ETH in 2d CFTs, condensation of zeros in virasoro blocks

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Thermalization in isolated quantum systems

thermal state: micro-canonical ensemble

$$\rho^{\mathrm{micro}} = N^{-1} \sum |E_n\rangle\langle E_n| \qquad E_n \in [E - \Delta E, E + \Delta E]$$

unitary evolution for isolated quantum systems

$$\Psi(t) = U(t)\Psi_0$$

pure states —> pure states ~~**thermalization**~~

Eigenstate Thermalization Hypothesis (ETH)

Deutsch'91 Srednicki'94'99 Rigol et al' 08

energy eigenstate ~ **micro-canonical**

observables

Eigenstate Thermalization Hypothesis (ETH)

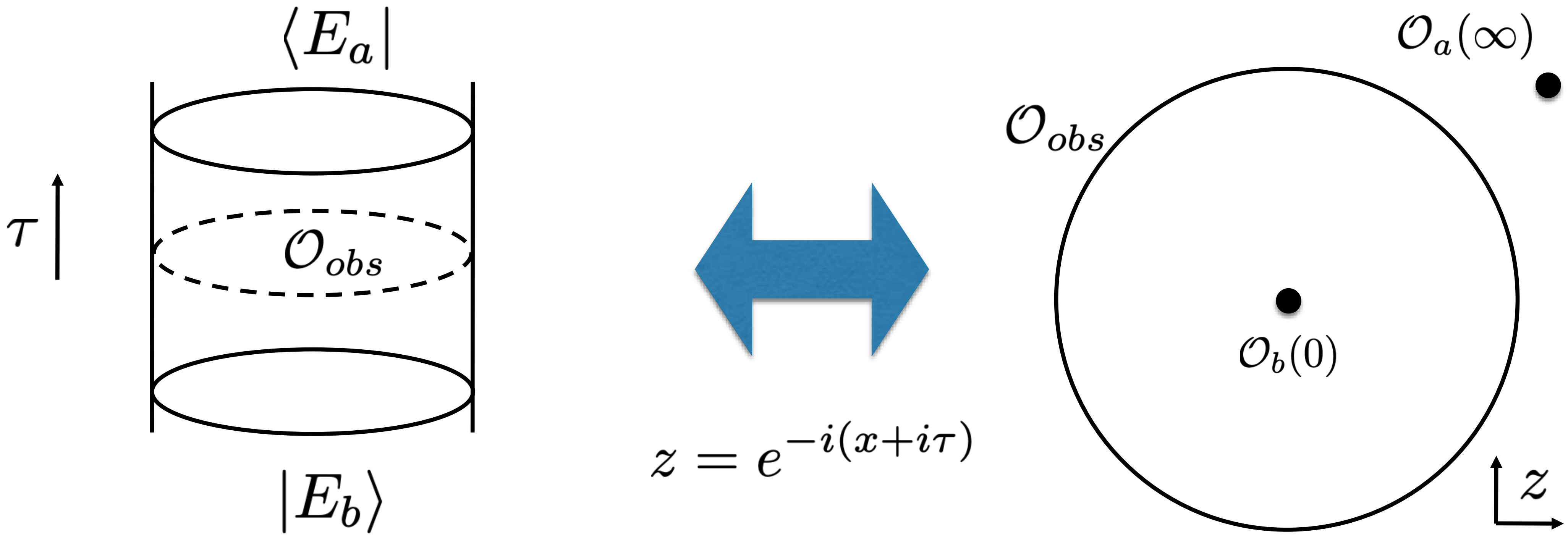
Deutsch’91 Srednicki’94’99 Rigol et al’ 08

energy eigenstate $\sim$ **micro-canonical**

observables

$$\langle E_a|\mathcal{O}_{obs}|E_b\rangle = f_{\mathcal{O}}(E)\delta_{ab} + e^{-S(E)/2}R_{ab}$$

in 2d CFTs: state-operator correspondence:



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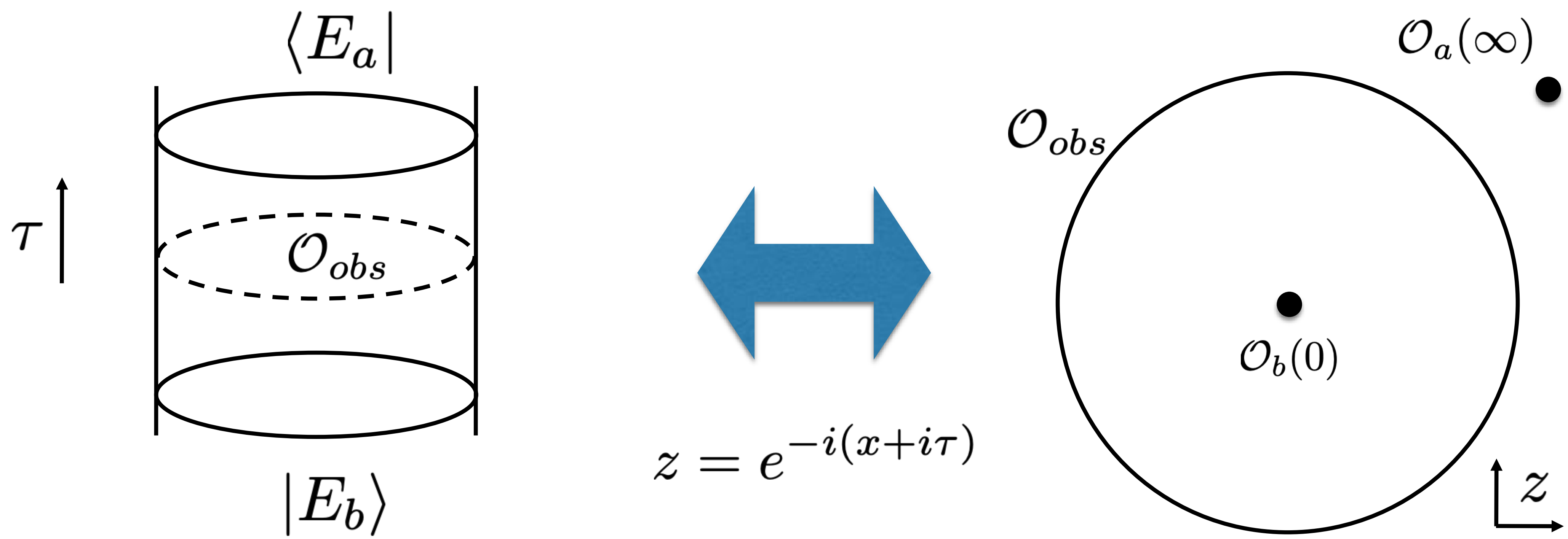
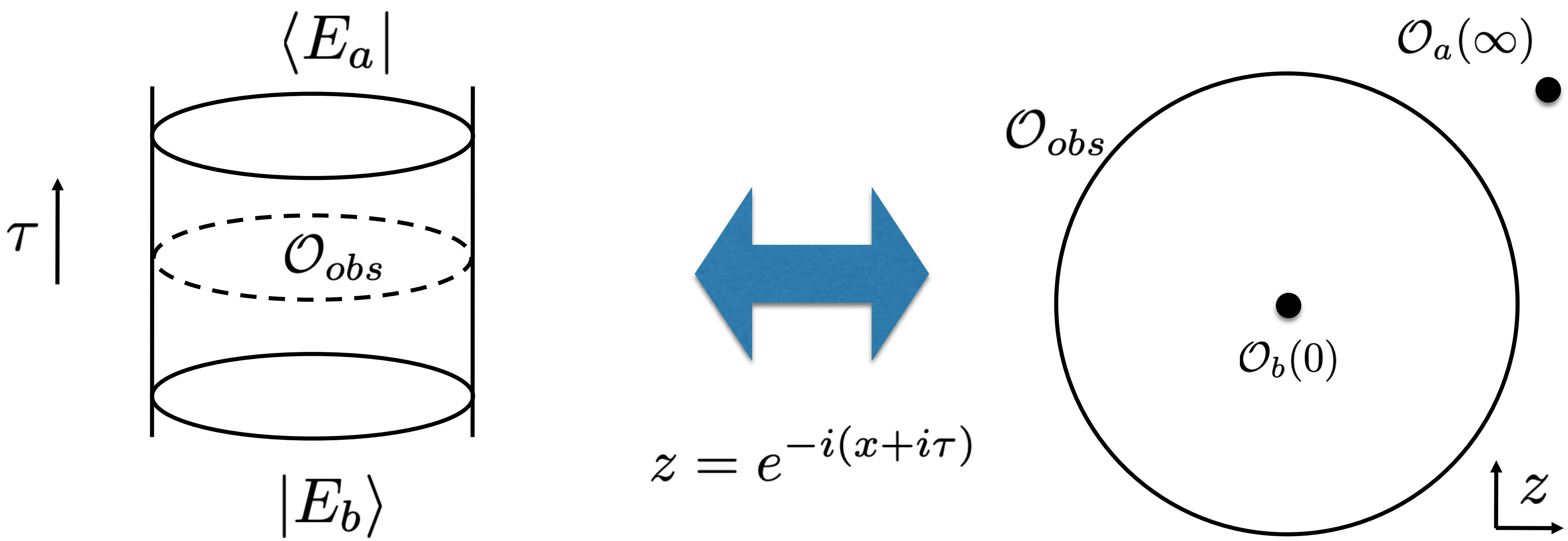


