

THE DEGENERATE MONGE-AMPÈRE EQUATIONS WITH THE NEUMANN CONDITION

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ABSTRACT. In this paper, we study *a priori* derivative estimates (up to the second order) of solutions for the Monge-Ampère equation $\det D^2u = f(x)$ with the Neumann boundary value condition, which are independent of $\inf f$. Based on these uniform estimates, the existence and uniqueness of the global $C^{1,1}$ solution to the Neumann problem of the degenerate Monge-Ampère equation are established under the assumption $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$.

1. Introduction. In this paper, we study the following degenerate Monge-Ampère equation

$$\det D^2u = f(x), \quad \text{in } \Omega, \quad (1.1)$$

associated to the Neumann boundary value condition

$$D_\nu u = \varphi(x, u), \quad \text{on } \partial\Omega, \quad (1.2)$$

where Ω is a bounded and uniformly convex domain in \mathbb{R}^n , D^2u denotes the Hessian matrix of second order derivatives of the unknown function $u : \Omega \rightarrow \mathbb{R}$, $f : \Omega \rightarrow \mathbb{R}^+ \cup \{0\}$ is a nonnegative function and $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$, ν denotes the unit inner normal vector field on $\partial\Omega$ and φ is a given function on $\partial\Omega \times \mathbb{R}$. As usual, we shall use x and z to denote the points in Ω and \mathbb{R} respectively.

We say that the Monge-Ampère equation (1.1) is nondegenerate if

$$f(x) \geq f_0, \quad (1.3)$$

holds for all $x \in \Omega$, where f_0 is a positive constant. If f_0 is replaced by 0, we say that the Monge-Ampère equation (1.1) is degenerate.

For the nondegenerate case, Monge-Ampère equations with Dirichlet boundary value condition can be referred to [5, 8]. For the nondegenerate Monge-Ampère equations with Neumann boundary value condition, Lions, Trudinger and Urbas obtained the global C^2 regularity of solutions in [16] under some proper conditions

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of f and φ . Ma and Qiu established the global C^2 regularity of solutions in [19] for the k -Hessian equations which contained the Monge-Ampère equations as a special case.

For semilinear oblique derivative problem $\det D^2u = f(x, u, Du)$ in Ω , $D_\beta u + \varphi(x, u) = 0$ on $\partial\Omega$, where β is unit vector field on $\partial\Omega$ satisfying $\beta \cdot \nu > 0$, Urbas [22] and Wang [25] proved the existence of classical convex solution in two dimensional space under the restrictions of β and φ_z , (see condition (1.6) in [22] and condition (2.7) in [25]). For $n \geq 3$, an example of Pogorelov [23, 25] shows that the solution of the oblique boundary value problem may not be smooth even though β is strictly oblique and smooth. When β is a small perturbation of ν , Urbas [23] and Li [17] used different methods to prove the existence and uniqueness of classical convex solution for the oblique boundary value problem in high dimensional space.

The global C^2 regularity for the Neumann problem of the Monge-Ampère type equations $\det[D^2u - A(x, u, Du)] = B(x, u, Du)$ with a matrix function A and a positive scalar function B is established by Jiang, Trudinger and Xiang in [13], which includes the standard Monge-Ampère equation as a special case. The existence of the classical solutions is proved by the method of continuity.

For the degenerate Monge-Ampère equations with the Neumann boundary value condition, *a priori* derivative estimates up to order two is proved by Trudinger in [20]. However, the argument in [20] only suits for the case when the domain is a ball. Naturally, one can ask whether the *a priori* estimates hold for a uniformly convex domain or not. This is one motivation of our paper.

The assumption that $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$ has been used for the study of degenerate Monge-Ampère equations, see [9, 10, 1]. It is known that the $C^{1,1}$ regularity of the solution is the best that can be expected for the degenerate Monge-Ampère equations. The global $C^{1,1}$ regularity for solutions was obtained under the homogeneous and inhomogeneous Dirichlet boundary conditions in [9] and [10] respectively. The interior $C^{1,1}$ regularity for solutions of the degenerate Monge-Ampère equation was studied by Błocki in [1]. By Example 3 in [24], the exponent $\frac{1}{n-1}$ in the assumption $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$ is optimal. Parallel to the Dirichlet problem in [9, 10], we aim to establish the global $C^{1,1}$ regularity of the solution to the Neumann problem for degenerate Monge-Ampère equation under the same assumption $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$. This is another motivation of the current paper.

We now formulate the main results.

Theorem 1.1. *Let $u \in C^4(\Omega) \cap C^{1,1}(\bar{\Omega})$ be a strongly convex solution of the Neumann problem (1.1)-(1.2) in a $C^{3,1}$ domain $\Omega \subset \mathbb{R}^n$, which is bounded and uniformly convex. Suppose that f is a positive function in Ω , $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$. If f vanishes at some boundary point, assume further that there exist positive constants δ and C such that*

$$|Df^{\frac{1}{n-1}}(x)| \leq Cf^{\frac{1}{2(n-1)}}(x) \tag{1.4}$$

holds for any $x \in \Omega$ with $\text{dist}(x, \partial\Omega) \leq \delta$. Assume that $\varphi \in C^{2,1}(\bar{\Omega} \times \mathbb{R})$ and φ is non-decreasing in z for all $(x, z) \in \partial\Omega \times \mathbb{R}$. Then, we have

$$\sup_{\Omega} |D^2u| \leq C, \tag{1.5}$$

where C depends on n , Ω , $|u|_{1,\Omega}$, $\|f^{\frac{1}{n-1}}\|_{C^{1,1}(\bar{\Omega})}$ and $\|\varphi\|_{C^{2,1}(\Omega \times (-M, M))}$, where $M = \sup_{\Omega} |u|$.

As in [1], a function u is called strongly convex if locally there exists $\lambda > 0$ such that $u - \lambda|x|^2$ is convex. The constant C on the right hand side of (1.5) is independent of $\inf_{\Omega} f$. In condition (1.4), $\text{dist}(x, \partial\Omega)$ denotes the distance from the point x to $\partial\Omega$. Note that in Theorem 1.1, the function φ in (1.2) has been extended from $\partial\Omega \times \mathbb{R}$ to $\bar{\Omega} \times \mathbb{R}$.

Remark 1. Our techniques for the estimate (1.5) rest strongly on the uniform convexity of the domain Ω , which is used in the double normal and the double tangential derivative estimates on $\partial\Omega$.

The uniform second derivative estimates in Theorem 1.1 together with the uniform lower order derivative estimates can lead to the following existence result for the Neumann problem (1.1)-(1.2).

