]$. For $z_1\,, z_2 \in \mathcal{Z}(\mathbb{C}[S_L])$ and $\sigma \in \mathbb{C}[S_L]$

$$m^{\mathfrak{ad}}[z_1](m^{\mathfrak{L}}[z_2](\sigma)) = m^{\mathfrak{ad}}[z_1](m^{\mathfrak{R}}[z_2](\sigma)) = m^{\mathfrak{L}}[z_2](m^{\mathfrak{ad}}[z_1](\sigma)) = m^{\mathfrak{R}}[z_2](m^{\mathfrak{ad}}[z_1](\sigma)). \tag{2.71}$$

Equivalently as linear operators on $\mathbb{C}[S_L]$,

$$m^{\mathfrak{ad}}[z_1] m^{\mathfrak{L}}[z_2] = m^{\mathfrak{L}}[z_2] m^{\mathfrak{ad}}[z_1] \tag{2.72}$$

It follows from (2.63) and (2.56) that the centre acts on the Kronecker basis as

$$m^{\mathfrak{L}}[z_1](\mathcal{Q}^{R,\Lambda,\tau}_{K}) = m^{\mathfrak{R}}[z_1](\mathcal{Q}^{R,\Lambda,\tau}_{K}) = \frac{\chi^R(z_1)}{d_R}\, \mathcal{Q}^{R,\Lambda,\tau}_{K}\,, \qquad m^{\mathfrak{ad}}[z_2](\mathcal{Q}^{R,\Lambda,\tau}_{K}) = \frac{\chi^\Lambda(z_2)}{d_\Lambda}\, \mathcal{Q}^{R,\Lambda,\tau}_{K} \tag{2.73}$$

In other words, the Kronecker basis simultaneously diagonalises the left/right action and the adjoint action of the centre $\mathcal{Z}(\mathbb{C}[S_L])$.

The state label I, J in the matrix units Q^R_{IJ} and K in the Kronecker basis $\mathcal{Q}^{R,\Lambda,\tau}_{K}$ can be determined by the action of non-central elements such as the Young-Jucys-Murphy elements. We will elaborate on this in section 4.

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2.3 Permutation centraliser algebra $\mathcal{A}(\mu)$

Consider the multi-matrix invariants $\mathcal{O}_g(\vec{a})$ in (2.31) which consist of M types of $N \times N$ matrices. We fix the field content of $\mathcal{O}_g(\vec{a})$ as having μ_1 X^1's, μ_2 X^2's and so on up to μ_M X^M's. Each multi-matrix invariant corresponds to a choice of the field contents $\mu = (\mu_1, \dots, \mu_M)$ within the range

$$\mu_\ell \in \{0, 1, \dots, L\} \quad \text{such that} \quad \sum_{\ell=1}^{M} \mu_\ell = L. \tag{2.74}$$

For simplicity we assume that the matrix size is larger than or equal to the operator length, $N \geq L$.

Using the conjugation symmetry (2.3), we may rearrange $\vec{a}$ as

$$\vec{a}_\mu = (\underbrace{1, 1, \dots, 1}_{\mu_1}, \underbrace{2, 2, \dots, 2}_{\mu_2}, \dots, \underbrace{M, M, \dots, M}_{\mu_M}). \tag{2.75}$$

We denote the multi-matrix invariant for the field content $\mu = (\mu_1, \dots, \mu_M)$ by

$$\mathcal{O}_g(\vec{a}_\mu) = \mathrm{tr}_{V_N^{\otimes L}} \Big(\mathcal{L}(g)\, (X^1)^{\otimes \mu_1} \otimes (X^2)^{\otimes \mu_2} \otimes \dots \otimes (X^M)^{\otimes \mu_M} \Big). \tag{2.76}$$

This $\mathcal{O}_g(\vec{a}_\mu)$ is invariant under conjugation by the subgroup,

$$\mathcal{O}_g(\vec{a}_\mu) = \mathcal{O}_{hgh^{-1}}(\vec{a}_\mu), \qquad \forall (g,h) \in S_L \times S_\mu \,, \tag{2.77}$$

where S_μ is called the Young subgroup indexed by μ,

$$S_\mu = S_{\mu_1} \times S_{\mu_2} \times \dots \times S_{\mu_M} \subseteq S_L \,. \tag{2.78}$$

The sum over the orbits of the equivalence class of S_μ in (2.77) defines an interesting subalgebra of $\mathbb{C}[S_L]$ called permutation centraliser algebra [27],

$$\mathcal{A}(\mu) = \mathcal{A}(\mu_1, \mu_2, \dots, \mu_M) = \Big\{ \sigma \in \mathbb{C}[S_L] \,\Big|\, h \sigma h^{-1} = \sigma \;\; \text{for all } h \in S_\mu \Big\} \tag{2.79}$$

One may also write $\mathcal{A}(\mu) = \mathbb{C}[S_L]^{S_\mu}$, where G^H is the H-invariant subspace of G.

For any $g \in S_L$ we define the S_μ-orbit

$$\{g\}_\mu = \{ hgh^{-1} \,|\, \forall h \in S_\mu \}. \tag{2.80}$$

Let $l = \dim \mathcal{A}(\mu)$ and $g_1, g_2, \dots, g_l$ be a complete set of representatives. That is, a set of elements satisfying

$$\{g_1\}_\mu \cup \{g_2\}_\mu \dots \cup \dots \{g_l\}_\mu = S_L \tag{2.81}$$

and $\{g_i\}_\mu \cap \{g_j\}_\mu$ is empty if $i \neq j$. Let us define a linear operator $P_\mu : S_L \to A(\mu)$ by

$$P_\mu(g) = \frac{1}{|\mathrm{Stab}(g)|} \sum_{\gamma \in S_\mu} \gamma g \gamma^{-1} \tag{2.82}$$

where Stab(g) is the subgroup of S_μ that leaves g invariant under conjugation.[1] The algebra $A(\mu)$ has a linear basis labelled by the S_μ-orbits,

$$\{P_\mu(g_1), P_\mu(g_2), \dots, P_\mu(g_l)\} \tag{2.83}$$

that we call the orbit basis. It follows that

$$\mathcal{O}_{P_\mu(\sigma)}(\vec{a}_\mu) = |\mathrm{Orb}(\sigma)|\mathcal{O}_\sigma(\vec{a}_\mu), \qquad \sigma \in \mathbb{C}[S_L] \tag{2.84}$$

where $|\mathrm{Orb}(\sigma)|$ is the number of different elements in $\mathbb{C}[S_L]$ which belong to the same equivalence class of $\mathcal{A}(\mu)$. We have $|\mathrm{Orb}(\sigma)||\mathrm{Stab}(\sigma)| = |S_\mu|$ according to the orbit-stabilizer theorem.

There are two interesting cases where $\mathcal{A}(\mu)$ reduces to well-known algebras. The first case is

$$\mathcal{A}(L) = \left\{\sigma \in \mathbb{C}[S_L] \,\middle|\, h\sigma h^{-1} = \sigma \ \text{ for all } h \in S_L\right\} = \mathcal{Z}(\mathbb{C}[S_L]) \tag{2.85}$$

which corresponds to the centre of the symmetric group algebra and describes the half-BPS operators of $\mathcal{N} = 4$ SYM. The second case is

$$\mathcal{A}(\underbrace{1, 1, \dots, 1}_{L}) = \mathbb{C}[S_L] \tag{2.86}$$

which is the symmetric group algebra discussed in section 2.2.

The dimension of $\mathcal{A}(\mu)$ is equal to the number of multi-matrix invariants at large N, i.e. for operators with $L \leq N$.

$$\sigma \in \mathcal{A}(\mu) \quad \overset{\text{large } N}{\Longleftrightarrow} \quad \mathcal{O}_\sigma(\vec{a}_\mu) \in \Big\{M\text{-matrix invariants of degree } L \leq N \text{ and field content } \mu\Big\}. \tag{2.87}$$

The restricted Schur basis of multi-matrix operators [10, 11] corresponds to a restricted basis in $\mathcal{A}(\mu)$ with labels $R \vdash L, r_i \vdash \mu_i$, in addition to multiplicity labels $1 \leq \nu_\mp \leq g(r_1, r_2; R)$. The finite N constraint is the condition $\ell(R) \leq N$ on this basis. Equivalently the space of finite N multi-matrix invariants is in 1-1 correspondence with a quotient of $\mathcal{A}(\mu)$ defined by setting to zero all the elements of the restricted basis with $l(R) > N$. In the next two sub-sections we will arrive at the restricted basis in $\mathcal{A}(\mu)$ by starting with the Artin-Wedderburn basis in $\mathbb{C}[S_L]$ and projecting with branching coefficients for the subgroup $S_\mu \to S_L$.

The covariant basis [9, 12] corresponds to a basis in $\mathcal{A}(\mu)$ with labels $R, \Lambda \vdash L$, along with a branching multiplicity coefficient for branching of the irreducible representation of S_L labelled by Λ into the trivial representation of S_μ, and a Clebsch-Gordan (or Kronecker) multiplicity for $V_\Lambda^{S_L}$ in the tensor product $V_R^{S_L} \otimes V_R^{S_L}$. The finite N constraint is again the condition $\ell(R) \leq N$ on this basis. Equivalently the space of finite N multi-matrix invariants is in 1-1 correspondence with a quotient of $\mathcal{A}(\mu)$ defined by setting to zero all the elements of the covariant basis with $\ell(R) > N$.

[1]The stabilizer subgroup Stab(σ) is also called Automorphism group, denoted by Aut(σ) in [28].

2.4 Restricted Schur basis

We review the restricted Schur basis [19] emphasising the connection with the Artin-Wedderburn basis for $\mathbb{C}[S_L]$ in section 2.2.

Given an irreducible representation of S_L, we can restrict $S_L \to S_\mu$ and look at the corresponding decomposition into irreducible representations of S_μ

$$V_R^{S_L} \cong \bigoplus_{r_1 \vdash \mu_1} \cdots \bigoplus_{r_M \vdash \mu_M} \left(V_{r_1}^{S_{\mu_1}} \otimes \cdots \otimes V_{r_M}^{S_{\mu_M}} \otimes V_{R \to (r_1 \ldots r_M)} \right). \tag{2.88}$$

Here $V_{R\to(r_1\ldots r_M)}$ is the multiplicity space of this decomposition, and its dimension is equal to the Littlewood-Richardson coefficient

$$\dim V_{R\to(r_1\ldots r_M)} = g(r_1 \ldots r_m; R). \tag{2.89}$$

Concretely, we can write an orthonormal basis of states for the irreducible representation of S_μ as

$$\left| \begin{matrix} r_1 \ldots r_M \\ i_1 \ldots i_M \end{matrix} \; \nu \right\rangle = \sum_{I=1}^{d_R} B^{R\to(r_1\ldots r_M)}_{I\to(i_1\ldots i_M),\nu} \left| \begin{matrix} R \\ I \end{matrix} \right\rangle \tag{2.90}$$

where $i_k = 1, \ldots, d_{r_k}$ and $\nu = 1, \ldots, g(r_1 \ldots r_M; R)$. The transformation matrix $B^{R\to(r_1\ldots r_M)}_{I\to(i_1\ldots i_M),\nu}$ is called the branching coefficients, whose properties are discussed in detail in appendix A.1. The restricted Schur basis is built from this decomposition as

$$\begin{aligned} \mathcal{A}(\mu) \;&\cong\; \mathrm{Span}_{\mathbb{C}} \left(Q^{R,(r_1\ldots r_M)}_{\nu_+\nu_-} \,\middle|\, R \vdash L, r_k \vdash \mu_k \,, \nu_\pm \in \{1, \ldots, g(R; r_1, \ldots, r_M)\} \right) \\ Q^{R,(r_1\ldots r_M)}_{\nu_+\nu_-} &= \sum_{I,J} \sum_{i_1,\ldots,i_M} B^{R\to(r_1\ldots r_M)}_{I\to(i_1\ldots i_M),\nu_-} \, Q^R_{IJ} \, B^{R\to(r_1\ldots r_M)}_{J\to(i_1\ldots i_M),\nu_+} \end{aligned} \tag{2.91}$$

Tracing over the representations $(r_1 \otimes \cdots \otimes r_M)$ ensures that the restricted Schur basis satisfies

$$h \, Q^{R,(r_1\ldots r_M)}_{\nu_+,\nu_-} \, h^{-1} = Q^{R,(r_1\ldots r_M)}_{\nu_+\nu_-}, \qquad \forall h \in S_\mu. \tag{2.92}$$

The numerical coefficient of $Q^{R,(r_1\ldots r_M)}_{\nu_+\nu_-}$ in front of $g^{-1} \in S_L$ is called the restricted Schur character [19],

$$\chi^{R,(r_1\ldots r_M)}_{\nu_+\nu_-}(g) = \frac{d_R}{|S_L|} \sum_{I,J} \sum_{i_1\ldots i_M} B^{R\to(r_1\ldots r_M)}_{I\to(i_1\ldots i_M);\nu_-} \, D^R_{JI}(g) \, B^{R\to(r_1\ldots r_M)}_{J\to(i_1\ldots i_M);\nu_+} \, . \tag{2.93}$$

Note that the restricted Schur character is invariant under the permutation of $(r_1 \,, \ldots, r_M)$. The restricted Schur basis is a matrix unit of $\mathcal{A}(\mu)$ as they satisfy the relation [27, 53]

$$Q^{R,(r_1\ldots r_M)}_{\nu_+\nu_-} \, Q^{S,(s_1\ldots s_M)}_{\xi_+\xi_-} = \delta^{RS} \left(\prod_{k=1}^M \delta^{r_k s_k} \right) \delta_{\nu_-\xi_+} \, Q^{R,(r_1\ldots r_M)}_{\nu_+\xi_-} \, . \tag{2.94}$$

The δ-function inner product of the restricted Schur basis can be computed as

$$\begin{aligned} \delta\Big(\mathcal{S}(Q^{R,(r_1\ldots r_M)}_{\nu_+\nu_-}) \, Q^{S,(s_1\ldots s_M)}_{\xi_+\xi_-} \Big) &= \delta^{RS} \left(\prod_{k=1}^M \delta^{r_k s_k} \right) \delta_{\nu_+\xi_+} \, \delta(Q^{R,(r_1\ldots r_M)}_{\nu_-\xi_-}) \\ &= \delta^{RS} \left(\prod_{k=1}^M \delta^{r_k s_k} \, d_{r_k} \right) \delta_{\nu_+\xi_+} \, \delta_{\nu_-\xi_-} \end{aligned} \tag{2.95}$$

where we used the real unitarity of R,

$$\mathcal{S}(Q^{R,(r_1\dots r_M)}_{\nu_+\nu_-}) = \sum_{I,J}\sum_{i_1,\dots,i_M} B^{R\to(r_1\dots r_M)}_{I\to(i_1\dots i_M),\nu_-} D^R_{IJ}(\sigma)\, B^{R\to(r_1\dots r_M)}_{J\to(i_1\dots i_M),\nu_+}\,\sigma = Q^{R,(r_1\dots r_M)}_{\nu_-\nu_+}\,. \tag{2.96}$$

In fact, this basis is a Artin-Wedderburn decomposition of $\mathcal{A}(\mu)$ [21]

$$\mathcal{A}(\mu) \cong \bigoplus_{\substack{R\vdash L\\ r_i\vdash\mu_i}} \mathrm{Mat}(V_{R\to(r_1\dots r_M)}) \tag{2.97}$$

and it gives the formula

$$\dim\mathcal{A}(\mu) = \sum_{\substack{R\vdash L\\ r_i\vdash\mu_i}} g(r_1,\dots,r_M;R)^2. \tag{2.98}$$

With the help of (2.87), we write the corresponding multi-matrix invariants as

$$\mathcal{O}^{R,(r_1\dots r_m)}_{\nu_+\nu_-}(\vec{a}_\mu) = \mathrm{tr}_{V_N^{\otimes L}}\left(\mathcal{L}(Q^{R(r_1\dots r_M)}_{\nu_+\nu_-})\,(X^1)^{\otimes\mu_1}\otimes(X^2)^{\otimes\mu_2}\otimes\cdots\otimes(X^M)^{\otimes\mu_M}\right) \tag{2.99}$$

which is called the restricted Schur operators. The operators $\mathcal{O}^{R,(r_1\dots r_m)}_{\nu_+\nu_-}(\vec{a}_\mu)$ form a basis of multi-matrix invariants for $L\le N$. At finite N, only the subset of operators with $\ell(R)\le N$ remain linearly independent [5, 19]. The restricted Schur operators form an orthonormal basis with respect to the two-point functions of $\mathcal{N}=4$ SYM at zero coupling. The operators above the cutoff $\ell(R)>N$ have zero norm. Further discussion will be given in section 3.5 about the two-point functions at finite N.

Let us investigate the properties of the restricted Schur basis as in section 2.2. The following basic observations were made in [21]. First, it follows from (2.91) that for $z\in\mathcal{Z}(\mathbb{C}[S_L])$

$$z\,Q^{R,(r_1\dots r_M)}_{\nu_+\nu_-} = \frac{\chi^R(z)}{d_R}\,Q^{R,(r_1\dots r_M)}_{\nu_+\nu_-}. \tag{2.100}$$

Secondly, the irreducible representations r_k can be determined in a similar manner. For this, the relevant subalgebras are the centres of the group algebras $\mathbb{C}[S_{\mu_k}]\subseteq\mathbb{C}[S_L]$. As we will now show, for any $z_k\in\mathcal{Z}(\mathbb{C}[S_{\mu_k}])$

$$z_k\,Q^{R,(r_1\dots r_M)}_{\nu_+\nu_-} = \frac{\chi^{r_k}(z_k)}{d_{r_k}}\,Q^{R,(r_1\dots r_M)}_{\nu_+\nu_-}\,. \tag{2.101}$$

To prove this, consider

$$z_k = \sum_{\gamma\in S_{\mu_k}} c_\gamma(z_k)\,\gamma \;\in\; \mathcal{Z}(\mathbb{C}[S_{\mu_k}]) \tag{2.102}$$

and compute its left action on the restricted Schur basis

$$\begin{aligned} z_k Q^{R,(r_1\dots r_M)}_{\nu_+\nu_-} &= \sum_{\gamma\in S_{\mu_k}}\sum_{\sigma\in S_L}\sum_{I,J,K}\sum_{i_1\dots i_M} c_\gamma(z_k) B^{R\to(r_1\dots r_M)}_{I\to(i_1\dots i_M);\nu_-} D^R_{JK}(\sigma) D^R_{KI}(\gamma) B^{R\to(r_1\dots r_M)}_{J\to(i_1\dots i_M);\nu_+}\sigma^{-1}\\ &= \sum_{\gamma\in S_{\mu_k}}\sum_{\sigma\in S_L}\sum_{J,K}\sum_{\substack{i_1\dots i_M\\ j_1\dots j_M}} c_\gamma(z_k) B^{R\to(r_1\dots r_M)}_{K\to(j_1\dots j_M);\nu_-} D^{r_k}_{j_k i_k}(\gamma)\left(\prod_{\ell\neq k}^{M}\delta_{j_\ell i_\ell}\right) D^R_{JK}(\sigma) B^{R\to(r_1\dots r_M)}_{J\to(i_1\dots i_M);\nu_+}\sigma^{-1} \end{aligned} \tag{2.103}$$

where we used the equivariance property of branching coefficients (A.11). Since the central element $z_k \in \mathcal{Z}(\mathbb{C}[S_{\mu_k}])$ satisfies $D^{r_k}_{j_k i_k}(z_k) = \chi^{r_k}(z_k)\,\delta_{j_k i_k}/d_{r_k}$, we find

$$\begin{aligned} z_k\, Q^{R,(r_1\dots r_M)}_{\nu_+\nu_-} &= \frac{\chi^{r_k}(z_k)}{d_{r_k}} \sum_{J,K} \sum_{i_1\dots i_M} B^{R\to(r_1\dots r_M)}_{K\to(i_1\dots i_M);\nu_-}\, Q^R_{KJ} B^{R\to(r_1\dots r_M)}_{J\to(i_1\dots i_M);\nu_+} \\ &= \frac{\chi^{r_k}(z_k)}{d_{r_k}}\, Q^{R,(r_1\dots r_M)}_{\nu_+\nu_-} \end{aligned} \tag{2.104}$$

which is (2.101). The actions of $\mathcal{Z}(\mathbb{C}[S_L]), \mathcal{Z}(\mathbb{C}[S_{\mu_1}]), \dots, \mathcal{Z}(\mathbb{C}[S_{\mu_M}])$ all commute and can be simultaneously diagonalised. The corresponding eigenspaces, which we will construct in section 3 are

$$\mathcal{A}^{R,r_1,\dots,r_M}(\mu) = \mathrm{Span}\Big(Q^{R,(r_1\dots r_M)}_{\nu_+\nu_-} \,|\, \nu_+,\nu_- = 1,\dots g(r_1,\dots,r_M;R) \Big). \tag{2.105}$$

2.5 Covariant basis

In a similar manner we will review the covariant basis [9, 12], which can be viewed as generalisation of the Kronecker basis of $\mathbb{C}[S_L]$.

Recall that the Kronecker basis was organised in terms of irreducible representations $V^{S_L}_\Lambda$ of the adjoint action of S_L. Restricting to the adjoint action of S_μ gives the following decomposition

$$V^{S_L}_\Lambda \cong \bigoplus_{r_1\vdash\mu_1} \cdots \bigoplus_{r_M\vdash\mu_M} \Big(V^{S_{\mu_1}}_{r_1} \otimes \cdots \otimes V^{S_{\mu_M}}_{r_M} \otimes V_{\Lambda\to(r_1\dots r_M)} \Big) \tag{2.106}$$

where $V_{\Lambda\to(r_1\dots r_M)}$ is the multiplicity space. Since we are interested in the states invariant under the adjoint action of S_μ, we pick out the trivial representation as

$$V^{S_L}_\Lambda \Big|_{S_\mu\text{-inv}} \cong V^{S_\mu}_{\mathrm{triv}(\mu)} \otimes V_{\Lambda\to([\mu_1]\dots[\mu_M])}\,, \qquad V^{S_\mu}_{\mathrm{triv}(\mu)} \equiv V^{S_{\mu_1}}_{[\mu_1]} \otimes \cdots \otimes V^{S_{\mu_M}}_{[\mu_M]} \tag{2.107}$$

where $\mathrm{triv}(\mu) = ([\mu_1]\dots[\mu_M])$ is the trivial representation of S_μ. The multiplicity space has the dimension

$$\dim V_{\Lambda\to([\mu_1]\dots[\mu_M])} = g([\mu_1],\dots,[\mu_M];\Lambda) \equiv K_{\Lambda\mu}, \tag{2.108}$$

which is also known as the Kostka number. In concrete terms, we take orthonormal bases of states and express (2.106) as

$$\left| \begin{matrix} r_1 \dots r_M \\ k_1 \dots k_M \end{matrix}\ \beta \right\rangle = \sum_{K=1}^{d_\Lambda} B^{\Lambda\to(r_1\dots r_M)}_{K\to(k_1\dots k_M),\beta} \left| \begin{matrix} \Lambda \\ K \end{matrix} \right\rangle \tag{2.109}$$

with $\beta = 1,\dots,K_{\Lambda\mu}$ as in (2.90). We are interested in the S_μ-invariant subspace as in (2.107). Let us write the branching coefficients (2.90) to the trivial representation as

$$\left| \begin{matrix} [\mu_1] \dots [\mu_M] \\ 1 \ \dots \ 1 \end{matrix}\ \beta \right\rangle = \sum_{K=1}^{d_\Lambda} B^{\Lambda\to\mathrm{triv}(\mu)}_{K,\beta} \left| \begin{matrix} \Lambda \\ K \end{matrix} \right\rangle, \qquad B^{\Lambda\to\mathrm{triv}(\mu)}_{K,\beta} \equiv B^{\Lambda\to([\mu_1]\dots[\mu_M])}_{K\to(1\dots1),\beta}\,. \tag{2.110}$$

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From (2.110) we obtain the covariant basis for $\mathcal{A}(\mu)$,

$$\begin{aligned}\mathcal{A}(\mu) &\cong \mathrm{Span}_{\mathbb{C}}\left(\mathcal{Q}_\beta^{R,\Lambda,\mu,\tau}\,\middle|\, R\vdash L, \Lambda\vdash L, \tau\in\{1,\dots,C(R,R,\Lambda)\}, \beta\in\{1,\dots,K_{\Lambda\mu}\}\right)\\ \mathcal{Q}_\beta^{R,\Lambda,\mu,\tau} &= \sum_K B_{K,\beta}^{\Lambda\to\mathrm{triv}(\mu)}\,\mathcal{Q}_K^{R,\Lambda,\tau}\end{aligned} \tag{2.111}$$

which implies

$$\dim\mathcal{A}(\mu) = \sum_{R,\Lambda\vdash L} C(R,R,\Lambda)K_{\Lambda\mu}\,. \tag{2.112}$$

The corresponding matrix invariants

$$\mathcal{O}_\beta^{R,\Lambda,\mu,\tau}(\vec{a}_\mu) = \mathrm{tr}_{V_N^{\otimes L}}\left(\mathcal{L}(\mathcal{Q}_\beta^{R,\Lambda,\mu,\tau})\,(X^1)^{\otimes\mu_1}\otimes(X^2)^{\otimes\mu_2}\otimes\cdots\otimes(X^M)^{\otimes\mu_M}\right) \tag{2.113}$$

is called the covariant basis of operators. Again, the covariant basis of operators diagonalises the two-point functions of $\mathcal{N}=4$ SYM at zero coupling. When $L>N$ we need a cut-off on R given by $\ell(R)\le N$. We may impose an extra condition $\ell(\Lambda)\le M$ to the equation (2.111), because the Kotska number $K_{\Lambda\mu}$ vanishes if $\ell(\Lambda)>M$.[2]

The real unitarity of R gives

$$\mathcal{S}(\mathcal{Q}_\beta^{R,\Lambda,\mu,\tau}) = \mathcal{Q}_\beta^{R,\Lambda,\mu,\tau} \tag{2.114}$$

and the δ-function inner product of the covariant basis is

$$\delta\left(\mathcal{S}(\mathcal{Q}_\beta^{R,\Lambda,\mu,\tau})\,\mathcal{Q}_{\beta'}^{R',\Lambda',\mu',\tau'}\right) = \frac{d_R}{|S_L|}\,\delta^{RR'}\delta^{\Lambda\Lambda'}\,\delta^{\tau\tau'}\,\delta^{\mu\mu'}\,\delta_{\beta\beta'}\,. \tag{2.115}$$

This result follows from (2.69) and the orthogonality relation of the branching coefficients (A.10).