$$\langle E_a|\mathcal{O}_{obs}|E_b\rangle \propto \langle \mathcal{O}_b(0)\mathcal{O}_{obs}\mathcal{O}_a(\infty)\rangle$$

in 2d CFTs: state-operator correspondence:



$$\langle E_a|\mathcal{O}_{obs}|E_b\rangle \propto \langle \mathcal{O}_b(0)\mathcal{O}_{obs}\mathcal{O}_a(\infty)\rangle$$

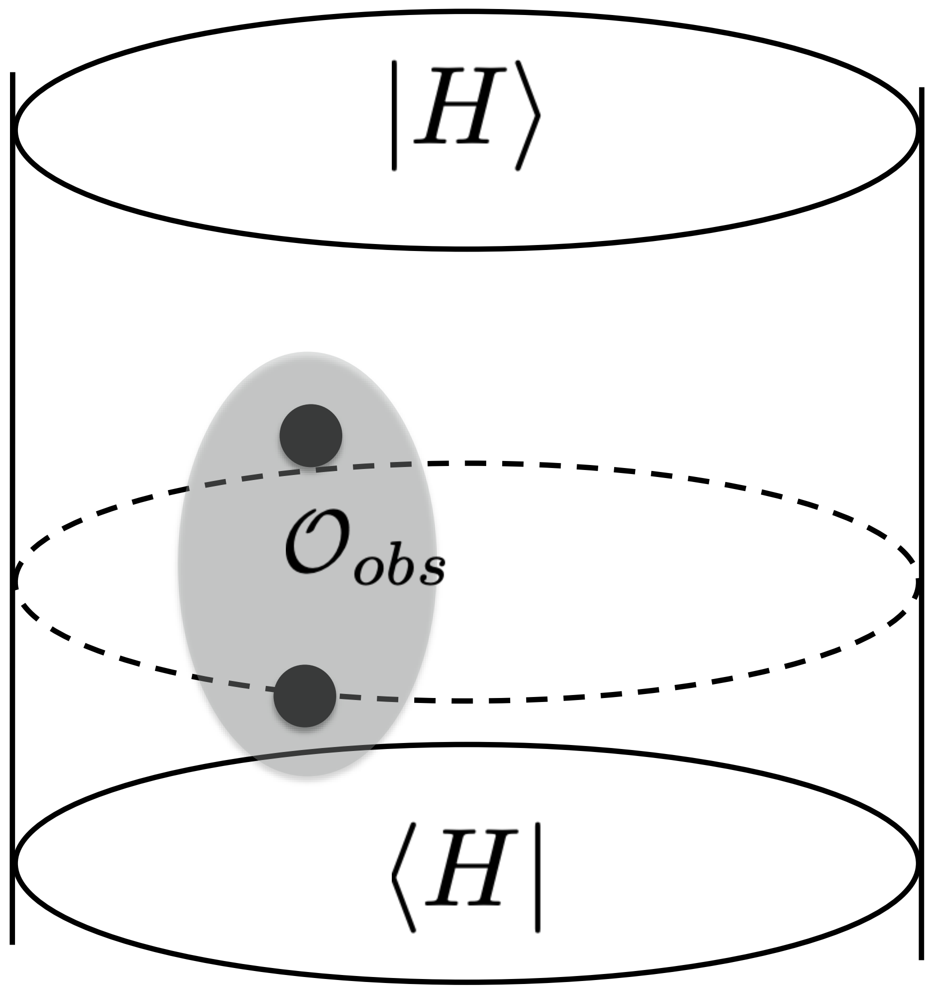
when $\mathcal{O}_{obs}$ **is a single primary operator**

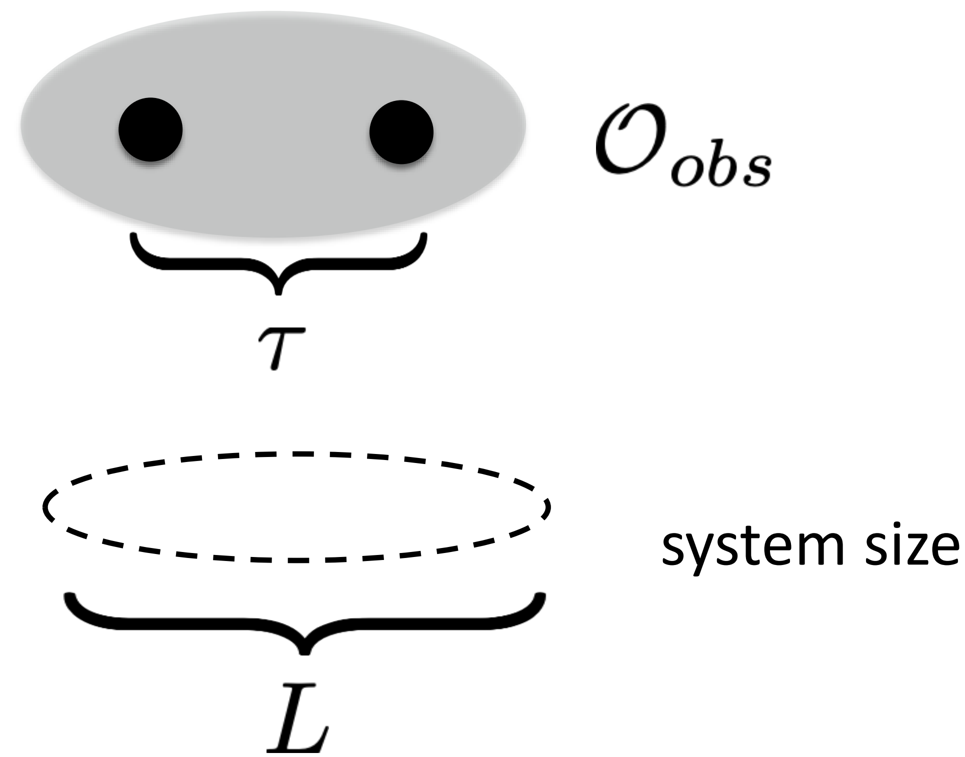
ETH → **constraints on OPE coefficients**