Theorem 1.2. *Under the assumption of Theorem 1.1, assume that “ f is a non-negative function” and “ $\varphi_z \geq \gamma_0$ on $\partial\Omega \times \mathbb{R}$ for a positive constant γ_0 ” instead of “ f is a positive function” and “ φ is non-decreasing in z for all $(x, z) \in \partial\Omega \times \mathbb{R}$ ” respectively. Then there exists a unique convex solution $u \in C^{1,1}(\bar{\Omega})$ for the Neumann problem of the degenerate Monge-Ampère equation, (1.1)-(1.2).*

Remark 2. In Theorems 1.1 and 1.2, we need to assume the condition (1.4) for $f^{\frac{1}{n-1}}$ in $\bar{\Omega}$. In general, when $f^{\frac{1}{n-1}}(\bar{\Omega}) \in C^{1,1}(\bar{\Omega})$ is a nonnegative function, then $f^{\frac{1}{2(n-1)}}$ is locally Lipschitz in Ω by Lemma 3.1 in [2]. This implies that (1.4) holds in any $\Omega' \subset\subset \Omega$ and may not hold globally in Ω . For example, let Ω be a uniformly convex domain and $\nu(x_0)$ be the unit inner normal vector at the boundary point $x_0 \in \partial\Omega$, the function

$$f^{\frac{1}{n-1}}(x) := \nu(x_0) \cdot (x - x_0) \in C^{1,1}(\bar{\Omega}) \tag{1.6}$$

is positive in Ω and vanishes at $x_0 \in \partial\Omega$. However, it does not satisfy the condition (1.4) near $\partial\Omega$, since

$$|Df^{\frac{1}{n-1}}(x)| = |D[\nu(x_0) \cdot (x - x_0)]| \equiv 1 \tag{1.7}$$

in Ω and

$$f^{\frac{1}{2(n-1)}}(x) = [\nu(x_0) \cdot (x - x_0)]^{\frac{1}{2}} \rightarrow 0 \tag{1.8}$$

as $x \rightarrow x_0 \in \partial\Omega$. Therefore, by assuming (1.4), we exclude such type of functions $f^{\frac{1}{n-1}}$ in (1.6) in Theorems 1.1 and 1.2.

It is clear that Theorem 1.2 can be applied to the Monge-Ampère equation with homogenous right hand side as follows,

$$\det D^2u = c_0|x|^\alpha, \text{ in } \Omega, \tag{1.9}$$

where c_0 is a nonnegative constant, $\alpha \in [2(n-1), +\infty)$ is a constant. Then we have the following consequence of Theorem 1.2.

Corollary 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, uniformly convex $C^{3,1}$ domain. Assume that $\varphi_z \geq \gamma_0$ on $\partial\Omega \times \mathbb{R}$ for a positive constant γ_0 , then there exists a unique convex solution $u \in C^{1,1}(\bar{\Omega})$ of the Neumann problem of the degenerate Monge-Ampère equation (1.9)-(1.2).*

If $c_0 > 0$ and the origin $0 \in \bar{\Omega}$ in Corollary 1, the regularity of solution can only be globally $C^{1,1}$ in general, since 0 is a degenerate point of the equation. While if $c_0 > 0$ and the origin $0 \notin \bar{\Omega}$ in Corollary 1, there exists a unique $u \in C^{2,\alpha}(\bar{\Omega})$ for $\alpha \in (0, 1)$. This is because the equation (1.9) is nondegenerate and the classical existence result

in [16] is applicable in this case. Note that if the right hand side of (1.9) is replaced by $c_0|x-x_0|^\alpha$, then the above remarks still hold by replacing “ $0 \in \bar{\Omega}$ ” and “ $0 \notin \bar{\Omega}$ ” by “ $x_0 \in \bar{\Omega}$ ” and “ $x_0 \notin \bar{\Omega}$ ” respectively. For more examples of nonnegative f satisfying $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$, one can refer to [9].

Remark 3. The totally degenerate Monge-Ampère equation

$$\det D^2u = 0 \quad \text{in } \Omega, \quad (1.10)$$

is included as a special case of Corollary 1 by taking $c_0 = 0$ in (1.9). Thus, there exists a unique convex solution $u \in C^{1,1}(\bar{\Omega})$ of the equation (1.10) under the Neumann boundary value condition (1.2). For the Dirichlet problem of the totally degenerate Monge-Ampère equation (1.10), we refer the reader to [21, 6].

The main feature of this paper is to establish the *a priori* uniform derivative estimates, which are independent of $\inf_{\Omega} f$. The uniform C^0 estimate is established under the condition $\varphi_z \geq \gamma_0 > 0$ on $\partial\Omega \times \mathbb{R}$. The uniform C^1 estimate relies particularly on the convexity of the solution u . The assumptions $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$ and uniform convexity of Ω are only used for the second derivative estimates. The former assumption is used in both the reduction of the C^2 estimate to the boundary and also the double normal derivative estimate on $\partial\Omega$, while the latter assumption is used particularly for the boundary estimates as remarked in Remark 1. Note that it would be interesting to ask whether the technical assumption (1.4) (when f vanishes at boundary point) can be reduced or not, or to further study the generalizations of Theorems 1.1 and 1.2 here to the problems with general oblique boundary value condition. We will pursue these investigations in a sequel.

The paper is organized as follows. In Section 2, we shall derive uniform maximum modulus estimate and gradient estimate for convex solutions of the Monge-Ampère equations with the Neumann boundary value condition. In Section 3, we give the proof of the uniform second derivative estimate in Theorem 1.1, which is the core of this paper. We first prove the mixed tangential normal and the double normal derivative estimates on the boundary in §3.1. Here the double normal derivative estimate, Theorem 3.2, is proved by a proper barrier argument in a small neighbourhood of a boundary point, which is more complicated than the barrier argument for the nondegenerate case in [16]. Then in §3.2 the global second derivative estimate is obtained by using the same auxiliary function as in [16]. In Section 4, by using approximations, the existence of solutions in $C^{1,1}(\bar{\Omega})$ are obtained for the Neumann problem (1.1)-(1.2). The uniqueness of solutions in $C^{1,1}(\bar{\Omega})$ is discussed. The generalization of the main results to the prescribed Gauss curvature equation is also indicated.

2. Maximum modulus estimate and gradient estimate. In this section, we shall establish maximum modulus estimate for positive f and gradient estimate for nonnegative f for convex solutions of the Monge-Ampère equations with the Neumann boundary value condition (1.1)-(1.2). Furthermore, these estimates are independent of the lower bound of f .

We first formulate the result for the maximum modulus estimates.

Theorem 2.1. *Let Ω be a C^1 bounded domain in \mathbb{R}^n and $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a strongly convex solution of (1.1)-(1.2). Assume that $\varphi_z \geq \gamma_0 > 0$ on $\partial\Omega \times \mathbb{R}$ and $f > 0$ in Ω . Then we have*

$$N_0 - R_0 \cdot \text{diam } \Omega \leq u \leq N_1, \quad (2.1)$$

in $\bar{\Omega}$, where $N_0 = -(R_0 + \max_{\partial\Omega} |\varphi(x, 0)|)/\gamma_0$, $N_1 = \max_{\partial\Omega} |\varphi(x, 0)|/\gamma_0$ and $R_0 = (|\Omega| \max_{\bar{\Omega}} f/\omega_n)^{\frac{1}{n}}$ with ω_n being the volume of unit ball in \mathbb{R}^n .

Proof. We first show the upper bound of u in $\bar{\Omega}$. From the convexity of u , the maximum of u over $\bar{\Omega}$ must be obtained on $\partial\Omega$. Therefore, at the maximum point \bar{x} , we have $D_\nu u(\bar{x}) \leq 0$, namely $\varphi(\bar{x}, u(\bar{x})) \leq 0$. We can always assume $u(\bar{x}) > 0$, otherwise we have already obtained the upper bound of u in $\bar{\Omega}$ since $u(x) \leq u(\bar{x}) \leq 0$ for $x \in \bar{\Omega}$. By the mean value theorem, it follows from (1.2), $\varphi_z \geq \gamma_0 > 0$ and $u(\bar{x}) > 0$ that

$$0 \geq \varphi(\bar{x}, u(\bar{x})) = \varphi(\bar{x}, 0) + \varphi_z(\bar{x}, \tilde{u})u(\bar{x}) \geq -\max_{\partial\Omega} |\varphi(x, 0)| + \gamma_0 u(\bar{x}), \tag{2.2}$$

where $\tilde{u} = \theta u(\bar{x})$ for some $\theta \in (0, 1)$. Therefore, from (2.2) we have

$$u(x) \leq u(\bar{x}) \leq N_1, \quad \text{for } x \in \bar{\Omega}. \tag{2.3}$$

