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Abstract. We identify stable pairs and stable framed sheaves as epimorphisms and monomorphisms in the hearts of tilted t-structures under appropriate conditions. We then identify the moduli spaces with the corresponding Quot spaces. As a result, we obtain the projectivity of the Quot spaces in absolute cases. In addition, we prove a formula in a motivic Hall algebra, which relates together the Quot spaces under a tilt.

1. Introduction

Decorated sheaves are sheaves with additional structures. The most famous moduli space of decorated sheaves is Grothendieck's Quot scheme. However, quite often, the Quot scheme is oversized. To bypass this issue, variants of the Quot scheme are more suitable for some problems. One such variant is the moduli space of stable pairs. It was used by Thaddeus [20] to calculate the Verlinde numbers and by Pandharipande and Thomas [17] to study curve counting on Calabi–Yau 3-folds. For a more recent application towards strange duality, see [6, 7]. Here, we focus on two variants of quotient sheaves: stable pairs and framed sheaves.

Let X be a nonsingular projective variety with a fixed polarization over an algebraically closed field of characteristic 0. Let E_0 be a fixed coherent sheaf on X. Despite the overuse of the term, we call a sheaf equipped with a morphism

$$(E, \alpha\colon E_0 \to E)$$

a *pair*. A family of stability conditions (Definition 4.1) is defined for pairs. When the stability polynomial is large, a pair (E, α) is stable if and only if α is generically surjective. In this case, we say that the pair is *limit stable*. Limit

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stable pairs are also known as quotient husks in Kollár's work [13, Sec. 9] where he studied them in a relative setting. We will identify stable pairs with quotients in the heart of a certain tilt of the standard t-structure (Propositions 3.13 & 4.6).

For a fixed non-negative integer m, we define a torsion pair $(\mathcal{T}, \mathcal{F})$ on $\mathrm{Coh}(X)$ (see Lemma 3.1):

$$\mathcal{T} = \{E \in \mathrm{Coh}(X) \mid \dim E \leq m\},$$
$$\mathcal{F} = \{F \in \mathrm{Coh}(X) \mid \mathrm{Hom}(E, F) = 0, \ \forall E \in \mathcal{T}\}.$$

Let $\mathrm{Coh}^{\#}(X)$ denote the heart of the tilted t-structure (see (3)) with respect to the torsion pair above. Suppose $E_0 \in \mathcal{F}$, namely E_0 has no subsheaves of dimension $\leq m$. Let P be a polynomial of degree $m+1$. The moduli space of quotients of E_0 in $\mathrm{Coh}^{\#}(X)$ with Hilbert polynomial P is denoted as $\mathrm{Quot}^{\#}(E_0, P)$ and is called the *Quot space* [2, Defn. 11.2, Prop. 11.6]. We will prove the following result.

Theorem 1.1 (Theorem 3.8)**.** *The Quot space* $\mathrm{Quot}^{\#}(E_0, P)$ *is projective.*

There is a notion that is dual to pairs: *framed sheaves* [11]. Again, let E_0 be a fixed coherent sheaf. A framed sheaf is defined as a coherent sheaf E together with a map α:

$$(E, \alpha \colon E \to E_0).$$

There is also a family of stability conditions (Definition 4.7) on framed sheaves. We will identify stable framed sheaves with monomorphisms in a certain tilted heart (Proposition 4.10) and prove the projectivity of the corresponding Quot space (Theorem 4.9).

When X is a Calabu–Yau 3-fold, $E_0 \cong \mathcal{O}_X$, and E has dimension 1, the equivalence between stable pairs and quotients in a tilted heart has been obtained by Bridgeland [4] and used to derive a wall-crossing formula between the Donaldson–Thomas (DT) invariants and Pandharipande–Thomas (PT) invariants. As an immediate application of our results, we derive a formula in a motivic Hall algebra, which relates together the classes of the moduli space of limit stable pairs and the Quot scheme under certain assumptions (Theorem 5.6).

Over a nonsingular curve, stable pairs of the form $(E, \alpha \colon \mathcal{O}_X \to E)$ have been identified as epimorphisms in the corresponding tilted heart in [19]. A variant of the result on framed sheaves has also been obtained there.

Finally, we discuss another potential application. Suppose that we have a family of projective varieties parametrized by a smooth curve where the generic fiber is smooth and the special fiber is singular. Then we have a relative Quot space. If we can define an invariant on the Quot space over the generic fiber, then we can define an invariant over the singular fiber, by using the specialization map from the Quot space over the generic fiber to the one over the special fiber. Sometimes, we can expect that the relative Quot space is projective over the base curve (see [15, Rmk. 4.6]). If this is the case, it is easier to define the invariants.

We organize the paper as follows. In Section 2, we review some basic notions about t-structures. In Section 3, we compare quotient husks and quotients in a tilted heart and prove Theorem 1.1. In Section 4, we identify stable pairs as epimorphisms and framed sheaves as monomorphisms in differently tilted hearts. We also prove the projectivity of the corresponding Quot spaces. In Section 5, we review the formalism of motivic Hall algebras and derive the formula, which relates together the Quot spaces under a tilt.

2. *t*-structures and torsion pairs

In this section, we review the basic notions of t-structures, hearts, torsion pairs, and tilts, and prove an observation important for us.

Definition 2.1. A *t-structure* τ on a triangulated category $\mathcal{D}$ is a pair of strictly full subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ satisfying the following conditions:

(i) $\mathcal{D}^{\leq 0}[1] \subset \mathcal{D}^{\leq 0}$ and $\mathcal{D}^{\geq 0}[-1] \subset \mathcal{D}^{\geq 0}$;

(ii) $\mathrm{Hom}(F, G) = 0$ for every $F \in \mathcal{D}^{\leq 0}$ and $G \in \mathcal{D}^{\geq 1}$;

(iii) for every object $E \in \mathcal{D}$, there exists an exact triangle

$$\tau^{\leq 0}E \to E \to \tau^{\geq 1}E \to (\tau^{\leq 0}E)[1]$$

with $\tau^{\leq 0}E \in \mathcal{D}^{\leq 0}$ and $\tau^{\geq 1}E \in \mathcal{D}^{\geq 1}$.

Moreover, the t-structure is *bounded* if $\mathcal{D} = \cup_{n,m\in\mathbb{Z}} \mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq m}$.

Here, for $n \in \mathbb{Z}$, $\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n]$ and $\mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]$. Moreover, we let

$$\mathcal{D}^{[a,b]} := \mathcal{D}^{\geq a} \cap \mathcal{D}^{\leq b}. \tag{1}$$

The truncation functors $\tau^{\leq n}$ and $\tau^{\geq n}$ are defined as follows:

$$\tau^{\leq n}E = \tau^{\leq 0}(E[n])[-n] \text{ and } \tau^{\geq n}E = \tau^{\geq 0}(E[n])[-n].$$

The full subcategory $\mathcal{A} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is called the *heart* of the t-structure, which is an abelian category. The cohomology objects of an object $E \in \mathcal{D}$ with respect to the heart $\mathcal{A}$ are

$$\mathrm{H}^n_{\mathcal{A}}(E) := (\tau^{\leq n}\tau^{\geq n}E)[n].$$

If $\mathcal{A}$ is the heart of coherent sheaves, we simply write $\mathrm{H}^n(E)$.

Definition 2.2. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be a pair of triangulated categories equipped with t-structures. An exact functor $\Phi\colon \mathcal{D}_1 \to \mathcal{D}_2$ is *left* (resp. *right*) *t-exact* if $\Phi(\mathcal{D}_1^{\geq 0}) \subset \mathcal{D}_2^{\geq 0}$ (resp. $\Phi(\mathcal{D}_1^{\leq 0}) \subset \mathcal{D}_2^{\leq 0}$). The exact functor is *t-exact* if it is both left and right t-exact.

Definition 2.3. Let $\mathcal{A}$ be an abelian category. A pair of additive subcategories $(\mathcal{T}, \mathcal{F})$ of $\mathcal{A}$ is a *torsion pair* if $\mathrm{Hom}(\mathcal{T}, \mathcal{F}) = 0$ and every object $E \in \mathcal{A}$ fits into an exact sequence

$$0 \to T \to E \to F \to 0 \tag{2}$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

The subcategories $\mathcal{T}$ and $\mathcal{F}$ are closed under extensions. Moreover, $\mathcal{T}$ and $\mathcal{F}$ are respectively closed under taking quotients and sub-objects. Note that if $(\mathcal{T},\mathcal{F})$ is a torsion pair, then $\mathcal{F}=\mathcal{T}^{\perp}$ is the right orthogonal to $\mathcal{T}$ in $\mathcal{A}$.

Example 2.4. Let X be a Noetherian scheme of dimension n. For any integer $0 \le d < n$, we consider the full subcategory

$$\mathcal{T} = \mathrm{Coh}_{\le d}(X) := \{E \in \mathrm{Coh}(X) \mid \dim E \le d\}.$$

Here and henceforward, we use $\dim E$ to denote the dimension of the support of the sheaf E. (When $d=0$, we simply write $\mathrm{Coh}_0(X)$.) Since $\mathcal{T}$ is closed under extension, $(\mathcal{T},\mathcal{T}^{\perp})$ is a torsion pair in the abelian category $\mathrm{Coh}(X)$.