N. Lashkari, et al 2016

our interest: $\mathcal{O}_{obs} \propto \mathcal{O}_L(\tau)\mathcal{O}_L(0)$

bi-local observable $\mathcal{O}_{obs} \propto \mathcal{O}_L(\tau)\mathcal{O}_L(0)$





$|H\rangle \sim \rho_{\beta_H}$ "effective temperature"

$|H\rangle \sim \mathcal{O}_H|0\rangle$ dimension h_H

our interest: $\mathcal{O}_{obs} \propto \mathcal{O}_L(\tau)\mathcal{O}_L(0)$

total energy: $E \propto \frac{h_H}{L}$ **energy density:** $\mathcal{E} \propto \frac{h_H}{L^2}$

"effective temperature": $\mathcal{E}_T \propto cT^2 \to T_H L \propto \sqrt{\frac{h_H}{c}}$

Thermodynamic limit:

$L \to \mathcal{O}(1)$, $c \to \infty$, $h_H/c \gg 1$ **but finite**

$\beta_H \ll L$, τ/β_H **finite**

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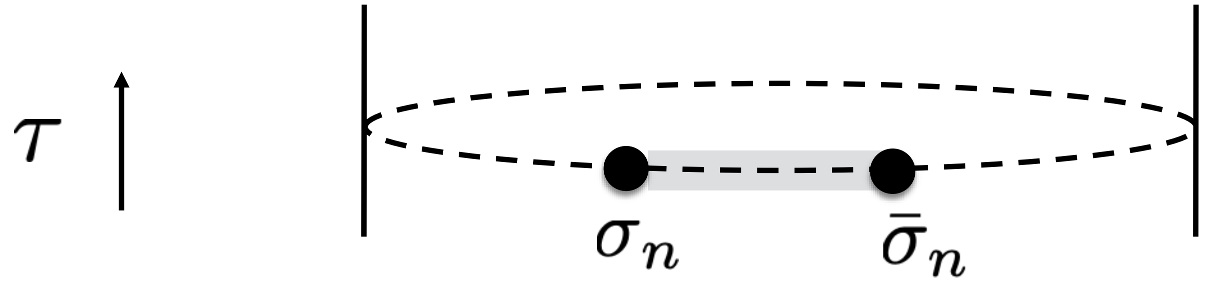

renyi-entropy

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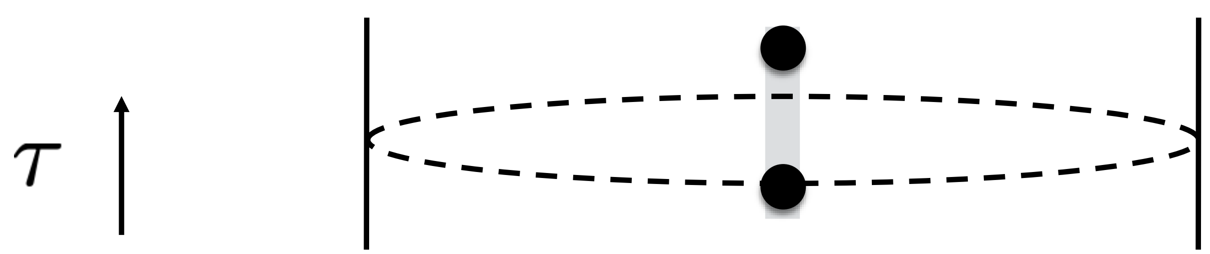


our interest

"forbidden singularities"

radial quantization: $x = 1 - e^{-\tau}$

$$\langle H|\mathcal{O}_{abs}|H\rangle \propto \langle \mathcal{O}_L(0)\mathcal{O}_L(x)\mathcal{O}_H(1)\mathcal{O}_H(\infty)\rangle \equiv f(x)$$

consequence of ETH:

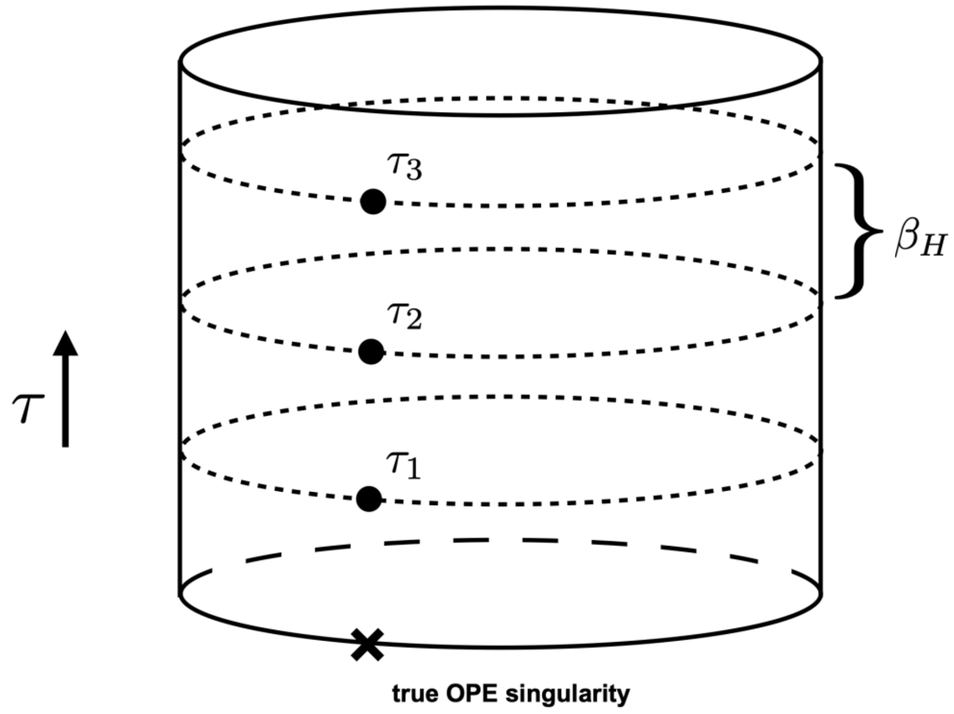
$$\langle H|\mathcal{O}_{obs}|H\rangle \approx \langle \mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle_{\beta_H}$$

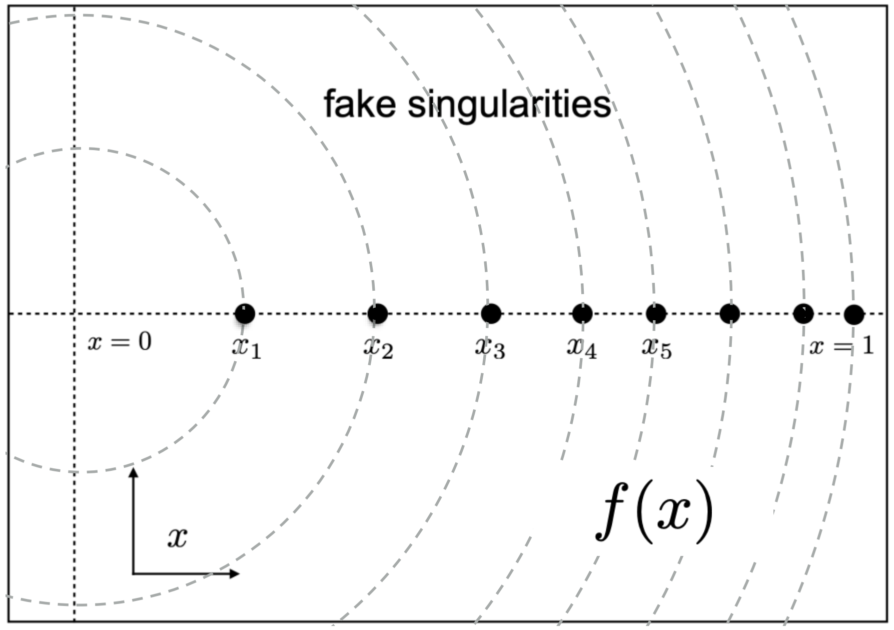
"forbidden singularities" in $f(x)$

"forbidden singularities"