Then we show the lower bound of u in $\bar{\Omega}$. Since $\det D^2u$ is the Jacobian of the gradient map $Du : \Omega \rightarrow \mathbb{R}^n$, we have

$$\int_{Du(\Omega)} 1dy = \int_{\Omega} \det D^2u dx = \int_{\Omega} f dx \leq |\Omega| \max_{\bar{\Omega}} f < \int_{B_R(0)} 1dy = \omega_n R^n, \tag{2.4}$$

for any positive constant $R > R_0$, where the equation (1.1) is used in the second equality, $B_R(0)$ is the ball centered at 0 with the radius R . The inequality in (2.4) implies that the set $B_R(0) - Du(\Omega)$ is not empty. We now choose a vector $p \in B_R(0) - Du(\Omega)$. Let w be an affine function such that $w \leq u$ in Ω , $Dw = p$ and $w(x_0) = u(x_0)$ for some $x_0 \in \bar{\Omega}$. Clearly, such a point x_0 can only exist on $\partial\Omega$. We divide the proof into two cases.

Case 1. $u(x_0) \leq 0$. Then at x_0 , we have

$$-D_\nu u \leq -D_\nu w \leq |\nu||Dw| \leq R. \tag{2.5}$$

Hence, by the mean value theorem, we obtain from (1.2), $\varphi_z \geq \gamma_0 > 0$ and $u(x_0) \leq 0$ that

$$-R \leq \varphi(x_0, u(x_0)) = \varphi(x_0, 0) + \varphi_z(x_0, \tilde{u})u(x_0) \leq \max_{\partial\Omega} |\varphi(x, 0)| + \gamma_0 u(x_0), \tag{2.6}$$

where $\tilde{u} = \tilde{\theta} u(x_0)$ for some $\tilde{\theta} \in (0, 1)$. Hence, we have

$$u(x_0) \geq -(R + \max_{\partial\Omega} |\varphi(x, 0)|)/\gamma_0. \tag{2.7}$$

Case 2. $u(x_0) > 0$. In this case, the inequality (2.7) automatically holds.

Combining Cases 1 and 2, the lower bound (2.7) for $u(x_0)$ always holds. Assuming that the minimum of u over $\bar{\Omega}$ is obtained at a point x_1 , from the relationship between w and u , we have for $x \in \bar{\Omega}$

$$\begin{aligned} u(x) &\geq u(x_1) \geq w(x_1) = w(x_0) + p \cdot (x_1 - x_0) = u(x_0) + p \cdot (x_1 - x_0) \\ &\geq -(R + \max_{\partial\Omega} |\varphi(x, 0)|)/\gamma_0 - R \cdot \text{diam}\Omega. \end{aligned} \tag{2.8}$$

Letting $R \rightarrow R_0$, we get

$$u(x) \geq N_0 - R_0 \cdot \text{diam}\Omega, \quad \text{for } x \in \bar{\Omega}. \tag{2.9}$$

Then the conclusion (2.1) of Theorem 2.1 holds by combining (2.3) and (2.9). \square

Remark 4. The maximum modulus estimates has been established in [16] under three structural conditions of φ which can be deduced by the condition $\varphi_z \geq \gamma_0 > 0$. We just show some supplement of the proof and get the precise constants R, N_0 and N_1 in (2.1) under the condition $\varphi_z \geq \gamma_0 > 0$ here.

For the Neumann problem of the Monge-Ampère equation, the result of global gradient estimate is already known in [16, 13, 14]. We state it here for later use.

Theorem 2.2. *Let Ω be a $C^{1,1}$ bounded domain in \mathbb{R}^n , $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a convex solution of (1.1)-(1.2), and $f \geq 0$ in Ω . Then we have*

$$\sup_{\Omega} |Du| \leq C, \tag{2.10}$$

where C depends on $\varphi, |u|_{0;\Omega}$ and Ω .

Theorem 2.2 is a special case of Theorem 2.2 of [16] when $\beta = \nu$ and $M = \max |\varphi(x, u)|$. We shall omit its proof. Alternatively, such global gradient estimate (2.10) can also be found in the context of more general Monge-Ampère type equations, see Lemma 3.2 in [13] and Theorem 1.1 in [14].

3. Second order derivative estimates. In this section, we prove the uniform second derivative estimate in Theorem 1.1. We first prove the mixed tangential normal derivative estimate and the double normal derivative estimate on the boundary in §3.1. The former estimate can be obtained by tangentially differentiating the Neumann boundary value condition. The latter estimate established in Theorem 3.2 is independent of $\inf f$, which relies on a subtle barrier argument since $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$. Then in §3.2 we construct a global auxiliary function to get the global estimates, where the key trick of [16] is used.

Since the assumption $\psi := f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$ will be used in the second derivative estimate, we recall the following key property under this assumption, which is proved in [2] and used in [7].

Lemma 3.1. *Let Ω is a bounded domain in \mathbb{R}^n and let $\psi \in C^{1,1}(\bar{\Omega})$ be a nonnegative function. Then $\sqrt{\psi}$ is locally Lipschitz in Ω . For almost every $x \in \Omega$, we have*

$$\left| D\sqrt{\psi}(x) \right| \leq \max \left\{ \frac{|D\psi(x)|}{2\text{dist}(x, \partial\Omega)}, \frac{1 + \sup_{\Omega} \lambda_{\max}(D^2\psi)}{2} \right\},$$

where $\lambda_{\max}(D^2\psi)$ denotes the maximum eigenvalue of the Hessian matrix of ψ .

For f in Theorem 1.1, by Lemma 3.1, we know that $\sqrt{\psi} = f^{\frac{1}{2(n-1)}}$ is locally Lipschitz in Ω . In the case when $f > 0$ in Ω and $f = 0$ at some boundary point, from condition (1.4) we have $|Df^{\frac{1}{2(n-1)}}| \leq C$ in $\Omega_{\delta} := \{x \in \Omega \mid \text{dist}(x, \partial\Omega) \leq \delta\}$, which shows that $\sqrt{\psi} = f^{\frac{1}{2(n-1)}}$ is Lipschitz in Ω_{δ} . Thus, $\sqrt{\psi} = f^{\frac{1}{2(n-1)}}$ is globally Lipschitz in $\bar{\Omega}$ in this case. While in the case when $f > 0$ in $\bar{\Omega}$, from $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$ it is obvious that $\sqrt{\psi} = f^{\frac{1}{2(n-1)}}$ is globally Lipschitz in $\bar{\Omega}$. In conclusion, $f^{\frac{1}{2(n-1)}} \in C^{0,1}(\bar{\Omega})$ under the assumptions of Theorem 1.1, which implies that

$$\frac{|Df(x)|}{f(x)} \leq C f^{-\frac{1}{2(n-1)}}(x) \tag{3.1}$$

can hold in Ω for a uniform constant C . Actually, by a direct calculation, we can also have a relaxed version of the estimate (3.1),

$$\frac{|Df(x)|}{f(x)} = (n-1)|Df^{\frac{1}{n-1}}(x)|f^{-\frac{1}{n-1}}(x) \leq C f^{-\frac{1}{n-1}}(x), \quad \text{in } \Omega, \tag{3.2}$$

for the constant C depending on $\|f^{\frac{1}{n-1}}\|_{C^{0,1}(\bar{\Omega})}$ and n .