Let $\mathcal{A} \subset \mathcal{D}$ be the heart of a t-structure and $(\mathcal{T},\mathcal{F})$ a torsion pair in $\mathcal{A}$. We can *tilt* the t-structure on $\mathcal{D}$ to obtain a new t-structure $(\mathcal{D}^{\#,\le 0},\mathcal{D}^{\#,\ge 0})$ on $\mathcal{D}$ by setting

$$\begin{aligned} \mathcal{D}^{\#,\le 0} &:= \{E \in \mathcal{D}^{\le 1} \mid \mathrm{H}^1_{\mathcal{A}}(E) \in \mathcal{T}\} \text{ and} \\ \mathcal{D}^{\#,\ge 0} &:= \{E \in \mathcal{D}^{\ge 0} \mid \mathrm{H}^0_{\mathcal{A}}(E) \in \mathcal{F}\}. \end{aligned} \tag{3}$$

The heart of the resulting t-structure is the extension closure $\mathcal{A}^{\#}=\langle \mathcal{F},\mathcal{T}[-1]\rangle$. Moreover, the shift $\mathcal{A}^{\#}[1]$ can be described as

$$\mathcal{A}^{\#}[1] = \left\{E \in \mathcal{D} \mid \mathrm{H}^0_{\mathcal{A}}(E) \in \mathcal{T},\ \mathrm{H}^{-1}_{\mathcal{A}}(E) \in \mathcal{F},\ \mathrm{H}^i_{\mathcal{A}}(E)=0 \text{ for } i \ne 0,-1\right\}.$$

We refer to [9] for details about tilting.

The following observation is important for us.

Lemma 2.5. *Let $\mathcal{A}$ be the heart of a bounded t-structure on a triangulated category $\mathcal{D}$ and $(\mathcal{T},\mathcal{F})$ a torsion pair of $\mathcal{A}$. Let $\mathcal{A}^{\#}$ be the corresponding tilted heart.*

(i) *If $E_0 \in \mathcal{F}$, then a morphism $\alpha\colon E_0 \to E$ in $\mathcal{A}^{\#}$ is an epimorphism if and only if α is a morphism in $\mathcal{A}$ with $E \in \mathcal{F}$ and the cokernel* coker (α) *taken in $\mathcal{A}$ lies in $\mathcal{T}$.*

(ii) *If $E_0 \in \mathcal{T}$, then a morphism $\beta\colon E \to E_0$ in $\mathcal{A}^{\#}[1]$ is a monomorphism if and only if β is a morphism in $\mathcal{A}$ with $E \in \mathcal{T}$ and the kernel* $\ker(\beta)$ *taken in $\mathcal{A}$ lies in $\mathcal{F}$.*

Proof. We only show (ii), as the proof of (i) is similar (cf. [4, Lem. 2.3]). Given $E_0 \in \mathcal{T}$ and a short exact sequence in $\mathcal{A}^{\#}[1]$

$$0 \to E \xrightarrow{\beta} E_0 \to Q \to 0,$$

we have an exact sequence

$$0 \to \mathrm{H}^{-1}_{\mathcal{A}}(E) \to 0 \to \mathrm{H}^{-1}_{\mathcal{A}}(Q) \to \mathrm{H}^0_{\mathcal{A}}(E) \xrightarrow{\beta} E_0 \to \mathrm{H}^0_{\mathcal{A}}(Q) \to 0 \tag{4}$$

by taking cohomology with respect to the original t-structure. Note that an object $F \in \mathcal{D}$ lies in $\mathcal{A}^{\#}[1] \subset \mathcal{D}$ precisely when $\mathrm{H}^{-1}_{\mathcal{A}}(F) \in \mathcal{F}$, $\mathrm{H}^0_{\mathcal{A}}(F) \in \mathcal{T}$ and

$\mathrm{H}^i_{\mathcal{A}}(F) = 0$ for $i \neq 0, -1$. It follows that $\mathrm{H}^i_{\mathcal{A}}(E) = 0$ for all $i \neq 0$, and thus $E \in \mathcal{A} \cap \mathcal{A}^{\#}[1] = \mathcal{T}$. Moreover, $\ker(\beta) = \mathrm{H}^{-1}_{\mathcal{A}}(Q) \in \mathcal{F}$.

For the converse, take a morphism $\beta\colon E \to E_0$ in $\mathcal{A}$ with $E \in \mathcal{T}$, $\ker(\beta) \in \mathcal{F}$ and embed it in a distinguished triangle

$$E \xrightarrow{\beta} E_0 \to Q \to E[1]. \tag{5}$$

Since $\mathcal{T}$ is closed under taking quotients and $E_0 \in \mathcal{T}$, the same long exact sequence (4) then shows that $\mathrm{H}^0_{\mathcal{A}}(Q) \in \mathcal{T}$, $\mathrm{H}^{-1}_{\mathcal{A}}(Q) = \ker(\beta) \in \mathcal{F}$ and $\mathrm{H}^i_{\mathcal{A}}(Q) = 0$ for $i \neq 0, -1$. Thus $Q \in \mathcal{A}^{\#}[1]$. It follows that (5) defines a short exact sequence in $\mathcal{A}^{\#}[1]$, and hence β is a monomorphism in $\mathcal{A}^{\#}[1]$. □

3. Quot spaces and moduli of quotient husks

We construct a family of t-structures on the fibers of a projective morphism, and compare the Quot space in the sense of [2, Sec. 11] with the moduli space of quotient husks constructed by Kollár [13, Sec. 9].

Over a field k, let S be a k-scheme of finite type and

$$f\colon X \to S$$

be a flat projective morphism with a fixed relatively ample line bundle. Let m be a non-negative integer.

3.1. A family of tilted *t*-structures

On each fiber X_s for $s \in S$, let

$$\mathcal{T}_s = \{E \in \mathrm{Coh}(X_s) \mid \dim E \leq m\},$$
$$\mathcal{F}_s = \{F \in \mathrm{Coh}(X_s) \mid \mathrm{Hom}(E, F) = 0, \forall E \in \mathcal{T}_s\}.$$

Notice that $\mathcal{F}_s = \mathcal{T}_s^{\perp}$. Then, $(\mathcal{T}_s, \mathcal{F}_s)$ defines a torsion pair of $\mathrm{Coh}(X_s)$ (see Example 2.4). If we denote the standard t-structure on $\mathrm{D^b}(X_s)$ as $(D_s^{\leq 0}, D_s^{\geq 0})$, the tilted t-structure is defined by setting

$$\begin{aligned} D_s^{\#,\leq 0} &= \{E \in D_s^{\leq 1} \mid \mathrm{H}^1(E) \in \mathcal{T}_s\} \text{ and} \\ D_s^{\#,\geq 0} &= \{E \in D_s^{\geq 0} \mid \mathrm{H}^0(E) \in \mathcal{F}_s\}. \end{aligned} \tag{6}$$

The heart of the t-structure is $\mathcal{A}_s^{\#} = D_s^{\#,\leq 0} \cap D_s^{\#,\geq 0}$.

We show that the family of t-structures defined above integrates over S (see Lemma 3.3). Namely, they are pullbacks of a t-structure on X (see [2, Defn. 10.10]).

We have two full subcategories of $\mathrm{Coh}(X)$:

$$\mathcal{T} = \{E \in \mathrm{Coh}(X) \mid \forall s \in S,\ \dim E_s \leq m\},$$
$$\mathcal{F} = \{F \in \mathrm{Coh}(X) \mid \mathrm{Hom}(E, F) = 0,\ \forall E \in \mathcal{T}\}.$$

Here, E_s is the derived pullback of E to the fiber X_s. The pair $(\mathcal{T}, \mathcal{F})$ is the relative version of the one in the introduction. We say objects in $\mathcal{T}$ have *relative dimension* $\leq m$ *over* S for brevity.

Lemma 3.1. *For a fixed non-negative integer m, $(\mathcal{T}, \mathcal{F})$ as defined above is a torsion pair of* $\mathrm{Coh}(X)$.

The proof is similar to the absolute case.

Proof. For any coherent sheaf E, the sum of subsheaves of dimension $\leq m$ has dimension $\leq m$. We can take its maximal subsheaf E' whose support has a relative dimension $\leq m$ over S. Then the quotient E/E' belongs to $\mathcal{F}$. □

The torsion pair $(\mathcal{T}, \mathcal{F})$ induces the following tilted t-structure on $\mathrm{D^b}(X)$:

$$\begin{aligned} D^{\#,\leq 0} &= \{E \in D^{\leq 1} \mid \mathrm{H}^1(E) \in \mathcal{T}\} \text{ and} \\ D^{\#,\geq 0} &= \{E \in D^{\geq 0} \mid \mathrm{H}^0(E) \in \mathcal{F}\}, \end{aligned} \tag{7}$$

where $(D^{\leq 0}, D^{\geq 0})$ is the standard t-structure on $\mathrm{D^b}(X)$.

We next recall the locality of t-structures following [18, p.119].

Definition 3.2. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on $\mathrm{D^b}(X)$ is S-*local* if, for every open subset $U \subset S$, there is a t-structure on $\mathrm{D^b}(f^{-1}(U))$ such that the restriction functor $\mathrm{D^b}(X) \to \mathrm{D^b}(f^{-1}(U))$ is t-exact.

According to [18, Thm. 2.3.2], the t-structure (7) is S-local.