ETH → **"emergent" periodicity in τ**

"thermal images" of OPE singularity:

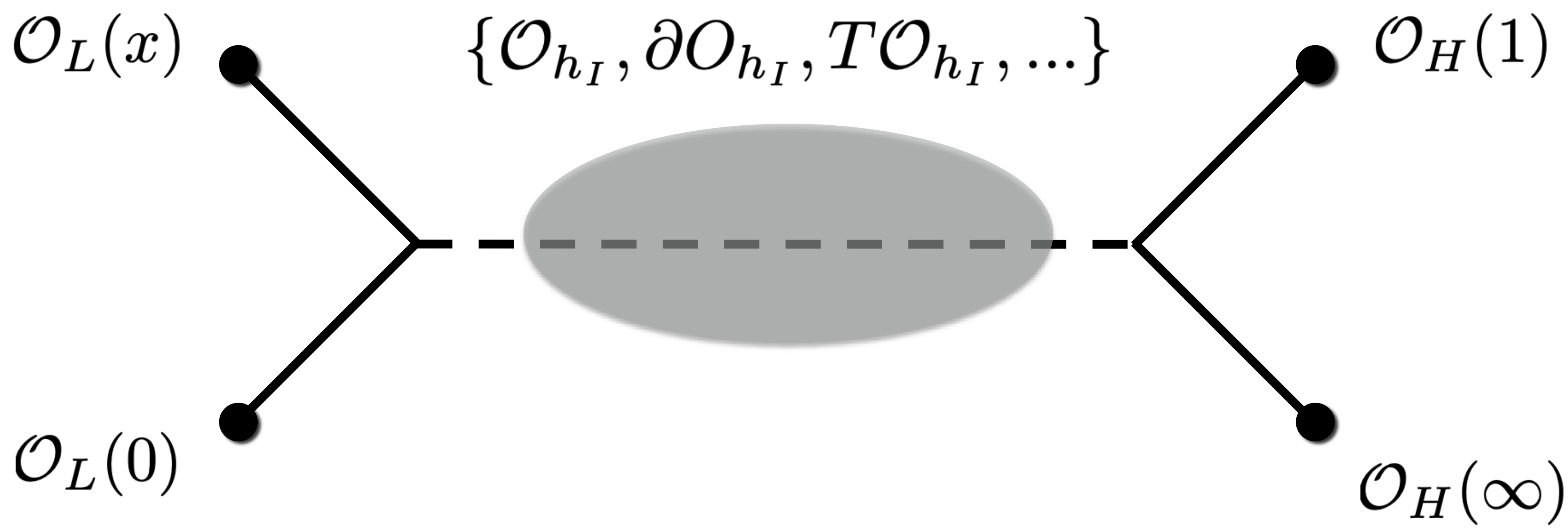




$$\tau_n = n\beta_H$$

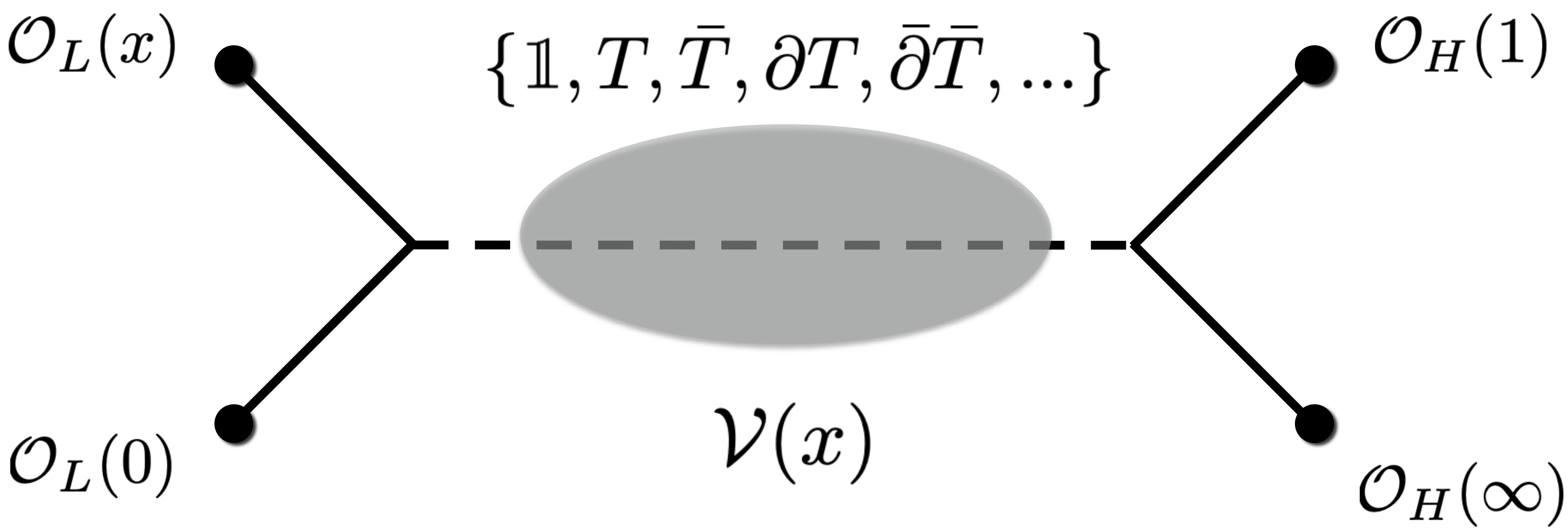
$$x_n = 1 - e^{-n\beta_H}$$

Block decomposition



$$f(x) = \sum_{h_I,\bar{h}_I} C^{LL}_{h_I,\bar{h}_I} C^{HH}_{h_I,\bar{h}_I} \mathcal{V}_{h_I}(x)\bar{\mathcal{V}}_{\bar{h}_I}(\bar{x}) \ , \ \ \mathcal{V}_{h_I}(x) \propto x^{-h_I}$$