The inequalities (3.1) and (3.2) show the relationships between $|D(\log f)|$ and $f^{-\frac{1}{2(n-1)}}$, and between $|D(\log f)|$ and $f^{-\frac{1}{n-1}}$, respectively. Both the estimates (3.1)

and (3.2) are used to obtain the global second order derivative estimate in this section. Note that the estimate (3.1) is only used in reducing the global second order derivative estimate to the corresponding estimate on the boundary.

3.1. Estimates on the boundary. In this subsection, we will establish the mixed tangential normal derivative bound and the double normal derivative bound on $\partial\Omega$. For the mixed tangential normal second order derivatives of u on $\partial\Omega$, we apply the tangential operator to the boundary value condition (1.2). The bound of double normal derivative of u on $\partial\Omega$ is obtained by constructing a suitable barrier function near a boundary point.

Supposing that $f > 0$ in Ω and D^2u is positive definite in Ω , we can write the equation (1.1) in the form

$$\log \det D^2u = \log f := \tilde{f}, \quad \text{in } \Omega. \tag{3.3}$$

Differentiating (3.3) in the direction $\xi \in \mathbb{R}^n$ once and twice respectively, we have

$$\sum_{i,j=1}^n u^{ij} D_\xi u_{ij} = D_\xi \tilde{f}, \tag{3.4}$$

and

$$\sum_{i,j=1}^n u^{ij} D_{\xi\xi} u_{ij} = \sum_{i,j,s,t=1}^n u^{is} u^{jt} D_\xi u_{ij} D_\xi u_{st} + D_{\xi\xi} \tilde{f} \geq D_{\xi\xi} \tilde{f}, \tag{3.5}$$

where $\{u^{ij}\}$ is the inverse of the Hessian matrix $D^2u = \{u_{ij}\}$, and the concavity of $\log \det$ is used in the inequality.

Mixed tangential normal derivative estimate on $\partial\Omega$:

We first estimate the mixed tangential normal second order derivatives of u on $\partial\Omega$. We introduce the tangential gradient operator $\delta = (\delta_1, \dots, \delta_n)$, where $\delta_i = (\delta_{ij} - \nu_i \nu_j) D_j$. Applying the tangential operator to the boundary condition (1.2), we have

$$(D_k u)(\delta_i \nu_k) + \nu_k (\delta_i D_k u) = \delta_i \varphi, \quad \text{on } \partial\Omega. \tag{3.6}$$

If τ is a direction tangential to $\partial\Omega$ at any point $y \in \partial\Omega$, we have

$$\begin{aligned} D_{\tau\nu} u(y) &= \sum_{i,k=1}^n \tau_i \nu_k D_{ik} u = \sum_{i,k=1}^n \tau_i \nu_k \delta_i D_k u = \sum_{i,k=1}^n \tau_i \delta_i \varphi - \tau_i (\delta_i \nu_k) D_k u \\ &= \sum_{i,k=1}^n \tau_i D_i \varphi - \tau_i (D_i \nu_k) D_k u, \end{aligned} \tag{3.7}$$

where the fact $\tau \cdot \nu = 0$ is used in the second and the last equality, and (3.6) is used to get the second line. Hence, we get the bound for the second derivative of u in the mixed tangential normal directions, namely

$$|D_{\tau\nu} u(y)| \leq C, \quad \text{for } y \in \partial\Omega. \tag{3.8}$$

Double normal derivative estimate on $\partial\Omega$:

We next deduce the double normal derivative of u on $\partial\Omega$. We may consider at any boundary point. Without loss of generality, we may take the boundary point to be the origin and the x_n -axis to be the interior normal.

We construct the auxiliary function

$$W(x) = \pm[D_\nu u - \varphi(x, u)] - \tilde{B}x_n + B|x|^2, \quad \text{in } S_\epsilon := \{x \in \Omega \mid x_n < \epsilon\}, \tag{3.9}$$

where \tilde{B} and B are positive constants to be determined, ϵ is a fixed small positive constant. Here, S_ϵ is an open subset in Ω , the unit inner normal vector $\nu =$

(ν_1, \dots, ν_n) and the function $\varphi(\cdot, u(\cdot))$ have been smoothly extended from $\partial\Omega$ to \bar{S}_ϵ . We define the linearized operator

$$L = \sum_{i,j=1}^n u^{ij} D_{ij}. \tag{3.10}$$

By the definition of L in (3.10), for any $x \in S_\epsilon$, we obtain

$$\begin{aligned} LW &= \pm \sum_{i,j,k=1}^n u^{ij} (D_{ij}\nu_k D_k u + D_j \nu_k D_{ik} u + \nu_k D_{ijk} u) \\ &\mp \sum_{i,j=1}^n u^{ij} (\varphi_{x_i x_j} + 2\varphi_{x_j z} u_i + \varphi_{zz} u_i u_j + \varphi_z u_{ij}) + 2B \sum_{i=1}^n u^{ii} \\ &\geq (2B - C) \sum_{i=1}^n u^{ii} - C \sum_{k=1}^n |D_k \tilde{f}| - C, \end{aligned} \tag{3.11}$$

where the first inequality is established by (3.4), C is a positive constant depending on the known data. In the following calculation, the constant C changes from line to line. By the arithmetic-geometric mean inequality, we have

$$\sum_{i=1}^n u^{ii} \geq n \left(\sup_{\Omega} f \right)^{-\frac{1}{n}}. \tag{3.12}$$

From (3.11) and (3.12), we have, for any $x \in S_\epsilon$,

$$LW \geq (2B - C) \sum_{i=1}^n u^{ii} - C f^{-\frac{1}{n-1}}, \tag{3.13}$$

for a further constant C , where the inequality (3.2) is used.

We shall decompose $S_\epsilon = S_\epsilon^1 \cup S_\epsilon^2$, where

$$S_\epsilon^1 := S_\epsilon \cap \{x \in \Omega \mid \Delta u \geq n\}, \quad S_\epsilon^2 := S_\epsilon \cap \{x \in \Omega \mid \Delta u < n\}.$$

If $\max_{\bar{S}_\epsilon} W = W(\bar{x})$ for some $\bar{x} \in \bar{S}_\epsilon$. We can split the discussion into the following

three cases according to the position of \bar{x} , namely $\bar{x} \in S_\epsilon^1$, or $\bar{x} \in S_\epsilon^2$, or neither.

Case 1. $\bar{x} \in S_\epsilon^1$. Since $\max_{\bar{S}_\epsilon} W = W(\bar{x})$ for $\bar{x} \in S_\epsilon^1$, then, at \bar{x} , we have

$$D_i W = 0, \text{ for } i = 1, \dots, n, \quad \text{and } (D_{ij} W)_{n \times n} \leq 0. \tag{3.14}$$

Therefore, we obtain

$$LW(\bar{x}) \leq 0, \tag{3.15}$$

where L is the operator in (3.10). We choose new coordinates y_1, \dots, y_n near \bar{x} , which is an orthonormal linear transform of $\bar{x}_1, \dots, \bar{x}_n$, such that the matrix $\left\{ \frac{\partial^2 u}{\partial y_\beta \partial y_\gamma} \right\}$ is diagonal at \bar{x} . Since $\sum_{\beta=1}^n \frac{\partial^2 u}{\partial y_\beta^2}(\bar{x}) = \Delta u(\bar{x}) \geq n$, there exists $1 \leq \sigma \leq n$, such that