Convention. Given a morphism $g\colon Y \to Z$ of schemes, we use g_* to denote the derived pushforward functor and g^* the derived pullback functor.

Let $\phi\colon T \to S$ be a morphism and consider the cartesian square

$$\begin{array}{ccc} X_T := T \times_S X & \xrightarrow{\phi'} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ T & \xrightarrow{\phi} & S. \end{array} \tag{8}$$

If the canonical morphism of functors $\phi^* f_* \to f'_* \phi'^*$ is an isomorphism, then the cartesian square (8) is *exact* and the morphism ϕ is *faithful* with respect to $f\colon X \to S$. Since we assume f to be flat, (8) is exact and ϕ is faithful with respect to f, according to [14, Cor. 2.23]. By [2, Thm. 5.7], there is a canonical way to pullback the t-structure (7) to $\mathrm{D^b}(X_T)$ for an arbitrary morphism ϕ. In particular, we have the following. Let $\phi\colon T = \{s\} \to S$ be the embedding of a closed point and denote the pullback t-structure on the fiber $X_T = X_s$ as

$$((D^{\#})_s^{\leq 0}, (D^{\#})_s^{\geq 0}). \tag{9}$$

Then, for $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ and $a \leq b$,

$$(D^{\#})_s^{[a,b]} = \left\{F \in \mathrm{D^b}(X_s) \mid \phi'_* F \in (D^{\#})^{[a,b]}\right\},$$

where the subcategory $(D^{\#})^{[a,b]}$ is defined by the tilted t-structure (7) as in (1). We have to refer the reader to [2, Sec. 5] for the general construction of the pullback t-structure, which is rather involved.

Lemma 3.3. *Suppose s is a (possibly non-closed) point of S. The pullback t-structures* (9) *agree with those* (6) *defined fiberwise, namely,* $(D^{\#})_s^{\leq 0} = D_s^{\#,\leq 0}$ *and* $(D^{\#})_s^{\geq 0} = D_s^{\#,\geq 0}$.

Proof. If s is a closed point, we can use [2, Thm. 5.7 (3)]. Let $T = \{s\}$ and ϕ be the closed embedding of $\{s\}$ into S. Take $a = -\infty$, $b = 0$, then $D_s^{(-\infty,0]} = (D^{\#})_s^{\leq 0}$ and $D^{(-\infty,0]} = D^{\#,\leq 0}$. Let $i_s \colon X_s \to X$ be the inclusion of the fiber X_s in X. We have

$$\begin{aligned}(D^{\#})_s^{\leq 0} &= \{F \in \mathrm{D^b}(X_s) \mid i_{s*}F \in D^{\#,\leq 0}\}\\ &= \{F \in \mathrm{D^b}(X_s) \mid F \in D^{\leq 1}, \mathrm{H}^1(F) \in \mathcal{T}\} = D_s^{\#,\leq 0}.\end{aligned}$$

The other equality can be shown similarly.

For a non-closed point s, we can still apply [2, Theorems 5.3, 5.6, 5.7] and obtain the result. □

Remark 3.4. Suppose $X = S \times Y \to S$ is a trivial family with fiber Y over an algebraic closed field k, the t-structure (7) may *not* be the pullback of the one on $\mathrm{D^b}(Y)$, which is similar to (6). Namely, it is not constant in the sense of [1, 18], as illustrated in the following example.

Example 3.5. Let $Y = \mathbb{P}^1$ and $S = \mathbb{P}^1$. Let $\Delta \cong \mathbb{P}^1 \subset X = S \times Y$ be the diagonal. Then $\mathcal{O}_\Delta[-1]$ lies in the heart $D^{\#,\leq 0} \cap D^{\#,\geq 0}$. However, it does not lie in the heart of the pullback of the corresponding t-structure, according to [18, Lem. 3.3.2].

3.2. Quotients and quotient husks

Let $F_0 \in \mathcal{F} \subset \mathrm{Coh}(X)$ be flat over S such that the restriction

$$F_{0s} \in \mathcal{F}_s.$$

Let T be an S-scheme and F_{0T} denote the pullback of F_0 to X_T.

Definition 3.6. A *family of epimorphisms* in the tilted hearts of the family of t-structures (6), parameterized by T, is a morphism $F_{0T} \to F$ in $\mathrm{D^b}(X_T)$ satisfying the following conditions:

(a) F is flat over T, i.e. the derived pullback F_t of F to a fiber X_t lies in the tilted heart $\mathrm{Coh}^{\#}(X_t)$ of the induced t-structure on $\mathrm{D^b}(X_t)$;
(b) The morphism $F_{0t} \to F_t$ in $\mathrm{Coh}^{\#}(X_t)$ is an epimorphism for all $t \in T$.

Remark 3.7. In this setting, F is automatically a sheaf because its derived pullback F_t to each fiber is a quotient of F_{0t} in $\mathrm{Coh}^{\#}(X_t)$. By Lemma 2.5, it must be a sheaf in $\mathcal{F}_t$.

Let

$$\underline{\mathrm{Quot}}^{\#}_f(F_0, P) \colon (S\text{-Schemes})^{\mathrm{op}} \to \mathrm{Sets}$$

be the functor that sends an S-scheme T to the set of families of epimorphisms in the tilted hearts $\mathrm{Coh}^{\#}(X_t)$ parametrized by T where the quotients have

Hilbert polynomial P. If the family of t-structures universally satisfies openness of flatness, a property which we will define and show in a moment, then the functor $\underline{\mathrm{Quot}}^{\#}_f(F_0, P)$ is represented by an algebraic space $\mathrm{Quot}^{\#}_f(F_0, P)$ locally of finite presentation over S [2, Prop. 11.6].

We show the following at the end of the section.

Theorem 3.8. *Let k be an algebraic closed field of characteristic 0 and $f\colon X \to S = \operatorname{Spec} k$ be a nonsingular projective scheme. Let P be a polynomial of degree $m+1$. Then, the Quot space $\mathrm{Quot}^{\#}_f(E_0, P)$ is projective.*

We say that the family (9) of t-structures *universally satisfies openness of flatness* (cf. [2, Defn. 10.4 & Rmk. 10.9]) if for every $T \to S$ and every T-perfect complex $E \in \mathrm{D}(X_T)$, the set

$$\{t \in T \mid E_t \in \mathrm{Coh}^{\#}(X_t)\}$$

is open. For the definition of T-perfectness, see [2, Defn. 8.1].

Proposition 3.9. *The family* (9) *universally satisfies openness of flatness.*

Proof. The proposition can be reduced to the case where T is an affine Noetherian scheme [2, Lem. 10.7]. Suppose we are given a T-perfect complex $E \in \mathrm{D}(X_T)$. First, notice that $E \in \mathrm{D}^{\mathrm{b}}(X_T)$ [2, Lem. 8.3]. So, we assume that E is a finite complex of coherent sheaves. For a point $t \in T$, we have the following spectral sequence

$$E_2^{p,q} = \mathcal{T}or_{-p}^{\mathcal{O}_T}(\mathrm{H}^q(E), \kappa(t)) = \mathrm{H}^p(\mathrm{H}^q(E) \overset{\mathrm{L}}{\otimes} \kappa(t)) \Rightarrow \mathrm{H}^{p+q}(E_t),$$

see e.g. [10, (3.10), p.81]. Here, we use H^i to denote the cohomology sheaf of a complex. Notice that $E_2^{p,q} = 0$ unless $p \le 0$. Suppose there is a point $t \in T$ such that the derived restriction E_t lies in $\mathrm{Coh}^{\#}(X_t)$. In particular, $\mathrm{H}^i(E_t) = 0$ unless $i = 0, 1$. Let n be the largest integer such that $\mathrm{H}^n(E) \neq 0$. We assume $n > 1$. Then $\mathrm{H}^n(E_t) = 0$ implies that $\mathrm{H}^n(E) \otimes \kappa(t) = 0$. Thus, there is a neighborhood $U \subset T$ of t such that $\mathrm{H}^n(E)|_{X_U} = 0$. Inductively, we can shrink U if necessary so that $\mathrm{H}^i(E)|_{X_U} = 0$ for all $i > 1$. (We take the liberty to shrink the neighborhood U whenever necessary, without mentioning.) So, we can assume $E_2^{p,q} = 0$ for $q \ge 2$ over U. The assumption that E_t lies in the tilted heart implies that $\mathrm{H}^1(E_t) \in \mathcal{T}_t$, that is, it has dimension $\le m$. Thus, $\mathrm{H}^1(E) \otimes \kappa(t) \in \mathcal{T}_t$, which in turn implies that $\mathcal{T}or_1(\mathrm{H}^1(E), \kappa(t)) \in \mathcal{T}_t$. This last term is a subsheaf of $\mathrm{H}^0(E_t) \in \mathcal{F}_t$, therefore it has to be zero. Hence, $\mathrm{H}^1(E)$ is flat near t. Since $\mathrm{H}^{-1}(E_t) = 0$, $\mathrm{H}^{-1}(E) \otimes \kappa(t) = 0$ and $\mathcal{T}or_1(\mathrm{H}^0(E), \kappa(t)) = 0$, which implies $\mathrm{H}^0(E)$ is flat near t. Therefore, for every $u \in U$, $\mathrm{H}^0(E_u) \cong \mathrm{H}^0(E) \otimes \kappa(u)$, and since $\mathrm{H}^0(E_t)$ has no subsheaves of dimension $\le m$, neither does $\mathrm{H}^0(E_u)$. Hence, $\mathrm{H}^0(E_u)$ lies in $\mathcal{F}_u$. On the other hand, we can inductively prove that for all $i < 0$ and $u \in U$, $\mathrm{H}^i(E) \otimes \kappa(u) = 0$. From the spectral sequence, we have proven that for every $u \in U$, $E_u \in \mathrm{Coh}^{\#}(X_u)$. The family of t-structures universally satisfies openness of flatness. □

We next recall the notion of a *quotient husk* [13, Definitions 9.33 & 9.39].