It immediately follows from the fact that we started with the Kronecker basis that,

$$z\,\mathcal{Q}_\beta^{R,\Lambda,\mu,\tau} = \frac{\chi^R(z)}{d_R}\,\mathcal{Q}_\beta^{R,\Lambda,\mu,\tau},\quad z\in\mathcal{Z}(\mathbb{C}[S_L]) \tag{2.116}$$

$$m^{\mathfrak{ad}}[z](\mathcal{Q}_\beta^{R,\Lambda,\mu,\tau}) = \frac{\chi^\Lambda(z)}{d_\Lambda}\,\mathcal{Q}_\beta^{R,\Lambda,\mu,\tau},\quad z\in\mathcal{Z}(\mathbb{C}[S_L]). \tag{2.117}$$

We give a name to the corresponding eigenspaces

$$\mathcal{A}^{R,\Lambda}(\mu) = \mathrm{Span}\left(\mathcal{Q}_\beta^{R,\Lambda,\mu,\tau}\,|\,\tau\in\{1,\dots,C(R,R,\Lambda)\}, \beta\in\{1,\dots,K_{\Lambda\mu}\}\right). \tag{2.118}$$

They will be constructed in section 3.

[2]The symbol $K_{\Lambda\mu}$ counts the number of semi-standard Young tableau of shape λ and contents μ. In other words, one must fill the numbers $1\dots M$ in the Young diagram Λ. If $\ell(\Lambda)>M$ the Young tableau cannot be semi-standard.

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2.6 General covariant basis

Recall that the multi-matrix invariants in (2.31) respect the global $\mathrm{U}(M)$ symmetry. The tuple $\vec{a} = (a_1, \ldots, a_L)$ belongs to the tensor product $V_M^{\otimes L}$, which decomposes into

$$V_M^{\otimes L} = \bigoplus_{\substack{\Lambda \vdash L \\ \ell(\Lambda) \leq M}} \left(V_\Lambda^{S_L} \otimes V_\Lambda^{\mathrm{U}(M)} \right) \tag{2.119}$$

under the Schur-Weyl duality of $S_L \times \mathrm{U}(M)$ symmetry. By taking an explicit basis of states, we can rewrite it as

$$|a_1, \ldots, a_L\rangle_{\mathrm{U}(M)} = \sum_{\substack{\Lambda \vdash L \\ \ell(\Lambda) \leq M}} \sum_{M_\Lambda = 1}^{\mathrm{Dim}_M(\Lambda)} \sum_{K=1}^{d_\Lambda} C^{\Lambda\,\Lambda}_{K\,M_\Lambda}(\vec{a}) \left| \begin{matrix} \Lambda \\ K \end{matrix} \right\rangle \left| \begin{matrix} \Lambda \\ M_\Lambda \end{matrix} \right\rangle_{\mathrm{U}(M)} \tag{2.120}$$

where $C^{\Lambda\,\Lambda}_{K\,M_\Lambda}(\vec{a})$ is the Clebsch-Gordan coefficient of $S_L \times \mathrm{U}(M)$, whose properties will be explained in detail in appendix A.3. Following [12], we define the general covariant basis of operators as

$$\mathcal{O}^{R,\Lambda,M_\Lambda,\tau} = \sum_{a_1,\ldots,a_L=1}^{M} \sum_K C^{\Lambda\,\Lambda}_{K\,M_\Lambda}(\vec{a}) \ \mathrm{tr}_{V_N^{\otimes L}} \left(\mathcal{L}(\mathcal{Q}^{R,\Lambda,\tau}_K)\, \mathcal{O}(\vec{a}) \right) \tag{2.121}$$

where Λ is the irreducible representation of $\mathrm{SU}(M)$ with its dimension equal to $\mathrm{Dim}_M(\Lambda)$. The general covariant basis respects the global $\mathrm{U}(M)$ symmetry, and hence the name 'covariant'.

The parameters $\mu = (\mu_1, \ldots, \mu_M)$ in (2.113) represent the $\mathrm{U}(1)$ charges of the Cartan generators in $\mathrm{U}(M)$.[3] If the states with different M_Λ carry the same $\mathrm{U}(1)^M$ charges, the number of such states is counted by the Kotska number. Therefore, we can interchangeably specify a state in Λ by M_Λ or (μ, β). Concretely we can write

$$\left| \begin{matrix} \Lambda \\ M_\Lambda \end{matrix} \right\rangle_{\mathrm{U}(M)} = \sum_{\substack{\mu_1,\ldots,\mu_M \geq 0 \\ \mu_1 + \cdots + \mu_M = L}} \sum_{\beta=1}^{K_{\Lambda\mu}} \mathcal{B}^{\Lambda \to (\mu_1 \ldots \mu_M)}_{M_\Lambda \to \beta} \left| \begin{matrix} \Lambda \\ \mu, \beta \end{matrix} \right\rangle_{\mathrm{U}(M)} \tag{2.122}$$

where $\mathcal{B}^{\Lambda \to (\mu_1 \ldots \mu_M)}_{M_\Lambda \to \beta}$ is the coefficients of some unitary transformation. This equation implies

$$\mathrm{Dim}_M(\Lambda) = \sum_{\substack{\mu_1,\ldots,\mu_M \geq 0 \\ \mu_1 + \cdots + \mu_M = L}} K_{\Lambda\mu} \,. \tag{2.123}$$

We stress that the multiplicity space of $\mathrm{U}(M) \to \mathrm{U}(1)^M$ has the same dimension as the multiplicity space of the restriction $S_L \to S_\mu$ in (2.108),

$$\dim V^{\mathrm{U}(M) \to \mathrm{U}(1)^M}_{\Lambda \to \mu} = \dim V_{\Lambda \to ([\mu_1] \ldots [\mu_M])} = K_{\Lambda\mu} \tag{2.124}$$

which can be explained by Schur-Weyl duality [39].

[3] Our $\mathrm{U}(1)$ charges take non-negative integer values.

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By using (2.122), we can relate the original covariant basis and the general covariant basis as

$$\mathrm{Span}\left(\mathcal{O}^{R,\Lambda,\mu,\tau}_{\beta}(\vec{a}_\mu)\,\Big|\,\beta\in\{1,\ldots,K_{\Lambda\mu}\}\right)=\mathrm{Span}\left(\sum_{M_\Lambda}\mathcal{B}^{\Lambda\to(\mu_1\ldots\mu_M)}_{M_\Lambda\to\beta}\mathcal{O}^{R,\Lambda,M_\Lambda,\tau}\,\Big|\,\beta\in\{1,\ldots,K_{\Lambda\mu}\}\right). \tag{2.125}$$

In other words, the original covariant basis is a fixed-charge projection of the general covariant basis.

3 Eigenvalue method for $\mathcal{A}(\mu_1,\mu_2)$

Having described the general representation theoretic structure of the restricted Schur and covariant bases of the PCA $\mathcal{A}(\mu_1,\cdots,\mu_M)$ we now turn our attention to concrete construction algorithms. These involve integer matrix algorithms, in particular Hermite normal forms. We will see that these constructions give finite N integer orthogonal bases (with respect to the free two-point function in $\mathcal{N}=4$ SYM). Similar integer matrix methods were used in [42] to describe and construct representation theoretic subspace of algebras related to tensor invariants and Kronecker coefficients.

For concreteness, we will consider the case of $M=2, L=\mu_1+\mu_2\leq 14$ and $\mu=(\mu_1,\mu_2)$, but the generalisation is straightforward. In this section, we describe the mathematical aspects of the construction, while a detailed description of the code is given in appendix B. For specified μ_1,μ_2, the code produces as output:

- A restricted basis $\mathcal{A}(\mu_1,\mu_2)$, labelled by Young diagrams $R\vdash\mu_1+\mu_2=L, r_1\vdash\mu_1, r_2\vdash\mu_2$, presented as $g(r_1,r_2;R)^2$ linear combinations of multi-traces, where $g(r_1,r_2;R)$ is the Littlewood-Richardson coefficient for the triple.
- A covariant basis of $\mathcal{A}(\mu_1,\mu_2)$, labelled by Young diagrams $R\vdash L$ and $\Lambda\vdash L$, with $\ell(\Lambda)\leq 2$, presented as $C(R,R,\Lambda)K_{\Lambda\mu}$ with $\mu=(\mu_1,\mu_2)$ linear combinations of multi-traces, where $C(R,R,\Lambda)$ is the Kronecker coefficient (or Clebsch-Gordan multiplicity) for the triple of Young diagrams.

For $N>L$, these span the space of 2-matrix invariants and diagonalise the free field inner product. For $N<L$, the basis of 2-matrix invariants is obtained by keeping only basis states of $\mathcal{A}(\mu_1,\mu_2)$ with $\ell(R)\leq N$. The basis states of $\mathcal{A}(\mu_1,\mu_2)$ labelled by Young diagrams with $\ell(R)>N$ span the vector space of multi-traces with (μ_1,μ_2) copies of the two matrices which vanish by finite N trace relations (which are also known as Mandelstam identities in the physics literature [54–56]).

The finite N relations for the 2-matrix invariants are also encoded in 2-variable partition functions, or Hilbert series, for this problem. The form of these partition functions rapidly increase in complexity as a function of N, and are explicitly available for N up to 7 in the mathematics and physics literature [57–60]. These partition functions are of interest in the thermodyamics of the 2-matrix system which captures the deconfinement property of $\mathcal{N}=4$ SYM with U(N) gauge group and is important in the AdS/CFT correspondence. Discussion of the thermodynamic implications of finite N relations in the context of these partition functions are available in [60–62]. The description of the basis of null states given by the code for a fixed value of L, contains information related to the finite N relations for all $N < L$. For $L = 14$, this includes N up to 13. This illustrates that the algebraic method presented here is a substantial explicit information about finite N relations in the 2-matrix system.

3.1 Regular representation of $\mathcal{A}(\mu_1, \mu_2)$

To decompose $\mathcal{A}(\mu_1, \mu_2)$ we will study its regular representation and a set of relevant eigenvalue systems. The regular representation of $\mathcal{A}(\mu_1, \mu_2)$ is defined as follows.

Let $g_1, \ldots, g_l \in S_L$ be a complete set of representatives, which forms an orbit basis of $\mathcal{A}(\mu_1, \mu_2)$, defined in (2.81). We apply $\sigma \in \mathcal{A}(\mu_1, \mu_2)$ to $P_{\mu_1,\mu_2}(g_i)$ from left, and expand it in the orbit basis as

$$m^{\mathfrak{L}}[\sigma](P_{\mu_1,\mu_2}(g_i)) = \sigma P_{\mu_1,\mu_2}(g_i) \equiv \sum_{j=1}^{l} m^{\mathfrak{L}}_{ji}[\sigma] P_{\mu_1,\mu_2}(g_j). \tag{3.1}$$

The matrix elements $m^{\mathfrak{L}}_{ji}[\sigma]$ define the (left) regular representation of $\mathcal{A}(\mu_1, \mu_2)$. Borrowing the notation in section 2.1.1, we define the adjoint action of $\sigma = \sum_h c_h(\sigma)\, h$ on the orbit basis as

$$m^{\mathfrak{ad}}[\sigma](P_{\mu_1,\mu_2}(g_i)) = \sum_{h \in S_L} c_h(\sigma)\, h\, P_{\mu_1,\mu_2}(g_i)\, h^{-1} \equiv \sum_j m^{\mathfrak{ad}}_{ji}[\sigma] P_{\mu_1,\mu_2}(g_j) \tag{3.2}$$

We can extract the matrix elements from (3.1) or (3.2) using the δ-function as in (2.11), keeping in mind that we should sum over the conjugacy class of S_μ when working with $\mathcal{A}(\mu_1, \mu_2)$.

When σ is an integer linear combination of the basis elements $P_{\mu_1,\mu_2}(g_i)$, the corresponding representation matrices $m^{\mathfrak{L}}_{ji}[\sigma], m^{\mathfrak{ad}}_{ji}(\sigma)$ are all integer matrices. This follows from the fact that the structure constants C_{ijk} are integers

$$P_{\mu_1,\mu_2}(g_i)\, P_{\mu_1,\mu_2}(g_j) = \sum_{k=1}^{l} C_{ijk}\, P_{\mu_1,\mu_2}(g_k) \tag{3.3}$$

To show that C_{ijk} are integers, we rewrite $P_\mu(g_i)$ as

$$P_\mu(g_i) = \frac{1}{|\mathrm{Stab}(g_i)|} \sum_{\gamma \in S_\mu} \gamma g_i \gamma^{-1} = \sum_{a=1}^{|\mathrm{Orb}(g_i)|} g_{i,a} \tag{3.4}$$

with $g_i = g_{i,1}$. By using the multiplication rule of S_L, we find

$$P_\mu(g_i)\, P_\mu(g_j) = \sum_{a=1}^{|\mathrm{Orb}(g_i)|} \sum_{b=1}^{|\mathrm{Orb}(g_j)|} g_{i,a}\, g_{j,b} = \sum_{a,b} \sum_{k=1}^{l} \sum_{c=1}^{|\mathrm{Orb}(g_k)|} \delta\Big(g_{i,a}\, g_{j,b}\, g_{k,c}^{-1}\Big)\, g_{k,c}\,. \tag{3.5}$$

The sum over the δ-functions can be written as

$$\begin{aligned} \sum_{a,b} \delta\Big(g_{i,a}\, g_{j,b}\, g_{k,c}^{-1}\Big) &= \frac{1}{|\mathrm{Stab}(g_i)|\,|\mathrm{Stab}(g_j)|} \sum_{\gamma_i,\gamma_j \in S_\mu} \delta\Big(\gamma_i\, g_i\, \gamma_i^{-1}\, \gamma_j\, g_j\, \gamma_j^{-1}\, g_{k,c}^{-1}\Big) \\ &= \frac{1}{|\mathrm{Stab}(g_i)|\,|\mathrm{Stab}(g_j)|} \sum_{\gamma_i,\eta \in S_\mu} \delta\Big(g_i\, \eta\, g_j\, \eta^{-1}\, \gamma_i^{-1} g_{k,c}^{-1} \gamma_i\Big) \\ &= \frac{|\mathrm{Stab}(g_k)|}{|\mathrm{Stab}(g_i)|\,|\mathrm{Stab}(g_j)|} \sum_{\eta \in S_\mu} \sum_{c} \delta\Big(g_i\, \eta\, g_j\, \eta^{-1}\, g_{k,c}^{-1}\Big) \end{aligned} \tag{3.6}$$

with $\eta = \gamma_i^{-1}\gamma_j$. This equation shows that the quantity

$$C_{ijk} \equiv \sum_{a,b} \delta\Big(g_{i,a}\, g_{j,b}\, g_{k,c}^{-1}\Big) \tag{3.7}$$

is independent of c. The equation (3.5) simplifies as

$$P_\mu(g_i)\, P_\mu(g_j) = \sum_{k=1}^{l} C_{ijk} \sum_{c=1}^{|\mathrm{Orb}(g_k)|} g_{k,c} = \sum_{k=1}^{l} C_{ijk}\, P_{m,n}(g_k) \tag{3.8}$$

which is (3.3). From (3.7) we find that C_{ijk} is an integer.

3.2 Generating sets of central elements and integer eigenvalues

Having defined the representation matrices corresponding to left and adjoint actions, we will now formulate the eigenspaces $\mathcal{A}^{R,r_1,r_2}(\mu_1,\mu_2), \mathcal{A}^{R,\Lambda}(\mu_1,\mu_2)$ defined in (2.105), (2.118) as the intersections of kernels of explicit integer matrices acting on $\mathcal{A}(\mu_1,\mu_2)$.

Define the following elements in $\mathcal{A}(\mu_1,\mu_2)$

$$T_2^{(L)} = \sum_{1\le i<j\le L} (ij), \qquad T_2^{(\mu_1)} = \sum_{1\le a<b\le \mu_1} (ab), \qquad T_2^{(\mu_2)} = \sum_{\mu_1+1\le p<q\le L} (pq) \tag{3.9}$$
$$T_3^{(L)} = \sum_{1\le i<j<k\le L} (ijk)+(ikj), \quad T_3^{(\mu_1)} = \sum_{1\le a<b<c\le \mu_1} (abc)+(acb), \quad T_3^{(\mu_2)} = \sum_{\mu_1+1\le p<q<r\le L} (pqr)+(prq).$$

We call $T_p^{(k)}$ Casimir operators, because they are the central elements we encountered in (2.100) and (2.101),

$$T_p^{(L)} \in \mathcal{Z}(\mathbb{C}[S_L]), \qquad T_p^{(\mu_1)} \in \mathcal{Z}(\mathbb{C}[S_{\mu_1}]), \qquad T_p^{(\mu_2)} \in \mathcal{Z}(\mathbb{C}[S_{\mu_2}]). \tag{3.10}$$

As discussed in section 2.1.2, only the generating set of Casimir operators is needed to distinguish different representations. Furthermore, the elements in (3.9) are integer combinations of the basis elements $P_{\mu_1,\mu_2}(g_i)$ and therefore the corresponding $m^{\mathfrak{L}}_{ji}, m^{\mathfrak{ad}}_{ji}$ are integer matrices.

First, consider the case of $\mathcal{A}^{R,r_1,r_2}(\mu_1,\mu_2)$ related to the restricted Schur basis. We will use the following shorthand notation for these eigenvalues,

$$\widehat{\chi}_p^R = \frac{\chi^R(T_p^{(L)})}{d_R}, \quad \widehat{\chi}_p^{r_1} = \frac{\chi^{r_1}(T_p^{(\mu_1)})}{d_{r_1}}, \quad \widehat{\chi}_p^{r_2} = \frac{\chi^{r_2}(T_p^{(\mu_2)})}{d_{r_2}}, \qquad p = 2,3 \tag{3.11}$$

which are normalised characters. The normalised characters of symmetric groups are known to have integer values [63]. Now we have a collection of integer matrices with integer eigenvalues acting on (the regular representation of) $\mathcal{A}(\mu_1,\mu_2)$. According to the discussion in section 2.1.2, these matrices and corresponding eigenvalues are sufficient, for $\mu_1+\mu_2 \le 14$, to distinguish all the subspaces $\mathcal{A}^{R,r_1,r_2}(\mu_1,\mu_2)$ in the sense that

$$\mathcal{A}^{R,r_1,r_2}(\mu_1,\mu_2) \cong \mathrm{Ker}\Big(m^{\mathfrak{L}}[T_2^{(L)}] - \widehat{\chi}_2^R\Big) \cap \mathrm{Ker}\Big(m^{\mathfrak{L}}[T_3^{(L)}] - \widehat{\chi}_3^R\Big) \cap \mathrm{Ker}\Big(m^{\mathfrak{L}}[T_2^{(\mu_1)}] - \widehat{\chi}_2^{r_1}\Big) \cap$$
$$\mathrm{Ker}\Big(m^{\mathfrak{L}}[T_3^{(\mu_1)}] - \widehat{\chi}_3^{r_1}\Big) \cap \mathrm{Ker}\Big(m^{\mathfrak{L}}[T_2^{(\mu_2)}] - \widehat{\chi}_2^{r_2}\Big) \cap \mathrm{Ker}\Big(m^{\mathfrak{L}}[T_3^{(\mu_2)}] - \widehat{\chi}_3^{r_2}\Big) \tag{3.12}$$

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where $\mathrm{Ker}(M)$ for a linear operator M is the subspace of vectors x satisfying $Mx = 0$.

For completeness, let us now describe the situation for $\mathcal{A}^{R,\Lambda}$ related to the covariant basis, which is analogous. From (2.117) we know that the matrices $m^{\mathfrak{ad}}_{ij}[T_2^{(L)}], m^{\mathfrak{ad}}_{ij}[T_3^{(L)}]$ have eigenvalues given by normalised characters

$$\widehat{\chi}^{\Lambda}_p = \frac{\chi^{\Lambda}(T_p^{(L)})}{d_\Lambda}, \quad p = 2, 3. \tag{3.13}$$

As before, $\Lambda \vdash L \leq 14$ is uniquely determined by these eigenvalues and it follows that

$$\begin{aligned} \mathcal{A}^{R,\Lambda}(\mu_1,\mu_2) \cong \mathrm{Ker}\Big(m^{\mathfrak{L}}[T_2^{(L)}] - \widehat{\chi}^R_2\Big) \cap \mathrm{Ker}\Big(m^{\mathfrak{L}}[T_3^{(L)}] - \widehat{\chi}^R_3\Big) \cap \\ \mathrm{Ker}\Big(m^{\mathfrak{ad}}[T_2^{(L)}] - \widehat{\chi}^\Lambda_2\Big) \cap \mathrm{Ker}\Big(m^{\mathfrak{ad}}[T_3^{(L)}] - \widehat{\chi}^\Lambda_3\Big). \end{aligned} \tag{3.14}$$

3.3 Hermite normal forms and kernels of integer matrices

Computing the intersection of kernels of square matrices corresponds to computing the kernel of a single rectangular matrix. For example, given a pair of $l \times l$ matrices M_1, M_2 we find the intersection $\mathrm{Ker}(M_1) \cap \mathrm{Ker}(M_2)$ by finding the kernel of the matrix

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix}. \tag{3.15}$$

This follows since $Mx = 0$ if and only if $M_1 x = M_2 x = 0$ and this generalises to the intersection of kernels of several matrices. Therefore, the space (3.12) corresponds to solutions to

$$\begin{bmatrix} m^{\mathfrak{L}}[T_2^{(L)}] - \widehat{\chi}^R_2 \\ m^{\mathfrak{L}}[T_3^{(L)}] - \widehat{\chi}^R_3 \\ m^{\mathfrak{L}}[T_2^{(\mu_1)}] - \widehat{\chi}^{r_1}_2 \\ m^{\mathfrak{L}}[T_2^{(\mu_1)}] - \widehat{\chi}^{r_1}_2 \\ m^{\mathfrak{L}}[T_2^{(\mu_2)}] - \widehat{\chi}^{r_2}_2 \\ m^{\mathfrak{L}}[T_3^{(\mu_2)}] - \widehat{\chi}^{r_2}_3 \end{bmatrix} x = 0 \tag{3.16}$$

We will now give a procedure for solving this equation over the integers. In other words, for fixed (R, r_1, r_2) we will give a procedure for constructing an integer basis for the space satisfying the simultaneous equations. We will see that this can be turned into an integer orthogonal basis of two-matrix invariants related to the restricted Schur basis. The method works identically for the system of equations

$$\begin{bmatrix} m^{\mathfrak{L}}[T_2^{(L)}] - \widehat{\chi}^R_2 \\ m^{\mathfrak{L}}[T_3^{(L)}] - \widehat{\chi}^R_3 \\ m^{\mathfrak{ad}}[T_2^{(L)}] - \widehat{\chi}^\Lambda_2 \\ m^{\mathfrak{ad}}[T_3^{(L)}] - \widehat{\chi}^\Lambda_3 \end{bmatrix} x = 0 \tag{3.17}$$

corresponding to (3.14), which gives an integer orthogonal basis related to the covariant basis for two-matrix invariants.