$u_{\sigma\sigma}(\bar{x}) = \frac{\partial^2 u}{\partial y_\sigma^2}(\bar{x}) \geq 1$. At the point \bar{x} , we have

$$\begin{aligned} \sum_{i=1}^n u^{ii}(\bar{x}) &= \text{tr}\{u^{ij}(\bar{x})\} = \text{tr}\left\{ \{u_{ij}(\bar{x})\}^{-1} \right\} = \text{tr}\left\{ \{u_{\gamma\beta}(\bar{x})\}^{-1} \right\} = \text{tr}\{u^{\gamma\beta}(\bar{x})\} \\ &= \sum_{\beta=1}^n u^{\beta\beta}(\bar{x}) \geq \sum_{\beta \neq \sigma}^n u^{\beta\beta}(\bar{x}) \geq (n-1) \left(\prod_{\beta \neq \sigma} u^{\beta\beta}(\bar{x}) \right)^{\frac{1}{n-1}} \end{aligned}$$

$$\begin{aligned} &= (n-1) \left(\prod_{\beta=1}^n u^{\beta\beta}(\bar{x}) \right)^{\frac{1}{n-1}} (u^{\sigma\sigma}(\bar{x}))^{-\frac{1}{n-1}} \\ &= (n-1) (\det(u^{\gamma\beta}(\bar{x})))^{\frac{1}{n-1}} (u^{\sigma\sigma}(\bar{x}))^{-\frac{1}{n-1}} \geq (n-1) f^{-\frac{1}{n-1}}(\bar{x}). \end{aligned} \tag{3.16}$$

Since $\bar{x} \in S_\epsilon^1 \subset S_\epsilon$, the inequality in (3.13) also holds at \bar{x} . Inserting (3.16) into (3.13), by choosing $B > Cn/2(n-1)$, we obtain

$$LW(\bar{x}) \geq \left(2B - \frac{Cn}{n-1} \right) \sum_{i=1}^n u^{ii} > 0, \tag{3.17}$$

which leads to a contradiction with (3.15). Therefore, we get $\bar{x} \notin S_\epsilon^1$.

Case 2. $\bar{x} \in S_\epsilon^2$. Since $\max_{S_\epsilon} W = W(\bar{x})$, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in S_\epsilon^2$, we obtain

$$D_n W(\bar{x}) = 0. \tag{3.18}$$

For the unit inner normal vector $\nu = (\nu_1, \dots, \nu_n)$, we must have

$$0 \leq |\nu_i| \leq 1, \text{ for } i = 1, \dots, n. \tag{3.19}$$

By a calculation, we have at \bar{x} ,

$$0 = D_n W = \pm \left[\sum_{i=1}^n (\nu_i D_{in} u + D_n \nu_i D_i u) - \varphi_{x_n} - \varphi_z D_n u \right] - \tilde{B} + 2B\bar{x}_n. \tag{3.20}$$

From (3.19) and (3.20), we have at \bar{x} ,

$$\tilde{B} \pm \left[\varphi_{x_n} + \varphi_z D_n u - \sum_{i=1}^n D_n \nu_i D_i u \right] - 2B\bar{x}_n = \pm \sum_{i=1}^n \nu_i D_{in} u \leq \sum_{i=1}^n |D_{in} u|. \tag{3.21}$$

Choosing \tilde{B} sufficiently large, such that

$$\tilde{B} > (1 + \sup_{\Omega} |Du|) \|\varphi\|_{C^1(\bar{\Omega})} + \sup_{\Omega} |Du| \sum_{i=1}^n \sup_{\Omega} |D_n \nu_i| + 2B \text{diam} \Omega + n(n+1), \tag{3.22}$$

we can get, at \bar{x} ,

$$\tilde{B} \pm \left[\varphi_{x_n} + \varphi_z D_n u - \sum_{i=1}^n D_n \nu_i D_i u \right] - 2B\bar{x}_n > n(n+1). \tag{3.23}$$

For the right hand side of (3.21), we have

$$\sum_{i=1}^n |D_{in} u| \leq \sum_{i=1}^n \sqrt{u_{ii} u_{nn}} \leq \sum_{i=1}^n \frac{u_{ii} + u_{nn}}{2} \leq \frac{n+1}{2} \Delta u, \tag{3.24}$$

at \bar{x} , where the first inequality holds by the positivity of $D^2 u$, the second inequality holds from the arithmetic-geometric mean inequality. By (3.21), (3.23) and (3.24), we can get $\Delta u \geq 2n$ at \bar{x} , which is a contradiction with $\bar{x} \in S_\epsilon^2$. Therefore, we get $\bar{x} \notin S_\epsilon^2$.

Case 3. $\bar{x} \in \partial S_\epsilon$. On $\partial S_\epsilon \cap \{x \in \Omega \mid x_n = \epsilon\}$, we have

$$W = \pm(D_\nu u - \varphi(x, u)) - \tilde{B}\epsilon + B|x|^2 \leq -\tilde{B}\epsilon + C \leq 0, \tag{3.25}$$

where we choose $\tilde{B} > C/\epsilon$ such that the last inequality is established. On $\partial S_\epsilon \cap \partial\Omega$, by the boundary condition (1.2) and the uniform convexity of the domain Ω , we have

$$W = -\tilde{B}x_n + B|x|^2 \leq -\tilde{B}\kappa|x|^2 + B|x|^2 \leq 0, \quad (3.26)$$

for some positive constant κ , where we choose $\tilde{B} > B/\kappa$ such that the last inequality holds. Since $W(\bar{x}) = \max_{\bar{S}_\epsilon} W = \max_{\partial S_\epsilon} W \leq 0$, we have

$$W(x) \leq 0, \quad \text{for all } x \in S_\epsilon. \quad (3.27)$$

Since $W(0) = 0$, we have

$$D_\nu W(0) \leq 0. \quad (3.28)$$

Thus, by the construction of W in (3.9) (“ \pm ”), we have

$$|D_{\nu\nu}u(0)| \leq C. \quad (3.29)$$

Since the above argument for (3.29) can be carried out for any boundary point on $\partial\Omega$, we then have

$$|D_{\nu\nu}u| \leq C, \quad \text{on } \partial\Omega. \quad (3.30)$$

Remark 5. The auxiliary function W in (3.9) is critical in the proof of the estimate (3.30). In the above proof, the constants B and \tilde{B} in the auxiliary function W are successively chosen as follows. We first fix a large constant $B > Cn/2(n-1)$ such that (3.17) holds. We then choose a larger constant \tilde{B} depending B such that (3.22), $\tilde{B} > C/\epsilon$ and $\tilde{B} > B/\kappa$ hold. Then the required inequalities (3.23), (3.25) and (3.26) hold accordingly.

We now summarize the double normal derivative estimate on $\partial\Omega$ in the following theorem.

Theorem 3.2. *Let $u \in C^4(\Omega) \cap C^{1,1}(\bar{\Omega})$ be a strongly convex solution of the Neumann problem (1.1)-(1.2) in a $C^{3,1}$ domain $\Omega \subset \mathbb{R}^n$, which is bounded and uniformly convex. Suppose that f is a positive function in Ω , $\varphi \in C^{1,1}(\bar{\Omega} \times \mathbb{R})$ and $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$. Then, we have*

$$|D_{\nu\nu}u(x)| \leq C, \quad (3.31)$$

on $\partial\Omega$, where C depends on n , Ω , $|u|_{1;\Omega}$, $\|f^{\frac{1}{n-1}}\|_{C^{1,1}(\bar{\Omega})}$ and $\|\varphi\|_{C^{1,1}(\Omega \times (-M, M))}$, where $M = \sup_\Omega |u|$.

Note that in the proof of Theorem 3.2, we use the inequality (3.2) rather than (3.1). Therefore, here we do not need to assume the condition (1.4) in the double normal derivative estimate on $\partial\Omega$.