Definition 3.10. A *husk* of a coherent sheaf F on a normal scheme Y is a homomorphism $q\colon F \to E$ such that

(a) q is an isomorphism on all n-dimensional points, where $n = \dim F$;
(b) E is pure of dimension n.

A *quotient husk* of a fixed coherent sheaf E_0 is a homomorphism $q\colon E_0 \to E$ such that it factors as $E_0 \to F \to E$ where the first arrow is an epimorphism and the second arrow is a husk.

Definition 3.11. Let $f\colon X \to S$ be a morphism of schemes, and E_0 be a coherent sheaf on X. A *quotient husk* of E_0 over S is a coherent sheaf together with a morphism $q\colon E_0 \to E$ such that

(a) E is pure and flat over S;
(b) the homomorphism $q_s\colon E_{0s} \to E_s$ is a quotient husk on each fiber X_s.

Here, we say a sheaf E on X is *pure over S* if for every $s \in S$, the restriction E_s is pure of the same dimension.

We view a quotient husk over S as a family of quotient husks parametrized by S. Let

$$\underline{\mathrm{QHusk}}_f(E_0, P)\colon (S\text{-Schemes})^{\mathrm{op}} \to \mathrm{Sets}$$

denote the moduli functor of quotient husks where $E_0 \in \mathrm{Coh}(X)$ and P is a fixed polynomial with rational coefficients. It sends an S-scheme T to the set of families of quotient husks $f_T^* E_0 \to E_T$ on X_T such that when restricted to each fiber X_t for $t \in T$, the Hilbert polynomial of E_t is P. Then, we have the following existence theorem on the moduli space of quotient husks [13, Thm. 9.42]:

Theorem 3.12 ([13]). *The moduli functor $\underline{\mathrm{QHusk}}_f(E_0, P)$ is represented by an algebraic space $\mathrm{QHusk}_f(E_0, P)$, which is proper and separated over S.*

The following proposition shows that a family of quotient husks is equivalent to a family of epimorphisms with respect to the family of t-structures (6).

Proposition 3.13. *Assume E_0, E are two coherent sheaves flat over S, and the restriction E_s has $(m+1)$-dimensional support, then a homomorphism $q\colon E_0 \to E$ of sheaves is a family of quotient husks if and only if it is a family of quotients with respect to the family of t-structures* (6).

Proof. Suppose $q\colon E_0 \to E$ is a family of quotient husks, then for every point $s \in S$, the restriction $q_s\colon E_{0s} \to E_s$ is a quotient husk on X_s. In particular, E_s is a pure sheaf with $(m+1)$-dimensional support. This implies E_s is contained in $\mathcal{F}_s$. The fact that q_s is surjective at all $(m+1)$-dimensional points implies its cokernel is supported in a locus of dimension m or less. Together with Lemma 2.5, this shows that q_s is an epimorphism in $\mathrm{Coh}^{\#}(X_s)$.

Conversely, if q is a family of quotients, then on each fiber, the homomorphism q_s factors as $E_{0s} \to \mathrm{Im}(q_s) \to E_s$, where the first arrow is an epimorphism. By Lemma 2.5, the cokernel of q_s is in $\mathcal{T}_s$ so it is supported in a locus of dimension $\leq m$. Therefore, the second arrow above has to be an isomorphism at all $(m+1)$-dimensional points. Also, according to Lemma 2.5, E_s is contained in $\mathcal{F}_s$ and does not contain any subsheaf supported on a locus of dimension m or less. This implies E_s is pure and E is pure over S. This finishes the proof that q_s is a quotient husk. □

Let $E_0 = F_0$, which is defined at the beginning of the subsection, and P be a polynomial of degree $m+1$. In particular, $E_0 \in \mathcal{F}$ is flat over S such that $E_{0s} \in \mathcal{F}_s$. Then, the functor $\underline{\mathrm{Quot}}^{\#}_f(E_0, P)$ is the same as the functor $\underline{\mathrm{QHusk}}_f(E_0, P)$, that is, we have the following proposition.

Proposition 3.14. *Given an S-scheme T,*

$$\underline{\mathrm{Quot}}^{\#}_f(E_0, P)(T) = \underline{\mathrm{QHusk}}_f(E_0, P)(T).$$

Proof. Suppose we are given an element $\alpha_T : E_{0T} \to E$ in $\mathrm{Quot}^{\#}_f(E_0, P)(T)$. For $t \in T$, the restriction $\alpha_t : E_{0t} \to E_t$ lies in the heart $\mathrm{Coh}^{\#}(X_t)$ of the pullback t-structure. By Lemma 2.5, $E_t \in \mathrm{Coh}(X_t)$. Thus, E is a coherent sheaf flat over T. On the other hand, α_t has cokernel (taken in $\mathrm{Coh}(X_t)$) in $\mathcal{T}_t$. Therefore, α_T is a family of quotient husks.

Given a family of quotient husks $\alpha_T : E_{0T} \to E$, the restriction α_t is a quotient husk. By Proposition 3.13 and Lemma 2.5, E is flat over T and α_t is a quotient in $\mathrm{Coh}^{\#}(X_t)$. □

Now, we provide the proof of Theorem 3.8.

Proof of Theorem 3.8. By identifying the moduli functors, we know the Quot space is isomorphic to the moduli space of quotient husks:

$$\mathrm{Quot}^{\#}_f(E_0, P) \cong \mathrm{QHusk}_f(E_0, P).$$

Furthermore, in this set-up, $\mathrm{QHusk}_f(E_0, P)$ is isomorphic to the moduli space of limit stable pairs $(E, \alpha : E_0 \to E)$ [15, Lem. 2.10], whose projectivity is obtained via a geometric invariant theoretic construction [15, Thm. 1.1]. □

Remark 3.15. Quotient husks are also known as limit stable pairs (see next section). By carrying out a GIT construction [15, Rmk. 4.6], one would be able to obtain the projectivity of the Quot space over a general base S in characteristic 0.

Remark 3.16. Let $R = \mathbb{C}[t]$ and $C = \operatorname{Spec} R$. Let η be its generic point and 0 its closed point. Let X_η be the generic fiber, and X_0 be the central fiber. It can be shown that derived categories and hearts over the subscheme X_η are equivalent to quotients of the corresponding categories over the total space X. However, because flat limits are not unique, we do not expect a specialization functor. But the valuative criteria [13, 9.31] of separatedness and properness of the moduli space of quotient husks provide a specialization map on the level of moduli spaces:

$$\mathrm{QHusk}^{X_\eta}(\iota_\eta^* E_0) \to \mathrm{QHusk}^{X_0}(\iota_0^* E_0).$$

4. Stable pairs and framed sheaves

For stable pairs with respect to a smaller stability condition, we can also identify them as quotients in the heart of a tilted t-structure. On the other hand, we study framed sheaves, which can be identified as monomorphisms in a tilted heart.

Over an algebraically closed field k of characteristic 0, let X be a nonsingular projective variety of dimension n with a fixed polarization $\mathcal{O}_X(1)$. Let $E_0 \in \mathrm{Coh}(X)$ be fixed. Let $P \in \mathbb{Q}[m]$ be a fixed polynomial of degree d, which is used as a Hilbert polynomial.

Given a coherent sheaf E on $(X, \mathcal{O}_X(1))$, we denote its Hilbert polynomial by P_E, its multiplicity by r_E and its reduced Hilbert polynomial by $p_E = P_E/r_E$.

4.1. Stable pairs

Let $\delta \in \mathbb{Q}[m]$ be a polynomial with a positive leading coefficient. We consider homomorphisms of the form $E_0 \to E$.

Definition 4.1. A *pair*

$$(E, \alpha\colon E_0 \to E)$$

with $\alpha \neq 0$ is *δ-stable* if E is pure and for every subsheaf $F \subset E$,

(i) $p_F + \delta/r_F < p_E + \delta/r_E$ if $\operatorname{im}\alpha \subset F$,
(ii) $p_F < p_E + \delta/r_E$ otherwise.

We can replace the strong inequalities by weak inequalities to define δ-semistability. Stability can be equivalently defined in terms of quotients.

Two δ-semistable pairs (E, α) and (E', α) are *isomorphic* if there is an isomorphism $\phi\colon E \to E'$ such that $\alpha' = \phi \circ \alpha$. Let

$$\underline{S}_{E_0}(P, \delta)\colon (k\text{-Schemes})^{\mathrm{op}} \to \mathrm{Sets}$$

denote the moduli functor of isomorphism classes of δ-semistable pairs. We have the following existence result [15, Thm. 1.1].