Block decomposition

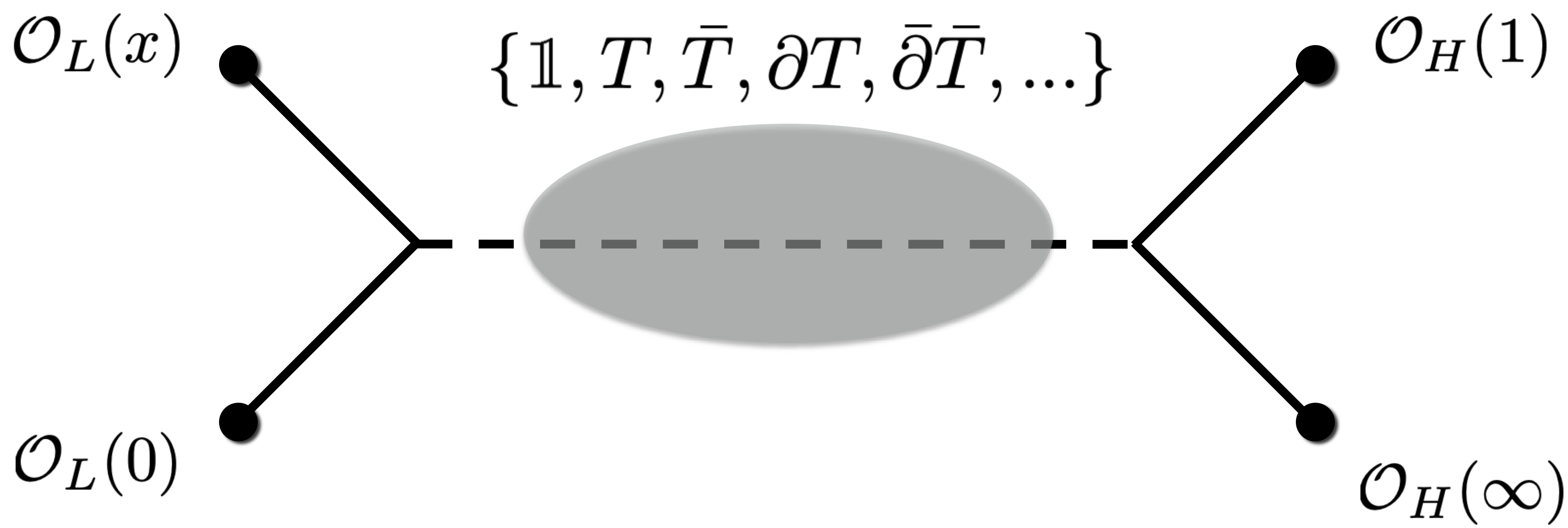


for CFTs with large c, sparse spectrum

dominance of the Virasoro vacuum block $|x| < 1$**:**

$$f(x) \approx \mathcal{V}(x)\bar{\mathcal{V}}(\bar{x})$$

Block decomposition



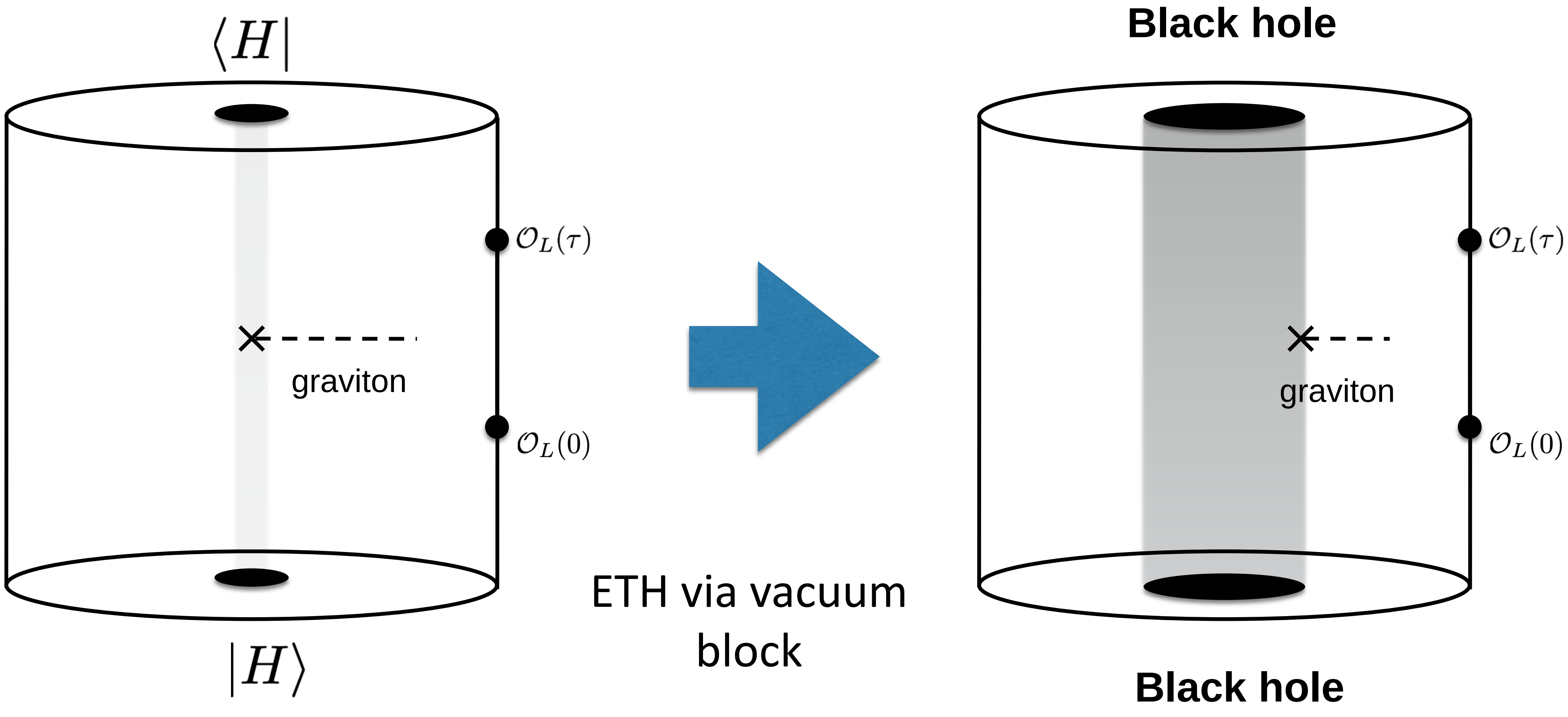
$c \to \infty,\ h_H = \frac{c}{6}\epsilon_H,\ h_L = \frac{c}{6}\epsilon_L$, **leading order in** ϵ_L

$\mathcal{V}(x)$ **is singular at** $x_n = 1 - e^{-n\beta_H}$, L. Fitzpatrick, et al'13

ETH encoded in the Virasoro vacuum blocks!

holographic picture

AdS3/CFT2: vacuum block $\equiv$ bulk graviton exchange



black hole information paradox

pure state $|H\rangle$ → mixed state $|BH\rangle$

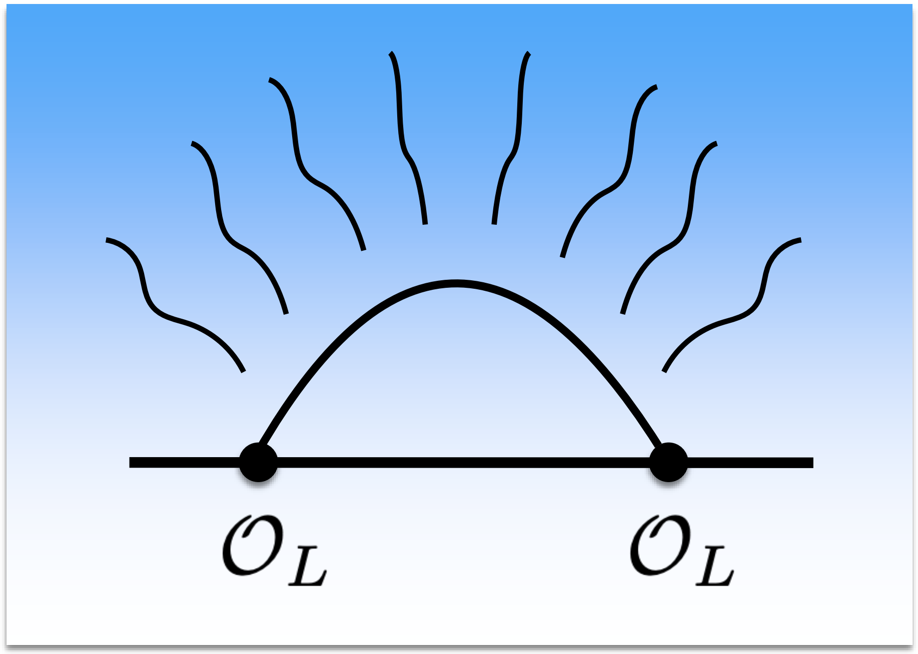
$f(x)$ analytic away from OPE singularities → additional singularities