In this subsection, we have proved the mixed tangential normal derivative estimate (3.8), as well as the double normal derivative estimate (3.31). Combining these estimates, we have already proved

$$|D_{\xi\nu}u| \leq C, \quad \text{for any } \xi \in \mathbb{S}^{n-1}, \quad (3.32)$$

on $\partial\Omega$, where ν is the unit inner normal vector field on $\partial\Omega$.

3.2. Global estimates. In this subsection, we construct a global barrier function, which is used to reduce the global estimates to the boundary estimates. By the trick in [16] and the boundary estimates in §3.2, we can obtain the global second derivative estimate in Theorem 1.1.

Proof of Theorem 1.1. We employ the auxiliary function

$$G(x, \xi) = D_{\xi\xi}u(x) - v'(x, \xi) + A|x|^2, \quad (3.33)$$

where v' is given by

$$v'(x, \xi) = 2(\xi \cdot \nu)\xi'_i(D_i\varphi - D_k u D_i \nu_k). \quad (3.34)$$

Here ν is a $C^{2,1}(\bar{\Omega})$ extension of the inner unit normal vector field on $\partial\Omega$, $\xi \in \mathbb{S}^{n-1}$, ξ' is given by

$$\xi' = \xi - (\xi \cdot \nu)\nu, \quad (3.35)$$

and A is a positive constant to be determined. We can denote v' in the following form

$$v'(x, \xi) = a_k D_k u + b, \quad (3.36)$$

where

$$a_k = 2(\xi \cdot \nu)(\varphi_z \xi'_k - \xi'_i D_i \nu_k), \quad b = 2(\xi \cdot \nu)\xi'_k \varphi_{x_k}. \quad (3.37)$$

We claim that G only attains its maximum on $\partial\Omega$. Assume by contradiction that G attains its maximum at $x_0 \in \Omega$ and $\xi = \xi_0$. At x_0 , we obtain

$$D_i G = 0, \text{ for } i = 1, \dots, n, \quad (D_{ij}G)_{n \times n} \leq 0. \quad (3.38)$$

Differentiating (3.33) once and twice respectively, we have

$$D_i G = D_{i\xi\xi}u - D_i a_k D_k u - a_k D_{ik}u - D_i b + 2A x_i, \quad (3.39)$$

$$D_{ij}G = D_{ij\xi\xi}u - D_{ij}a_k D_k u - D_i a_k D_{jk}u - D_j a_k D_{ik}u - a_k D_{ijk}u - D_{ij}b + 2A \delta_{ij}. \quad (3.40)$$

Then, at x_0 , it follows from the convexity of u and (3.38) that

$$LG \leq 0, \quad (3.41)$$

where L is defined in (3.10). By (3.40), we have

$$\begin{aligned} LG &= \sum_{i,j=1}^n u^{ij} D_{ij\xi\xi}u - \sum_{i,j,k=1}^n a_k u^{ij} D_{ijk}u - 2 \sum_{i,j,k=1}^n u^{ij} D_{ik}u D_j a_k \\ &\quad - \sum_{i,j,k=1}^n u^{ij} D_{ij}a_k D_k u - \sum_{i,j=1}^n u^{ij} D_{ij}b + 2A \sum_{i=1}^n u^{ii} \\ &\geq (2A-C) \sum_{i=1}^n u^{ii} - C + D_{\xi\xi} \tilde{f} - \sum_{i,j,k=1}^n a_k u^{ij} D_{ijk}u \\ &\geq (2A-C) \sum_{i=1}^n u^{ii} + D_{\xi\xi} \tilde{f} - C \sum_{k=1}^n |D_k \tilde{f}| \\ &= (2A-C) \sum_{i=1}^n u^{ii} + \left[\frac{D_{\xi\xi} f}{f} - \frac{n-2}{n-1} \left(\frac{D_{\xi} f}{f} \right)^2 \right] - \frac{1}{n-1} \left(\frac{D_{\xi} f}{f} \right)^2 - C \sum_{k=1}^n \frac{|D_k f|}{f}, \end{aligned} \quad (3.42)$$

where the first inequality is valid by (3.5), the second inequality holds by (3.4) and (3.12). Here the constant C changes from line to line. Using $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$ and (3.1), then

$$\frac{D_{\xi\xi}f}{f} - \frac{n-2}{n-1} \left(\frac{D_{\xi}f}{f}\right)^2 \geq -Cf^{-\frac{1}{n-1}}, \tag{3.43}$$

and

$$-\frac{1}{n-1} \left(\frac{D_{\xi}f}{f}\right)^2 \geq -Cf^{-\frac{1}{n-1}}, \tag{3.44}$$

hold in Ω , respectively. Using (3.43), (3.44) and (3.2), we get from (3.42) that

$$LG \geq (2A - C) \sum_{i=1}^n u^{ii} - Cf^{-\frac{1}{n-1}}. \tag{3.45}$$

Without loss of generality, we can assume that $\{u_{ij}\}$ is diagonal at x_0 . We can assume that $D_{\xi_0\xi_0}u(x_0) \geq n$, otherwise we can get the estimates. Since $\xi_0 = (\xi_{01}, \dots, \xi_{0n})$ is a unit vector, we have

$$\Delta u \geq \sum_{i=1}^n u_{ii}\xi_{0i}^2 = \sum_{i,j=1}^n u_{ij}\xi_{0i}\xi_{0j} = D_{\xi_0\xi_0}u \geq n, \tag{3.46}$$

at x_0 . Then we may assume that $u_{11}(x_0) \geq 1$. Accordingly, we have, at x_0 ,

$$\sum_{i=1}^n u^{ii} \geq (n-1) \left(\prod_{i=2}^n u^{ii}\right)^{\frac{1}{n-1}} = (n-1) \left(\prod_{i=1}^n u^{ii}\right)^{\frac{1}{n-1}} (u_{11})^{\frac{1}{n-1}} \geq (n-1)f^{-\frac{1}{n-1}}. \tag{3.47}$$

Here since $\{u_{ij}\}$ is already diagonal at x_0 , the deduction of (3.47) is a bit simpler than that of (3.16). By (3.45) and (3.47), we have

$$LG \geq (2A - C) \sum_{i=1}^n u^{ii}, \tag{3.48}$$

for a further constant C . Taking $A > C/2$, we obtain at x_0 ,

$$LG > 0, \tag{3.49}$$

which contradicts with (3.41). Thus, the claim is proved.

Therefore, G attains its maximum at a point $x_0 \in \partial\Omega$. The estimation of the rest of the Hessian D^2u splits into two stages according to different directions of ξ .

(i). ξ tangential. Applying the tangential gradient operator δ_i to the boundary condition (1.2) twice, we obtain

$$D_k u \delta_i \delta_j \nu_k + \delta_i \nu_k \delta_j D_k u + \delta_j \nu_k \delta_i D_k u + \nu_k \delta_i \delta_j D_k u = \delta_i \delta_j \varphi, \quad \text{on } \partial\Omega. \tag{3.50}$$

Hence, at x_0 , we have

$$\begin{aligned} D_{\xi\xi\nu}u &= \nu_k \xi_i \xi_j D_{ijk}u \geq -2(\delta_i \nu_k) D_{jk}u \xi_i \xi_j + \varphi_z D_{ij}u \xi_i \xi_j - C \\ &\geq -2(\delta_i \nu_k) D_{jk}u \xi_i \xi_j - C \geq 2\kappa u_{\xi\xi} - C, \end{aligned} \tag{3.51}$$

where the first inequality is valid by $\xi \cdot \nu = 0$, (3.8) and (3.31), the second inequality is established by $\varphi_z \geq 0$ and the convexity of u , the third inequality is using the fact that $(\delta_i \nu_k) \leq -\kappa I$ for some positive constant κ by virtue of the uniform convexity of Ω .