Computing an integer basis for $\mathrm{Ker}(M)$ uses Hermite normal forms, which are the analog of the more familiar echelon forms, adapted to integer matrix problems, as we will now

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describe. We now make some general remarks about Hermite normal forms. The fundamental result on Hermite normal forms is the following (see [64, Theorem 2.4.3]). Let M be an $m \times n$ integer matrix, then there exists a unimodular $n \times n$ matrix U — an integer matrix with $\det(U) = \pm 1$ — such that

$$MU = H = \begin{bmatrix} 0 \; h \end{bmatrix}, \tag{3.18}$$

where h is an invertible, upper triangular integer matrix and 0 represents are rectangular matrix of zeros. By upper triangular we mean $h_{ij} = 0$ for $i > j$. Further conditions are put on h to make it unique, see [64, Definition 2.4.2]. The matrix H is called the Hermite normal form of M. Note that, despite H being unique, U is not unique in general. In particular, H is invariant under permutations of the zero columns; multiplication of any zero column by ± 1; addition of any scalar multiple of a zero column to any other column. These operations can be implemented by right multiplication with unimodular matrices V and we get

$$H = HV = MUV \equiv MU' \tag{3.19}$$

where $U' = UV$ is another unimodular matrix giving an Hermite decomposition of $M = H(U')^{-1}$. A Hermite decomposition gives a basis for the kernel of an integer matrix M as follows [64, Proposition 2.4.9]. Let $H = MU$ be a Hermite decomposition such that the first r columns of H are equal to zero

$$[\underbrace{0}_{r} \; h] = MU. \tag{3.20}$$

Then the first r columns of U form a basis for the kernel of M. In other words, define the vectors $v_i^{(s)} = U_{si}$, then

$$\mathrm{Ker}(M) = \mathrm{Span}(v^{(1)}, v^{(2)}, \ldots, v^{(r)}). \tag{3.21}$$

The number of independent solutions r is precisely equal to the dimensions of the subalgebra, namely

$$\dim \mathcal{A}^{R,r_1,r_2}(\mu_1, \mu_2) = g(r_1, r_2; R)^2 \tag{3.22}$$

for the restricted Schur basis and

$$\dim \mathcal{A}^{R,\Lambda}(\mu_1, \mu_2) = C(R, R, \Lambda) K_{\Lambda(\mu_1,\mu_2)} \tag{3.23}$$

for the covariant basis. In appendix B.2 we will explain an algorithm to compute an Hermite normal form.

3.4 Simple example

As a simple example to illustrate the procedure, we will consider the Schur case $\mathcal{A}(3, 0) = \mathcal{Z}(\mathbb{C}[S_3])$ in detail. We have a set of representatives given by the identity permutation $g_1 = (1)(2)(3)$ and $g_2 = (12)(3), g_3 = (123)$ with orbit basis

$$\begin{aligned} P_{3,0}(g_1) &= (1)(2)(3), \\ P_{3,0}(g_2) &= (12)(3) + (13)(2) + (23)(1), \\ P_{3,0}(g_3) &= (123) + (132). \end{aligned} \tag{3.24}$$

We will only need the left action of $T_2^{(3)} = P_{3,0}(g_2)$ in this basis, it is straightforward to compute

$$T_2^{(3)} P_{3,0}(g_1) = P_{3,0}(g_2), \tag{3.25}$$

$$T_2^{(3)} P_{3,0}(g_2) = 3P_{3,0}(g_1) + 3P_{3,0}(g_3), \tag{3.26}$$

$$T_2^{(3)} P_{3,0}(g_3) = 2P_{3,0}(g_2), \tag{3.27}$$

and equivalently, the left action matrix in this basis is

$$m^{\mathfrak{L}}[T_2^{(3)}] = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 3 & 0 \end{pmatrix} \tag{3.28}$$

We will only consider the eigenspace corresponding to $R = [3]$. This has normalised eigenvalue $\widehat{\chi}^R(T_2^{(3)}) = 3$ and so we are looking for the kernel of the matrix

$$m^{\mathfrak{L}}[T_2^{(3)}] - 3I_3 = \begin{pmatrix} -3 & 3 & 0 \\ 1 & -3 & 2 \\ 0 & 3 & -3 \end{pmatrix} \tag{3.29}$$

It is easy to confirm that the vector $(1, 1, 1)$ lies in the kernel and forms an integer basis (this is derived in appendix B.3 using an integer matrix algorithm). In the orbit basis, this corresponds to

$$P_{3,0}(g_1) + P_{3,0}(g_2) + P_{3,0}(g_3) = \sum_{g \in S_3} g \tag{3.30}$$

or in terms of traces

$$\mathrm{Tr}(Z)^3 + 3\,\mathrm{Tr}(Z)\,\mathrm{Tr}\Big(Z^2\Big) + 2\,\mathrm{Tr}\Big(Z^3\Big). \tag{3.31}$$

3.5 Orthogonality from planar to finite N

We will show that the finite N two-point functions in $\mathcal{N} = 4$ SYM at zero coupling are proportional to the planar two-point functions in any operator bases which are labeled by the irreducible representation R under the left/right action of the centre of $\mathbb{C}[S_L]$.

The finite N free-field two-point functions in the permutation basis is given by

$$\langle \mathcal{O}_{g_1}[Z,W]\mathcal{O}_{g_2}[Z,W]^\dagger \rangle = \langle \mathcal{O}_{g_1}[Z,W]\mathcal{O}_{g_2^{-1}}[Z^\dagger,W^\dagger] \rangle = \sum_{\gamma \in S_{\mu_1}\times S_{\mu_2}} \sum_{g_3 \in S_L} \delta(g_1 \gamma g_2^{-1} \gamma^{-1} g_3) N^{\mathcal{C}_{g_3}}, \tag{3.32}$$

where the operators have μ_1 Z's and μ_2 W's, and $\mathcal{C}_g$ is the number of cycles in $g \in S_L$. The two-point functions vanish if $\mathcal{O}_{g_1}[Z,W]$ and $\mathcal{O}_{g_2}[Z,W]$ have different (μ_1,μ_2)'s. This defines an inner product on $\mathcal{A}(\mu_1,\mu_2)$. Let $\sigma_1,\sigma_2 \in \mathcal{A}(\mu_1,\mu_2)$, then

$$(\sigma_1,\sigma_2)_{\text{finite}\,N} = \sum_{\gamma \in S_{\mu_1}\times S_{\mu_2}} \sum_{g \in S_L} \delta(\sigma_1 \gamma \mathcal{S}(\sigma_2)\gamma^{-1} g) N^{\mathcal{C}_g} = \mu_1! n! \sum_{g \in S_L} \delta(\sigma_1 \mathcal{S}(\sigma_2) g) N^{\mathcal{C}_g}, \tag{3.33}$$

where $\mathcal{S}(\sigma_2)$ is defined in (2.12). Clearly, this inner product is symmetric, $(\sigma_1,\sigma_2)_{\text{finite}\,N} = (\sigma_2,\sigma_1)_{\text{finite}\,N}$.

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The planar inner product is defined by the large N limit

$$(\sigma_1,\sigma_2)_{\text{planar}} = \lim_{N\to\infty} \frac{1}{N^L}(\sigma_1,\sigma_2)_{\text{finite}\,N} = \mu_1!\mu_2!\,\delta(\sigma_1\mathcal{S}(\sigma_2)). \tag{3.34}$$

The planar inner product is essentially the same as the δ-function inner product computed in (2.115). Conversely, the finite N inner product can be obtained by deforming the planar inner product

$$(\sigma_1,\sigma_2)_{\text{finite}\,N} = (\Omega\,\sigma_1,\sigma_2)_{\text{planar}}\,, \tag{3.35}$$

where we defined $\Omega \in \mathbb{C}[S_L]$ by

$$\Omega = \sum_{g\in S_L} N^{\mathcal{C}_g} g = L! \sum_{R\vdash L} \frac{\text{Dim}_N(R)}{d_R} P^R. \tag{3.36}$$

Here P^R is the projector to the representation R in (2.45), and $\text{Dim}_N(R)$ is the $\text{U}(N)$ dimension of the representation R as in (2.37). See [5] for the derivation of the second equality. As we will now see, orthogonalisation with respect to the planar limit is very closely related to orthogonalisation in the finite N case.

The left and adjoint actions of $T_p \in \mathcal{Z}(\mathbb{C}[S_L])$ used in (3.16), (3.17) are self-adjoint with respect to the planar as well as finite N inner products. For the left action we have

$$(T_p\,\sigma_1,\sigma_2)_{\text{planar}} = \mu_1!\mu_2!\,\delta(T_p\,\sigma_1\mathcal{S}(\sigma_2)) = \mu_1!\mu_2!\,\delta(\sigma_1\mathcal{S}(T_p\sigma_2)) = (\sigma_1,T_p\,\sigma_2)_{\text{planar}}, \tag{3.37}$$

where we have used $T_p = \mathcal{S}(T_p)$ because every conjugacy class of S_L contains its inverse elements and $\mathcal{S}(\sigma\tau) = \mathcal{S}(\tau)\mathcal{S}(\sigma)$ for $\sigma,\tau \in \mathbb{C}[S_L]$. For the adjoint action, define C_p to be the conjugacy class C_ρ with $\rho = (p, 1^{L-p}) \vdash L$, then we have

$$\begin{aligned}
(m^{\mathfrak{ad}}[T_p](\sigma_1),\sigma_2)_{\text{planar}} &= \sum_{g\in C_p}(g\sigma_1 g^{-1},\sigma_2)_{\text{planar}} = \mu_1!\mu_2! \sum_{g\in C_p} \delta(g\,\sigma_1 g^{-1}\,\mathcal{S}(\sigma_2)) \\
&= \mu_1!\mu_2! \sum_{g\in C_p} \delta(\sigma_1\,\mathcal{S}(g^{-1}\sigma_2 g)) = \mu_1!\mu_2! \sum_{g\in C_p} \delta(\sigma_1\,\mathcal{S}(g\sigma_2 g^{-1})) \\
&= (\sigma_1, m^{\mathfrak{ad}}[T_p](\sigma_2))_{\text{planar}}.
\end{aligned} \tag{3.38}$$

In the penultimate step we used the fact that if $g \in C_p$ then so is g^{-1} and therefore we can relabel the sum under the transformation $g \mapsto g^{-1}$. The proofs are similar for the finite N case.

3.5.1 Restricted Schur basis

Let the eigenvectors of the first equation in (3.16) be $\mathsf{Q}_A^{R,r_1 r_2}$, where A corresponds to the multiplicity in a fixed eigenspace of dimension $g(r_1,r_2;R)^2$. Explicitly, we write them as linear combinations of the restricted Schur basis

$$\mathsf{Q}_A^{R,r_1 r_2} = \sum_{\nu_+,\nu_-=1}^{g(R;r_1,r_2)} \mathsf{N}_{A,\nu_+\nu_-}^{R,r_1,r_2}\, Q_{\nu_+\nu_-}^{R,r_1,r_2} \tag{3.39}$$

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where $\mathsf{N}^{R,r_1,r_2}_{A,\nu_+\nu_-}$ is a numerical coefficient. With the help of (2.95), the planar inner product is given by

$$(\mathsf{Q}^{R,r_1,r_2}_A, \mathsf{Q}^{S,s_1,s_2}_B)_{\text{planar}} = \delta^{RS}\delta^{r_1 s_1}\delta^{r_2 s_2}\, d_{r_1} d_{r_2}\, g^{R,r_1,r_2}_{AB}\,, \qquad g^{R,r_1,r_2}_{AB} \equiv \sum_{\nu_+,\nu_-=1}^{g(R;r_1,r_2)} \mathsf{N}^{R,r_1,r_2}_{A,\nu_+\nu_-}\, \mathsf{N}^{R,r_1,r_2}_{B,\nu_+\nu_-} \tag{3.40}$$

where g^{R,r_1,r_2}_{AB} is a Gram (real symmetric) matrix of dimensions $g(r_1,r_2;R)^2$. Note that the inner product must vanish if $(R,r_1,r_2) \neq (S,s_1,s_2)$, because they label the eigenvectors of the self-adjoint operators $(T^{(L)}_p, T^{(\mu_1)}_p, T^{(\mu_2)}_p)$.

To compute the finite N inner product, consider the deformation by Ω in (3.35). Because Ω is central, using (2.100) we get

$$\Omega\, \mathsf{Q}^{R,r_1,r_2}_A = \frac{\chi^R(\Omega)}{d_R}\, \mathsf{Q}^{R,r_1,r_2}_A\,. \tag{3.41}$$

The normalised character is readily computed using equation (3.36)

$$\frac{\chi^R(\Omega)}{d_R} = \frac{L!}{d_R}\sum_{S\vdash L}\frac{\text{Dim}_N(S)}{d_S}\chi^R(P^S) = \frac{L!}{d_R}\sum_{S\vdash L}\frac{\text{Dim}_N(S)}{d_S} d_S\,\delta^{RS} = L!\,\frac{\text{Dim}_N(R)}{d_R}. \tag{3.42}$$

Consequently,

$$(\mathsf{Q}^{R,r_1,r_2}_A, \mathsf{Q}^{S,s_1,s_2}_B)_{\text{finite}\,N} = (\Omega\,\mathsf{Q}^{R,r_1,r_2}_A, \mathsf{Q}^{S,s_1,s_2}_B)_{\text{planar}} = L!\,\frac{\text{Dim}_N(R)}{d_R}\,(\mathsf{Q}^{R,r_1,r_2}_A, \mathsf{Q}^{S,s_1,s_2}_B)_{\text{planar}} \tag{3.43}$$

which shows that the finite N inner product is proportional to the planar inner product in the restricted Schur basis. Note that all the N-dependence is captured by the factor $\text{Dim}_N(R)$.

3.5.2 Covariant basis

Analogous statements hold for the eigenvectors of (3.17), which we denote by $\mathsf{Q}^{R,\Lambda,\mu}_A$ where $A \in \{1,\ldots,C(R,R,\Lambda)K_{\Lambda\mu}\}$. Explicitly, they are linear combinations of covariant bases elements

$$\mathsf{Q}^{R,\Lambda,\mu}_A = \sum_{\tau=1}^{C(R,R,\Lambda)}\sum_{\beta=1}^{K_{\Lambda\mu}} \mathsf{N}^{R,\Lambda,\tau,\mu,\beta}_A\, \mathcal{Q}^{R,\Lambda,\mu,\tau}_\beta\,. \tag{3.44}$$

By using (2.115) we find that the planar inner product is given by

$$(\mathsf{Q}^{R,\Lambda,\mu}_A, \mathsf{Q}^{S,\Lambda',\mu}_B)_{\text{planar}} = \frac{d_R}{|S_L|}\,\delta^{RS}\delta^{\Lambda\Lambda'}\, G^{R,\Lambda,\mu}_{AB}\,, \qquad G^{R,\Lambda,\mu}_{AB} \equiv \sum_{\tau=1}^{C(R,R,\Lambda)}\sum_{\beta=1}^{K_{\Lambda\mu}} \mathsf{N}^{R,\Lambda,\tau,\mu,\beta}_A\, \mathsf{N}^{R,\Lambda,\tau,\mu,\beta}_B \tag{3.45}$$

where $G^{R,\Lambda,\mu}_{AB}$ is a Gram matrix of dimensions $C(R,R,\Lambda)\,K_{\Lambda\mu}$. As before, the inner product vanishes if $(R,\Lambda) \neq (S,\Lambda')$. From (2.116)

$$\Omega\,\mathsf{Q}^{R,\Lambda,\mu}_A = \frac{\chi^R(\Omega)}{d_R}\mathsf{Q}^{R,\Lambda}_A. \tag{3.46}$$

This gives

$$(\mathsf{Q}^{R,\Lambda,\mu}_A, \mathsf{Q}^{S,\Lambda',\mu}_B)_{\text{finite}\,N} = (\Omega\,\mathsf{Q}^{R,\Lambda,\mu}_A, \mathsf{Q}^{S,\Lambda',\mu}_B)_{\text{planar}} = L!\,\frac{\text{Dim}_N(R)}{d_R}\,(\mathsf{Q}^{R,\Lambda,\mu}_A, \mathsf{Q}^{S,\Lambda',\mu}_B)_{\text{planar}}\,. \tag{3.47}$$

3.5.3 Gram-Schmidt orthogonalisation

The orbit basis is orthogonal with respect to the planar inner product (3.34) since

$$\delta(P_{\mu_1,\mu_2}(g_i)\,\mathcal{S}(P_{\mu_1,\mu_2}(g_j))) = \sum_{\substack{g\in\{g_i\}_{\mu_1,\mu_2}\\ h\in\{g_j\}_{\mu_1,\mu_2}}} \delta(gh^{-1}) = |\mathrm{Orb}(g_i)|\delta_{ij}. \tag{3.48}$$

It follows that

$$(P_{\mu_1,\mu_2}(g_i), P_{\mu_1,\mu_2}(g_j))_{\text{planar}} = \mu_1!\mu_2!\,|\mathrm{Orb}(g_i)|\,\delta_{ij}\,, \tag{3.49}$$

which is integer valued.

As we have seen, the integer eigenvectors Q^{R,r_1,r_2}_A or $\mathsf{Q}^{R,\Lambda,\mu}_A$ are not automatically orthogonal. Integer orthogonal eigenvectors are readily constructed using Gram-Schmidt orthogonalisation. We simply define

$$\begin{aligned}
&\widehat{\mathsf{Q}}^{R,r_1,r_2}_1 = \mathsf{Q}^{R,r_1,r_2}_1,\\
&\widehat{\mathsf{Q}}^{R,r_1,r_2}_A = \mathsf{Q}^{R,r_1,r_2}_A - \sum_{B=1}^{A-1} \frac{(\mathsf{Q}^{R,r_1,r_2}_A, \widehat{\mathsf{Q}}^{R,r_1,r_2}_B)_{\text{planar}}}{(\widehat{\mathsf{Q}}^{R,r_1,r_2}_B, \widehat{\mathsf{Q}}^{R,r_1,r_2}_B)_{\text{planar}}}\,\widehat{\mathsf{Q}}^{R,r_1,r_2}_B\,, \quad A\in\{2,\dots,g(r_1,r_2;R)^2\}
\end{aligned} \tag{3.50}$$

and it follows that

$$(\widehat{\mathsf{Q}}^{R,r_1,r_2}_A, \widehat{\mathsf{Q}}^{R,r_1,r_2}_B)_{\text{planar}} \propto \delta_{AB} \quad \Rightarrow \quad (\widehat{\mathsf{Q}}^{R,r_1,r_2}_A, \widehat{\mathsf{Q}}^{R,r_1,r_2}_B)_{\text{finite N}} \propto L!\,\frac{\mathrm{Dim}_N(R)}{d_R}\,\delta_{AB}. \tag{3.51}$$

Similarly for the eigenvectors $\mathsf{Q}^{R,\Lambda,\mu}_A$. Note that, even if Q^{R,r_1,r_2}_A, $\mathsf{Q}^{R,\Lambda,\mu}_A$ are integer vectors, the orthogonalised vectors $\widehat{\mathsf{Q}}^{R,r_1,r_2}_A$, $\widehat{\mathsf{Q}}^{R,\Lambda,\mu}_A$ are in general rational. This is easily remedied by multiplying each basis element by an appropriate factor.

3.5.4 General basis

We have discovered that the finite N inner product is proportional to the planar inner product in a basis diagonalising the left action of the centre of $\mathbb{C}[S_L]$. The planar inner products are proportional to the Gram matrix, g^{R,r_1,r_2}_{AB} in (3.40) and $G^{R,\Lambda,\mu}_{AB}$ in (3.45). If we orthogonalise the Gram matrices, for example using Gram-Schmidt (3.50), we obtain orthogonal finite N bases.

In both (3.43) and (3.47), the proportionality constant only depends on the label $R\vdash L$. Let $\mathsf{Q}^R_\alpha \in \mathcal{A}(\mu_1,\mu_2)$ be solutions to the eigenvalue equations only for the left action of $T^{(L)}_p$. Written explicitly, they are

$$\mathsf{Q}^R_\alpha = \sum_{\substack{r_1\vdash\mu_1\\ r_2\vdash\mu_2}} \sum_{A=1}^{g(R;r_1,r_2)^2} \mathsf{M}^{r_1r_2}_{\alpha A}\,\mathsf{Q}^{R,r_1r_2}_A = \sum_{\substack{\Lambda\vdash L\\ \ell(\Lambda)\le 2}} \sum_{A=1}^{C(R,R,\Lambda)K_{\Lambda\mu}} \mathsf{M}^{\Lambda}_{\alpha A}\,\mathsf{Q}^{R,\Lambda,\mu}_A \tag{3.52}$$

for some numerical constants $\mathsf{M}^{r_1r_2}_{\alpha A}$ and $\mathsf{M}^{\Lambda}_{\alpha A}$. If we define the Gram matrix $\mathcal{G}^R_{\alpha\beta}$ by

$$(\mathsf{Q}^R_\alpha\,,\mathsf{Q}^S_\beta)_{\text{planar}} = \delta^{RS}\mathcal{G}^R_{\alpha\beta} \tag{3.53}$$

from arguments similar to those above we find

$$(\mathsf{Q}^R_\alpha\,,\mathsf{Q}^S_\beta)_{\text{finite}\,N} = L!\,\frac{\text{Dim}_N(R)}{d_R}(\mathsf{Q}^R_\alpha\,,\mathsf{Q}^S_\beta)_{\text{planar}}\,. \tag{3.54}$$

Now the dimensions of the Gram matrix $\mathcal{G}^R_{\alpha\beta}$ is

$$\sum_{\substack{\Lambda\vdash L\\ \ell(\Lambda)\leq 2}} C(R,R,\Lambda)K_{\Lambda\mu} = \sum_{\substack{r_1\vdash\mu_1\\ r_2\vdash\mu_2}} g(r_1,r_2;R)^2. \tag{3.55}$$

Orthogonalising $\mathcal{G}^R_{\alpha\beta}$ gives a finite N orthogonal basis, which is a linear combination of the restricted Schur or covariant basis.

4 Eigenvalue method for $\mathbb{C}[S_L]$

In section 3 we developed eigenvalue systems for $\mathcal{A}(\mu)$ which give bases of multi-matrix invariants related to the restricted Schur and covariant bases. Similarly, in this section we develop two eigensystems for $\mathcal{A}(1,\ldots,1) = \mathbb{C}[S_L]$. The first one gives the Artin-Wedderburn decomposition, and the second gives the Kronecker decomposition of $\mathbb{C}[S_L]$.

4.1 Decomposition of the regular representation

We will construct and diagonalise a set of mutually commuting Hermitian operators in order to decompose the regular representation of $\mathbb{C}[S_L]$ denoted by $\mathcal{V}_{\text{reg}}$. A set of commuting operators

$$\{m^X[O_1],\ldots,m^X[O_\Omega]\}\ \in\ \text{End}\,(\mathcal{V}_{\text{reg}}), \qquad X\in\{\mathfrak{L},\mathfrak{R},\mathfrak{ad}\} \tag{4.1}$$

is called complete if their eigenvalues specify a unique vector in $\mathcal{V}_{\text{reg}}$ up to normalisation. That is, for

$$\sum_{j=1}^{L!} m^X_{ij}[O_w](\alpha_j) = \lambda^{(\alpha)}_w\,\alpha_i \tag{4.2}$$

and any pair of eigenstates $\alpha,\beta\in\mathcal{V}_{\text{reg}}$

$$\alpha\neq\beta \quad\Leftrightarrow\quad \{\lambda^{(\alpha)}_1,\ldots,\lambda^{(\alpha)}_\Omega\}\neq\{\lambda^{(\beta)}_1,\ldots,\lambda^{(\beta)}_\Omega\}. \tag{4.3}$$

We are interested in finding a set of commuting operators $\{m^X[O_1],\ldots,m^X[O_\Omega]\}$ whose eigenstates correspond to the two decompositions of $\mathbb{C}[S_L]$ discussed in section 2.2. These decompositions allow us to construct orthogonal bases of operators of $\mathcal{N}=4$ SYM for the case of $\mathcal{A}(1,\ldots,1)=\mathbb{C}[S_L]$. This is a good starting point for considering the general case of $\mathcal{A}(\mu)$.

One natural choice of the commuting operators comes from $\mathbb{C}[S_L]$ itself. The elements

$$\mathcal{J}_1 = 0, \qquad \mathcal{J}_k = \sum_{i=1}^{k-1}(i,k), \qquad (k=2,3,\ldots,L) \tag{4.4}$$

are known as Young-Jucys-Murphy elements [65, 66] They generate a maximal commuting subalgebra,[4] denoted $\mathcal{M}(\mathbb{C}[S_L])$, of the full group algebra and $\mathcal{Z}(\mathbb{C}[S_L])$ is a subalgebra

$$\mathcal{Z}(\mathbb{C}[S_L])\ \subset\ \mathcal{M}(\mathbb{C}[S_L])\ \subset\ \mathbb{C}[S_L]. \tag{4.5}$$

[4]This subalgebra is called Gelfand-Tsetlin algebra in [67].

They satisfy $[\mathcal{J}_k\,, \mathcal{J}_\ell] = 0$ for any k, ℓ [65, 66] and the following relation with $s_k = (k, k+1) \in S_L$

$$s_k\, \mathcal{J}_{k+1} = \mathcal{J}_k\, s_k + 1. \tag{4.6}$$

In the regular representation, every element of $\mathbb{C}[S_L]$ corresponds to an element of $\mathrm{End}(\mathcal{V}_{\rm reg})$ and the YJM elements correspond to a complete set of commuting matrices.