"baby version" of the black hole information paradox

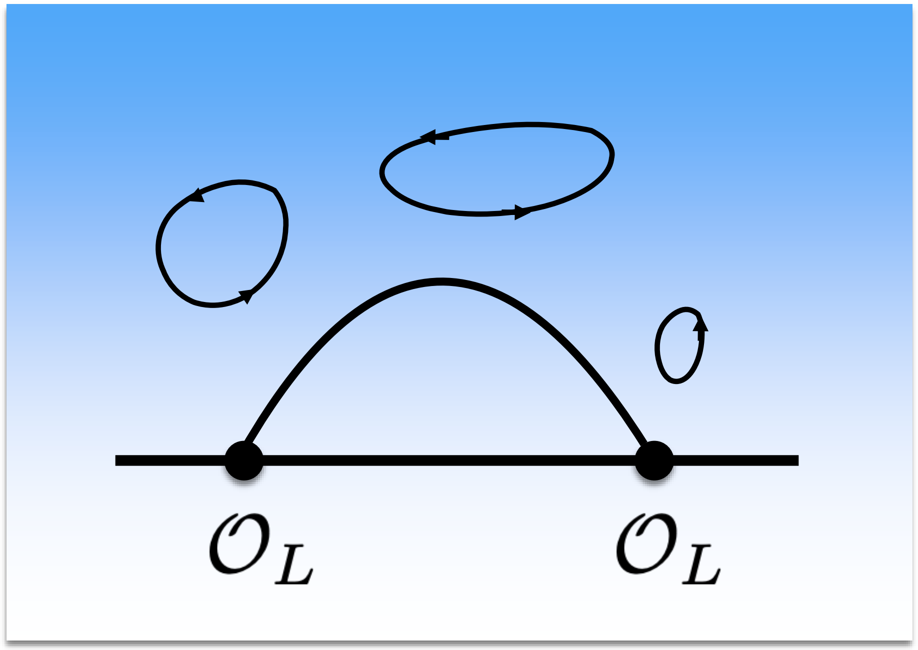
L. Fitzpatrick, et al'16

The Goal:

go beyond the thermodynamic limit, study corrections to both sides of ETH



$\epsilon_L \propto h_L/c$



$1/c$

resolving "forbidden singularities".

Outline

- ETH at leading order — “forbidden singularities”
- Resolution by “probe” corrections
- Resolution by finite c corrections
- Real time dynamics
- Conclusions/Future directions

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- **ETH at leading order — “forbidden singularities”**
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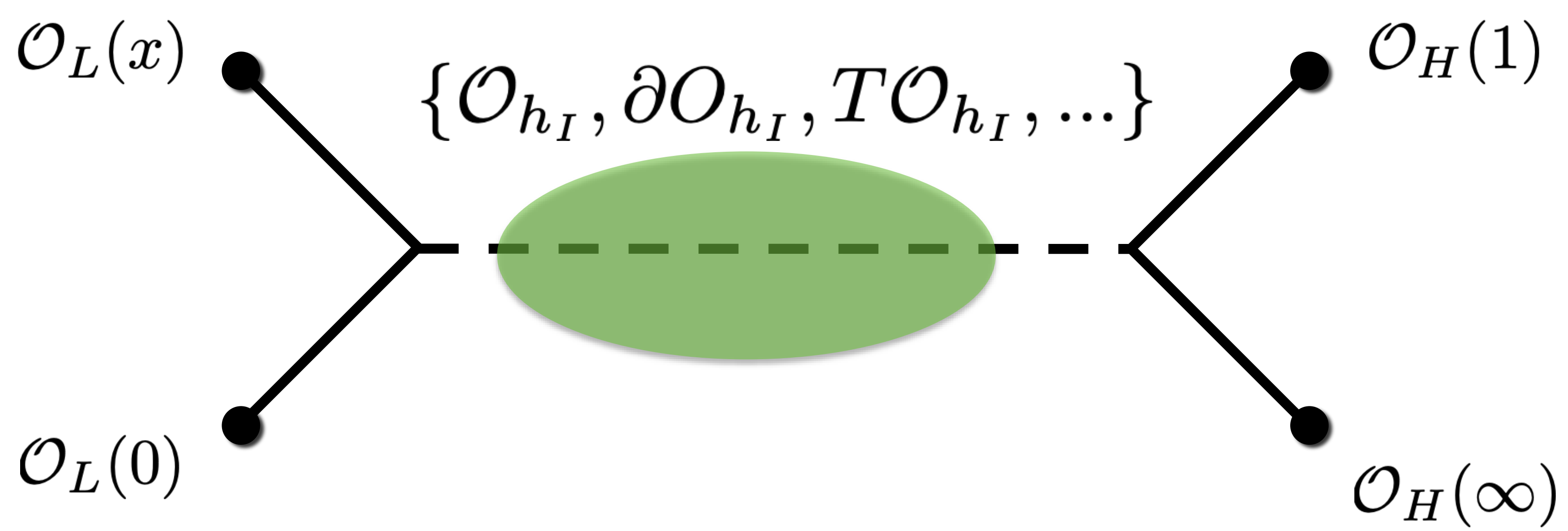
The method of monodromy



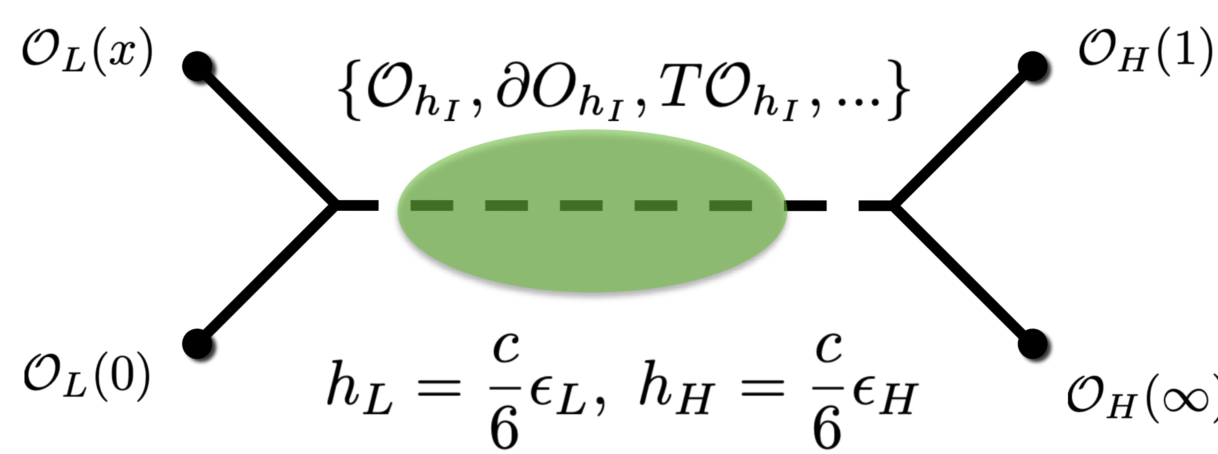
$$f(x) = \sum_{h_I,\bar{h}_I} C^{LL}_{h_I,\bar{h}_I} C^{HH}_{h_I,\bar{h}_I} \mathcal{V}_{h_I}(x) \bar{\mathcal{V}}_{\bar{h}_I}(\bar{x})$$

compute the virasoro block: $\mathcal{V}_{h_I}(x)$, $c \to \infty, h_i = \frac{c}{6}\epsilon_i$