Since G attains its maximum at x_0 , we have

$$0 \geq D_{\nu}G = D_{\xi\xi\nu}u - a_k D_{k\nu}u - (D_{\nu}a_k) D_k u - D_{\nu}b + 2A(x \cdot \nu) \geq D_{\xi\xi\nu}u - C, \tag{3.52}$$

at x_0 , where (3.32) is used to obtain the inequality. Therefore, from (3.52), we obtain at x_0 ,

$$D_{\xi\xi\nu}u \leq C. \tag{3.53}$$

Combining (3.51) and (3.53), we have

$$D_{\xi\xi}u(x_0) \leq C. \tag{3.54}$$

(ii). ξ non-tangential. We write

$$\xi = \alpha\tau + \beta\nu, \tag{3.55}$$

where $\alpha = \xi \cdot \tau$, $|\tau| = 1$, $\tau \cdot \nu = 0$, $\beta = \xi \cdot \nu \neq 0$ and

$$\alpha^2 + \beta^2 = 1. \tag{3.56}$$

Therefore, at x_0 , we have

$$D_{\xi\xi}u = \alpha^2 D_{\tau\tau}u + \beta^2 D_{\nu\nu}u + 2\alpha\beta D_{\tau\nu}u = \alpha^2 D_{\tau\tau}u + \beta^2 D_{\nu\nu}u + v'(x, \xi). \tag{3.57}$$

From (3.57), we obtain

$$G(x_0, \xi) = \alpha^2 G(x_0, \tau) + \beta^2 G(x_0, \nu) \leq \alpha^2 G(x_0, \xi) + \beta^2 G(x_0, \nu), \tag{3.58}$$

where $G(x_0, \tau) \leq G(x_0, \xi)$ is used. Hence, we get

$$G(x_0, \xi) \leq G(x_0, \nu). \tag{3.59}$$

Using (3.33), we get

$$D_{\xi\xi}u(x_0) \leq C + D_{\nu\nu}u(x_0) \leq C, \tag{3.60}$$

where (3.31) in Theorem 3.2 is used again.

Combining (i) and (ii), we now get the estimate for $D_{\xi\xi}u$ at the maximum point x_0 of G and the direction ξ . Then the conclusion (1.5) of Theorem 1.1 can be readily obtained. \square

Remark 6. If u_{ii} has a positive lower bound for some i , $1 \leq i \leq n$, we can build the relationship between $\sum_{i=1}^n u^{ii}$ and $f^{-\frac{1}{n-1}}$ by using equation (1.1), see (3.16) and (3.47) for example. Otherwise, we should divide the discussion into two cases as in case 1 and case 2 in §3.1, which is inspired by the argument in [9] for the Dirichlet problem.

4. Existence and uniqueness. In this section, we will prove Theorem 1.2 and Corollary 1. Combining the estimates in Section 2 and Section 3, the existence of $C^{1,1}$ convex solutions of the problem (1.1)-(1.2) is established by smooth approximations. In the approximation scheme, the right hand term of the equation may not satisfy the regularity assumption. Therefore, it is necessary to make the mollification such that the right hand term satisfies the regularity assumption as in Theorem 1.1. The uniqueness of convex solutions in $C^{1,1}(\bar{\Omega})$ under the condition $\varphi_z \geq \gamma_0 > 0$ will also be discussed.

Proof of Theorem 1.2. Suppose $f \geq 0$ and $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$. For any $\varepsilon > 0$, let

$$g_\varepsilon = f^{\frac{1}{n-1}} + \varepsilon, \quad \text{and} \quad f_\varepsilon = g_\varepsilon^{n-1}. \tag{4.1}$$

We may assume $f_\varepsilon \in C^3(\bar{\Omega})$. Otherwise, we can use the smoothing process. Namely that for all $\rho > 0$, define

$$\tilde{g}_{\varepsilon,\rho} = g_\varepsilon * \phi_\rho, \quad \text{and} \quad \tilde{f}_{\varepsilon,\rho} = \tilde{g}_{\varepsilon,\rho}^{n-1}, \tag{4.2}$$

where $\phi_\rho(x) = \phi(|x|/\rho)$, $\phi(A) \in C_0^\infty(\mathbb{R})$, $\phi(A) \geq 0$ and $\int_{\mathbb{R}} \phi(A)dA = 1$. We have $f_{\varepsilon,\rho} \in C^3(\bar{\Omega})$. We claim that $f_{\varepsilon,\rho}^{\frac{1}{n-1}} = \tilde{g}_{\varepsilon,\rho} \in C^{1,1}(\bar{\Omega})$. Indeed, we have

$$\begin{aligned} |\tilde{g}_{\varepsilon,\rho}| &= |g_\varepsilon * \phi_\rho| \leq (|f|^{\frac{1}{n-1}} + 1), \\ |D\tilde{g}_{\varepsilon,\rho}| &= |Dg_\varepsilon * \phi_\rho| \leq |f^{\frac{1}{n-1}}|_{C^{0,1}(\bar{\Omega})} * \phi_\rho = |f^{\frac{1}{n-1}}|_{C^{0,1}(\bar{\Omega})}, \end{aligned} \tag{4.3}$$

and

$$|\Delta\tilde{g}_{\varepsilon,\rho}| = |\Delta g_\varepsilon * \phi_\rho| \leq |f^{\frac{1}{n-1}}|_{C^{1,1}(\bar{\Omega})} * \phi_\rho = |f^{\frac{1}{n-1}}|_{C^{1,1}(\bar{\Omega})}, \tag{4.4}$$

for $\rho \leq \delta$, $x \in \{x \in \Omega | \text{dist}(x, \partial\Omega) \leq \delta\}$, where δ is a small positive constant. If f vanishes at some boundary point, since f satisfies condition (1.4) near the boundary, $f_{\varepsilon,\rho}$ also satisfies condition (1.4) with the same constants δ and C (independent of ε and ρ).

Now we have $f_{\varepsilon,\rho}^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$ uniformly for all $\varepsilon > 0$, $0 < \rho \leq \delta$. Since $\varphi_z \geq \gamma_0 > 0$, we consider the following problem

$$\begin{cases} \det D^2u = f_{\varepsilon,\rho}, & \text{in } \Omega, \\ D_\nu u = \varphi(x, u), & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

By the result in [16], there exists $u_{\varepsilon,\rho} \in C^4(\Omega)$ for $\varepsilon > 0$ and $0 < \rho \leq \delta$. Furthermore, by Theorem 1.1, Theorem 2.1 and Theorem 2.2, we have

$$\|u_{\varepsilon,\rho}\|_{C^2(\bar{\Omega})} \leq C, \tag{4.6}$$

where the constant C is independent of ε, ρ . First, letting $\rho \rightarrow 0$, by convexity of $u_{\varepsilon,\rho}$, (passing to a subsequence if necessary), we obtain a convex solution u_ε , with

$$\begin{cases} \det D^2u_\varepsilon = f_\varepsilon, & \text{in } \Omega, \\ D_\nu u_\varepsilon = \varphi(x, u_\varepsilon), & \text{on } \partial\Omega, \end{cases} \tag{4.7}$$

and $\|u_\varepsilon\|_{C^2(\bar{\Omega})} \leq C$, where the constant C is independent of ε . Now letting $\varepsilon \rightarrow 0$, the same argument gives a convex solution u , with

$$\|u\|_{C^{1,1}(\bar{\Omega})} \leq C. \tag{4.8}$$

The uniqueness of convex solutions in $C^{1,1}(\Omega) \cap C^1(\bar{\Omega})$ can be proved under the condition $\varphi_z \geq \gamma_0 > 0$. Suppose that $u, v \in C^{1,1}(\Omega) \cap C^1(\bar{\Omega})$ are two convex solutions. We claim that

$$u \geq v, \quad \text{in } \bar{\Omega}. \tag{4.9}$$

Indeed, for $\tau > 0$ we suppose that $v - (u - \frac{\tau}{2}|x|^2)$ attains its positive maximum at a point $x_0 \in \Omega$, namely

$$v(x_0) - \left[u(x_0) - \frac{\tau}{2}|x_0|^2 \right] = \max_{\bar{\Omega}} \left[v - \left(u - \frac{\tau}{2}|x|^2 \right) \right] > 0. \tag{4.10}$$