The YJM elements are not central in general, but power symmetric polynomials

$$\sum_{k=1}^{L} \mathcal{J}_k, \quad \sum_{k=1}^{L} \mathcal{J}_k^2, \dots \tag{4.7}$$

are related to central elements by [65, 68, 69]

$$T_2 = \sum_{i<j}^{L} (i,j) = \sum_{k}^{L} \mathcal{J}_k \tag{4.8}$$

$$T_3 = \sum_{i<j<k}^{L} \Big\{ (i,j,k) + (j,i,k) \Big\} = \sum_{k}^{L} \mathcal{J}_k^2 - \frac{L(L-1)}{2}\,, \qquad (L \geq 3). \tag{4.9}$$

Because the YJM elements generate a maximal commuting subalgebra, the eigenvalues of the corresponding matrices uniquely specify a standard Young tableau. More precisely, the left or right eigenvalue of $\mathcal{J}_k$ is equal to the content of the box k inside the Young tableau I of shape R,

$$\mathcal{J}_k \left| \begin{matrix} R \\ I \end{matrix} \right\rangle = \texttt{Cont}_I^R(k) \left| \begin{matrix} R \\ I \end{matrix} \right\rangle \qquad \Longleftrightarrow \qquad D^R_{JI}(\mathcal{J}_k) = \delta_{JI}\, \texttt{Cont}_I^R(k) \tag{4.10}$$

where the content function $\texttt{Cont}$ assigns $(x-y)$ for the box k found at the x-th column and the y-th row,

$$\texttt{Cont}\left(\begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & \\ \cline{1-4} & & \\ \cline{1-3} \end{array} \right) = \begin{array}{|c|c|c|c|c|} \hline 0 & 1 & 2 & 3 & 4 \\ \hline -1 & 0 & 1 & 2 \\ \cline{1-4} -2 & -1 & 0 \\ \cline{1-3} \end{array}\,. \tag{4.11}$$

One can also show that the adjoint actions of $\mathcal{J}_k$ have the same eigenvalues as in the left or right actions by using (4.6).[5]

The relations (4.8) and (4.9) make it clear that the eigenbasis $\{ \left| \begin{smallmatrix} R \\ I \end{smallmatrix} \right\rangle \}$ in (4.10) diagonalises the central elements T_p, and their eigenvalues depend only on the Young diagram R. Furthermore, it was shown that this eigenbasis coincides with the Young seminormal representation by computing the action of s_k [66, 67]. To obtain the Young-Yamanouchi orthonormal representation, which is real and unitary, we need to rescale the basis elements of $\{ \left| \begin{smallmatrix} R \\ I \end{smallmatrix} \right\rangle \}$ in the Young seminormal representation.

[5] See the argument in [70].

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4.2 Eigenvalue system for the Artin-Wedderburn decomposition

Recall that the Artin-Wedderburn decomposition of $\mathbb{C}[S_L]$ gives the matrix unit denoted by Q^R_{IJ}. The matrix unit satisfies the relation

$$Q^R_{IJ}\, Q^S_{KL} = \delta^{RS}\, \delta_{JK}\, Q^R_{IL} \,. \tag{4.12}$$

As shown in (2.53) and (2.54), the left and right actions of $\mathbb{C}[S_L]$ on the matrix unit can be written as the multiplication of the irreducible representation matrix. Let us identify the subscripts of Q^R_{IJ} as a basis of Young seminormal representations. If we apply the YJM elements from left and right, we obtain

$$\begin{aligned} \mathcal{J}_k\, Q^R_{IJ} &= \sum_K D^R_{KI}(\mathcal{J}_k)\, Q^R_{KJ} = \texttt{Cont}^R_I(k)\, Q^R_{IJ} \\ Q^R_{IJ}\, \mathcal{J}_\ell &= \sum_K Q^R_{IK}\, D^R_{JK}(\mathcal{J}_\ell) = \texttt{Cont}^R_J(k)\, Q^R_{IK} \end{aligned} \tag{4.13}$$

where we used (4.10).

The matrix unit Q^R_{IJ} satisfies the following set of eigenvalue system,

$$\begin{aligned} T_p\, Q^R_{IJ} = Q^R_{IJ}\, T_p &= \frac{\chi^R(T_p)}{d_R}\, Q^R_{IJ} \\ \mathcal{J}_k\, Q^R_{IJ} &= \texttt{Cont}^R_I(k)\, Q^R_{IJ} \\ Q^R_{IJ}\, \mathcal{J}_k &= \texttt{Cont}^R_J(k)\, Q^R_{IJ} \,. \end{aligned} \tag{4.14}$$

Conversely, we can reconstruct the matrix unit by solving this eigensystem up to an overall normalisation, which is fixed by (4.12). Since the matrix unit should be expanded as

$$Q^R_{IJ} = \frac{d_R}{|S_L|} \sum_{g\in S_L} D^R_{JI}(g)\, g^{-1} \tag{4.15}$$

we can also reconstruct the matrix of the Young seminormal representations $D^R_{JI}(g)$ by reading off the coefficients in (4.15).

We have seen that a complete basis of $\mathbb{C}[S_L]$ is given by the matrix units $\{Q^R_{IJ}\}$, and the matrix units can be reconstructed as the solution of the eigensystem (4.14). This means that the following permutations

$$\Big\{ m^{\mathfrak{L}}[T_p] = m^{\mathfrak{R}}[T_p]\,, m^{\mathfrak{L}}[\mathcal{J}_k]\,, m^{\mathfrak{R}}[\mathcal{J}_k] \,\Big|\, p \vdash L, k \in \{2,3,\dots,L\} \Big\} \tag{4.16}$$

form a complete set of commuting operators in $\mathrm{End}\,(\mathcal{V}_{\rm reg})$.

Here are some remarks. The eigenvalue method for modules was first developed by Murphy [66] for constructing the Young seminormal representation, which was further refined in [67]. We extended this method to construct complete bases of $\mathbb{C}[S_L]$, corresponding to Artin-Wedderburn and Kronecker decompositions.

The matrix units Q^R_{IJ} is called the Young seminormal units, when the subscripts I, J diagonalise the action of YJM elements.[6] The Young seminormal units can also be constructed

[6]In this sense, the Young seminormal units can be written as $Q^R_{IJ} = \left|\begin{matrix} R \\ I \end{matrix}\right\rangle \left\langle \begin{matrix} R \\ J \end{matrix}\right|$, which automatically satisfy (4.12).

by a recursive method in [71]. However, the recursive method is not as efficient as our eigenvalue method, because it takes a long time to execute the iterative steps.

The efficiency of our algorithm comes from the fact that we only need the minimal generating set of the centre instead of T_p for all p in (4.16). This should be contrasted with Murphy's approach where he used to distinguish all basis elements of irreducible representations.

4.3 Eigenvalue system for the Kronecker decomposition

The Kronecker basis $\mathcal{Q}^{R,\Lambda,\tau}_K$ satisfies the following set of eigenvalue system,

$$\begin{aligned} T_p\, \mathcal{Q}^{R,\Lambda,\tau}_K = \mathcal{Q}^{R,\Lambda,\tau}_K\, T_p &= \frac{\chi^R(T_p)}{d_R}\, \mathcal{Q}^{R,\Lambda,\tau}_K \\ m^{\mathfrak{ad}}[T_q](\mathcal{Q}^{R,\Lambda,\tau}_K) &= \frac{\chi^\Lambda(T_1)}{d_\Lambda}\, \mathcal{Q}^{R,\Lambda,\tau}_K \\ m^{\mathfrak{ad}}[\mathcal{J}_k](\mathcal{Q}^{R,\Lambda,\tau}_K) &= \mathtt{Cont}^\Lambda_K(k)\, \mathcal{Q}^{R,\Lambda,\tau}_K \end{aligned} \tag{4.17}$$

where the last equation follows from (2.63) and (4.10).

Conversely, we take a solution of the eigenvalue system (4.17). We fix the overall normalisation, which is fixed by the δ-function inner product (2.69). For each $R,\Lambda)$, the dimension of the solution space should be equal to $C(R,R,\Lambda)$. We take a real orthogonal basis with respect to the inner product, and use τ to label its components. In this way, we can reconstruct the Kronecker basis. Since the Kronecker basis should be expanded as

$$\mathcal{Q}^{R,\Lambda,\tau}_K = \frac{d_R}{|S_L|} \sum_{g\in S_L} \sum_{I,J=1}^{d_R} S^{\tau}{}^{\Lambda}_{K}{}^{R}_{I}{}^{R}_{J}\, D^R_{JI}(g)\, g^{-1} = \sum_{I,J=1}^{d_R} S^{\tau}{}^{\Lambda}_{K}{}^{R}_{I}{}^{R}_{J}\, Q^R_{IJ} \tag{4.18}$$

we can determine the Clebsch-Gordan coefficients as

$$S^{\tau}{}^{\Lambda}_{K}{}^{R}_{I}{}^{R}_{J} = \delta\Big(\mathcal{S}(Q^R_{IJ})\, \mathcal{Q}^{R,\Lambda,\tau}_K\Big). \tag{4.19}$$

Related to this is the fact that in general the set

$$\Big\{ m^{\mathfrak{L}}[T_p] = m^{\mathfrak{R}}[T_p]\,, m^{\mathfrak{ad}}[T_q]\,, m^{\mathfrak{ad}}[\mathcal{J}_k] \,\Big|\, p \vdash L, q \vdash L, k \in \{2,3,\dots,L\} \Big\} \tag{4.20}$$

forms an *incomplete* set of commuting operators in $\mathrm{End}\,(\mathcal{V}_{\mathrm{reg}})$ due to the extra multiplicity label τ.

5 Integer eigenvalue system for general rational finite groups

It is known that characters and normalised characters of finite groups are algebraic integers [51, chapter 6.5]. A finite group G is called rational if all irreducible representations have rational-valued characters. Since rational algebraic integers are ordinary integers, it follows that rational groups have integer characters and integer normalised characters. In this section, we will generalise the above integer constructions to any pair of rational groups (G,H) where H is a subgroup of G and can be used to define a corresponding centraliser algebra

$$\mathbb{C}[G]^H = \{\sigma \in \mathbb{C}[G] \,:\, h\sigma = \sigma h \quad \forall h \in H\}. \tag{5.1}$$

We will start the presentation with a discussion of general pairs (G,H) of group and subgroup, and subsequently specialise to the case where these are both rational groups.

5.1 Two decompositions of $\mathbb{C}[G]$

The Artin-Wedderburn decomposition holds for general group algebras (see [51, chapter 6.2] or [41, chapter 3–17]). Let $\mathrm{Rep}(G)$ be a complete set of unitary irreducible representations of G and for every $R \in \mathrm{Rep}(G)$ let d_R be the dimension. The elements

$$Q^R_{IJ} = \frac{d_R}{|G|} \sum_{g \in G} D^R_{JI}(g^{-1}) g, \quad R \in \mathrm{Rep}(G),\ I, J \in \{1, \ldots, d_R\} \tag{5.2}$$

form a basis of matrix units of $\mathbb{C}[G]$. As before, we have

$$g_1 Q^R_{IJ} = \sum_{K=1}^{d_R} D^R_{KI}(g_1) Q^R_{KJ}, \qquad Q^R_{IJ} g_2^{-1} = \sum_{K=1}^{d_R} D^R_{JK}(g_2^{-1}) Q^R_{IK} = \sum_{K=1}^{d_R} D^{\overline{R}}_{KJ}(g_2) Q^R_{IK}. \tag{5.3}$$

where we have used unitarity in the last line of the second equation and $\overline{R}$ is the complex conjugate representation of R. Therefore, as a representation of $G \times G$,

$$\mathbb{C}[G] \cong \bigoplus_{R \in \mathrm{Rep}(G)} V^G_R \otimes \overline{V}^G_R. \tag{5.4}$$

This implies the famous counting formula

$$|G| = \sum_{R \in \mathrm{Rep}(G)} d_R^2. \tag{5.5}$$

Restricting $G \times G$ to the diagonal subgroup of elements $(g_1, g_2) = (g, g)$ and decomposing into irreducible representations of this diagonal subgroup gives the Kronecker decomposition of $\mathbb{C}[G]$, generalising our discussion for $\mathbb{C}[S_L]$. As a representation of the diagonal subgroup we have

$$V^G_R \otimes \overline{V}^G_R \cong \bigoplus_{\Lambda \in \mathrm{Rep}(G)} V^G_\Lambda \otimes V_{R,\overline{R},\Lambda}, \tag{5.6}$$

where the multiplicity space

$$V_{R,\overline{R},\Lambda} = \mathrm{Hom}_G(V^G_\Lambda, V^G_R \otimes \overline{V}^G_R), \quad \dim V_{R,R,\Lambda} = C(R, \overline{R}, \Lambda) \tag{5.7}$$

has dimension given by Clebsch-Gordan multiplicities. Therefore, we get

$$\mathbb{C}[G] \cong \bigoplus_{R,\Lambda \in \mathrm{Rep}(G)} V^G_\Lambda \otimes V_{R,\overline{R},\Lambda} \tag{5.8}$$

which implies the counting formula

$$|G| = \sum_{R,\Lambda \in \mathrm{Rep}(G)} d_\Lambda\, C(R, \overline{R}, \Lambda). \tag{5.9}$$

The corresponding basis has the form

$$\mathbb{C}[G] = \mathrm{Span}(\mathcal{Q}^{R,\Lambda,\tau}_K \mid R, \Lambda \in \mathrm{Rep}(G), K \in \{1, \ldots, d_\Lambda\}, \tau \in \{1, \ldots, C(R, R, \Lambda)\}\}. \tag{5.10}$$

The centre of a general group algebra $\mathcal{Z}(\mathbb{C}[G])$ has a basis labelled by conjugacy classes. Let $CL(G)$ be the set of conjugacy classes of G and for every $C \in CL(G)$ define

$$T_C = \sum_{g\in C} g\,. \tag{5.11}$$

The elements form a basis for the centre

$$\mathcal{Z}(\mathbb{C}[G]) = \mathrm{Span}\Big(T_C \,\Big|\, C \in CL(G)\Big). \tag{5.12}$$

The matrix units diagonalise the left/right action of the centre

$$m^{\mathfrak{L}}[T_C](Q^R_{IJ}) = \frac{\chi^R(T_C)}{d_R}\, Q^R_{IJ}, \tag{5.13}$$

and the "Kronecker basis", in addition, diagonalises the adjoint action of the centre,

$$m^{\mathfrak{ad}}[T_C](\mathcal{Q}^{R,\Lambda,\tau}_{\ \ K}) = \frac{\chi^\Lambda(T_C)}{d_\Lambda}\, \mathcal{Q}^{R,\Lambda,\tau}_{\ \ K}. \tag{5.14}$$

As previously mentioned, the above eigenvalues are integers for all rational groups G.

5.2 Two decompositions of $\mathbb{C}[G]^H$

The construction of orbit bases of $\mathbb{C}[G]^H$ is completely analogous to $\mathcal{A}(\mu)$ and follows by replacing

$$S_L \;\to\; G, \qquad S_\mu \;\to\; H, \qquad \mathcal{A}(\mu) \;\to\; \mathbb{C}[G]^H \tag{5.15}$$

where H is any subgroup of G. An orbit basis of $\mathbb{C}[G]^H$ is defined by

$$P_H(g_i) = \frac{1}{|\mathrm{Stab}(g_i)|} \sum_{h\in H} h g_i h^{-1} \tag{5.16}$$

for a selection of orbit representatives $\{g_i\}$, as in section 2.3. The matrix elements of the left action are defined by

$$\sum_{J=1}^{l} m^{\mathfrak{L}}_{ji}[\sigma]\, P_H(g_j) = m^{\mathfrak{L}}[\sigma](P_H(g_i)) = \sigma P_H(g_i) \tag{5.17}$$

as in section 3.1. σ is an integer linear combination of the basis elements $P_H(g_i)$, the corresponding representation matrices $m^{\mathfrak{L}}_{ji}[\sigma], m^{\mathfrak{ad}}_{ji}[\sigma]$ are all integer matrices.

5.2.1 Restriction basis

Restricting G to H gives a decomposition of V^G_R into irreducible representations V^H_r of H,

$$V^G_R \cong \bigoplus_{r\in\mathrm{Rep}(H)} V^H_r \otimes V_{R\to r} \tag{5.18}$$

where $V_{R\to r} = \mathrm{Hom}_H(V_r, V_R)$ is the multiplicity space of dimension

$$\dim V_{R\to r} \equiv g(r; R). \tag{5.19}$$

This restriction defines a set of branching coefficients

$$B^{R\to r}_{I\to i,\nu}\,,\quad I\in\{1,\dots,d_R\}\quad i\in\{1,\dots,d_r\},\quad \nu\in\{1,\dots,g(r;R)\}. \tag{5.20}$$

Let us apply the restriction (5.18) to the Artin-Wedderburn decomposition of $\mathbb{C}[G]$ in (5.4). We obtain a Artin-Wedderburn decomposition of $\mathbb{C}[G]^H$

$$\mathbb{C}[G]^H \cong \bigoplus_{\substack{R\in\mathrm{Rep}(G)\\ r\in\mathrm{Rep}(H)}} V_{R\to r}\otimes \overline{V}_{R\to r} \tag{5.21}$$

which implies

$$\dim \mathbb{C}[G]^H = \sum_{\substack{R\in\mathrm{Rep}(G)\\ r\in\mathrm{Rep}(H)}} g(r;R)^2. \tag{5.22}$$

Following the construction of the restricted Schur basis, we combine the matrix units of $\mathbb{C}[G]$ and branching coefficients to define a set of elements

$$Q^{R,r}_{\nu_+\nu_-} = \sum_{I,J}\sum_i B^{R\to r}_{I\to i,\nu_-} Q^R_{IJ} B^{\overline{R}\to\overline{r}}_{J\to i,\nu_+}. \tag{5.23}$$

These elements satisfy

$$h\, Q^{R,r}_{\nu_+\nu_-} = Q^{R,r}_{\nu_+\nu_-} h, \quad \forall h\in H. \tag{5.24}$$

It is known that the centre of a centraliser algebra $\mathcal{Z}(\mathbb{C}[G]^H)$ is generated by $\mathcal{Z}(\mathbb{C}[G])$ together with $\mathcal{Z}(\mathbb{C}[H])$ [72]. Let $C\in CL(G)$ and $d\in CL(H)$, then we have the eigensystems

$$\begin{aligned} m^{\mathfrak{L}}[T_C](Q^{R,r}_{\nu_+\nu_-}) &= \frac{\chi^R(T_C)}{d_R}\, Q^{R,r}_{\nu_+\nu_-}\\ m^{\mathfrak{L}}[T_d](Q^{R,r}_{\nu_+\nu_-}) &= \frac{\chi^r(T_d)}{d_r}\, Q^{R,r}_{\nu_+\nu_-}. \end{aligned} \tag{5.25}$$

5.2.2 H-invariant Kronecker basis

As before, we can start from the "Kronecker basis" of $\mathbb{C}[G]$ to find a basis for $\mathbb{C}[G]^H$. Here, we restrict to the diagonal subgroup of elements $(h,h)\in H\subset G\times G$. Under this restriction, we have decompositions

$$V^G_\Lambda \cong \bigoplus_{r\in\mathrm{Rep}(H)} V^H_r \otimes V_{\Lambda\to r}. \tag{5.26}$$

In particular, we are interested in the trivial representation r_0 of H appearing in this decomposition since

$$\mathbb{C}[G]^H \cong \bigoplus_{R,\Lambda\in\mathrm{Rep}(G)} V_{R,\overline{R},\Lambda}\otimes V_{\Lambda\to r_0}. \tag{5.27}$$

This corresponds to a basis of elements

$$\mathcal{Q}^{R,\Lambda,H,\tau}_\beta,\quad H\subset G,\quad \tau\in\{1,\dots,C(R,\overline{R},\Lambda)\},\quad \beta\in\{1,\dots,g(r_0;\Lambda)\} \tag{5.28}$$

and gives the equality

$$\dim\mathbb{C}[G]^H = \sum_{R,\Lambda\in\mathrm{Rep}(G)} C(R,\overline{R},\Lambda)\, g(r_0;\Lambda) = \sum_{\substack{R\in\mathrm{Rep}(G)\\ r\in\mathrm{Rep}(H)}} g(r;R)^2. \tag{5.29}$$

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5.3 Integer orthogonal bases for rational groups

We now restrict to G, H rational groups with $H \subset G$. As mentioned before, normalised characters of rational groups are integers and the central elements (5.11) are integer combinations of orbit basis elements. Therefore, the matrices $m^{\mathfrak{L}}_{ji}[T_C], m^{\mathfrak{ad}}_{ji}[T_C]$ are integer matrices with integer eigenvalues. Following section 3 we obtain the integer eigenvalue system

$$(\mathbb{C}[G]^H)^{R,r} \cong \bigcap_{C,d} \left\{ \mathrm{Ker}\Big(m^{\mathfrak{L}}[T_C] - \widehat{\chi}^R_C\Big) \cap \mathrm{Ker}\Big(m^{\mathfrak{L}}[T_d] - \widehat{\chi}^r_C\Big) \right\} \tag{5.30}$$

and

$$(\mathbb{C}[G]^H)^{R,\Lambda} \cong \bigcap_{C,C'} \left\{ \mathrm{Ker}\Big(m^{\mathfrak{L}}[T_C] - \widehat{\chi}^R_C\Big) \cap \mathrm{Ker}\Big(m^{\mathfrak{ad}}[T_{C'}] - \widehat{\chi}^\Lambda_{C'}\Big) \right\}. \tag{5.31}$$

It suffices to use a subset of conjugacy classes C, C' in $CL(G)$ with the property that their normalised characters uniquely identify the irreducible representations $\mathrm{Rep}(G)$. Similarly it suffices to use a subset of conjugacy classes $d \in CL(H)$ which uniquely identify the irreducible representations in $\mathrm{Rep}(H)$. The integer eigenvectors can be found following the discussion in section 3.3. Because all characters are rational, $T_C = \mathcal{S}(T_C)$ and it follows that the operators $m^{\mathfrak{L}}[T_C]$, $m^{\mathfrak{L}}[T_d]$, $m^{\mathfrak{ad}}[T_C]$, $m^{\mathfrak{ad}}[T_d]$ are self-adjoint with respect to the δ-function inner product. This argument is identical to the proof in equation (3.37). Therefore the eigenvectors corresponding to different eigenvalues are orthogonal.

6 Conclusions

In this paper we developed the theory of multi-matrix invariants and constructions of bases of operators in $\mathcal{N} = 4$ SYM through their connection to permutation centraliser algebras $\mathcal{A}(\mu)$, which generalise the symmetric group algebras $\mathbb{C}[S_L]$. We discussed two decompositions of $\mathbb{C}[S_L]$, the Artin-Wedderburn and Kronecker decompositions, and reviewed the correspondence between elements of $\mathcal{A}(\mu)$ and multi-matrix invariants in section 2.3. Two orthogonal bases of multi-matrix invariants were well-studied in the literature, the restricted Schur and covariant bases. The restricted Schur basis decomposes $\mathcal{A}(\mu)$ into a direct sum of subspaces $\mathcal{A}^{R,r_1,\ldots,r_M}(\mu)$, while the covariant basis gives a direct sum of subspaces $\mathcal{A}^{R,\Lambda}(\mu)$. Section 2.4 explained how the restricted Schur basis (2.91) can be understood as descending from the Artin-Wedderburn decomposition of $\mathbb{C}[S_L]$. Analogously, section 2.5 explained how the covariant basis (2.111) comes from the Kronecker decomposition. In section 3, we derived an efficient algorithm for constructing finite N orthogonal bases of $\mathcal{N} = 4$ SYM operators which are integer linear combinatons of multi-traces. We set up integer eigensystems whose solutions give a basis for $\mathcal{A}^{R,r_1,r_2}(\mu)$ and $\mathcal{A}^{R,\Lambda}(\mu)$, which are realised as eigenspaces. The eigenspaces $\mathcal{A}^{R,r_1,r_2}(\mu)$ have dimensions equal to $g(r_1, r_2; R)^2$, while the eigenspaces $\mathcal{A}^{R,\Lambda}(\mu)$ have dimensions equal to $C(R, R, \Lambda)K_{\Lambda\mu}$. It was emphasised that these equations, summarised in (3.16), (3.17), can be solved by finding kernels of integer matrices using Hermite normal forms. In section 4, we considered generalised eigenvalue systems derived from the Artin-Wedderburn and Kronecker decompositions of $\mathbb{C}[S_L]$. In the case of the Artin-Wedderburn decomposition, we constructed the matrix units of $\mathbb{C}[S_L]$ in the Young seminormal representation. For the case of the Kronecker decomposition, we obtained the

Kronecker basis and associated Clebsch-Gordan coefficients. The implementation in `SageMath` is explained in appendix B, and many examples are given in appendices C and D.

The integer orthogonal bases up to $L = 10$ can be easily determined by running our code in a laptop, and our code is attached to this paper as supplementary material. The code needs minor modification beyond $L = 14$ because more Casimir operators $T_4, T_5, \ldots$ should be added to the eigensystem from the discussion of k_* in section (2.1.2).

It is an open problem to systematically determine the multiplicity indices, $\nu_\mp$ in (2.105) and τ, β in (2.118). Resolving $\nu_\mp$ involves finding a subset of elements in $\mathcal{A}(\mu)$ that generate a maximal commutative subalgebra of $\mathcal{A}(\mu)$. It is interesting that the τ multiplicity in the covariant basis has already appeared in the Kronecker decomposition of $\mathbb{C}[S_L]$. This Clebsch-Gordan multiplicity should be explained from the structural properties of another PCA called $\mathcal{K}(n)$, which has appeared in the study of tensor models. The multiplicity problem can be addressed by developing results in [21, 27, 73, 74]. It is straightforward to generalise our algorithm to M-matrix models with $M > 2$. We will consider these problems in more detail in upcoming work [75].

The representation theoretic approach we have used in this paper provides a comprehensive characterization of the trace relations for finite N by imposing the constraint $\ell(R) \leq N$ for a Young diagram label R. We have focused on the two-matrix problem here, but the approach also works with multiple fields [9, 12, 48]. An interesting avenue to explore is the application of this systematic description of finite N relations in the context of the construction of $\frac{1}{16}$-BPS operators (for related comments and steps in this direction see [76–78]).