Theorem 4.2 ([15])**.** *There is a projective coarse moduli space $S_{E_0}(P, \delta)$ of S-equivalence classes of δ-semistable pairs with Hilbert polynomial P. It contains an open subscheme $S^{\mathrm{s}}_{E_0}(P, \delta)$ as the fine moduli space of δ-stable pairs.*

Here, S-equivalence is similar to the one in the theory of sheaves, see [15, p.132].

Definition 4.3. When $\deg \delta \geq \dim E$, a δ-stable pair is called a *limit stable pair.*

This is the same as a quotient husk in the absolute setting [15, Lem. 2.10]. We next consider δ-stable pairs (E,α) with $P_E = P$ for a small δ:

$$\deg\delta < \deg P = d.$$

Let $r = r_E$ and

$$\lambda = \frac{P+\delta}{r}.$$

We define two full subcategories of $\mathrm{Coh}(X)$:

$$\mathcal{T}^\lambda = \left\{ E\in\mathrm{Coh}(X) \;\middle|\; \begin{array}{l} \dim E\le d \text{ and} \\ \forall \text{ quotient sheaf } E\twoheadrightarrow G, \\ \dim G<d \text{ or } \ p_G>\lambda \end{array} \right\} \quad \text{and} \tag{10}$$

$$\mathcal{F}^\lambda = \left\{ E\in\mathrm{Coh}(X) \;\middle|\; \begin{array}{l} \forall \text{ nontrivial subsheaf } F\subset E \\ \text{with } \dim F\le d, \\ F \text{ is pure of dim. } d \text{ and } \ p_F\le\lambda \end{array} \right\}. \tag{11}$$

Lemma 4.4. *The pair $(\mathcal{T}^\lambda,\mathcal{F}^\lambda)$ forms a torsion pair of* $\mathrm{Coh}(X)$.

Proof. Clearly, $\mathcal{T}^\lambda$ and $\mathcal{F}^\lambda$ contain 0, and their hom-sets are abelian groups such that the compositions are bilinear. To show that they are additive categories, it is enough to show they are closed under extensions and hence admit finite coproducts.

Suppose E is an extension of E'' by E'. Note that the inequality about dimension is preserved under extensions. For $E'',E'\in\mathcal{T}^\lambda$, let G be a quotient sheaf of E such that $\dim G\ge d$. We can form the following commutative diagram of exact sequences.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & G'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $p_{G'},p_{G''}>\lambda$, $p_G>\lambda$. Therefore, $E\in\mathcal{T}^\lambda$.

For $E'',E'\in\mathcal{F}^\lambda$, let $F\subset E$ be a nontrivial subsheaf with dimension $\le d$. We can also form a commutative diagram like the one above. The subsheaf F is pure of dimension d, otherwise it would induce a nontrivial subsheaf of dimension $<d$ in either E' or E'', leading to a contradiction. Furthermore, $p_F\le\lambda$. Thus, $E\in\mathcal{F}^\lambda$.

It is clear that $\mathrm{Hom}(\mathcal{T}^\lambda, \mathcal{F}^\lambda) = 0$ by the inequalities of reduced Hilbert polynomials.

For any $E \in \mathrm{Coh}(X)$, there exists $T \subset E$ such that $T \in \mathcal{T}^\lambda$ and $E/T \in \mathcal{F}^\lambda$. Indeed, we can take the torsion filtration of E: $0 \subset T_0 \subset T_1 \subset \cdots \subset T_n = E$ (see [12, Defn. 1.1.4]). Then T_d/T_{d-1} is pure of dimension d. Consider the Harder-Narasimhan filtration of T_d/T_{d-1}:

$$0 = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_t = T_d/T_{d-1}.$$

If all the reduced Hilbert polynomials $p_{H_i/H_{i-1}}$ are not larger than λ, let $s = 0$. Otherwise, let s be the largest integer i such that $p_{H_i/H_{i-1}} > \lambda$. Therefore, the pre-image T of $H_s \subset T_d/T_{d-1}$ in T_d has the desired property. □

Then, we denote the heart of the tilted t-structure as $\mathrm{Coh}^{\lambda,\#}(X)$.

Theorem 4.5. *Suppose $E_0 \in \mathcal{F}^\lambda$ is fixed. We further assume that δ is not a critical value; namely, all δ-semistable pairs are δ-stable. Then, the Quot space $\mathrm{Quot}^{\lambda,\#}(E_0, P)$, which parameterizes quotients of E_0 with Hilbert polynomial P in the heart $\mathrm{Coh}^{\lambda,\#}(X)$, is isomorphic to the moduli space $S^{\mathrm{s}}_{E_0}(P,\delta) = S_{E_0}(P,\delta)$ of δ-stable pairs. Furthermore, the Quot space is projective.*

Similar to Theorem 3.8, the key to proving this theorem is identifying the two moduli functors. It is enough to prove the following proposition.

Proposition 4.6. *With assumptions and notation as in the previous theorem and E having Hilbert polynomial P, $\alpha\colon E_0 \to E$ is an epimorphism in $\mathrm{Coh}^{\lambda,\#}(X)$ if and only if $(E, \alpha\colon E_0 \to E)$ is a δ-stable pair.*

Proof. Suppose that α is an epimorphism in $\mathrm{Coh}^{\lambda,\#}(X)$. By the assumption $E_0 \in \mathcal{F}^\lambda$ and Lemma 2.5, we know that α is a morphism in $\mathrm{Coh}(X)$ with $E \in \mathcal{F}^\lambda$ and $\mathrm{coker}\,(\alpha) \in \mathcal{T}^\lambda$. Given a quotient $q\colon E \twoheadrightarrow G$ in $\mathrm{Coh}(X)$, if $q \circ \alpha = 0$, then G is a quotient of $\mathrm{coker}\,(\alpha)$. Thus, $p_G > \lambda$. If $q \circ \alpha \neq 0$, let $F = \ker q$. Then $p_F \leq \lambda$. Since δ is not critical, this is a strict inequality. Therefore, $\lambda < (P_G + \delta)/r_G$. We have shown (E, α) is a δ-stable pair.

Conversely, suppose $(E, \alpha\colon E_0 \to E)$ is a δ-stable pair. Then, for every subsheaf $F \subset E$, $p_F < \lambda$. Thus, $E \in \mathcal{F}^\lambda$. On the other hand, $\mathrm{coker}\,(\alpha)$ has dimension $\leq d$. Given a quotient G of $\mathrm{coker}\,(\alpha)$, it is also a quotient of E, and the composition $E_0 \to E \to G$ is zero. Thus, $p_G > \lambda$. Hence, $\mathrm{coker}\,(\alpha) \in \mathcal{T}^\lambda$. Again, by Lemma 2.5, α is an epimorphism in $\mathrm{Coh}^{\lambda,\#}(X)$. □

4.2. Framed sheaves

Let $\tau \in \mathbb{Q}[m]$ be another polynomial with a positive leading coefficient. We have a notion that is dual to pairs: *framed sheaves.* They are homomorphisms of the form

$$E \to E_0$$

where E_0 is again fixed.

Definition 4.7. A *framed sheaf* is a coherent sheaf E with Hilbert polynomial $P_E = P$, together with a nonzero *framing* $\alpha\colon E \to E_0$. It is *τ-stable* if $\ker\alpha$ is zero or pure of dimension d, and for every nonzero subsheaf $F \subset E$ of dimension d,

(i) $p_F < p_E - \tau/r_E$ if $F \subset \ker\alpha$,
(ii) $p_F - \tau/r_F < p_E - \tau/r_E$ otherwise.

Two framed sheaves $\alpha\colon E \to E_0$ and $\alpha'\colon E' \to E_0$ are isomorphic if there is an isomorphism $\psi\colon E \to E'$ such that $\alpha' \circ \psi = \alpha$. Let

$$\underline{F}_{E_0}(P,\tau)\colon (k\text{-Schemes})^{\mathrm{op}} \to \mathrm{Sets}$$

denote the moduli functor of isomorphic classes of τ-semistable frame sheaves. We have the following existence result [11, Thm. 0.1].

Theorem 4.8 ([11,12])**.** *There is a projective coarse moduli space $F_{E_0}(P,\tau)$ of S-equivalence classes of τ-semistable framed sheaves with Hilbert polynomial P. It contains an open subscheme $F^{\mathrm{s}}_{E_0}(P,\tau)$ as the fine moduli space of τ-stable framed sheaves.*

When $\deg\tau \geq d$, the moduli space is isomorphic to a Quot scheme. (See [11, Lem. 1.7] and the discussion immediately after its proof.) Therefore, we again consider a small stability parameter τ:

$$\deg\tau < \deg P = d.$$

Now, we let

$$\lambda = \frac{P-\tau}{r}.$$

Let $\mathcal{T}_\lambda$ and $\mathcal{F}_\lambda$ be as in (10) and (11), and let $\mathrm{Coh}^{\lambda,\#}(X)$ be the corresponding tilted heart. Let

$$\underline{\mathrm{Quot}}^{\lambda,\#[1]}(E_0, P_{E_0} - P)\colon (k\text{-Schemes})^{\mathrm{op}} \to \mathrm{Sets}$$

be the moduli functor of quotients of E_0 with Hilbert polynomial $P_{E_0} - P$ in the heart $\mathrm{Coh}^{\lambda,\#}(X)[1]$. Let $\mathrm{Quot}^{\lambda,\#[1]}(E_0, P_{E_0} - P)$ denote the corresponding Quot space.