The method of monodromy

Step 1:

solve the complex ODE



$$\Psi_h''(z) + T(z)\Psi_h(z) = 0$$

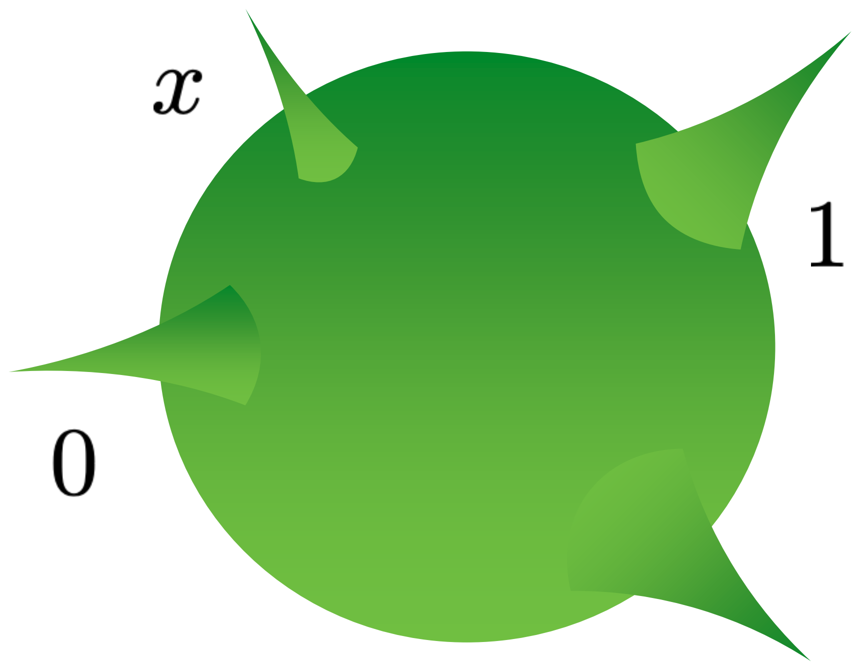
$$T(z) = \frac{\epsilon_L}{z^2} + \frac{\epsilon_L}{(z-x)^2} + \frac{\epsilon_H}{(1-z)^2} + \frac{2\epsilon_L}{z(1-z)} - \frac{p_x x(1-x)}{z(z-x)(1-z)}$$

→ two independent solutions $\{\Psi_h^+(z), \Psi_h^-(z)\}$

The method of monodromy

Step 2:

the ODE has regular singularities at $z = \{0, x, 1, \infty\}$



The method of monodromy

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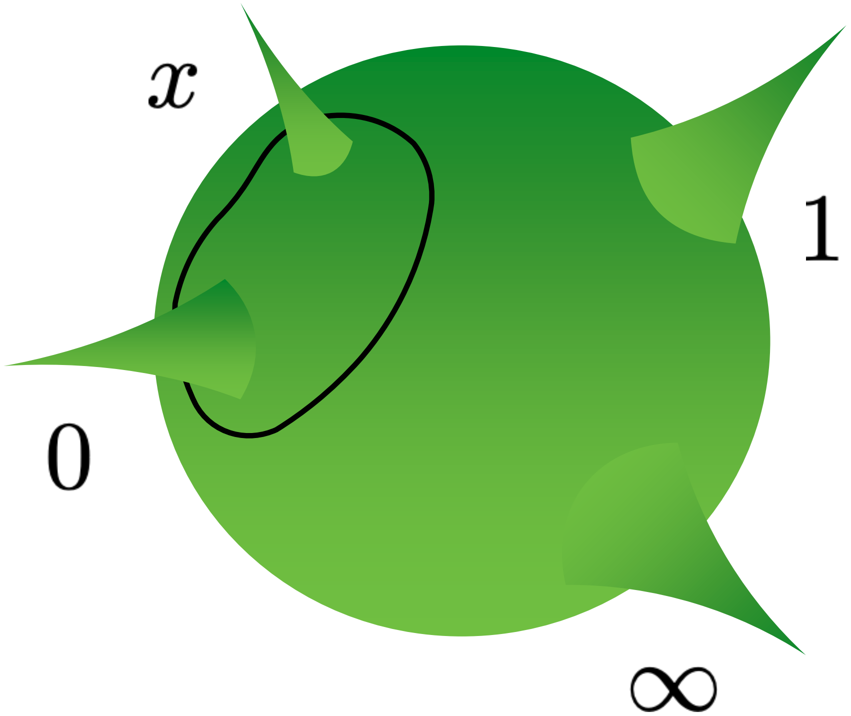

"Non-trivial monodromy matrix"

$$\begin{pmatrix} \Psi_h^+ \\ \Psi_h^- \end{pmatrix}_{\circlearrowleft} = \hat{M} \begin{pmatrix} \Psi_h^+ \\ \Psi_h^- \end{pmatrix}$$

The method of monodromy

Step 3:

View $\hat{M}_{0x}$ as a function $\hat{M}_{0x}(\epsilon_L, \epsilon_H, p_x, x)$

Solve the monodromy equation:

$$\boxed{\mathrm{tr}\hat{M}_{0x} = -2\cos(\pi\Lambda_h)\ , \quad h = \frac{c}{24}(1 - \Lambda_h^2)} \quad \Rightarrow \quad p_x(\epsilon_L, \epsilon_H, x)$$

integrate $\ln \mathcal{V}_h(x) = -\frac{c}{6}\int^x p_x$

ETH at leading order — “forbidden singularities”

Heavy-Light limit: $\epsilon_H \propto h_H/c$ **large;** $\epsilon_L \propto h_L/c$ **small**

ETH at leading order — “forbidden singularities”

Perturbative solution in ϵ_L

ETH at leading order — “forbidden singularities”

Perturbative solution in ϵ_L

expand in power series:

$$\Psi^{\pm}(z) = \Psi_0^{\pm}(z) + \epsilon_L \Psi_1^{\pm}(z) + \epsilon_L^2 \Psi_2^{\pm}(z) + \ldots$$

$$p_x = \epsilon_L p_x^0 + \epsilon_L^2 p_x^1 + \ldots$$

$$T(z) = T_0(z) + \epsilon_L T_1(z) + \epsilon_L^2 T_2(z) + \ldots$$

$$T_0(z) = \frac{\epsilon_H}{(1-z)^2}, \quad T_1(z) = \frac{1}{z^2} + \frac{1}{(z-x)^2} + \frac{2}{z(1-z)} - \frac{p_x^0 x(1-x)}{z(z-x)(1-z)}$$

$$T_n(z) = -\frac{p_x^n x(1-x)}{z(z-x)(1-z)}$$

ETH at leading order — "forbidden singularities"

Perturbative solution in ϵ_L

Monodromy equation for the vacuum block:

$$\mathrm{tr}\hat{M}_{0x}(p_x) = 2$$

$$\Downarrow$$

$$\epsilon_L M^0_{0x} + \epsilon_L^2 M^1_{0x} + \epsilon_L^3 M^2_{0x} + \ldots = 0$$

ETH at leading order — "forbidden singularities"

Perturbative solution in ϵ_L

Solving the monodromy equation order by order:

$$M^0_{0x}(p^0_x) = 0$$ $p^0_x = ?$ L. Fitzpatrick, et al'13

$$M^1_{0x}(p^0_x, p^1_x) = 0$$ $p^1_x = ?$ L. Fitzpatrick, et al'16

M. Beccaria, et al'16

$$M^2_{0x}(p^0_x, p^1_x, p^2_x) = 0$$ $p^2_x = ?$

... ...

$$p_x = \epsilon_L p^0_x + \epsilon_L^2 p^1_x + \ldots$$

ETH at leading order — "forbidden singularities"

Leading order solution:

L. Fitzpatrick, et al'13

$$M^0_{0x} \propto 1 - i\alpha_H + (x-1)^{i\alpha_H}(1+i\alpha_H) + \left[(1-x)^{i\alpha_H} - 1\right](x-1)p^0_x$$

$$p^0_x = \frac{i\alpha_H - 1 + (1-x)^{i\alpha_H}(1+i\alpha_H)}{\left[(1-x)^{i\alpha_H} - 1\right](x-1)} \qquad \alpha_H = \sqrt{4\epsilon_H - 1}$$