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A Notation

In this appendix we define the notation used for branching coefficients and Clebsch-Gordan coefficients for S_L and $\mathrm{U}(M)$. They are used to define the decompositions in sections 2.2.2, 2.5, 2.4 and 2.6.

A.1 Branching coefficients

We review branching coefficients for S_L to its subgroup, which are used to define the restricted Schur basis in section 2.4 and the covariant basis in section 2.5. A Young subgroup of S_L is defined by a list of positive integers $\mu = (\mu_1, \ldots, \mu_M)$ such that

$$\sum_{k}^{M} \mu_k = L. \tag{A.1}$$

The Young subgroup corresponding to μ is defined as

$$S_\mu = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_M} \subseteq S_L \tag{A.2}$$

Suppose that the restriction of the irreducible representation R under $S_\mu \subset S_L$ decomposes into the direct sum of $(r_1 \otimes \cdots \otimes r_M)$, where $r_k \vdash \mu_k$ for $k = 1, \dots, M$. The basis of states in R decomposes as

$$\left| \begin{matrix} R \\ I \end{matrix} \right\rangle = \sum_{r_1 \vdash \mu_1} \cdots \sum_{r_M \vdash \mu_M} \sum_{\nu=1}^{g(r_1,\dots,r_M;R)} \sum_{i_1=1}^{d_{r_1}} \cdots \sum_{i_M=1}^{d_{r_M}} B^{R \to (r_1 \dots r_M)}_{I \to (i_1 \dots i_M), \nu} \left| \begin{matrix} r_1 \dots r_M \\ i_1 \dots i_M \end{matrix} \; \nu \right\rangle \tag{A.3}$$

where $B^{R \to (r_1 \dots r_M)}_{I \to (i_1 \dots i_M), \nu}$ is called the branching coefficient and ν is called the multiplicity label. The symbol $g(R; r_1, \dots, r_M)$ is the Littlewood-Richardson coefficient defined by

$$g(r_1, \dots, r_M; R) = \frac{1}{|S_\mu|} \sum_{\alpha_1 \in S_{\mu_1}} \cdots \sum_{\alpha_M \in S_{\mu_M}} \chi^R(\alpha_1 \otimes \cdots \otimes \alpha_M) \prod_{k=1}^{M} \chi^{r_k}(\alpha_k). \tag{A.4}$$

We introduce the standard inner product in the restricted basis,

$$\left\langle \begin{matrix} r_1 \dots r_M \\ i_1 \dots i_M \end{matrix} \; \nu \middle| \begin{matrix} s_1 \dots s_M \\ j_1 \dots j_M \end{matrix} \; \xi \right\rangle = \delta_{\nu\xi} \Big(\prod_{k=1}^{M} \delta^{r_k s_k} \, \delta_{i_k j_k} \Big), \qquad (i_k, j_k \in \{1, \dots, d_{r_k}\}). \tag{A.5}$$

The original basis and the restricted basis have the same dimensions,

$$d_R = \sum_{r_1 \vdash \mu_1} \cdots \sum_{r_M \vdash \mu_M} \left(g(r_1, \dots, r_M; R) \prod_{k=1}^{M} d_{r_k} \right) \tag{A.6}$$

which is consistent with (2.88). This shows that the transformation (A.3) relates two complete orthonormal bases, and therefore unitary. Because S_L and S_μ are real groups, we can choose the branching coefficients to be real, giving

$$B^{R \to (r_1 \dots r_M)}_{I \to (i_1 \dots i_M), \nu} = \left\langle \begin{matrix} r_1 \dots r_M \\ i_1 \dots i_M \end{matrix} \; \nu \middle| \begin{matrix} R \\ I \end{matrix} \right\rangle = \left\langle \begin{matrix} R \\ I \end{matrix} \middle| \begin{matrix} r_1 \dots r_M \\ i_1 \dots i_M \end{matrix} \; \nu \right\rangle. \tag{A.7}$$

The inverse transformation of (A.3) is[7]

$$\left| \begin{matrix} r_1 \dots r_M \\ i_1 \dots i_M \end{matrix} \; \nu \right\rangle = \sum_{I=1}^{d_R} B^{R \to (r_1 \dots r_M)}_{I \to (i_1 \dots i_M), \nu} \left| \begin{matrix} R \\ I \end{matrix} \right\rangle. \tag{A.8}$$

The restricted basis (A.3) forms an irreducible representation of S_μ for each multiplicity label ν. As such, a generic element $h \equiv h_1 \otimes \cdots \otimes h_M \in S_\mu$ acts diagonally on the multiplicity space as

$$h \left| \begin{matrix} r_1 \dots r_M \\ i_1 \dots i_M \end{matrix} \; \nu \right\rangle = \sum_{j_1=1}^{d_{r_1}} \cdots \sum_{j_M=1}^{d_{r_M}} D^{r_1}_{j_1 i_1}(h_1) \dots D^{r_M}_{j_M i_M}(h_M) \left| \begin{matrix} r_1 \dots r_M \\ j_1 \dots j_M \end{matrix} \; \nu \right\rangle. \tag{A.9}$$

[7] Note that the restricted basis depends on R through the multiplicity label, $\nu \in \{1, \dots, g(r_1 \dots r_M; R)\}$.

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The branching coefficients satisfy the completeness relations [22, 41]

$$\sum_I B^{R\to(r_1\ldots r_M)}_{I\to(i_1\ldots i_M),\nu} B^{R\to(s_1\ldots s_M)}_{I\to(j_1\ldots j_M),\xi} = \sum_I \left\langle \begin{matrix} r_1 \ldots r_M \\ i_1 \ldots i_M \end{matrix} \nu \middle| \begin{matrix} R \\ I \end{matrix} \right\rangle \left\langle \begin{matrix} R \\ I \end{matrix} \middle| \begin{matrix} s_1 \ldots s_M \\ j_1 \ldots j_M \end{matrix} \xi \right\rangle = \delta_{\nu\xi} \left(\prod_{k=1}^M \delta^{r_k s_k} \delta_{i_k j_k} \right) \tag{A.10}$$

$$\sum_{r_1\ldots r_M} \sum_{\nu} \sum_{i_1\ldots i_M} B^{R\to(r_1\ldots r_M)}_{J\to(i_1\ldots i_M),\nu} B^{R\to(r_1\ldots r_M)}_{I\to(i_1\ldots i_M),\nu} = \sum_{r_1\ldots r_M} \sum_{\nu} \sum_{i_1\ldots i_M} \left\langle \begin{matrix} R \\ J \end{matrix} \middle| \begin{matrix} r_1 \ldots r_M \\ i_1 \ldots i_M \end{matrix} \nu \right\rangle \left\langle \begin{matrix} r_1 \ldots r_M \\ i_1 \ldots i_M \end{matrix} \nu \middle| \begin{matrix} R \\ I \end{matrix} \right\rangle = \delta_{IJ} .$$

They also exhibit an equivariance property

$$\sum_I D^R_{JI}(h) \, B^{R\to(r_1\ldots r_M)}_{I\to(i_1\ldots i_M),\nu} = \sum_{j_1\ldots j_M} B^{R\to(r_1\ldots r_M)}_{J\to(j_1\ldots j_M),\nu} D^{r_1}_{j_1 i_1}(h_1) \ldots D^{r_M}_{j_M i_M}(h_M). \tag{A.11}$$

A.2 CG coefficients of S_L

We revire the Clebsch-Gordan (CG) coefficients for S_L , which are used to define the Kronecker basis in section 2.2.2 and covariant basis in 2.5.

Let $V^{S_L}_{R_1}, V^{S_L}_{R_2}$ be the irreducible representations of S_L . We define the tensor product representation by

$$\sigma \left| \begin{matrix} R_1 \\ I_1 \end{matrix} \right\rangle \otimes \left| \begin{matrix} R_2 \\ I_2 \end{matrix} \right\rangle = D^{R_1}_{J_1 I_1}(\sigma) \, D^{R_2}_{J_2 I_2}(\sigma) \left| \begin{matrix} R_1 \\ J_1 \end{matrix} \right\rangle \otimes \left| \begin{matrix} R_2 \\ J_2 \end{matrix} \right\rangle \tag{A.12}$$

generalising (2.40). This equation implies that the tensor produce representation is decomposed into irreducible representations of S_L as

$$V^{S_L}_{R_1} \otimes V^{S_L}_{R_2} \cong \bigoplus_{\Lambda \vdash L} V^{S_L}_{\Lambda} \otimes V_{R_1,R_2,\Lambda} , \qquad \dim V_{R_1,R_2,\Lambda} = C(R_1, R_2, \Lambda) \tag{A.13}$$

where $C(R_1, R_2, \Lambda)$ is the Kronecker coefficient defined by

$$C(R_1, R_2, \Lambda) = \frac{1}{|S_L|} \sum_{g \in S_L} \chi^{R_1}(g) \chi^{R_2}(g) \chi^{\Lambda}(g). \tag{A.14}$$

By equating the dimensions of the representations, we obtain

$$d_{R_1} \, d_{R_2} = \sum_{\Lambda \vdash L} C(\Lambda, R_1, R_2) \, d_\Lambda \, . \tag{A.15}$$

By taking a complete basis of S_L, we define the CG coefficients from the irreducible decomposition as,

$$\left| \begin{matrix} R_1 \\ I_1 \end{matrix} \right\rangle \otimes \left| \begin{matrix} R_2 \\ I_2 \end{matrix} \right\rangle = \sum_{\Lambda \vdash L} \sum_{\tau=1}^{C(\Lambda,R_1,R_2)} \sum_{K=1}^{d_\Lambda} S^{\tau}{}^{\Lambda}_{K}{}^{R_1}_{I_1}{}^{R_2}_{I_2} \left| \begin{matrix} \Lambda \\ K \end{matrix} \tau \right\rangle \tag{A.16}$$

whose inverse relation is

$$\left| \begin{matrix} \Lambda \\ k \end{matrix} \tau \right\rangle = \sum_{R_1 \vdash L} \sum_{R_2 \vdash L} \sum_{I_1=1}^{d_{R_1}} \sum_{I_2=1}^{d_{R_2}} S^{\tau}{}^{\Lambda}_{K}{}^{R_1}_{I_1}{}^{R_2}_{I_2} \left| \begin{matrix} R_1 \\ I_1 \end{matrix} \right\rangle \otimes \left| \begin{matrix} R_2 \\ I_2 \end{matrix} \right\rangle . \tag{A.17}$$

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Since S_L is a real group, we can choose the CG coefficients to be real. From (A.16) and (A.17) we find the orthogonality relations [22, 41]

$$\sum_{I_1,I_2} S^{\tau}{}^{\Lambda}_{K}{}^{R_1}_{I_1}{}^{R_2}_{I_2}\, S^{\tau'}{}^{\Lambda'}_{K'}{}^{R_1}_{I_1}{}^{R_2}_{I_2} = \delta^{\Lambda\Lambda'}\delta^{\tau\tau'}\delta_{KK'}\,, \quad \sum_{\tau,\Lambda,K} S^{\tau}{}^{\Lambda}_{K}{}^{R_1}_{I_1}{}^{R_2}_{I_2}\, S^{\tau}{}^{\Lambda}_{K}{}^{R'_1}_{J_1}{}^{R'_2}_{J_2} = \delta^{R_1R'_1}\delta^{R_2R'_2}\delta_{I_1J_1}\delta_{I_2J_2}\,. \tag{A.18}$$

The CG coefficients satisfy the equivariance property

$$\sum_{J_1,J_2} S^{\tau}{}^{\Lambda}_{K}{}^{R_1}_{J_1}{}^{R_2}_{J_2}\, D^{R_1}_{J_1I_1}(\sigma)\, D^{R_2}_{J_2I_2}(\sigma) = \sum_{K'} D^{\Lambda}_{KK'}(\sigma)\, S^{\tau}{}^{\Lambda}_{K'}{}^{R_1}_{I_1}{}^{R_2}_{I_2}\,. \tag{A.19}$$

A.3 CG coefficients of U(M)

The CG coefficients for U(N) are important in defining the general covariant basis in section 2.6. Let V_M be the fundamental representation of U(M), equipped with the standard inner product

$$\langle a|b\rangle_{\mathrm{U}(M)} = \delta_{ab}\,, \qquad a,b \in \{1,2,\ldots,M\}. \tag{A.20}$$

Under the Schur-Weyl duality, the tensor product $V_M^{\otimes L}$ decomposes as

$$V_M^{\otimes L} = \bigoplus_{\Lambda\vdash L} (V_\Lambda^{\mathrm{U}(M)} \otimes V_\Lambda^{S_L}). \tag{A.21}$$

If we write down explicit components, we find

$$|\vec a\rangle \equiv |a_1\,,\ldots\,,a_L\rangle_{\mathrm{U}(M)} = \sum_{\substack{\Lambda\vdash L\\ \ell(\Lambda)\le M}} \sum_{M_\Lambda=1}^{\mathrm{Dim}_M(\Lambda)} \sum_{K=1}^{d_\Lambda} C^{\Lambda\,\Lambda}_{K\,M_\Lambda}(\vec a) \left|\begin{matrix}\Lambda\\ K\end{matrix}\right\rangle \left|\begin{matrix}\Lambda\\ M_\Lambda\end{matrix}\right\rangle_{\mathrm{U}(M)} \tag{A.22}$$

where K labels a component of the irreducible representation Λ of S_L, and M_Λ labels a component of the irreducible representation Λ of U(M). The symbol $C^{\Lambda\,\Lambda}_{K\,M_\Lambda}(\vec a)$ in (A.22) is the CG coefficient of $S_L\times\mathrm{U}(M)$ used in (2.121). By comparing the dimensions in (A.22), we obtain

$$M^L = \sum_{\substack{\Lambda\vdash L\\ \ell(\Lambda)\le M}} d_\Lambda\, \mathrm{Dim}_M(\Lambda). \tag{A.23}$$

The CG coefficients satisfy the orthogonality relations

$$\sum_{\vec a} C^{\Lambda\,\Lambda}_{K\,M_\Lambda}(\vec a)\, C^{\Lambda'\,\Lambda'}_{K'\,M_{\Lambda'}}(\vec a) = \delta^{\Lambda\Lambda'}\,\delta_{M_\Lambda M_{\Lambda'}}\,\delta_{KK'}\,, \qquad \sum_{\Lambda,M_\Lambda,K} C^{\Lambda\,\Lambda}_{K\,M_\Lambda}(\vec a)\, C^{\Lambda\,\Lambda}_{K\,M_\Lambda}(\vec b) = \prod_{i=1}^{L} \delta_{a_i\,b_i}\,. \tag{A.24}$$

B Algorithm: decomposition of $\mathcal{A}(\mu_1, \mu_2)$

In this appendix we outline the algorithm used to construct an integer basis for $\mathcal{A}^{R,\Lambda}$, $\mathcal{A}^{R,r_1,r_2}$ using the ingredients introduced in the main text. We used the `GAP` package in `SageMath` [79, 80] to run the code. The code produces a text file enumerating the integer orthogonal basis elements in terms of linear combinations of multi-traces. In this appendix we write $\mathcal{A}(\mu_1,\mu_2) = \mathcal{A}(m,n)$ in order not to deviate from the notation in the code. The code comes in two files:

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- Amn_Restriction_Decomposition.sage
- Amn_Covariant_Decomposition.sage

and can be run in the terminal by executing, for example,

```
sage Amn_Covariant_Decomposition.sage L n N
```
(B.1)

The first parameter determines $L = m + n$ and the second parameter determines n. The last parameter N divides the output into two files. The first file contains a basis of operators for $L \leq N$, and the second file contains a basis for the finite N trace relations. The output is independent of N if $N \geq L$, in which case the second file is empty.

B.1 Outline of SageMath code

In this part we will give an outline of the SageMath code. Each file/algorithm consists of three parts: (1) Constructing the orbit basis for $\mathcal{A}(m, n)$. (2) constructing the representation matrices of central elements $T_2, T_3, \ldots$; (3) computing the intersection of kernels in section 3.

1. The first step of the algorithm is to construct $\mathcal{A}(m, n)$ using the orbit basis described in 2.3. The orbits are given by $S_m \times S_n$ acting on S_L by conjugation and can be constructed in Sage as follows.

   (a) Construct the groups $S_m \times S_n$ and S_{m+n}

   ```
   SmSn=libgap.DirectProduct(libgap.SymmetricGroup(m),libgap.SymmetricGroup(n))
   Smn=libgap.SymmetricGroup(m+n)
   ```
   (B.2)

   (b) Compute the orbits

   ```
   orbit_basis = libgap.OrbitsDomain(SmSn, Smn)
   ```
   (B.3)

   The variable orbit_basis is a set of orbits, each orbit contains elements of S_L related by conjugation of $S_m \times S_n$. We also collect a set of representatives of each orbit

   ```
   basis_keys = [ b[0] for b in orbit_basis ]
   ```
   (B.4)

   This gives the orbit basis of $\mathcal{A}(m, n)$ — each orbit or representative corresponds to a basis element of $\mathcal{A}(m, n)$. In the rest of this section we will refer to the representative elements as g_i, where g_i = basis_keys[i − 1] in the code.

2. The second step is to compute the left and adjoint action of $T_2, T_3 \in \mathcal{Z}(\mathbb{C}[S_L])$ on $\mathcal{A}(m, n)$. We will compute the matrix elements $m^L_{ji}[T_2]$ given by (3.1), and this can be understood in terms of actions on the orbit sets above. Focusing on the left action of T_2, we compute the following

   (a) GAP gives the conjugacy classes as a list

   ```
   T2 = libgap.ConjugacyClass(Smn, libgap.eval("(1, 2)")).List()
   ```
   (B.5)

(b) Construct an empty matrix of size $\dim \mathcal{A}(m,n) = \texttt{len(orbits)}$

$$\texttt{T2\_left\_matrix} = \texttt{zero\_matrix(ZZ, len(orbit\_basis))} \tag{B.6}$$

The input ZZ forces the matrix to have integer entries.

(c) The left action on $P_{m,n}(g_i)$ is computed in parallel for every representative. We introduce the notation $\{T_2 g\}$ for the set of elements with non-zero coefficients appearing in the expansion of the product $T_2 g$. The action on an orbit basis element is determined by the action on representatives because

$$\begin{aligned} T_2 P_{m,n}(g_i) &= \frac{1}{|\mathrm{Stab}(g_i)|} \sum_{h \in S_m \times S_n} h T_2 g_i h^{-1} \\ &= \frac{1}{|\mathrm{Stab}(g_i)|} \sum_{h \in S_m \times S_n} \sum_{g \in \{T_2 g_i\}} h g h^{-1} \\ &= \frac{1}{|\mathrm{Stab}(g_i)|} \sum_{g \in \{T_2 g_i\}} |\mathrm{Stab}(g)| P_{m,n}(g) \\ &= \sum_{g \in \{T_2 g_i\}} \frac{|\mathrm{Orb}(g_i)|}{|\mathrm{Orb}(g)|} P_{m,n}(g) \end{aligned} \tag{B.7}$$

where the last step uses the orbit-stabilizer theorem $|\mathrm{Stab}(g)||\mathrm{Orb}(g)| = |S_m \times S_n|$. Therefore, as an intermediate step, it is useful to compute

$$\texttt{t2\_orb} = \texttt{tuple(t2} * \texttt{g for t2 in T2)}, \tag{B.8}$$

t2_orb corresponds to the set $\{T_2 g_i\}$. Now, for every $h \in \{T_2 g_i\}$ that is also in the orbit $\{g_j\}_{m,n}$ we get a contribution to the coefficient in front of $P_{m,n}(g_j)$ in the expansion of $T_2 P_{m,n}(g_i)$ in the orbit basis. It follows that the expansion is given by the vector

```
t2_left_vec = list(
        len(orbit)/len(orb) * sum(1 for x in t2_orb if x in orb)
        for orb in orbits
        )
```
(B.9)

where len(orbit) is the length of the orbit $\{g_i\}_{m,n}$. The sum computes the order of the intersection of $\{T_2 g_i\}$ and $\{g_j\}_{m,n}$ for all $j \in \{1, \dots, \dim \mathcal{A}(m,n)\}$.

sum(1 for x in t2_orb if x in orb) = |{Elements in t2_orb that are also in orb}| (B.10)

t2_left_vec corresponds to the ith column in the representation of T_2 acting on $\mathcal{A}(m,n)$. We append it to the matrix by setting

$$\texttt{T2\_left\_matrix[i} - \texttt{1]} = \texttt{t2\_left\_vec} \tag{B.11}$$

for every representative.

(d) Lastly, transpose `T2_left_matrix`.

Similar procedures can be used to compute the left actions of T_3, adjoint actions of T_2, T_3 and so on. We call the matrix corresponding to the adjoint action `T2_adjoint_matrix` and similarly for T_3.

3. Having computed the representation matrices we want to compute the kernels in section 3. We will focus on the covariant case, the restricted Schur case is analogous. For every R, Λ we do the following

(a) Compute the normalised characters/eigenvalues $\widehat{\chi}_2^R, \widehat{\chi}_2^\Lambda$. This is done using the functions `T2eigval(), T3eigval()` which simply computes (4.8), (4.9),

$$\begin{aligned} \texttt{R1} &= \texttt{T2eigval(R)} \\ \texttt{R2} &= \texttt{T3eigval(R)} \\ \texttt{Lambda1} &= \texttt{T2eigval(Lambda)} \\ \texttt{Lambda2} &= \texttt{T3eigval(Lambda)} \end{aligned} \tag{B.12}$$

(b) Construct a tuple of matrices

$$\begin{aligned} \texttt{EV\_matrices} = \texttt{tuple}(&\texttt{R1} - \texttt{T2\_left\_matrix}, \\ &\texttt{R2} - \texttt{T3\_left\_matrix}, \\ &\texttt{Lambda1} - \texttt{T2\_adjoint\_matrix}, \\ &\texttt{Lambda2} - \texttt{T3\_adjoint\_matrix}) \end{aligned} \tag{B.13}$$

(c) Construct a new matrix made out of blocks, that is a 4-by-1 block matrix

$$\texttt{simul\_kernel\_matrix} = \texttt{block\_matrix}(\texttt{ZZ}, \texttt{len}(\texttt{EV\_matrices}), 1, \texttt{EV\_matrices}) \tag{B.14}$$

where `ZZ` defines the block matrix to be over the integers.

(d) Compute a basis for the kernel

$$\texttt{kernel\_basis} = \texttt{simul\_kernel\_matrix.right\_kernel\_matrix}() \tag{B.15}$$

The matrix `kernel_basis` is a matrix with rows giving a basis for the kernel of `simul_kernel_matrix` or equivalently a basis for $\mathcal{A}^{R,\Lambda}(m,n)$. For the benefit of the reader, the next subsection contains a review of a simple algorithm for computing this matrix. Lastly, we run the Gram-Schmidt process on the matrix `kernel_basis` and clear the denominators of the vectors to get integer orthogonal bases for $\mathcal{A}^{R,\Lambda}(m,n)$. The procedure is very similar for the space $\mathcal{A}^{R,r_1,r_2}(m,n)$.

Note that the kernel (B.15) contains a non-trivial solution if and only if[8]

$$C(R,R,\Lambda) > 0, \qquad \ell(\Lambda) \leq 2, \qquad K_{\Lambda\mu} > 0. \tag{B.16}$$

[8]In the restricted Schur basis, the eigenspace has non-zero dimension if and only if $g(R; r_1, r_2) > 0$.

Therefore, it is sufficient to compute the eigenvalues and kernels in those cases. In the code, the set of non-zero pairs (R, Λ) are computed using

```
sectors = tuple((R,Lambda) for R in Partitions(L) for Lambda in Partitions(L,max_length=2)
                if len(R) <= N and kroncoeff(R,R,Lambda) > 0 and
                symmetrica.kostka_number(Lambda,[L-i,i]) > 0)
```
(B.17)

here `kroncoeff` is a function that computes the Kronecker multiplicity $C(R, R, \Lambda)$ and `kostka_number` computes the Kostka number $K_{\Lambda\mu}$.

We encounter some problems for $L > 10$. First, output files are huge, typically larger than 1MB. Second, we need to solve memory shortage to run the code. A workstation with 1TB of memory can reach $L = 12$; computing $L > 12$ will require further refinement of the algorithm or some tricks for the memory management for parallel evaluation.

B.2 Integer kernels

The following algorithm/procedure gives a basis for the kernel of a $p \times q$ matrix M [64, Algorithm 2.4.5]:

1. Set $i = p, j = q, k = q$, $U = I_q$ the $q \times q$ identity matrix. If $p \leq q$ set $l = 1$ else $l = p - q + 1$.

2. If $j \neq 1$ go top step 3. Otherwise, go top step 4.

3. Reduce j by one. If $M_{ij} = 0$ go to step 2. Otherwise do the following:

   (a) Set $a = M_{ik}, b = M_{ij}$, compute d, u, v such that

   $$d = \gcd(a, b) = ua + vb. \tag{B.18}$$

   It is technically necessary for $|u|, |v|$ to be minimal, see the remark below [64, Algorithm 2.4.5] for more details.