Theorem 4.9. *Suppose $E_0 \in \mathcal{T}^\lambda$ is fixed. We further assume that τ is not a critical value; namely, all τ-semistable framed sheaves are τ-stable. The Quot space $\mathrm{Quot}^{\lambda,\#[1]}(E_0, P_{E_0} - P)$ is isomorphic to the moduli space $F^{\mathrm{s}}_{E_0}(P,\tau) = F_{E_0}(P,\tau)$ of τ-stable framed sheaves. In particular, the Quot space is projective.*

The key is again to identify the two moduli functors, which is a variant of [19, Lem. 5.5].

Proposition 4.10. *With assumptions and notation as in the previous theorem and supposing E has the fixed Hilbert polynomial P, $\alpha\colon E \to E_0$ is a monomorphism in $\mathrm{Coh}^{\lambda,\#}(X)[1]$ if and only if $(E, \alpha\colon E \to E_0)$ is a τ-stable framed sheaf.*

The proof is similar to that of Proposition 4.6. For completeness, we include it here.

Proof. Suppose that $\alpha\colon E \to E_0$ is a monomorphism in $\mathrm{Coh}^{\lambda,\#}(X)[1]$. By the assumption $E_0 \in \mathcal{T}^\lambda$ and Lemma 2.5, we know that α is a morphism in $\mathrm{Coh}(X)$ with $E \in \mathcal{T}^\lambda$ and $\ker(\alpha) \in \mathcal{F}^\lambda$. Given a subsheaf $F \subset E$, $p_{E/F} > \lambda$, because $E \in \mathcal{T}^\lambda$. Then, $p_F - \tau/r_F < \lambda$. If $F \subset \ker \alpha$, then $p_F \leq \lambda$, which is actually a strict inequality, since we assume τ is not critical. Therefore, $(E, \alpha\colon E \to E_0)$ is τ-stable.

Conversely, we assume that $(E, \alpha\colon E \to E_0)$ is τ-stable. First, τ-stability implies that for any dimension d quotient sheaf Q of E, $p_Q - \tau/r_Q > \lambda$ or $p_Q > \lambda$. Therefore, $E \in \mathcal{T}^\lambda$. On the other hand, the τ-stability also implies that if nonzero, $\ker(\alpha)$ is pure of dimension d and has reduced Hilbert polynomial $\leq \lambda$. Furthermore, $\ker(\alpha) \in \mathcal{F}^\lambda$. Again by Lemma 2.5, α is a monomorphism in $\mathrm{Coh}^{\lambda,\#}(X)[1]$. □

Proof of Theorem 4.9. Proposition 4.10 identifies the moduli functor $\underline{F}_{E_0}(P,\tau)$ with $\underline{\mathrm{Quot}}^{\lambda,\#[1]}(E_0, P_{E_0} - P)$. Therefore, the corresponding moduli spaces are isomorphic. The projectivity follows from that of $F_{E_0}(P,\tau)$ (Theorem 4.8). □

5. Change of Quot space under tilting

In this section, we prove a formula relating the moduli space of quotient husks and Grothendieck's Quot scheme, which parameterizes quotient sheaves supported in dimension no more than one. We follow Bridgeland's treatment of Hall algebra identities in [4, Sec. 6].

5.1. The stack of pairs

We first modify the stack of sheaves with sections, which was constructed in [4, Sec. 2.3]. Let X be a nonsingular projective variety over $\mathbb{C}$ and $E_0 \in \mathrm{Coh}(X)$ be fixed. We denote by $\mathcal{M}$ the stack of coherent sheaves on X. It is an Artin stack, locally of finite type over $\mathbb{C}$. There is another stack $\mathcal{M}(E_0)$ with a morphism $q\colon \mathcal{M}(E_0) \to \mathcal{M}$ parameterizing pairs $(E, \alpha\colon E_0 \to E)$. Indeed, the objects of $\mathcal{M}(E_0)$ lying over a scheme S are pairs (E, α) consisting of an S-flat coherent sheaf E on $S \times X$ together with $\alpha\colon E_{0S} \to E$ where E_{0S} denotes the pullback of E_0 under the projection $S \times X \to X$. Let $f\colon T \to S$ be a morphism of schemes and $f_X = f \times \mathrm{id}_X$. Given an object (F, β) lying over T, a morphism $\theta\colon (F,\beta) \to (E,\alpha)$ lying over f is an isomorphism $\theta\colon f_X^* E \to F$ on $T \times X$ with $\theta \circ f_X^* \alpha = \beta \circ \kappa$, where the morphism $\kappa\colon f_X^* E_{0S} \to E_{0T}$ denotes the canonical isomorphism of pullbacks. The morphism q of stacks is defined by forgetting the data of the morphism α in the obvious way.

By an easy modification of the argument of [4, Lem. 2.4], we have the following lemma.

Lemma 5.1. *The stack $\mathcal{M}(E_0)$ is an Artin stack, and the morphism q is representable and of finite type.*

The following lemma is a result of the fibers of the morphism q.

Lemma 5.2. *Let $E_0 \in \mathrm{Coh}(X)$ be fixed. There is a stratification of $\mathcal{M}$ by locally closed substacks $\mathcal{M}_r \subset \mathcal{M}$ such that the objects of $\mathcal{M}_r(\mathbb{C})$ are $E \in \mathrm{Coh}(X)$ with $\hom(E_0, E) = r$. The pullback of $q\colon \mathcal{M}(E_0) \to \mathcal{M}$ to $\mathcal{M}_r$ is a locally trivial fibration in the Zariski topology, with fiber $\mathbb{C}^r$.*

Proof. Let S be a scheme. Given an S-flat coherent sheaf E on $S \times X$, we write $\mathfrak{h}om(E_{0S}, E)$ for the set-valued covariant funtor on $(S\text{-Schemes})^{\mathrm{op}}$, which associates to any S-scheme $f\colon T \to S$ the set $\mathrm{Hom}(f_X^* E_{0S}, f_X^* E)$ of $\mathcal{O}_{T\times X}$-linear morphism. By a standard limit argument (cf. [8, (8.5.2), (8.8.2), (8.9.1), (11.2.6)]), we may assume that S is Noetherian. According to the results of Grothendieck (see [16, Thm. 5.8] and references therein), there is a coherent sheaf $G(E_{0S}, E)$ on S such that the funtor $\mathfrak{h}om(E_{0S}, E)$ is represented by the linear scheme

$$\mathbf{Spec}\,(\mathrm{Sym}_{\mathcal{O}_S} G(E_{0S}, E)).$$

Then, the remaining proof is essentially the same as in [4, Lem. 2.5]. □

5.2. Motivic Hall algebra

We now recall the notion of motivic Hall algebras. We refer to [4, 5] for a more detailed discussion.

We denote the subcategory $\mathrm{Coh}_{\le 1}(X)$ of $\mathrm{Coh}(X)$ by $\mathcal{C}$. This corresponds to an open and closed substack $\mathcal{C} \subset \mathcal{M}$ by the usual abuse of notation. There exists a stack $\mathcal{C}^{(2)}$ of short exact sequences in the category $\mathcal{C}$. It comes with three distinguished morphisms a_1, a_2, and $b\colon \mathcal{C}^{(2)} \to \mathcal{C}$. These morphisms correspond to sending a short exact sequence $0 \to A_1 \to B \to A_2 \to 0$ to the sheaves A_1, A_2, and B respectively. We remark that (a_1, a_2) is of finite type [5, Lem. 4.2].

The *motivic Hall algebra*, denoted by $\mathrm{H}(\mathcal{C})$, is the relative Grothendieck group $K(\mathrm{St}/\mathcal{C})$ over the stack $\mathcal{C}$. By definition, it is defined to be the complex vector space spanned by isomorphism classes of symbols $[\mathcal{X} \to \mathcal{C}]$ where $\mathcal{X}$ is an Artin stack of finite type over $\mathbb{C}$ with affine geometric stabilizers, modulo three relations: the scissor relations for finite disjoint stacks, geometric bijection relations and Zariski fibration relations (see [5, Defn. 3.10]).

It is equipped with a *noncommutative* product $*$ given explicitly by the rule

$$[\mathcal{X}_1 \xrightarrow{f_1} \mathcal{C}] * [\mathcal{X}_2 \xrightarrow{f_2} \mathcal{C}] = [\mathcal{Z} \xrightarrow{b\circ h} \mathcal{C}],$$

where h is defined by the Cartesian diagram

$$\begin{array}{ccccc} \mathcal{Z} & \xrightarrow{\quad h \quad} & \mathcal{C}^{(2)} & \xrightarrow{\quad b \quad} & \mathcal{C}. \\ \big\downarrow & & \big\downarrow{\scriptstyle (a_1,a_2)} & & \\ \mathcal{X}_1 \times \mathcal{X}_2 & \xrightarrow{f_1\times f_2} & \mathcal{C}\times\mathcal{C} & & \end{array}$$

The unit is given by $1 = [\mathrm{Spec}\,\mathbb{C} \to \mathcal{C}]$, which corresponds to the zero object in $\mathcal{C}$ and the product $*$ is associative [5, Thm. 4.3].