If $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$, at the point $x_0 \in \Omega$, we must have

$$D^2v + \tau I \leq D^2u, \tag{4.11}$$

where I denotes the $n \times n$ identity matrix. Therefore, we have

$$f(x_0) = \det D^2v(x_0) < \det[D^2v + \tau I](x_0) \leq \det D^2u(x_0) = f(x_0), \tag{4.12}$$

which leads to a contradiction. Then, $v - (u - \frac{\tau}{2}|x|^2)$ can only take its positive maximum on $\partial\Omega$, namely

$$\sup_{\bar{\Omega}} \left[v - \left(u - \frac{\tau}{2}|x|^2 \right) \right] \leq \sup_{\partial\Omega} \left[v - \left(u - \frac{\tau}{2}|x|^2 \right) \right]^+. \tag{4.13}$$

Letting $\tau \rightarrow 0$ in (4.13), we get

$$\sup_{\Omega}(v - u) \leq \sup_{\partial\Omega}(v - u)^+, \tag{4.14}$$

in the case when $u, v \in C^2(\Omega) \cap C^1(\bar{\Omega})$. While in the general case when $u, v \in C^{1,1}(\Omega) \cap C^1(\bar{\Omega})$, we can use the Bony’s maximum principle in [3] so that (4.11) and (4.12) hold in the limit sense. We still get the same inequality (4.14).

Now the function $v - u$ can only attain its positive maximum at a point $x_0 \in \partial\Omega$. Using the Neumann boundary value condition and $\varphi_z \geq \gamma_0 > 0$, we have

$$0 \geq D_\nu(v - u)(x_0) = \varphi(x_0, v(x_0)) - \varphi(x_0, u(x_0)) > 0, \tag{4.15}$$

which is a contradiction and implies

$$v - u \leq 0, \quad \text{on } \partial\Omega. \tag{4.16}$$

Combining (4.14) and (4.16), we have proved the claim (4.9). By exchanging u and v , we can also have $u \leq v$ in $\bar{\Omega}$. We now complete the proof of the uniqueness of convex solutions in $C^{1,1}(\Omega) \cap C^1(\bar{\Omega})$. Therefore, the convex solution of the problem (1.1)-(1.2) in the function space $C^{1,1}(\bar{\Omega})$ is also unique.

In conclusion, the proof of Theorem 1.2 is completed. □

Corollary 1 is a direct consequence of Theorem 1.2.

Proof of Corollary 1. When $c_0 > 0$, it is obvious that $c_0|x|^\alpha \geq 0$ and $(c_0|x|^\alpha)^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$ for $\alpha \geq 2(n - 1)$. If $0 \in \partial\Omega$, by taking

$$C = \frac{\alpha}{n - 1} c_0^{\frac{1}{2(n-1)}} (\text{diam}\Omega)^{\frac{\alpha}{2(n-1)} - 1}, \tag{4.17}$$

we can check that

$$\left| D(c_0|x|^\alpha)^{\frac{1}{n-1}} \right| \leq C (c_0|x|^\alpha)^{\frac{1}{2(n-1)}} \tag{4.18}$$

holds for any $x \in \Omega$.

When $c_0 = 0$, the equation is totally degenerate. In this case, the right hand side is automatically nonnegative and the $n - 1$ root of the right hand side belongs to $C^{1,1}(\bar{\Omega})$. Moreover, condition (1.4) also holds since both sides of (1.4) are zero.

Therefore, in both cases, the result of Theorem 1.2 can be applied. Thus, Corollary 1 is proved. □

Finally, we add some remarks to finish this paper.

Remark 7. For the uniqueness of convex solutions in Theorem 1.2, we can alternatively prove by using either the theory of generalized solutions or the theory of viscosity solutions. For the uniqueness of generalized solutions for the problem (1.1)-(1.2), one can refer to [25]. For the uniqueness of viscosity solutions, one can refer to Section 4 of [12]. Indeed, in the proof of Theorem 4.3(i) in [12], the idea of Ishii and Lions in [11] and the Jensen’s approximation are used, which are particularly fit for the Neumann type boundary value condition.

Remark 8. In [12], the existence and uniqueness of solutions in $C^{1,1}(\Omega) \cap C^{0,1}(\bar{\Omega})$ of the semilinear oblique derivative problem for the general degenerate augmented Hessian equations is studied, namely

$$\begin{cases} F[D^2u - A(x, u, Du)] = B(x, u, Du), & \text{in } \Omega, \\ D_\beta u = \varphi(x, u), & \text{on } \partial\Omega, \end{cases} \tag{4.19}$$

where Ω is a bounded domain, F is a general concave elliptic operator in some cone $\Gamma \subset \mathbb{S}^n$, $A \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is a matrix valued function which is uniformly regular, $B \in C^{1,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is a nonnegative scalar function, β is unit vector filed on $\partial\Omega$ satisfying $\beta \cdot \nu > 0$. Note that the results in the current paper are not covered by the results in [12], since the matrix $A \equiv 0$ in (1.1) does not satisfy the uniform regular condition. Furthermore, if $F = \det^{\frac{1}{n}}$ and B is independent of u and Du , then the condition $B \in C^{1,1}(\bar{\Omega})$ in [12] can be seen as $f^{\frac{1}{n}} \in C^{1,1}(\bar{\Omega})$, where f satisfies the equation $\det[D^2u - A(x, u, Du)] = f(x)$ in Ω . The assumption $f^{\frac{1}{n}} \in C^{1,1}(\bar{\Omega})$ appears natural for obtaining the C^2 estimates for the Monge-Ampère equation in [4, 15, 23]. While comparing with the condition $f^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$, the condition $f^{\frac{1}{n}} \in C^{1,1}(\bar{\Omega})$ is more restrictive, see [9, 10]. Therefore, the results of this paper are different from those in [12].

Remark 9. The results of this paper can be directly extended to the equation $\det D^2u = f(x)g(u, Du)$ in Ω with $f \geq 0$ in $\bar{\Omega}$, $g > 0$ and $g_z \geq 0$ in $\mathbb{R} \times \mathbb{R}^n$. When $f(x) = K(x) \geq 0$ and $g = (1 + |Du|^2)^{\frac{n+2}{2}}$, it is the prescribed Gauss curvature equation

$$\det D^2u = K(x)(1 + |Du|^2)^{\frac{n+2}{2}}, \quad \text{in } \Omega. \quad (4.20)$$

Then there exists a unique convex solution $u \in C^{1,1}(\bar{\Omega})$ of the Neumann problem (4.20)-(1.2) for K satisfying the same conditions as f in Theorem 1.2 and $\int_{\Omega} K < \omega_n$, where ω_n is the volume of unit ball in \mathbb{R}^n .

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