   (b) We introduce an auxiliary column vector B. For every $r \in \{1, \ldots, p\}$ set

   $$B_r = uM_{rk} + vM_{rj} \tag{B.19}$$
   $$M_{rj} = \frac{a}{d}M_{rj} - \frac{b}{d}M_{rk} \tag{B.20}$$
   $$M_{rk} = B_r \tag{B.21}$$

   and analogously for $r = 1, \ldots, q$

   $$B_r = uU_{rk} + vU_{rj} \tag{B.22}$$
   $$U_{rj} = \frac{a}{d}U_{rj} - \frac{b}{d}U_{rk} \tag{B.23}$$
   $$U_{rk} = B_r \tag{B.24}$$

4. Set $a = M_{ik}$ and do the following steps:

(a) If $a < 0$, then for every $r = 1, \dots, p$ set

$$M_{rk} = -M_{rk}, \quad U_{rk} = -U_{rk}, \quad a = -a. \tag{B.25}$$

(b) If $a = 0$, increase k by one. Otherwise, each $r = k+1, \dots, q$

$$q = \lfloor M_{ir}/a \rfloor \tag{B.26}$$

$$M_{sr} = M_{sr} - qM_{sk}, \quad \text{for every } s \in \{1, \dots, p\} \tag{B.27}$$

$$U_{sr} = U_{sr} - qU_{sk}, \quad \text{for every } s \in \{1, \dots, q\} \tag{B.28}$$

$$\tag{B.29}$$

(c) If $i = l$ output the $k-1$ first column of U and terminate the algorithm. Otherwise, set $i = i-1, k = k-1, j = k$ and go to step 2.

The above procedure is not the one used in practice, for reasons of efficiency, `SageMath` will choose an appropriate algorithm based on the input matrix.

B.3 Simple example: integer kernel

In section 3.4 we illustrated a simple example of applying the eigenvalue method to $\mathcal{A}(3,0) = \mathcal{Z}(\mathbb{C}[S_3])$ and $R = [3]$. This appendix contains the details of computing the integer eigenvectors using the algorithm in the previous subsection.

We apply the algorithm on the augmented matrix

$$m^{\mathfrak{L}}[T_2^{(3)}] - 3I_3 = \left(\begin{array}{ccc} -3 & 3 & 0 \\ 1 & -3 & 2 \\ 0 & 3 & -3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right) \tag{B.30}$$

Starting at the third row, we want turn all columns to the left of -3 to zero. We do this by adding the third column to the second column giving

$$\left(\begin{array}{ccc} -3 & 3 & 0 \\ 1 & -1 & 2 \\ 0 & 0 & -3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{array}\right) \tag{B.31}$$

Note that multiplying the upper matrix in (B.30) by the lower matrix of (B.31) on the right, gives the upper matrix of (B.31). There is nothing left to clear to the left of -3, but we switch the sign by multiplying the third column by -1 giving

$$\left(\begin{array}{ccc} -3 & 3 & 0 \\ 1 & -1 & -2 \\ 0 & 0 & 3 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{array}\right) \tag{B.32}$$

There is nothing to the right of the third column and so no further reduction is needed.

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Now consider the second row. We want to clear all columns to the left of -1. We do this by adding the second column to the first giving

$$\left(\begin{array}{ccc} 0 & 3 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \\ \hline 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & -1 \end{array}\right) \tag{B.33}$$

Since -1 is not positive, we multiply the second column by -1

$$\left(\begin{array}{ccc} 0 & -3 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 3 \\ \hline 1 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{array}\right) \tag{B.34}$$

We have already found a basis for the kernel here, the vector $(1, 1, 1)$. But for completeness we carry out the full algorithm which says to further reduce to the right of the second column. Let $q = \lfloor -2/1 \rfloor = -2$, we remove $q = -2$ times the second column from the third column to get

$$\left(\begin{array}{ccc} 0 & -3 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \\ \hline 1 & 0 & 0 \\ 1 & -1 & -2 \\ 1 & -1 & -3 \end{array}\right) \tag{B.35}$$

We now consider the first row and column, and the algorithm says to go to step 4. Because the entry is already zero we do not have to do anything and the algorithm terminates. As previously mentioned, a basis for the kernel is given by the first column of the augmented matrix. That is, the vector $(1, 1, 1)$.

C Data: integer orthogonal bases for $\mathcal{A}(\mu_1, \mu_2)$

In this appendix we give examples of solutions to the eigensystems in section 3, giving integer orthogonal bases for two-matrix invariants in $\mathcal{N} = 4$ SYM.

C.1 Half-BPS operators

The multi-matrix invariants for $\mathcal{A}(L, 0)$ describe the half-BPS operators of $\mathcal{N} = 4$ SYM. These operators are labelled by the Young diagram R only, and the restricted Schur and covariant operators become identical.[9] Following the methods in section 3 we obtain the

[9]The label Λ is always given by the totally symmetric representation $[L]$.

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integer basis of operators of $\mathcal{A}(L,0)$, which can be denoted by

$$\mathsf{O}^R[Z] = \mathrm{tr}_{V_N^{\otimes L}}\Big(\mathcal{L}(\mathsf{Q}^R)\, Z^{\otimes L}\Big) = \sum_{g\in T_p} |T_p|\, \tilde{c}^R(g)\, \mathcal{O}_g[Z] \tag{C.1}$$

where T_p is a sum over the conjugacy class of S_L having the cycle type $p \vdash L$. We can view Q^R as the restricted Schur or covariant basis whose multiplicity labels are all trivial,

$$\mathsf{Q}^R \propto \begin{cases} Q^{R,(r_1,r_2)}_{\nu_+\nu_-}\,, & (r_1,r_2,\nu_+,\nu_-) = (R,\emptyset,1,1) \\ \mathcal{Q}^{R,\Lambda,(L,0),\tau}_{\beta}\,, & (\Lambda,\tau,\beta) = ([L],1,1). \end{cases} \tag{C.2}$$

Below we will present the explicit form of $\mathsf{O}^R[Z]$. These data suggest that the coefficients in (C.1) are equal to the S_L characters,

$$|T_p|\,\tilde{c}^R(g) = \chi^R(T_p). \tag{C.3}$$

Since the normalised characters $\chi^R(T_p)/d_R$ are known to be integers [42], the coefficients have the common factor equal to the dimensions of the S_L representations,

$$\mathrm{GCD}_{g\in T_p}\{\tilde{c}^R(g)\} = d_R\,. \tag{C.4}$$

C.1.1 $\mathcal{A}(2,0)$

$$\begin{aligned} R =&[2], && \mathrm{Tr}(Z)\,\mathrm{Tr}(Z) + \mathrm{Tr}\Big(Z^2\Big) \\ R =&[1,1], && \mathrm{Tr}(Z)\,\mathrm{Tr}(Z) - \mathrm{Tr}\Big(Z^2\Big) \end{aligned}$$

C.1.2 $\mathcal{A}(3,0)$

$$\begin{aligned} R =&[3], && \mathrm{Tr}(Z)^3 + 3\,\mathrm{Tr}(Z)\,\mathrm{Tr}\Big(Z^2\Big) + 2\,\mathrm{Tr}\Big(Z^3\Big) \\ R =&[2,1], && 2\Big\{\mathrm{Tr}(Z)^3 - \mathrm{Tr}\Big(Z^3\Big)\Big\} \\ R =&[1,1,1], && \mathrm{Tr}(Z)^3 - 3\,\mathrm{Tr}(Z)\,\mathrm{Tr}\Big(Z^2\Big) + 2\,\mathrm{Tr}\Big(Z^3\Big) \end{aligned}$$

C.1.3 $\mathcal{A}(4,0)$

$$\begin{aligned} R =&[4], && \mathrm{Tr}(Z)^4 + 6\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}\Big(Z^2\Big) + 8\,\mathrm{Tr}(Z)\,\mathrm{Tr}\Big(Z^3\Big) + 3\,\mathrm{Tr}\Big(Z^2\Big)\,\mathrm{Tr}\Big(Z^2\Big) + 6\,\mathrm{Tr}\Big(Z^4\Big) \\ R =&[3,1], && 3\Big\{\mathrm{Tr}(Z)^4 + 2\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}\Big(Z^2\Big) - \mathrm{Tr}\Big(Z^2\Big)\,\mathrm{Tr}\Big(Z^2\Big) - 2\,\mathrm{Tr}\Big(Z^4\Big)\Big\} \\ R =&[2,2], && 2\Big\{\mathrm{Tr}(Z)^4 - 4\,\mathrm{Tr}(Z)\,\mathrm{Tr}\Big(Z^3\Big) + 3\,\mathrm{Tr}\Big(Z^2\Big)\,\mathrm{Tr}\Big(Z^2\Big)\Big\} \\ R =&[2,1,1], && 3\Big\{\mathrm{Tr}(Z)^4 - 2\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}\Big(Z^2\Big) - \mathrm{Tr}\Big(Z^2\Big)\,\mathrm{Tr}\Big(Z^2\Big) + 2\,\mathrm{Tr}\Big(Z^4\Big)\Big\} \\ R =&[1,1,1,1], && \mathrm{Tr}(Z)^4 - 6\,\mathrm{Tr}(Z)^2\,\mathrm{Tr}\Big(Z^2\Big) + 8\,\mathrm{Tr}(Z)\,\mathrm{Tr}(ZZZ) + 3\,\mathrm{Tr}\Big(Z^2\Big)\,\mathrm{Tr}\Big(Z^2\Big) - 6\,\mathrm{Tr}\Big(Z^4\Big) \end{aligned}$$

C.2 Restricted basis

We will present the operators which are proportional to the restricted Schur operators (2.99),

$$\begin{aligned} \mathsf{O}^{R,r_1,r_2}_A(\vec{a}_{\mu_1,\mu_2}) &= \mathrm{tr}_{V_N^{\otimes L}}\Big(\mathcal{L}(\mathsf{Q}^{R,r_1,r_2}_A)\, Z^{\otimes\mu_1}\otimes W^{\otimes\mu_2}\Big) \\ \mathsf{Q}^{R,r_1,r_2}_A &= \sum_{\nu_+,\nu_-=1}^{g(R;r_1,r_2)} \mathsf{N}^{R,(r_1,r_2)}_{A,\nu_+\nu_-}\, Q^{R,(r_1,r_2)}_{\nu_+\nu_-}\,. \end{aligned} \tag{C.5}$$

where Q_A^{R,r_1,r_2} is an integer eigenvector for the eigenvalue system of the restricted Schur basis (3.12), obtained according to the methods described in section 3.3.

C.2.1 $\mathcal{A}(1,1)$

$$(R,r_1,r_2) = ([2],[1],[1]), \qquad \mathrm{Tr}(Z)\,\mathrm{Tr}(W) + \mathrm{Tr}(ZW)$$
$$(R,r_1,r_2) = ([1,1],[1],[1]), \qquad \mathrm{Tr}(Z)\,\mathrm{Tr}(W) - \mathrm{Tr}(ZW)$$

C.2.2 $\mathcal{A}(2,1)$

$$(R,r_1,r_2) = ([3],[2],[1]), \qquad \mathrm{Tr}(Z)^2\,\mathrm{Tr}(W) + 2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(ZW) + \mathrm{Tr}\left(Z^2\right)\mathrm{Tr}(W) + 2\,\mathrm{Tr}\left(Z^2W\right)$$
$$(R,r_1,r_2) = ([2,1],[2],[1]), \qquad 2\left\{\mathrm{Tr}(Z)^2\,\mathrm{Tr}(W) - \mathrm{Tr}(Z)\,\mathrm{Tr}(ZW) + \mathrm{Tr}\left(Z^2\right)\mathrm{Tr}(W) - \mathrm{Tr}\left(Z^2W\right)\right\}$$
$$(R,r_1,r_2) = ([2,1],[1,1],[1]), \qquad 2\left\{\mathrm{Tr}(Z)^2\,\mathrm{Tr}(W) + \mathrm{Tr}(Z)\,\mathrm{Tr}(ZW) - \mathrm{Tr}\left(Z^2\right)\mathrm{Tr}(W) - \mathrm{Tr}\left(Z^2W\right)\right\}$$
$$(R,r_1,r_2) = ([1,1,1],[1,1],[1]), \qquad \mathrm{Tr}(Z)^2\,\mathrm{Tr}(W) - 2\,\mathrm{Tr}(Z)\,\mathrm{Tr}(ZW) - \mathrm{Tr}\left(Z^2\right)\mathrm{Tr}(W) + 2\,\mathrm{Tr}\left(Z^2W\right)$$

C.2.3 $\mathcal{A}(3,1)$

$$\begin{aligned}
&(R,r_1,r_2) = ([4],[3],[1]),\\
&\quad 6\,\mathrm{Tr}(WZZZ) + 6\,\mathrm{Tr}(WZZ)\,\mathrm{Tr}(Z) + 3\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)^2 + \mathrm{Tr}(W)\,\mathrm{Tr}(Z)^3 + 3\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(ZZ)\\
&\quad + 3\,\mathrm{Tr}(W)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(ZZ) + 2\,\mathrm{Tr}(W)\,\mathrm{Tr}(ZZZ)\\
&(R,r_1,r_2) = ([3,1],[3],[1]),\\
&\quad 3\Big\{-2\,\mathrm{Tr}(WZZZ) - 2\,\mathrm{Tr}(WZZ)\,\mathrm{Tr}(Z) - \mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)^2 + \mathrm{Tr}(W)\,\mathrm{Tr}(Z)^3 - \mathrm{Tr}(WZ)\,\mathrm{Tr}(ZZ)\\
&\quad + 3\,\mathrm{Tr}(W)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(ZZ) + 2\,\mathrm{Tr}(W)\,\mathrm{Tr}(ZZZ)\Big\}\\
&(R,r_1,r_2) = ([3,1],[2,1],[1]),\\
&\quad 6\Big\{-2\,\mathrm{Tr}(WZZZ) + \mathrm{Tr}(WZZ)\,\mathrm{Tr}(Z) + 2\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)^2 + \mathrm{Tr}(W)\,\mathrm{Tr}(Z)^3 - \mathrm{Tr}(WZ)\,\mathrm{Tr}(ZZ)\\
&\quad - \mathrm{Tr}(W)\,\mathrm{Tr}(ZZZ)\Big\}\\
&(R,r_1,r_2) = ([2,2],[2,1],[1]),\\
&\quad 2\Big\{-3\,\mathrm{Tr}(WZZ)\,\mathrm{Tr}(Z) + \mathrm{Tr}(W)\,\mathrm{Tr}(Z)^3 + 3\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(ZZ) - \mathrm{Tr}(W)\,\mathrm{Tr}(ZZZ)\Big\},\\
&(R,r_1,r_2) = ([2,1,1],[2,1],[1]),\\
&\quad 6\Big\{2\,\mathrm{Tr}(WZZZ) + \mathrm{Tr}(WZZ)\,\mathrm{Tr}(Z) - 2\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)^2 + \mathrm{Tr}(W)\,\mathrm{Tr}(Z)^3 - \mathrm{Tr}(WZ)\,\mathrm{Tr}(ZZ)\\
&\quad - \mathrm{Tr}(W)\,\mathrm{Tr}(ZZZ)\Big\}\\
&(R,r_1,r_2) = ([2,1,1],[1,1,1],[1]),\\
&\quad 3\Big\{2\,\mathrm{Tr}(WZZZ) - 2\,\mathrm{Tr}(WZZ)\,\mathrm{Tr}(Z) + \mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)^2 + \mathrm{Tr}(W)\,\mathrm{Tr}(Z)^3 - \mathrm{Tr}(WZ)\,\mathrm{Tr}(ZZ)\\
&\quad - 3\,\mathrm{Tr}(W)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(ZZ) + 2\,\mathrm{Tr}(W)\,\mathrm{Tr}(ZZZ)\Big\}\\
&(R,r_1,r_2) = ([1,1,1,1],[1,1,1],[1]),\\
&\quad - 6\,\mathrm{Tr}(WZZZ) + 6\,\mathrm{Tr}(WZZ)\,\mathrm{Tr}(Z) - 3\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)^2 + \mathrm{Tr}(W)\,\mathrm{Tr}(Z)^3 + 3\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(ZZ)\\
&\quad - 3\,\mathrm{Tr}(W)\,\mathrm{Tr}(Z)\,\mathrm{Tr}(ZZ) + 2\,\mathrm{Tr}(W)\,\mathrm{Tr}(ZZZ)
\end{aligned}$$

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C.2.4 $\mathcal{A}(2,2)$

$$(R,r_1,r_2)=([4],[2],[2]),$$
$$4\,\mathrm{Tr}(WWZZ)+2\,\mathrm{Tr}(WZ)^2+2\,\mathrm{Tr}(WZWZ)+4\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZZ)+4\,\mathrm{Tr}(WWZ)\,\mathrm{Tr}(Z)$$
$$+4\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\,\mathrm{Tr}(Z)^2+\mathrm{Tr}(WW)\,\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\,\mathrm{Tr}(ZZ)+\mathrm{Tr}(WW)\,\mathrm{Tr}(ZZ),$$
$$(R,r_1,r_2)=([3,1],[2],[2]),$$
$$-2\,\mathrm{Tr}(WZ)^2-2\,\mathrm{Tr}(WZWZ)+\mathrm{Tr}(W)^2\,\mathrm{Tr}(Z)^2+\mathrm{Tr}(WW)\,\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\,\mathrm{Tr}(ZZ)+\mathrm{Tr}(WW)\,\mathrm{Tr}(ZZ),$$
$$(R,r_1,r_2)=([3,1],[2],[1,1]),$$
$$2\Big\{-2\,\mathrm{Tr}(WWZZ)+2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZZ)-2\,\mathrm{Tr}(WWZ)\,\mathrm{Tr}(Z)+2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\,\mathrm{Tr}(Z)^2$$
$$-\mathrm{Tr}(WW)\,\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\,\mathrm{Tr}(ZZ)-\mathrm{Tr}(WW)\,\mathrm{Tr}(ZZ)\Big\},$$
$$(R,r_1,r_2)=([3,1],[1,1],[2]),$$
$$2\Big\{-2\,\mathrm{Tr}(WWZZ)-2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZZ)+2\,\mathrm{Tr}(WWZ)\,\mathrm{Tr}(Z)+2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\,\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(WW)\,\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)^2\,\mathrm{Tr}(ZZ)-\mathrm{Tr}(WW)\,\mathrm{Tr}(ZZ)\Big\},$$
$$(R,r_1,r_2)=([2,2],[2],[2]),$$
$$2\Big\{-2\,\mathrm{Tr}(WWZZ)+2\,\mathrm{Tr}(WZ)^2+2\,\mathrm{Tr}(WZWZ)-2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZZ)-2\,\mathrm{Tr}(WWZ)\,\mathrm{Tr}(Z)$$
$$-2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\,\mathrm{Tr}(Z)^2+\mathrm{Tr}(WW)\,\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\,\mathrm{Tr}(ZZ)+\mathrm{Tr}(WW)\,\mathrm{Tr}(ZZ)\Big\},$$
$$(R,r_1,r_2)=([2,2],[1,1],[1,1]),$$
$$2\Big\{2\,\mathrm{Tr}(WWZZ)+2\,\mathrm{Tr}(WZ)^2-2\,\mathrm{Tr}(WZWZ)-2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZZ)-2\,\mathrm{Tr}(WWZ)\,\mathrm{Tr}(Z)$$
$$+2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\,\mathrm{Tr}(Z)^2-\mathrm{Tr}(WW)\,\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)^2\,\mathrm{Tr}(ZZ)+\mathrm{Tr}(WW)\,\mathrm{Tr}(ZZ)\Big\},$$
$$(R,r_1,r_2)=([2,1,1],[2],[1,1]),$$
$$2\Big\{2\,\mathrm{Tr}(WWZZ)-2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZZ)+2\,\mathrm{Tr}(WWZ)\,\mathrm{Tr}(Z)-2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\,\mathrm{Tr}(Z)^2$$
$$-\mathrm{Tr}(WW)\,\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\,\mathrm{Tr}(ZZ)-\mathrm{Tr}(WW)\,\mathrm{Tr}(ZZ)\Big\},$$
$$(R,r_1,r_2)=([2,1,1],[1,1],[2]),$$
$$2\Big\{2\,\mathrm{Tr}(WWZZ)+2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZZ)-2\,\mathrm{Tr}(WWZ)\,\mathrm{Tr}(Z)-2\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\,\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(WW)\,\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)^2\,\mathrm{Tr}(ZZ)-\mathrm{Tr}(WW)\,\mathrm{Tr}(ZZ)\Big\},$$
$$(R,r_1,r_2)=([2,1,1],[1,1],[1,1]),$$
$$-2\,\mathrm{Tr}(WZ)^2+2\,\mathrm{Tr}(WZWZ)+\mathrm{Tr}(W)^2\,\mathrm{Tr}(Z)^2-\mathrm{Tr}(WW)\,\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)^2\,\mathrm{Tr}(ZZ)+\mathrm{Tr}(WW)\,\mathrm{Tr}(ZZ),$$
$$(R,r_1,r_2)=([1,1,1,1],[1,1],[1,1]),$$
$$-4\,\mathrm{Tr}(WWZZ)+2\,\mathrm{Tr}(WZ)^2-2\,\mathrm{Tr}(WZWZ)+4\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZZ)+4\,\mathrm{Tr}(WWZ)\,\mathrm{Tr}(Z)$$
$$-4\,\mathrm{Tr}(W)\,\mathrm{Tr}(WZ)\,\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\,\mathrm{Tr}(Z)^2-\mathrm{Tr}(WW)\,\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)^2\,\mathrm{Tr}(ZZ)+\mathrm{Tr}(WW)\,\mathrm{Tr}(ZZ)$$

C.2.5 $\mathcal{A}(4,1)$

$$(R,r_1,r_2)=([5],[4],[1]),$$
$$24\mathrm{Tr}(WZZZZ)+24\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)+12\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+4\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4$$
$$+12\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)+12\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2$$
$$+8\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)+8\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)$$
$$(R,r_1,r_2)=([4,1],[4],[1]),$$
$$4\Big\{-6\mathrm{Tr}(WZZZZ)-6\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)-3\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2-\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4$$

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$$-3\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)-3\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2$$
$$-2\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)+8\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,r_1,r_2)=([4,1],[3,1],[1])$,

$$12\Big\{-6\mathrm{Tr}(WZZZZ)+2\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)+5\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+3\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4$$
$$-3\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2$$
$$-2\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,r_1,r_2)=([3,2],[3,1],[1])$,

$$6\Big\{-4\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4+4\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$+2\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2+4\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,r_1,r_2)=([3,2],[2,2],[1])$,

$$4\Big\{-6\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4+6\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)$$
$$+3\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2-2\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)-4\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([3,1,1],[3,1],[1])$,

$$12\Big\{4\mathrm{Tr}(WZZZZ)+2\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)-2\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4+2\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)$$
$$-4\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2-2\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)$$
$$-2\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,r_1,r_2)=([3,1,1],[2,1,1],[1])$,

$$12\Big\{4\mathrm{Tr}(WZZZZ)-2\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4-2\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)$$
$$-4\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2+2\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)$$
$$+2\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,r_1,r_2)=([2,2,1],[2,2],[1])$,

$$4\Big\{6\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)-2\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4-6\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2$$
$$+2\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)-4\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([2,2,1],[2,1,1],[1])$,

$$6\Big\{4\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4+4\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$-2\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2-4\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)+2\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,r_1,r_2)=([2,1,1,1],[2,1,1],[1])$,

$$12\Big\{-6\mathrm{Tr}(WZZZZ)-2\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)+5\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2-3\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4$$
$$+3\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2$$
$$+2\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)+2\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,r_1,r_2)=([2,1,1,1],[1,1,1,1],[1])$,

$$4\Big\{-6\mathrm{Tr}(WZZZZ)+6\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)-3\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4$$
$$+3\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)-3\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-6\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2$$
$$+2\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)+8\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)-6\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,r_1,r_2)=([1,1,1,1,1],[1,1,1,1],[1])$,

$$24\mathrm{Tr}(WZZZZ)-24\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)+12\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2-4\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4$$
$$-12\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)+12\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-6\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2$$
$$-8\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)+8\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)-6\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)$$

C.2.6 $\mathcal{A}(3,2)$

$(R,r_1,r_2)=([5],[3],[2])$,

$$12\mathrm{Tr}(WWZZZ)+12\mathrm{Tr}(WZWZZ)+12\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+12\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+12\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$+6\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+6\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+12\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+6\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2$$
$$+6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+6\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)$$
$$+3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)+2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)$$