On the other hand, there is a natural grading on $\mathrm{H}(\mathcal{C})$ by the monoid Δ consisting of classes of sheaves supported in dimension ≤ 1. More precisely, let $N_1(X)$ denote the abelian group of cycles of dimension one modulo numerical equivalence. We define the monoid by

$$\Delta = \{(\beta, n) \in N_1(X) \oplus \mathbb{Z} \mid \beta > 0 \text{ or } \beta = 0 \text{ and } n \geq 0\}$$

(cf. [4, Sec. 2.1]). There are open and closed substacks $\mathcal{C}_\gamma \subset \mathcal{C}$, the stacks of objects of class $\gamma \in \Delta$. Thus elements of $\mathrm{H}(\mathcal{C})$ are naturally graded by the monoid Δ. An element $[f\colon \mathcal{X} \to \mathcal{C}]$ is *homogeneous* of degree $\gamma \in \Delta$ if f factors through the substack $\mathcal{C}_\gamma$.

5.3. Laurent subsets

Let us summarize Section 5.2 and 6.1 of [4]. A subset $S \subset \Delta$ is *Laurent* if for all $\beta \in N_1(X)$, the collection $\{n \in \mathbb{Z} \mid (\beta, n) \in S\}$ is bounded below. Let Φ denote the set of all Laurent subsets.

For the Δ-graded Hall algebra $\mathrm{H}(\mathcal{C})$, we can use Φ to define a new algebra, denoted by $\mathrm{H}(\mathcal{C})_\Phi$. Elements of this new algebra are of the form $a = \sum_{\gamma \in S} a_\gamma$ where $S \in \Phi$ and $a_\gamma \in \mathrm{H}(\mathcal{C})_\gamma \subset \mathrm{H}(\mathcal{C})$. There is a natural topology and product $*$ on $\mathrm{H}(\mathcal{C})_\Phi$ induced by projection operators (see [5, Sec. 5.2]).

To define a stability condition, we fix an ample divisor H on X. Given a class $\gamma = (\beta, n) \in \Delta$, we define the slope by $\mu(\gamma) = n(\beta \cdot H)^{-1} \in (-\infty, \infty]$. In particular, if $\beta = 0$, $\mu(\gamma) = \infty$, otherwise $\mu(\gamma) \in \mathbb{Q}$.

Given an interval $I \subset (-\infty, \infty]$, define $\mathrm{SS}(I) \subset \mathcal{C}$ to be the full subcategory consisting of zero objects together with those one-dimensional sheaves whose Harder-Narasimhan factors all have slope in I (see [4, Sec. 6.1]). We write $\mathrm{SS}(I) = \mathrm{SS}(\geq \mu)$ if $I = [\mu, \infty]$. Then the following lemma follows from Lemmas 5.3, 6.2, and (31) of [4].

Lemma 5.3. *In* $\mathrm{H}(\mathcal{C})_\Phi$*, the subcategory* $\mathrm{SS}([\mu, \infty))$ *defines an invertible element* $1_{\mathrm{SS}([\mu,\infty))}$.

5.4. Identities in the Laurent Hall algebra

Let $\mathcal{T} = \mathrm{Coh}_0(X) = \mathrm{SS}(\infty)$. Consider the torsion pair $(\mathcal{T}, \mathcal{F} \cap \mathcal{C})$ of $\mathcal{C}$, where $\mathcal{F} = \mathcal{T}^\perp$. Then $\mathcal{C}^\# = \langle \mathcal{F} \cap \mathcal{C}, \mathcal{T}[-1] \rangle$ is the tilt of $\mathcal{C}$.

For the Grothendieck's Quot scheme $\mathrm{Quot}(E_0)$ and the moduli space of quotient husks (or limit stable pairs, Proposition 3.14) $\mathrm{QHusk}(E_0)$, we introduce[1]

$$\mathrm{Quot}(E_0)_{\leq 1} := \mathrm{Quot}(E_0) \cap \mathcal{C} \quad \text{and}$$
$$\mathrm{QHusk}(E_0)_{\leq 1} := \mathrm{QHusk}(E_0) \cap \mathcal{C} \text{ if } E_0 \in \mathcal{F},$$

which parameterize quotients of E_0 supported in dimension ≤ 1. By the same argument of [4, Lem. 2.6], we can view $\mathrm{Quot}(E_0)_{\leq 1}$ and $\mathrm{QHusk}(E_0)_{\leq 1}$ as open substacks of the moduli stack $\mathcal{C}(E_0)$. In particular, the $\mathbb{C}$-valued points are

[1]The condition $E_0 \in \mathcal{F}$ implies that if $\alpha\colon E_0 \to E$ is an epimorphism in $\mathrm{Coh}(X)^\#$, then $E \in \mathcal{F} \subset \mathrm{Coh}(X)$ by Lemma 2.5.

morphisms $E_0 \to E$, which are epimorphisms in the categories $\mathcal{C}$ and $\mathcal{C}^{\#}$, respectively. The morphisms

$$\mathrm{Quot}(E_0)_{\leq 1} \to \mathcal{C} \text{ and } \mathrm{QHusk}(E_0)_{\leq 1} \to \mathcal{C},$$

which are the restrictions to $\mathrm{Quot}(E_0)_{\leq 1}$ and $\mathrm{QHusk}(E_0)_{\leq 1}$ of $q\colon \mathcal{C}(E_0) \to \mathcal{C}$, define elements $\mathcal{Q}_{\leq 1}$ and $\mathcal{Q}^{\#}_{\leq 1}$ of $\mathcal{H}(\mathcal{C})_\Phi$ by a similar argument to [4, Lem. 5.5].

Given a substack $i\colon \mathcal{N} \to \mathcal{C}$, we write $1_{\mathcal{N}} := [\mathcal{N} \to \mathcal{C}]$ in $\mathrm{H}(\mathcal{C})$. Pulling back the morphism $q\colon \mathcal{C}(E_0) \to \mathcal{C}$ to $\mathcal{N} \subset \mathcal{C}$ produces a stack $\mathcal{N}(E_0)$ with a morphism $\mathcal{N}(E_0) \to \mathcal{N}$ and hence an element $1^{E_0}_{\mathcal{N}} := [\mathcal{N}(E_0) \to \mathcal{C}]$ in $\mathrm{H}(\mathcal{C})$. By abuse of notation, we use the same symbol for an open substack of $\mathcal{C}$ and the corresponding full subcategory of $\mathcal{C}$ defined by its $\mathbb{C}$-valued points.

Following [4], we establish the torsion pair and Quot space identities in the next two lemmas.

Lemma 5.4. *The following identities hold in the Laurent Hall algebra* $\mathrm{H}(\mathcal{C})_\Phi$.

(a) $1_{\mathrm{SS}(\geq\mu)} = 1_{\mathcal{T}} * 1_{\mathrm{SS}([\mu,\infty))}$.

(b) $\lim_{\mu\to-\infty} \left(\mathcal{Q}_{\leq 1} * 1_{\mathrm{SS}(\geq\mu)} - 1^{E_0}_{\mathrm{SS}(\geq\mu)}\right) = 0$.

(c) $\lim_{\mu\to-\infty} \left(\mathcal{Q}^{\#}_{\leq 1} * 1_{\mathrm{SS}([\mu,\infty))} - 1^{E_0}_{\mathrm{SS}([\mu,\infty))}\right) = 0$ *if* $E_0 \in \mathcal{F}$.

The proof of Lemma 5.4 is essentially the same as in [4, Prop. 6.5], noticing the boundedness of the Quot scheme and the moduli space of stable pairs. We point out that the geometric bijection relations plays an essential role, and we need the assumption $E_0 \in \mathcal{F}$ of (c) to use Lemma 2.5 instead of [4, Lem. 2.3].

Lemma 5.5. *Assume that* E_0 *is locally free. There is an identity*

$$1^{E_0}_{\mathrm{SS}(\geq\mu)} = 1^{E_0}_{\mathcal{T}} * 1^{E_0}_{\mathrm{SS}([\mu,\infty))} \qquad \text{in } \mathrm{H}(\mathcal{C})_\Phi.$$

Proof. We have Cartesian squares

$$\begin{array}{ccccccc}
\mathcal{Y} & \xrightarrow{\quad p \quad} & \mathcal{X} & \xrightarrow{\quad j \quad} & \mathcal{C}^{(2)} & \xrightarrow{\ b\ } & \mathcal{C} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle (a_1,a_2)} & & \\
\mathcal{T}(E_0) \times \mathrm{SS}([\mu,\infty))(E_0) & \xrightarrow{(q,\mathrm{id})} & \mathcal{T} \times \mathrm{SS}([\mu,\infty))(E_0) & \xrightarrow{(i,q)} & \mathcal{C} \times \mathcal{C}. & &
\end{array}$$

Then, $1^{E_0}_{\mathcal{T}} * 1^{E_0}_{\mathrm{SS}([\mu,\infty))}$ is represented by the composite morphism $b \circ j \circ p\colon \mathcal{Y} \to \mathcal{C}$. Note that, by Lemma 5.2, the morphism of stacks $q\colon \mathcal{T}(E_0) \to \mathcal{T}$ is a Zariski fibration, with fiber over a sheaf T being the vector space $\mathrm{Hom}(E_0, T)$. By pulling back, the same is true for the map p.

Since the morphism (a_1, a_2) satisfies the iso-fibration property of [5, Lem. A.1], the groupoid of S-valued points of $\mathcal{X}$ is as follows. The objects are short exact sequences of S-flat sheaves on $S \times X$,

$$0 \to T \to E \xrightarrow{\gamma} F \to 0, \tag{12}$$

such that T and F define flat families of sheaves on X lying in the subcategories $\mathcal{T}$ and $\mathrm{SS}([\mu,\infty))$, respectively, together with a map $\alpha\colon E_{0S} \to F$. The morphisms are isomorphisms of short exact sequences commuting with the map α.