$(R,r_1,r_2)=([4,1],[3],[2])$,

$$6\Big\{2\mathrm{Tr}(WWZZZ)-8\mathrm{Tr}(WZWZZ)-8\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+2\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+2\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$-4\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)-4\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$+3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)+2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([4,1],[3],[1,1])$,

$$2\Big\{-6\mathrm{Tr}(WWZZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-6\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)+6\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-3\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2$$
$$+3\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-3\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)$$
$$+3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([4,1],[2,1],[2])$,

$$6\Big\{-4\mathrm{Tr}(WWZZZ)-2\mathrm{Tr}(WZWZZ)-2\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)-4\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+2\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$+2\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+2\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+4\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2$$
$$+4\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-2\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)$$
$$-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([3,2],[3],[2])$,

$$3\Big\{-4\mathrm{Tr}(WWZZZ)+4\mathrm{Tr}(WZWZZ)+4\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)-4\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-4\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$+2\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+2\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-2\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-2\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$+3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)+2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([3,2],[2,1],[2])$,

$$12\Big\{-\mathrm{Tr}(WWZZZ)+\mathrm{Tr}(WZWZZ)+\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)-\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-4\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$-\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)-\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+4\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+4\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)$$
$$-\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([3,2],[2,1],[1,1])$,

$$4\Big\{3\mathrm{Tr}(WWZZZ)-3\mathrm{Tr}(WZWZZ)+3\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)-3\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+3\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)$$
$$-3\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)-3\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2+3\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3$$
$$-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)+\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([3,1,1],[3],[1,1])$,

$$3\Big\{4\mathrm{Tr}(WWZZZ)-4\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+4\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2$$
$$-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+2\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)$$
$$+3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([3,1,1],[2,1],[2])$,

$$12\Big\{\mathrm{Tr}(WWZZZ)+3\mathrm{Tr}(WZWZZ)+3\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+2\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$-3\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)-3\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-2\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)$$
$$-\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([3,1,1],[2,1],[1,1])$,

$$12\Big\{\mathrm{Tr}(WWZZZ)+3\mathrm{Tr}(WZWZZ)-3\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)-\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-2\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$-3\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+3\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+2\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)$$
$$+\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([3,1,1],[1,1,1],[2])$,

$$3\Big\{4\mathrm{Tr}(WWZZZ)+4\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-4\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2$$
$$+2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-2\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)$$
$$-3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)+2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([2,2,1],[2,1],[2])$,

$$4\Big\{3\mathrm{Tr}(WWZZZ)-3\mathrm{Tr}(WZWZZ)-3\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+3\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)$$
$$+3\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)-3\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2-3\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3$$
$$-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([2,2,1],[2,1],[1,1])$,

$$12\Big\{-\mathrm{Tr}(WWZZZ)+\mathrm{Tr}(WZWZZ)-\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+4\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$

$$-\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-4\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+4\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)$$
$$+\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([2,2,1],[1,1,1],[1,1]),$

$$3\Big\{-4\mathrm{Tr}(WWZZZ)+4\mathrm{Tr}(WZWZZ)-4\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+4\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+4\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$+2\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)-2\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-2\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2+2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+2\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$+3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([2,1,1,1],[2,1],[1,1]),$

$$6\Big\{-4\mathrm{Tr}(WWZZZ)-2\mathrm{Tr}(WZWZZ)+2\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+4\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-2\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$+2\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)-2\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+4\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2-4\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+2\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)$$
$$+\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([2,1,1,1],[1,1,1],[2]),$

$$2\Big\{-6\mathrm{Tr}(WWZZZ)-6\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+6\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)+6\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-3\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2$$
$$-3\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+3\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)$$
$$-3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)+2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([2,1,1,1],[1,1,1],[1,1]),$

$$6\Big\{2\mathrm{Tr}(WWZZZ)-8\mathrm{Tr}(WZWZZ)+8\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-2\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$-4\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+4\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$+3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,r_1,r_2)=([1,1,1,1,1],[1,1,1],[1,1]),$

$$12\mathrm{Tr}(WWZZZ)+12\mathrm{Tr}(WZWZZ)-12\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)-12\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-12\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$+6\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)-6\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+12\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+6\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2$$
$$-6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-6\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)$$
$$-3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)$$

C.2.7 $\mathcal{A}(3,3)$

This is the first example of the restricted Schur basis with a non-trivial Littlewood-Richardson coefficient

$$g([2,1],[2,1];[3,2,1])=2. \tag{C.6}$$

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Below we show an integer basis of the four states with $(R, r_1, r_2) = ([3,2,1],[2,1],[2,1])$, which are orthogonal with respect to the δ-function inner product.

$$
\begin{aligned}
&(R,r_1,r_2)=([3,2,1],[2,1],[2,1]),\\
&12\Big\{\ \mathrm{Tr}(Z)^3\mathrm{Tr}(W)^3-\mathrm{Tr}(Z)^3\mathrm{Tr}(WWW)-3\mathrm{Tr}(Z)\mathrm{Tr}(ZZW)\mathrm{Tr}(W)^2\\
&\quad+3\mathrm{Tr}(Z)\mathrm{Tr}(ZZWWW)-3\mathrm{Tr}(Z)\mathrm{Tr}(ZW)^2\mathrm{Tr}(W)+3\mathrm{Tr}(Z)\mathrm{Tr}(ZWZWW)\\
&\quad+3\mathrm{Tr}(ZZ)\mathrm{Tr}(ZW)\mathrm{Tr}(W)^2-3\mathrm{Tr}(ZZ)\mathrm{Tr}(ZWWW)-\mathrm{Tr}(ZZZ)\mathrm{Tr}(W)^3\\
&\quad+\mathrm{Tr}(ZZZ)\mathrm{Tr}(WWW)-6\mathrm{Tr}(ZZW)\mathrm{Tr}(ZWW)+3\mathrm{Tr}(ZZWZW)\mathrm{Tr}(W)\\
&\quad+3\mathrm{Tr}(ZZWW)\mathrm{Tr}(ZW)\}\\
&(R,r_1,r_2)=([3,2,1],[2,1],[2,1]),\\
&12\Big\{\ \mathrm{Tr}(Z)^3\mathrm{Tr}(W)^3-\mathrm{Tr}(Z)^3\mathrm{Tr}(WWW)+9\mathrm{Tr}(Z)^2\mathrm{Tr}(ZW)\mathrm{Tr}(WW)\\
&\quad-9\mathrm{Tr}(Z)^2\mathrm{Tr}(ZWW)\mathrm{Tr}(W)+6\mathrm{Tr}(Z)\mathrm{Tr}(ZZW)\mathrm{Tr}(W)^2-6\mathrm{Tr}(Z)\mathrm{Tr}(ZZWWW)\\
&\quad-3\mathrm{Tr}(Z)\mathrm{Tr}(ZW)^2\mathrm{Tr}(W)+3\mathrm{Tr}(Z)\mathrm{Tr}(ZWZWW)-6\mathrm{Tr}(ZZ)\mathrm{Tr}(ZW)\mathrm{Tr}(W)^2\\
&\quad+6\mathrm{Tr}(ZZ)\mathrm{Tr}(ZWWW)-\mathrm{Tr}(ZZZ)\mathrm{Tr}(W)^3+\mathrm{Tr}(ZZZ)\mathrm{Tr}(WWW)\\
&\quad-9\mathrm{Tr}(ZZZW)\mathrm{Tr}(WW)+9\mathrm{Tr}(ZZZWW)\mathrm{Tr}(W)-6\mathrm{Tr}(ZZW)\mathrm{Tr}(ZWW)\\
&\quad+3\mathrm{Tr}(ZZWZW)\mathrm{Tr}(W)+3\mathrm{Tr}(ZZWW)\mathrm{Tr}(ZW)\}\\
&(R,r_1,r_2)=([3,2,1],[2,1],[2,1]),\\
&36\{\ \ \mathrm{Tr}(Z)\mathrm{Tr}(ZZW)\mathrm{Tr}(WW)-\mathrm{Tr}(Z)\mathrm{Tr}(ZZWW)\mathrm{Tr}(W)\\
&\quad-\mathrm{Tr}(Z)\mathrm{Tr}(ZW)\mathrm{Tr}(ZWW)+\mathrm{Tr}(Z)\mathrm{Tr}(ZWZW)\mathrm{Tr}(W)\\
&\quad-\mathrm{Tr}(ZZ)\mathrm{Tr}(ZW)\mathrm{Tr}(WW)+\mathrm{Tr}(ZZ)\mathrm{Tr}(ZWW)\mathrm{Tr}(W)\\
&\quad-\mathrm{Tr}(ZZW)\mathrm{Tr}(ZW)\mathrm{Tr}(W)+\mathrm{Tr}(ZZWWZW)\\
&\quad+\mathrm{Tr}(ZW)^3-\mathrm{Tr}(ZWZWZW)\Big\}\\
&(R,r_1,r_2)=([3,2,1],[2,1],[2,1]),\\
&36\{\ \ \mathrm{Tr}(Z)\mathrm{Tr}(ZZW)\mathrm{Tr}(WW)-\mathrm{Tr}(Z)\mathrm{Tr}(ZZWW)\mathrm{Tr}(W)-\mathrm{Tr}(Z)\mathrm{Tr}(ZW)\mathrm{Tr}(ZWW)\\
&\quad+\mathrm{Tr}(Z)\mathrm{Tr}(ZWZW)\mathrm{Tr}(W)-\mathrm{Tr}(ZZ)\mathrm{Tr}(ZW)\mathrm{Tr}(WW)+\mathrm{Tr}(ZZ)\mathrm{Tr}(ZWW)\mathrm{Tr}(W)\\
&\quad-\mathrm{Tr}(ZZW)\mathrm{Tr}(ZW)\mathrm{Tr}(W)+23\mathrm{Tr}(ZZWZWW)-22\mathrm{Tr}(ZZWWZW)\\
&\quad+\mathrm{Tr}(ZW)^3-\mathrm{Tr}(ZWZWZW)\Big\}
\end{aligned}
$$

C.3 Covariant basis

We will present the operators which are proportional to the covariant operators (2.113),

$$
\begin{aligned}
\mathsf{O}^{R,\Lambda,\mu,\tau}_{\beta}(\vec{a}_\mu) &= \mathrm{tr}_{V_N^{\otimes L}}\Big(\mathcal{L}(\mathsf{Q}^{R,\Lambda}_A)\, Z^{\otimes\mu_1}\otimes W^{\otimes\mu_2}\Big)\\
\mathsf{Q}^{R,\Lambda}_A &= \sum_{\tau=1}^{C(R,R,\Lambda)}\sum_{\beta=1}^{K_{\Lambda,(\mu_1,\mu_2)}} \mathsf{N}^{R,\Lambda,\tau}_{A,\beta}\, \mathcal{Q}^{R,\Lambda,(\mu_1,\mu_2),\tau}_{\beta}\,.
\end{aligned}
\tag{C.7}
$$

where $\mathsf{Q}^{R,\Lambda}_A$ is an integer eigenvector for the eigenvalue system of the covariant basis (3.14), obtained according to the methods described in section 3.3. The index β may be omitted because $K_{\Lambda\mu} = 0$ or 1 for $\mu = (\mu_1, \mu_2)$.

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C.3.1 $\mathcal{A}(1,1)$

$$(R,\Lambda)=([2],[2]),\qquad \mathrm{Tr}(Z)\mathrm{Tr}(W)+\mathrm{Tr}(ZW)$$
$$(R,\Lambda)=([1,1],[2]),\qquad \mathrm{Tr}(Z)\mathrm{Tr}(W)-\mathrm{Tr}(ZW)$$

C.3.2 A(2,1)

$$(R,\Lambda)=([3],[3]),\qquad \mathrm{Tr}(Z)^2\mathrm{Tr}(W)+2\mathrm{Tr}(Z)\mathrm{Tr}(ZW)+\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}(W)+2\mathrm{Tr}\left(Z^2W\right)$$
$$(R,\Lambda)=([2,1],[3]),\qquad 2\left\{\mathrm{Tr}(Z)^2\mathrm{Tr}(W)-\mathrm{Tr}\left(Z^2W\right)\right\}$$
$$(R,\Lambda)=([2,1],[2,1]),\qquad 2\left\{\mathrm{Tr}(Z)\mathrm{Tr}(ZW)-\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}(W)\right\}$$
$$(R,\Lambda)=([1,1,1],[3]),\qquad \mathrm{Tr}(Z)^2\mathrm{Tr}(W)-2\mathrm{Tr}(Z)\mathrm{Tr}(ZW)-\mathrm{Tr}\left(Z^2\right)\mathrm{Tr}(W)+2\mathrm{Tr}\left(Z^2W\right)$$

C.3.3 A(3,1)

$$(R,\Lambda)=([4],[4]),$$
$$6\mathrm{Tr}(WZZZ)+6\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+3\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(Z)^3+3\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)$$
$$+3\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)\mathrm{Tr}(ZZZ)$$
$$(R,\Lambda)=([3,1],[4]),$$
$$3\left\{-2\mathrm{Tr}(WZZZ)+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(Z)^3-\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)+\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)\right\}$$
$$(R,\Lambda)=([3,1],[3,1]),$$
$$6\left\{\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(ZZZ)\right\}$$
$$(R,\Lambda)=([2,2],[4]),$$
$$2\left\{-3\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(W)\mathrm{Tr}(Z)^3+3\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(ZZZ)\right\}$$
$$(R,\Lambda)=([2,1,1],[4]),$$
$$3\left\{2\mathrm{Tr}(WZZZ)-\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(Z)^3-\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)\right\}$$
$$(R,\Lambda)=([2,1,1],[3,1]),$$
$$6\left\{-\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+\mathrm{Tr}(W)\mathrm{Tr}(ZZZ)\right\}$$
$$(R,\Lambda)=([1,1,1,1],[4]),$$
$$-6\mathrm{Tr}(WZZZ)+6\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-3\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(Z)^3$$
$$+3\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-3\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)\mathrm{Tr}(ZZZ)$$

C.3.4 A(2,2)

$$(R,\Lambda)=([4],[4]),$$
$$4\mathrm{Tr}(WWZZ)+2\mathrm{Tr}(WZ)^2+2\mathrm{Tr}(WZWZ)+4\mathrm{Tr}(W)\mathrm{Tr}(WZZ)+4\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)$$
$$+4\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^2+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(ZZ)+\mathrm{Tr}(WW)\mathrm{Tr}(ZZ)$$
$$(R,\Lambda)=([3,1],[4]),$$
$$-4\mathrm{Tr}(WWZZ)-2\mathrm{Tr}(WZ)^2-2\mathrm{Tr}(WZWZ)+4\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)+3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(ZZ)-\mathrm{Tr}(WW)\mathrm{Tr}(ZZ)$$
$$(R,\Lambda)=([3,1],[3,1]),$$

$$2\Big\{-2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)+2\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZ)\Big\}$$

$(R,\Lambda)=([3,1],[2,2])$,

$$2\Big\{2\mathrm{Tr}(WWZZ)-2\mathrm{Tr}(WZ)^2-2\mathrm{Tr}(WZWZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(ZZ)+2\mathrm{Tr}(WW)\mathrm{Tr}(ZZ)\Big\}$$

$(R,\Lambda)=([2,2],[4])$,

$$2\Big\{2\mathrm{Tr}(WZ)^2-2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)-2\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^2+\mathrm{Tr}(WW)\mathrm{Tr}(ZZ)\Big\}$$

$(R,\Lambda)=([2,2],[2,2])$,

$$2\Big\{-2\mathrm{Tr}(WWZZ)+2\mathrm{Tr}(WZWZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(ZZ)\Big\}$$

$(R,\Lambda)=([2,1,1],[4])$,

$$4\mathrm{Tr}(WWZZ)-2\mathrm{Tr}(WZ)^2+2\mathrm{Tr}(WZWZ)-4\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)+3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^2$$
$$-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZ)-\mathrm{Tr}(WW)\mathrm{Tr}(ZZ)$$

$(R,\Lambda)=([2,1,1],[3,1])$,

$$2\Big\{2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)-2\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZ)\Big\}$$

$(R,\Lambda)=([2,1,1],[2,2])$,

$$2\Big\{2\mathrm{Tr}(WWZZ)+2\mathrm{Tr}(WZ)^2-2\mathrm{Tr}(WZWZ)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(ZZ)-2\mathrm{Tr}(WW)\mathrm{Tr}(ZZ)\Big\}$$

$(R,\Lambda)=([1,1,1,1],[4])$,

$$-4\mathrm{Tr}(WWZZ)+2\mathrm{Tr}(WZ)^2-2\mathrm{Tr}(WZWZ)+4\mathrm{Tr}(W)\mathrm{Tr}(WZZ)+4\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)$$
$$-4\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^2-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^2-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZ)+\mathrm{Tr}(WW)\mathrm{Tr}(ZZ)$$

C.3.5 A(4,1)

$(R,\Lambda)=([5],[5])$,

$$24\mathrm{Tr}(WZZZZ)+24\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)+12\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+4\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4$$
$$+12\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)+12\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2$$
$$+8\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)+8\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)$$

$(R,\Lambda)=([4,1],[5])$,

$$4\Big\{-6\mathrm{Tr}(WZZZZ)+3\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+2\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4-3\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)$$
$$+3\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)-2\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)+2\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,\Lambda)=([4,1],[4,1])$,

$$12\Big\{2\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$-\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2-2\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,\Lambda)=([3,2],[5])$,

$$-24\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)-12\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+4\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+5\mathrm{Tr}(W)\mathrm{Tr}(Z)^4$$
$$+12\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)+12\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2$$

$$+8\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)-8\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)-6\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)$$

$(R,\Lambda)=([3,2],[4,1])$,

$$24\Big\{-\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+3\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)$$
$$-2\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2-3\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)$$
$$-2\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)+\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,\Lambda)=([3,1,1],[5])$,

$$6\Big\{4\mathrm{Tr}(WZZZZ)+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4-4\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2\Big\},$$

$(R,\Lambda)=([3,1,1],[4,1])$,

$$24\Big\{-\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3-\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)$$
$$+\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)+\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,\Lambda)=([2,2,1],[5])$,

$$24\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)-12\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2-4\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+5\mathrm{Tr}(W)\mathrm{Tr}(Z)^4$$
$$-12\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)+12\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-6\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2$$
$$-8\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)-8\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)$$

$(R,\Lambda)=([2,2,1],[4,1])$,

$$24\Big\{-\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)-2\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+3\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)$$
$$+2\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2-3\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)$$
$$+2\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)+\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,\Lambda)=([2,1,1,1],[5])$,

$$4\Big\{-6\mathrm{Tr}(WZZZZ)+3\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2-2\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4+3\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)$$
$$-3\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)+2\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)+2\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,\Lambda)=([2,1,1,1],[4,1])$,

$$12\Big\{2\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)-2\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3-\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$-\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)+\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2+2\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)-2\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)\Big\}$$

$(R,\Lambda)=([1,1,1,1,1],[5])$,

$$24\mathrm{Tr}(WZZZZ)-24\mathrm{Tr}(WZZZ)\mathrm{Tr}(Z)+12\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)^2-4\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^3+\mathrm{Tr}(W)\mathrm{Tr}(Z)^4$$
$$-12\mathrm{Tr}(WZZ)\mathrm{Tr}(ZZ)+12\mathrm{Tr}(WZ)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-6\mathrm{Tr}(W)\mathrm{Tr}(Z)^2\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(ZZ)^2$$
$$-8\mathrm{Tr}(WZ)\mathrm{Tr}(ZZZ)+8\mathrm{Tr}(W)\mathrm{Tr}(Z)\mathrm{Tr}(ZZZ)-6\mathrm{Tr}(W)\mathrm{Tr}(ZZZZ)$$

C.3.6 A(3,2)

This is the first example of the covariant basis with a non-trivial Kronecker coefficient

$$C([3,1,1],[3,1,1],[3,2])=2. \tag{C.8}$$

Below we show an integer basis of the two states with $(R,\Lambda) = ([3,1,1],[3,2])$, which are orthogonal with respect to the δ-function inner product.

$$
\begin{aligned}
&(R,\Lambda)=([3,1,1],[3,2]),\\
&3\{\ \mathrm{Tr}(Z)^3\mathrm{Tr}(WW)-2\mathrm{Tr}(Z)^2\mathrm{Tr}(ZW)\mathrm{Tr}(W)+\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)\mathrm{Tr}(W)^2\\
&\quad+4\mathrm{Tr}(Z)\mathrm{Tr}(ZZWW)-4\mathrm{Tr}(Z)\mathrm{Tr}(ZWZW)-2\mathrm{Tr}(ZZ)\mathrm{Tr}(ZWW)\\
&\quad-2\mathrm{Tr}(ZZZ)\mathrm{Tr}(WW)+4\mathrm{Tr}(ZZW)\mathrm{Tr}(ZW)\}\\
&(R,\Lambda)=([3,1,1],[3,2]),\\
&12\{\ \mathrm{Tr}(Z)^2\mathrm{Tr}(ZWW)-\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)\mathrm{Tr}(WW)-2\mathrm{Tr}(Z)\mathrm{Tr}(ZZW)\mathrm{Tr}(W)\\
&\quad+\mathrm{Tr}(Z)\mathrm{Tr}(ZW)^2+\mathrm{Tr}(ZZZ)\mathrm{Tr}(W)^2+\mathrm{Tr}(ZZZWW)-\mathrm{Tr}(ZZWZW)\Big\}
\end{aligned}
$$

The remaining states are given as follows.

$$
\begin{aligned}
&(R,\Lambda)=([5],[5]),\\
&\quad 12\mathrm{Tr}(WWZZZ)+12\mathrm{Tr}(WZWZZ)+12\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+12\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+12\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)\\
&\quad+6\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+6\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+12\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+6\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2\\
&\quad+6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+6\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)\\
&\quad+3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)+2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\\
&(R,\Lambda)=([4,1],[5]),\\
&\quad 2\Big\{-6\mathrm{Tr}(WWZZZ)-6\mathrm{Tr}(WZWZZ)-6\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+3\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2\\
&\quad+6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+2\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-3\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)\\
&\quad+\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}\\
&(R,\Lambda)=([4,1],[4,1]),\\
&\quad 6\Big\{-6\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+4\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+2\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)\\
&\quad-2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+4\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3\\
&\quad-3\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)\Big\}\\
&(R,\Lambda)=([4,1],[3,2]),\\
&\quad 6\Big\{6\mathrm{Tr}(WWZZZ)-6\mathrm{Tr}(WZWZZ)-6\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+4\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)\\
&\quad-4\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)-2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2\\
&\quad+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+3\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+4\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)\\
&\quad+\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)+3\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}\\
&(R,\Lambda)=([3,2],[5]),\\
&\quad 12\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)-12\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-12\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)+6\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)-6\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)\\
&\quad-12\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-6\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2+6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+5\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3\\
&\quad+6\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)\\
&\quad-2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)+2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)
\end{aligned}
$$

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$$(R,\Lambda)=([3,2],[4,1]),$$

$$12\Big\{-3\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-2\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)-\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)$$
$$-2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+4\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3$$
$$+6\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$-2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-3\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$$(R,\Lambda)=([3,2],[3,2]),$$

$$12\Big\{-3\mathrm{Tr}(WWZZZ)+3\mathrm{Tr}(WZWZZ)-2\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)-\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+2\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)$$
$$-2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)\Big\}$$

$$(R,\Lambda)=([3,1,1],[5]),$$

$$6\Big\{2\mathrm{Tr}(WWZZZ)+2\mathrm{Tr}(WZWZZ)-2\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)$$
$$-\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)\Big\}$$

$$(R,\Lambda)=([3,1,1],[4,1]),$$

$$12\Big\{\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-2\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)-\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-2\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$$(R,\Lambda)=([3,1,1],[3,2]),$$

$$3\Big\{4\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+4\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-2\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$$(R,\Lambda)=([3,1,1],[3,2]),$$

$$12\Big\{\mathrm{Tr}(WWZZZ)-\mathrm{Tr}(WZWZZ)+\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)-2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2$$
$$-\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)\Big\}$$

$$(R,\Lambda)=([2,2,1],[5]),$$

$$-12\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+12\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+12\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)+6\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+6\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)$$
$$-12\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-6\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2-6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+5\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3$$
$$-6\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$-2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)$$

$$(R,\Lambda)=([2,2,1],[4,1]),$$

$$12\Big\{-3\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+3\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-2\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)+4\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)-\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)$$
$$+2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3$$
$$+6\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)-6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-3\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$$(R,\Lambda)=([2,2,1],[3,2]),$$

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$$12\Big\{3\mathrm{Tr}(WWZZZ)-3\mathrm{Tr}(WZWZZ)-2\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)+\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+2\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)$$
$$+2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)\Big\}$$

$(R,\Lambda)=([2,1,1,1],[5])$,

$$2\Big\{-6\mathrm{Tr}(WWZZZ)-6\mathrm{Tr}(WZWZZ)+6\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+3\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2$$
$$-6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+2\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+3\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)-3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$+\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)+\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,\Lambda)=([2,1,1,1],[4,1])$,

$$6\Big\{-6\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)+4\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)-2\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)+2\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)$$
$$+2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-4\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3$$
$$+3\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)-2\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)\Big\}$$

$(R,\Lambda)=([2,1,1,1],[3,2])$,

$$6\Big\{-6\mathrm{Tr}(WWZZZ)+6\mathrm{Tr}(WZWZZ)-6\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)+4\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)+4\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)$$
$$-4\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+2\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)-\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2-2\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2$$
$$+\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3+3\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)-4\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)$$
$$-\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)+3\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)\Big\}$$

$(R,\Lambda)=([1,1,1,1,1],[5])$,

$$12\mathrm{Tr}(WWZZZ)+12\mathrm{Tr}(WZWZZ)-12\mathrm{Tr}(WZ)\mathrm{Tr}(WZZ)-12\mathrm{Tr}(W)\mathrm{Tr}(WZZZ)-12\mathrm{Tr}(WWZZ)\mathrm{Tr}(Z)$$
$$+6\mathrm{Tr}(WZ)^2\mathrm{Tr}(Z)-6\mathrm{Tr}(WZWZ)\mathrm{Tr}(Z)+12\mathrm{Tr}(W)\mathrm{Tr}(WZZ)\mathrm{Tr}(Z)+6\mathrm{Tr}(WWZ)\mathrm{Tr}(Z)^2$$
$$-6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(Z)^2+\mathrm{Tr}(W)^2\mathrm{Tr}(Z)^3-\mathrm{Tr}(WW)\mathrm{Tr}(Z)^3-6\mathrm{Tr}(WWZ)\mathrm{Tr}(ZZ)+6\mathrm{Tr}(W)\mathrm{Tr}(WZ)\mathrm{Tr}(ZZ)$$
$$-3\mathrm{Tr}(W)^2\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+3\mathrm{Tr}(WW)\mathrm{Tr}(Z)\mathrm{Tr}(ZZ)+2\mathrm{Tr}(W)^2\mathrm{Tr}(ZZZ)-2\mathrm{Tr}(WW)\mathrm{Tr}(ZZZ)$$

D Data: orthonormal bases for $\mathbb{C}[S_L]$

In this appendix we give examples of solutions to the eigensystems in 4, giving the Artin-Wedderburn and Kronecker decomposition of $\mathbb{C}[S_L]$, respectively.