Recall that by Lemma 5.4 (a), $b \circ s \colon \mathcal{Z} \to \mathrm{SS}(\geq \mu)$ induces an equivalence on $\mathbb{C}$-valued points. On the other hand, consider a Cartesian diagram

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{h} & \mathrm{SS}(\geq \mu)(E_0) \subset \mathcal{C}(E_0) \\ \downarrow & & \downarrow q \\ \mathcal{Z} & \xrightarrow{b \circ s} & \mathrm{SS}(\geq \mu) \subset \mathcal{C}. \end{array}$$

Since $b \circ s$ is a geometric bijection, so is h as well. Thus, the element $1^{E_0}_{\mathrm{SS}(\geq\mu)}$ can be represented by the morphism $q \circ h$.

The groupoid of S-valued points of $\mathcal{W}$ can be represented by the short exact sequences (12) with a map $\delta \colon E_{0S} \to E$. Setting $\alpha = \gamma \circ \delta$ defines a morphism of stacks $\mathcal{W} \to \mathcal{X}$. It is easy to see that this is a Zariski fibration, with fiber over a $\mathbb{C}$-valued point of $\mathcal{X}$ represented by a sequence (12) with a map α being a vector space for $\mathrm{Hom}(E_0, T)$. Indeed, we have a long exact sequence

$$0 \to \mathrm{Hom}(E_0, T) \to \mathrm{Hom}(E_0, E) \to \mathrm{Hom}(E_0, F) \to \mathrm{Ext}^1(E_0, T)$$

on X. Since the support of T has dimension zero, so does that of $E_0^\vee \otimes T$. For a locally free sheaf E_0, we get

$$\mathrm{Ext}^1(E_0, T) \cong H^1(X, E_0^\vee \otimes T) = 0$$

due to the dimension. Since $\mathcal{W} \to \mathcal{X}$ has the same fibers $\mathrm{Hom}(E_0, T)$ as the map $p \colon \mathcal{Y} \to \mathcal{X}$, the result follows from the Zariski relation $[\mathcal{W} \to \mathcal{X} \to \mathcal{C}] = [\mathcal{Y} \to \mathcal{X} \to \mathcal{C}]$. ☐

We are now in a position to give the formula relating $\mathrm{Quot}(E_0)_{\leq 1}$ and $\mathrm{QHusk}(E_0)_{\leq 1}$.

Theorem 5.6. *Assume that $E_0 \in \mathcal{F}$ and it is locally free. There is an identity*

$$\mathcal{Q}_{\leq 1} * 1_{\mathcal{T}} = 1^{E_0}_{\mathcal{T}} * \mathcal{Q}^{\#}_{\leq 1} \quad \textit{in } \mathrm{H}(\mathcal{C})_\Phi.$$

Proof. By Lemma 5.4 (a) and Lemma 5.5, the expression (b) of Lemma 5.4 can be rewritten

$$\mathcal{Q}_{\leq 1} * 1_{\mathcal{T}} * 1_{\mathrm{SS}([\mu,\infty))} - 1^{E_0}_{\mathcal{T}} * 1^{E_0}_{\mathrm{SS}([\mu,\infty))} \to 0 \text{ as } \mu \to -\infty.$$

Multiplying (c) of Lemma 5.4 on the left by $1^{E_0}_{\mathcal{T}}$ gives

$$1^{E_0}_{\mathcal{T}} * \mathcal{Q}^{\#}_{\leq 1} * 1_{\mathrm{SS}([\mu,\infty))} - 1^{E_0}_{\mathcal{T}} * 1^{E_0}_{\mathrm{SS}([\mu,\infty))} \to 0 \text{ as } \mu \to -\infty.$$

Thus,

$$1^{E_0}_{\mathcal{T}} * \mathcal{Q}^{\#}_{\leq 1} * 1_{\mathrm{SS}([\mu,\infty))} - \mathcal{Q}_{\leq 1} * 1_{\mathcal{T}} * 1_{\mathrm{SS}([\mu,\infty))} \to 0 \text{ as } \mu \to -\infty.$$

By Lemma 5.3, we can multiply the inverse of $1_{\mathrm{SS}([\mu,\infty))}$ and deduce the result. ☐

Remark 5.7. In [21], Toda studied the higher rank DT/PT correspondence, via stable objects in the derived category of coherent sheaves. He applied the integration map to the moduli stacks. According to Behrend's result [3], the integrations are related to higher rank DT and PT invariants. The invariants are defined using the virtual fundamental classes, whose existence is guaranteed by the symmetric obstruction theories.

The moduli space of quotient husks/limit stable pairs can also be viewed as a version of the higher rank PT moduli space. Over a Calabi–Yau 3-fold, we can also apply the integration map. However, the question of whether the result is a deformation invariant remains, due to the absence of a result on a virtual fundamental class at the moment.

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References

[1] D. Abramovich and A. Polishchuk, *Sheaves of t-structures and valuative criteria for stable complexes*, J. Reine Angew. Math. **590** (2006), 89–130. https://doi.org/10.1515/CRELLE.2006.005
[2] A. Bayer, M. Lahoz, E. Macrì, H. Nuer, A. Perry, and P. Stellari, *Stability conditions in families*, Publ. Math. Inst. Hautes Études Sci. **133** (2021), 157–325. https://doi.org/10.1007/s10240-021-00124-6
[3] K. A. Behrend, *Donaldson-Thomas type invariants via microlocal geometry*, Ann. of Math. (2) **170** (2009), no. 3, 1307–1338. https://doi.org/10.4007/annals.2009.170.1307
[4] T. Bridgeland, *Hall algebras and curve-counting invariants*, J. Amer. Math. Soc. **24** (2011), no. 4, 969–998. https://doi.org/10.1090/S0894-0347-2011-00701-7
[5] T. Bridgeland, *An introduction to motivic Hall algebras*, Adv. Math. **229** (2012), no. 1, 102–138. https://doi.org/10.1016/j.aim.2011.09.003
[6] T. Goller and Y. Lin, *Rank-one sheaves and stable pairs on surfaces*, Adv. Math. **401** (2022), Paper No. 108322, 32 pp. https://doi.org/10.1016/j.aim.2022.108322
[7] T. Goller and Y. Lin, *Gaeta resolutions and strange duality over rational surfaces*, arXiv:2205.14827, 2022.
[8] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III*, Inst. Hautes Études Sci. Publ. Math. No. 28 (1966), 255 pp.
[9] D. Happel, I. Reiten, and O. SmaløSverre. *Tilting in Abelian Categories and Quasitilted Algebras*, Mem. Amer. Math. Soc. 120, no. 575, viii+88pp, 1996.
[10] D. Huybrechts, *Fourier-Mukai Transforms in Algebraic Geometry*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, Oxford, 2006. https://doi.org/10.1093/acprof:oso/9780199296866.001.0001
[11] D. Huybrechts and M. Lehn, *Framed modules and their moduli*, Internat. J. Math. **6** (1995), no. 2, 297–324. https://doi.org/10.1142/S0129167X9500050X

[12] D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, second edition, Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, 2010. https://doi.org/10.1017/CBO9780511711985
[13] J. Kollár, *Families of Varieties of General Type*, Cambridge Tracts in Mathematics, 231, Cambridge Univ. Press, Cambridge, 2023.
[14] A. G. Kuznetsov, *Hyperplane sections and derived categories*, Izv. Math. **70** (2006), no. 3, 447–547; translated from Izv. Ross. Akad. Nauk Ser. Mat. **70** (2006), no. 3, 23–128. https://doi.org/10.1070/IM2006v070n03ABEH002318
[15] Y. Lin, *Moduli spaces of stable pairs*, Pacific J. Math. **294** (2018), no. 1, 123–158. https://doi.org/10.2140/pjm.2018.294.123
[16] N. Nitsure, *Construction of Hilbert and Quot schemes*, Fundamental algebraic geometry, 105–137, Math. Surveys Monogr. 123, Amer. Math. Soc., Providence, RI, 2005.
[17] R. Pandharipande and R. P. Thomas, *Curve counting via stable pairs in the derived category*, Invent. Math. **178** (2009), no. 2, 407–447. https://doi.org/10.1007/s00222-009-0203-9
[18] A. Polishchuk, *Constant families of t-structures on derived categories of coherent sheaves*, Mosc. Math. J. **7** (2007), no. 1, 109–134, 167. https://doi.org/10.17323/1609-4514-2007-7-1-109-134
[19] F. Rota, *Some Quot schemes in tilted hearts and moduli spaces of stable pairs*, Internat. J. Math. **32** (2021), no. 13, Paper No. 2150098, 34 pp. https://doi.org/10.1142/S0129167X21500981
[20] M. Thaddeus, *Stable pairs, linear systems and the Verlinde formula*, Invent. Math. **117** (1994), no. 2, 317–353. https://doi.org/10.1007/BF01232244
[21] Y. Toda, *Hall algebras in the derived category and higher-rank DT invariants*, Algebr. Geom. **7** (2020), no. 3, 240–262. https://doi.org/10.14231/ag-2020-008

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