D.1 Matrix units of S_L

The matrix unit of S_L is defined by (2.48),

$$Q^R_{IJ} = \frac{d_R}{|S_L|}\sum_{g\in S_L} D^R_{JI}(g)\, g^{-1}. \tag{D.1}$$

In the following we construct Q^R_{IJ}, and thus determine the matrix elements of the irreducible representation of S_L.[10]

[10]Our construction gives the Young seminormal representation, which is real and unitary. See the discussion at the end of section 4.

We argued in the main text that by solving the eigenvalue system (4.14), the matrix unit can be determined uniquely up to the normalisation constant. If we follow the method of section 3.3 using the Hermite normal form, we obtain the eigenvectors whose components are all integers. Let us call such an eigenvector the integer basis, denoted by

$$\mathsf{Q}^R_{IJ} \equiv \mathsf{N}^R_{IJ}\, Q^R_{IJ} \tag{D.2}$$

where N^R_{IJ} is a normalisation constant. The integer basis satisfies the condition

$$\mathsf{Q}^R_{IJ} = \sum_{g \in \texttt{Basis}} c_g(\mathsf{Q}^R_{IJ})\, g, \qquad c_g(\mathsf{Q}^R_{IJ}) \in \mathbb{Z}\,, \tag{D.3}$$

and the product relation

$$\mathsf{Q}^R_{IJ}\, \mathsf{Q}^S_{KL} = \frac{\mathsf{N}^R_{IJ}\, \mathsf{N}^R_{JL}}{\mathsf{N}^R_{IL}}\, \delta^{RS}\, \delta_{JK}\, \mathsf{Q}^R_{IL}\,. \tag{D.4}$$

Note that the explicit data shown below may not give real and unitary matrices D^R in the sense that $D^R_{IJ}(\sigma^{-1}) \neq D^R_{JI}(\sigma)$.[11] As discussed at the end of section 4.1, we need to rescale the eigenvectors to obtain the Young-Yamanouchi orthogonal representation, which is real and unitary. However, the rescaling spoils the integer property. Other $\mathbb{Z}$-valued representation matrices, such as arising from the Young natural and Kazhdan-Lusztig constructions, also do not produce unitary matrices [81, 82].

We wrote a `Mathematica` code to compute the matrix units of S_L explicitly up to $L = 6$. Below we just present our results up to $L = 4$.

D.1.1 Case of S_2

One finds

$$\mathsf{Q}^{[2]} = () + (12)\,, \qquad \mathsf{Q}^{[1,1]} = () - (12), \qquad \mathsf{Q}^R\, \mathsf{Q}^S = 2\,\delta^{RS} Q^R \tag{D.5}$$

where we omit I, J because all representations are one-dimensional.

D.1.2 Case of S_3

We express the matrix unit Q^R_{IJ} as a pair of Young tableaux $\{I, J\}$ of shape R. We reorganise them into a vector

$$\begin{aligned}\mathscr{E}_\rho &\equiv \left(\mathsf{Q}^{[3]}_{11}, \mathsf{Q}^{[2,1]}_{11}, \mathsf{Q}^{[2,1]}_{12}, \mathsf{Q}^{[2,1]}_{21}, \mathsf{Q}^{[2,1]}_{22}, \mathsf{Q}^{[1,1,1]}_{11}\right) \\ &= \left(\left\{\ytableaushort{123}, \ytableaushort{123}\right\}, \left\{\ytableaushort{12,3}, \ytableaushort{12,3}\right\}, \left\{\ytableaushort{12,3}, \ytableaushort{13,2}\right\}, \left\{\ytableaushort{13,2}, \ytableaushort{12,3}\right\}, \left\{\ytableaushort{13,2}, \ytableaushort{13,2}\right\} \left\{\ytableaushort{1,2,3}, \ytableaushort{1,2,3}\right\} \right).\end{aligned} \tag{D.6}$$

We choose a basis of S_3 as

$$\texttt{Basis} = \{(), (23), (12), (123), (132), (13)\} \tag{D.7}$$

[11] In other words, we did not impose the real unitarity condition on the integer basis Q^R_{IJ} with a generic N^R_{IJ}.

and expand $\mathscr{E}_\rho$ as

$$\mathscr{E}_\rho = \sum_{g\in\texttt{Basis}} E_{\rho g}\, g. \tag{D.8}$$

The coefficient matrix in the integer basis is given by

$$E_{\rho g} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & -1 & 2 & -1 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 2 & 1 & -2 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix} \tag{D.9}$$

We can write down the product relation (D.4) explicitly as

$$\mathscr{E}_\rho\, \mathscr{E}_{\rho'} = \begin{pmatrix} 6\,\mathscr{E}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6\,\mathscr{E}_2 & 6\,\mathscr{E}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\,\mathscr{E}_2 & 6\,\mathscr{E}_3 & 0 \\ 0 & 6\,\mathscr{E}_4 & 2\,\mathscr{E}_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6\,\mathscr{E}_4 & 6\,\mathscr{E}_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6\,\mathscr{E}_6 \end{pmatrix}. \tag{D.10}$$

In order to reproduce the standard matrix unit relation (2.49), we need to adjust the normalisation so that $\mathsf{N}_{IJ}^R = 1$ for all eigenvectors. In this normalisation, we find

$$\widehat{\mathscr{E}}_\rho = \sum_{g\in\texttt{Basis}} \hat{E}_{\rho g}\, g, \qquad \hat{E}_{\rho g} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\ 0 & \frac{1}{2\sqrt{3}} & 0 & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ 0 & \frac{1}{2\sqrt{3}} & 0 & -\frac{1}{2\sqrt{3}} & \frac{1}{2\sqrt{3}} & -\frac{1}{2\sqrt{3}} \\ \frac{1}{3} & \frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} \end{pmatrix}. \tag{D.11}$$

For example,

$$\begin{aligned} \widehat{\mathscr{E}}_1 &= \left\{ \boxed{1}\boxed{2}\boxed{3}\,,\ \boxed{1}\boxed{2}\boxed{3} \right\} &&= \frac{() + (23) + (12) + (123) + (132) + (13)}{6} \\ \widehat{\mathscr{E}}_6 &= \left\{ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}\,,\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} \right\} &&= \frac{() - (23) - (12) + (123) + (132) - (13)}{6}\,. \end{aligned} \tag{D.12}$$

The product of $\widehat{\mathscr{E}}$'s is given by

$$\widehat{\mathscr{E}}_\rho\, \widehat{\mathscr{E}}_{\rho'} = \begin{pmatrix} \widehat{\mathscr{E}}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \widehat{\mathscr{E}}_2 & \widehat{\mathscr{E}}_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widehat{\mathscr{E}}_2 & \widehat{\mathscr{E}}_3 & 0 \\ 0 & \widehat{\mathscr{E}}_4 & \widehat{\mathscr{E}}_5 & 0 & 0 & 0 \\ 0 & 0 & 0 & \widehat{\mathscr{E}}_4 & \widehat{\mathscr{E}}_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & \widehat{\mathscr{E}}_6 \end{pmatrix} \tag{D.13}$$

which agrees with (2.49).

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D.1.3 Case of S_4

Define the vector of matrix units as

$$
\begin{aligned}
\mathscr{E}_\rho \equiv \Big(&\Big\{\young(1234),\young(1234)\Big\},\Big\{\young(123,4),\young(123,4)\Big\},\Big\{\young(123,4),\young(124,3)\Big\},\Big\{\young(123,4),\young(134,2)\Big\},\Big\{\young(124,3),\young(123,4)\Big\},\\
&\Big\{\young(124,3),\young(124,3)\Big\},\Big\{\young(124,3),\young(134,2)\Big\},\Big\{\young(134,2),\young(123,4)\Big\},\Big\{\young(134,2),\young(124,3)\Big\},\Big\{\young(134,2),\young(134,2)\Big\},\\
&\Big\{\young(12,34),\young(12,34)\Big\},\Big\{\young(12,34),\young(13,24)\Big\},\Big\{\young(13,24),\young(12,34)\Big\},\Big\{\young(13,24),\young(13,24)\Big\},\Big\{\young(12,3,4),\young(12,3,4)\Big\},\\
&\Big\{\young(12,3,4),\young(13,2,4)\Big\},\Big\{\young(12,3,4),\young(14,2,3)\Big\},\Big\{\young(13,2,4),\young(12,3,4)\Big\},\Big\{\young(13,2,4),\young(13,2,4)\Big\},\Big\{\young(13,2,4),\young(14,2,3)\Big\},\\
&\Big\{\young(14,2,3),\young(12,3,4)\Big\},\Big\{\young(14,2,3),\young(13,2,4)\Big\},\Big\{\young(14,2,3),\young(14,2,3)\Big\},\Big\{\young(1,2,3,4),\young(1,2,3,4)\Big\}\Big)
\end{aligned}
\tag{D.14}
$$

and choose a basis of S_4 as

$$
\begin{aligned}
\texttt{Basis} = \Big\{ &(), (34), (23), (234), (243), (24), \quad (12), (12)(34), (123), (1234), (1243), (124),\\
&(132), (1342), (13), (134), (13)(24), (1324), \quad (1432), (142), (143), (14), (1423), (14)(23) \Big\}.
\end{aligned}
\tag{D.15}
$$

The coefficient matrix in the integer normalisation is given by

$$
E_{\rho g} = \left(\begin{array}{cccccc|cccccc|cccccc|cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
3 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & 3 & -1 & -1 & -1 & 3 & -1 & 3 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1\\
0 & 2 & 0 & 2 & -1 & -1 & 0 & 2 & 0 & 2 & -1 & -1 & 0 & 2 & 0 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1\\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1\\
0 & 2 & 0 & -1 & 2 & -1 & 0 & 2 & 0 & -1 & 2 & -1 & 0 & -1 & 0 & -1 & -1 & -1 & 2 & -1 & 2 & -1 & -1 & -1\\
6 & 2 & -3 & -1 & -1 & 5 & 6 & 2 & -3 & -1 & -1 & 5 & -3 & -1 & -3 & -1 & -4 & -4 & -1 & 5 & -1 & 5 & -4 & -4\\
\hline
0 & 0 & 3 & 3 & 1 & 1 & 0 & 0 & 3 & 3 & 1 & 1 & -3 & -3 & -3 & -3 & -2 & -2 & -1 & -1 & -1 & -1 & 2 & 2\\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 & -1 & 1 & -1\\
0 & 0 & 3 & 1 & 3 & 1 & 0 & 0 & -3 & -1 & -3 & -1 & 3 & 1 & -3 & -1 & -2 & 2 & 3 & 1 & -3 & -1 & -2 & 2\\
2 & 2 & 1 & 1 & 1 & 1 & -2 & -2 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0\\
2 & 2 & -1 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 2 & 2 & -1 & -1 & -1 & -1 & 2 & 2\\
0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0\\
\hline
0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & -1 & -1 & 1 & 1 & 1 & 1 & -1 & -1 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0\\
2 & -2 & 1 & -1 & -1 & 1 & -2 & 2 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & 2 & -2 & 1 & -1 & -1 & 1 & -2 & 2\\
2 & -2 & -1 & 1 & 1 & -1 & 2 & -2 & -1 & 1 & 1 & -1 & -1 & 1 & -1 & 1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0\\
0 & 0 & 3 & -1 & -3 & 1 & 0 & 0 & 3 & -1 & -3 & 1 & -3 & 1 & -3 & 1 & 2 & 2 & 3 & -1 & 3 & -1 & -2 & -2\\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & -1 & 1 & 1 & 0 & 1 & 0 & 1 & -1 & -1\\
0 & 0 & 3 & -3 & -1 & 1 & 0 & 0 & -3 & 3 & 1 & -1 & 3 & -3 & -3 & 3 & 2 & -2 & -1 & 1 & 1 & -1 & 2 & -2\\
\hline
6 & -2 & 3 & -1 & -1 & -5 & -6 & 2 & -3 & 1 & 1 & 5 & -3 & 1 & 3 & -1 & -4 & 4 & 1 & 5 & -1 & -5 & 4 & -4\\
0 & 2 & 0 & 1 & -2 & -1 & 0 & -2 & 0 & -1 & 2 & 1 & 0 & -1 & 0 & 1 & 1 & -1 & 2 & 1 & -2 & -1 & -1 & 1\\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & -1 & 1 & 1 & -1\\
0 & 2 & 0 & -2 & 1 & -1 & 0 & -2 & 0 & 2 & -1 & 1 & 0 & 2 & 0 & -2 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1\\
3 & 1 & -3 & -1 & -1 & 1 & -3 & -1 & 3 & 1 & 1 & -1 & 3 & 1 & -3 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1\\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1
\end{array}\right).
\tag{D.16}
$$

The product $\mathscr{E}_\rho \, \mathscr{E}_{\rho'}$ is given by

$$\left(\begin{array}{cccccc|cccccc|cccccc|cccccc}
24\mathscr{E}_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 24\mathscr{E}_2 & 24\mathscr{E}_3 & 24\mathscr{E}_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 12\mathscr{E}_2 & 48\mathscr{E}_3 & 48\mathscr{E}_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 4\mathscr{E}_2 & 16\mathscr{E}_3 & 16\mathscr{E}_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 24\mathscr{E}_5 & 6\mathscr{E}_6 & 6\mathscr{E}_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 48\mathscr{E}_5 & 48\mathscr{E}_6 & 48\mathscr{E}_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 16\mathscr{E}_5 & 16\mathscr{E}_6 & 16\mathscr{E}_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 24\mathscr{E}_8 & 6\mathscr{E}_9 & 6\mathscr{E}_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 48\mathscr{E}_8 & 48\mathscr{E}_9 & 48\mathscr{E}_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 16\mathscr{E}_8 & 16\mathscr{E}_9 & 16\mathscr{E}_{10} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24\mathscr{E}_{11} & 24\mathscr{E}_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8\mathscr{E}_{11} & 24\mathscr{E}_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24\mathscr{E}_{13} & 8\mathscr{E}_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24\mathscr{E}_{13} & 24\mathscr{E}_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16\mathscr{E}_{15} & 16\mathscr{E}_{16} & 16\mathscr{E}_{17} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48\mathscr{E}_{15} & 48\mathscr{E}_{16} & 48\mathscr{E}_{17} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6\mathscr{E}_{15} & 6\mathscr{E}_{16} & 24\mathscr{E}_{17} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16\mathscr{E}_{18} & 16\mathscr{E}_{19} & 16\mathscr{E}_{20} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48\mathscr{E}_{18} & 48\mathscr{E}_{19} & 48\mathscr{E}_{20} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6\mathscr{E}_{18} & 6\mathscr{E}_{19} & 24\mathscr{E}_{20} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16\mathscr{E}_{21} & 16\mathscr{E}_{22} & 4\mathscr{E}_{23} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 48\mathscr{E}_{21} & 48\mathscr{E}_{22} & 12\mathscr{E}_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24\mathscr{E}_{21} & 24\mathscr{E}_{22} & 24\mathscr{E}_{23} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24\mathscr{E}_{24}
\end{array}\right)$$

D.2 Kronecker basis of S_L

Recall that the Kronecker basis of S_L is defined by (2.60),

$$\mathscr{Q}^{R,\Lambda,\tau}_{\ \ K} = \frac{d_R}{|S_L|} \sum_{g \in S_L} \sum_{I,J=1}^{d_R} S^{\tau}{}^{\Lambda}_{K}{}^{R}_{I}{}^{R}_{J} \, D^R_{JI}(g) \, g^{-1} = \frac{d_R}{|S_L|} \sum_{g \in S_L} \sum_{I,J=1}^{d_R} S^{\tau}{}^{\Lambda}_{K}{}^{R}_{I}{}^{R}_{J} \, D^R_{IJ}(g) \, g \tag{D.17}$$

We solve the eigenvalue system (4.17) using the Hermite normal form to compute the integer Kronecker basis, which is related to the original Kronecker basis by

$$\mathsf{Q}^{R,\Lambda,\tau}_{\ \ K} \equiv \mathsf{N}^{R,\Lambda,\tau}_{\ \ K} \, \mathscr{Q}^{R,\Lambda,\tau}_{\ \ K} \tag{D.18}$$

where $\mathsf{N}^{R,\Lambda,\tau}_{\ \ K}$ is a normalisation constant.

We will omit the multiplicity label τ , because $C(R,R,\Lambda) \in \{0,1\}$ for $L \le 4$.

D.2.1 Case of S_2

One finds

$$\mathsf{Q}^{[2],[2]} = () + (12) \, , \qquad \mathsf{Q}^{[1,1],[2]} = () - (12), \qquad \mathsf{Q}^{R,[2]} \mathsf{Q}^{S,[2]} = 2 \, \delta^{RS} \mathsf{Q}^{R,[2]} \tag{D.19}$$

where we omit K because all representations are one-dimensional. There is no basis element with $\Lambda = [1,1]$ because $C(R,R,[1,1]) = 0$ at $L = 2$.

D.2.2 Case of S_3

We express the matrix unit $\mathcal{Q}^{R,\Lambda,\tau}_{K}$ as a pair $\{R, K\}$, where R is a Young diagram and K is a Young tableau K of shape Λ. We reorganise them into a vector

$$\begin{aligned}\mathscr{F}_\rho \equiv & \left(\mathsf{Q}^{[1,1,1],[3]}_{1}, \mathsf{Q}^{[3],[3]}_{1}, \mathsf{Q}^{[2,1],[1,1,1]}_{1}, \mathsf{Q}^{[2,1],[3]}_{1}, \mathsf{Q}^{[2,1],[2,1]}_{1}, \mathsf{Q}^{[2,1],[2,1]}_{2}\right) \\ = & \left(\left\{\ydiagram{3}, \ytableaushort{123}\right\}, \left\{\ydiagram{2,1}, \ytableaushort{123}\right\}, \left\{\ydiagram{2,1}, \ytableaushort{12,3}\right\}, \left\{\ydiagram{2,1}, \ytableaushort{13,2}\right\}, \left\{\ydiagram{2,1}, \ytableaushort{1,2,3}\right\}, \left\{\ydiagram{1,1,1}, \ytableaushort{123}\right\}\right)\end{aligned} \tag{D.20}$$

and expand it in the basis of S_3 (D.7) as

$$\mathscr{F}_\rho = \sum_{g \in \texttt{Basis}} F_{\rho g}\, g. \tag{D.21}$$

The coefficient matrix in the integer basis is given by

$$F_{\rho g} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{pmatrix}. \tag{D.22}$$

For example,

$$\begin{aligned}\mathscr{F}_1 &= \left\{\ydiagram{3}, \ytableaushort{123}\right\} = () + (23) + (12) + (123) + (132) + (13). \\ \mathscr{F}_6 &= \left\{\ydiagram{1,1,1}, \ytableaushort{123}\right\} = () - (23) - (12) + (123) + (132) - (13).\end{aligned} \tag{D.23}$$

If we define $\mathscr{F}_\rho^{-1} = \sum_{g \in \texttt{Basis}} F_{\rho g}\, g^{-1}$ and compute the δ-function inner product, we find

$$\delta\left(\mathscr{F}_\rho^{-1} \mathscr{F}_{\rho'}\right) = \mathsf{M}_\rho\, \delta_{\rho\rho'}\,, \qquad \mathsf{M}_\rho = (6, 6, 6, 2, 2, 6)\,. \tag{D.24}$$

D.2.3 Case of S_4

Again, the multiplicity label τ is trivial at $L = 4$. Define the vector of Kronecker basis as

$$\begin{aligned}\mathscr{F}_\rho \equiv \Big(& \left\{\ydiagram{4}, \ytableaushort{1234}\right\}, \left\{\ydiagram{3,1}, \ytableaushort{1234}\right\}, \left\{\ydiagram{3,1}, \ytableaushort{123,4}\right\}, \left\{\ydiagram{3,1}, \ytableaushort{124,3}\right\}, \\ & \left\{\ydiagram{3,1}, \ytableaushort{134,2}\right\}, \left\{\ydiagram{3,1}, \ytableaushort{12,34}\right\}, \left\{\ydiagram{3,1}, \ytableaushort{13,24}\right\}, \left\{\ydiagram{3,1}, \ytableaushort{12,3,4}\right\}, \left\{\ydiagram{3,1}, \ytableaushort{13,2,4}\right\}, \\ & \left\{\ydiagram{3,1}, \ytableaushort{14,2,3}\right\}, \left\{\ydiagram{2,2}, \ytableaushort{1234}\right\}, \left\{\ydiagram{2,2}, \ytableaushort{12,34}\right\}, \left\{\ydiagram{2,2}, \ytableaushort{13,24}\right\}, \left\{\ydiagram{2,2}, \ytableaushort{1,2,3,4}\right\}, \\ & \left\{\ydiagram{2,1,1}, \ytableaushort{1234}\right\}, \left\{\ydiagram{2,1,1}, \ytableaushort{123,4}\right\}, \left\{\ydiagram{2,1,1}, \ytableaushort{124,3}\right\}, \left\{\ydiagram{2,1,1}, \ytableaushort{134,2}\right\}, \left\{\ydiagram{2,1,1}, \ytableaushort{12,34}\right\}, \\ & \left\{\ydiagram{2,1,1}, \ytableaushort{13,24}\right\}, \left\{\ydiagram{2,1,1}, \ytableaushort{12,3,4}\right\}, \left\{\ydiagram{2,1,1}, \ytableaushort{13,2,4}\right\}, \left\{\ydiagram{2,1,1}, \ytableaushort{14,2,3}\right\}, \left\{\ydiagram{1,1,1,1}, \ytableaushort{1234}\right\} \Big)\end{aligned} \tag{D.25}$$

and expand it in the basis of S_4 (D.15). The coefficient matrix in the integer basis is given by

$$F_{\rho g} = \left(\begin{array}{cccccc|cccccc|cccccc|cccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 1 & 1 & 0 & 0 & 1 & 1 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 1 & 0 & -1 & -1 & -1 & 0 & 0 & 1 & -1 & -1 \\
0 & 2 & -2 & 1 & 1 & 2 & -2 & 0 & -3 & 0 & 0 & 1 & -3 & 0 & -2 & 1 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 \\
0 & 2 & 1 & 1 & 1 & -1 & -2 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -2 & 1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & -1 & 2 & 4 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & -2 & -2 & 1 & 0 & 0 & -1 & -2 & -2 \\
\hline
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & -1 & 0 & -2 & 0 & 1 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & -1 & -2 & 0 & 1 & 0 & -1 & 0 & -2 & 1 & 2 & 1 & 0 & 2 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 3 & 2 & 2 & 1 & -3 & -2 & 0 & -1 & 0 & -2 & -2 & -1 & 1 & 0 & 2 & 0 \\
2 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & 2 & 0 & 0 & -1 & -1 & 0 & 0 & 2 \\
0 & 2 & -1 & 0 & 0 & -1 & 2 & 0 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 2 & -1 & 0 & 0 & -1 & 2 & 0 \\
\hline
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
3 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 1 & 0 & 0 & -1 & 1 & -1 \\
0 & 2 & -2 & -1 & -1 & 2 & -2 & 0 & 3 & 0 & 0 & -1 & 3 & 0 & -2 & -1 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 \\
0 & 2 & 1 & -1 & -1 & -1 & -2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 2 & -1 & -1 & 0 & 0 \\
0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
\hline
0 & 2 & -1 & 0 & 0 & -1 & 2 & -4 & 0 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & 2 & -2 & 1 & 0 & 0 & -1 & -2 & 2 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & -1 & 0 & 2 & 0 & 1 & 0 & 0 & 1 & 0 & -2 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 & -1 & 0 & -1 & 0 & 2 & -1 & 2 & 1 & 0 & -2 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 3 & -2 & -2 & 1 & -3 & 2 & 0 & -1 & 0 & 2 & 2 & -1 & 1 & 0 & -2 & 0 \\
1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1
\end{array}\right) \tag{D.26}$$

Their δ-function inner product is given by $\delta(\mathscr{F}_\rho^{-1}\mathscr{F}_{\rho'}) = \mathsf{M}_\rho\,\delta_{\rho\rho'}$, where

$$\mathsf{M}_\rho = (24, 24, 48, 24, 8, 48, \quad 16, 8, 24, 48, 24, 24, \quad 8, 8, 24, 48, 24, 8, \quad 48, 16, 8, 24, 48, 24)\,. \tag{D.27}$$

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has associated code in a code repository.

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