ETH at leading order — “forbidden singularities”

Leading order solution: L. Fitzpatrick, et al’13

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poles at $x = 0,\ 1,\ 1 - e^{-\frac{2\pi n}{\alpha_H}},\ n \in \mathbb{N}$

ETH at leading order — "forbidden singularities"

Leading order solution: L. Fitzpatrick, et al'13

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poles at $x = 0,\ 1,\ 1 - e^{-\frac{2\pi n}{\alpha_H}},\ n \in \mathbb{N}$

OPE singularities

ETH at leading order — "forbidden singularities"

Leading order solution: L. Fitzpatrick, et al'13

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poles at $x = 0,\ 1,\ 1 - e^{-\frac{2\pi n}{\alpha_H}},\ n \in \mathbb{N}$

"forbidden singularities"

ETH at leading order — "forbidden singularities"

Leading order solution:

L. Fitzpatrick, et al'13

$$M_{0x}^{0} \propto 1 - i\alpha_H + (x-1)^{i\alpha_H}(1+i\alpha_H) + \left[(1-x)^{i\alpha_H} - 1\right](x-1)p_x^0$$

$$p_x^0 = \frac{i\alpha_H - 1 + (1-x)^{i\alpha_H}(1+i\alpha_H)}{\left[(1-x)^{i\alpha_H} - 1\right](x-1)} \qquad \alpha_H = \sqrt{4\epsilon_H - 1}$$

→ $$\ln \mathcal{V}(x) \approx -\frac{c}{6}\int^x p_x^0$$ → $$x = 1 - e^{i(x+i\tau)}$$

ETH at leading order — "forbidden singularities"

Leading order solution:

L. Fitzpatrick, et al'13

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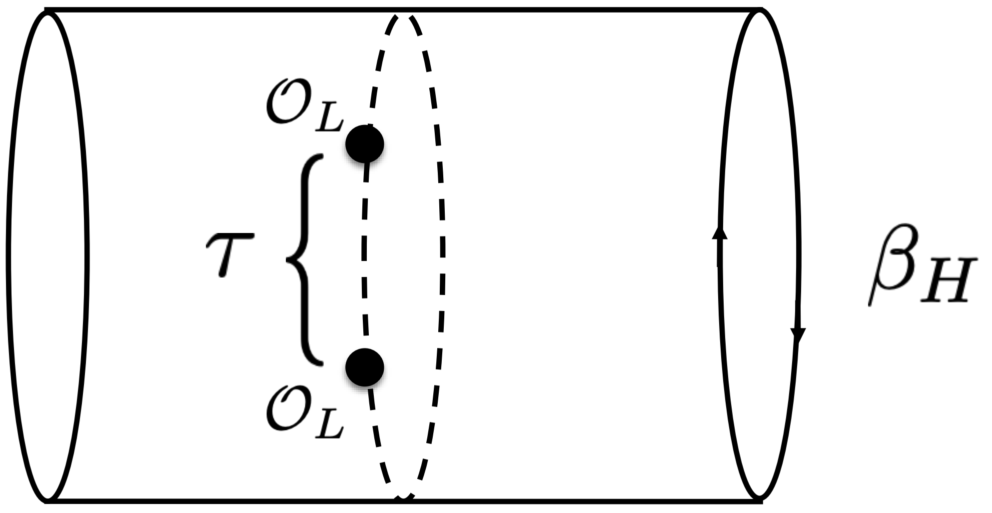
$$\ln \mathcal{V}(x) \approx -\frac{c}{6}\int^x p^0_x \quad \Rightarrow \quad x = 1 - e^{i(x+i\tau)}$$

$$\mathcal{V}(\tau) \approx \left[\frac{\beta_H}{\pi}\sin\left(\frac{\pi\tau}{\beta_H}\right)\right]^{-2h_L}$$

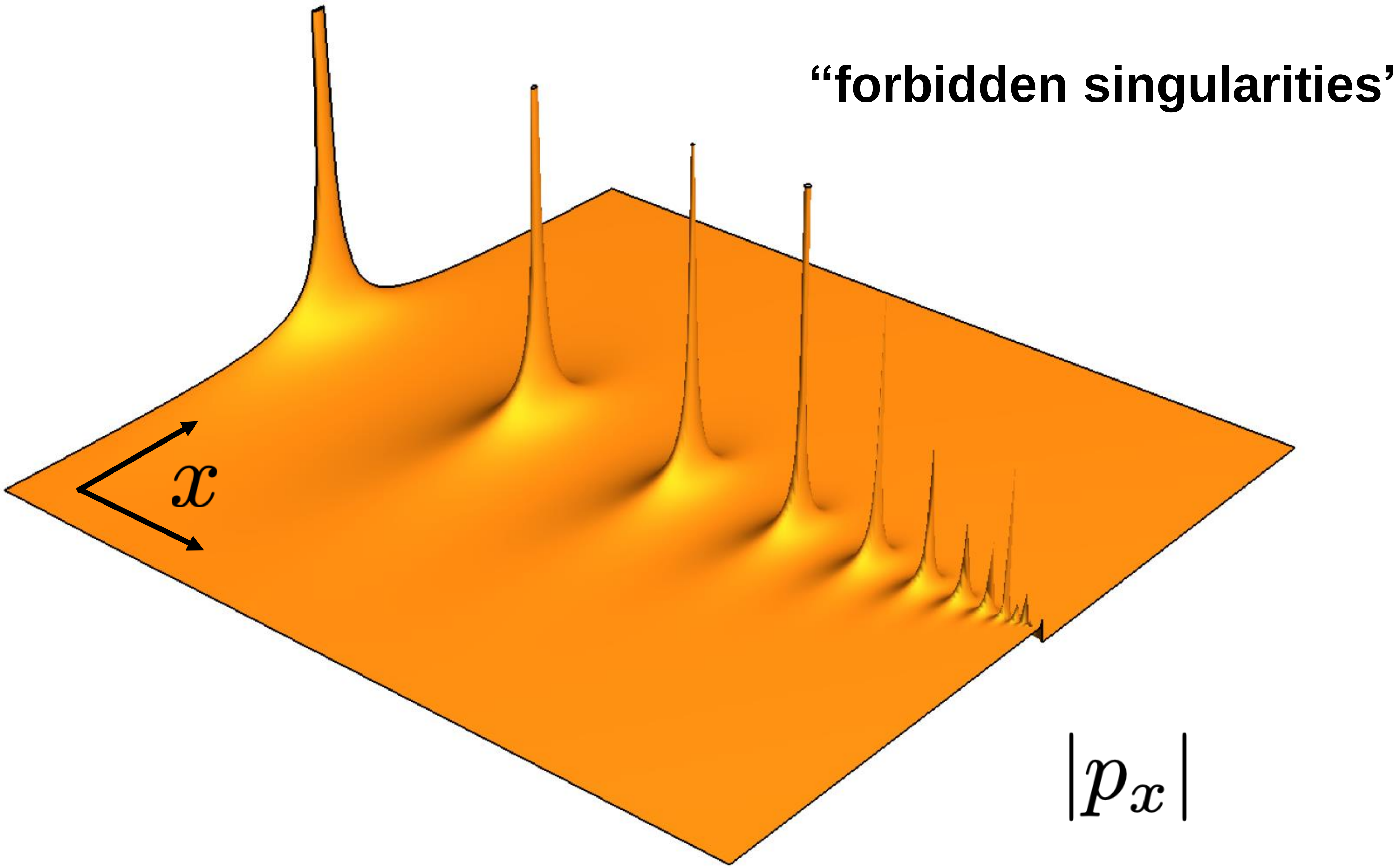
$$\beta_H = \frac{2\pi}{\alpha_H}$$



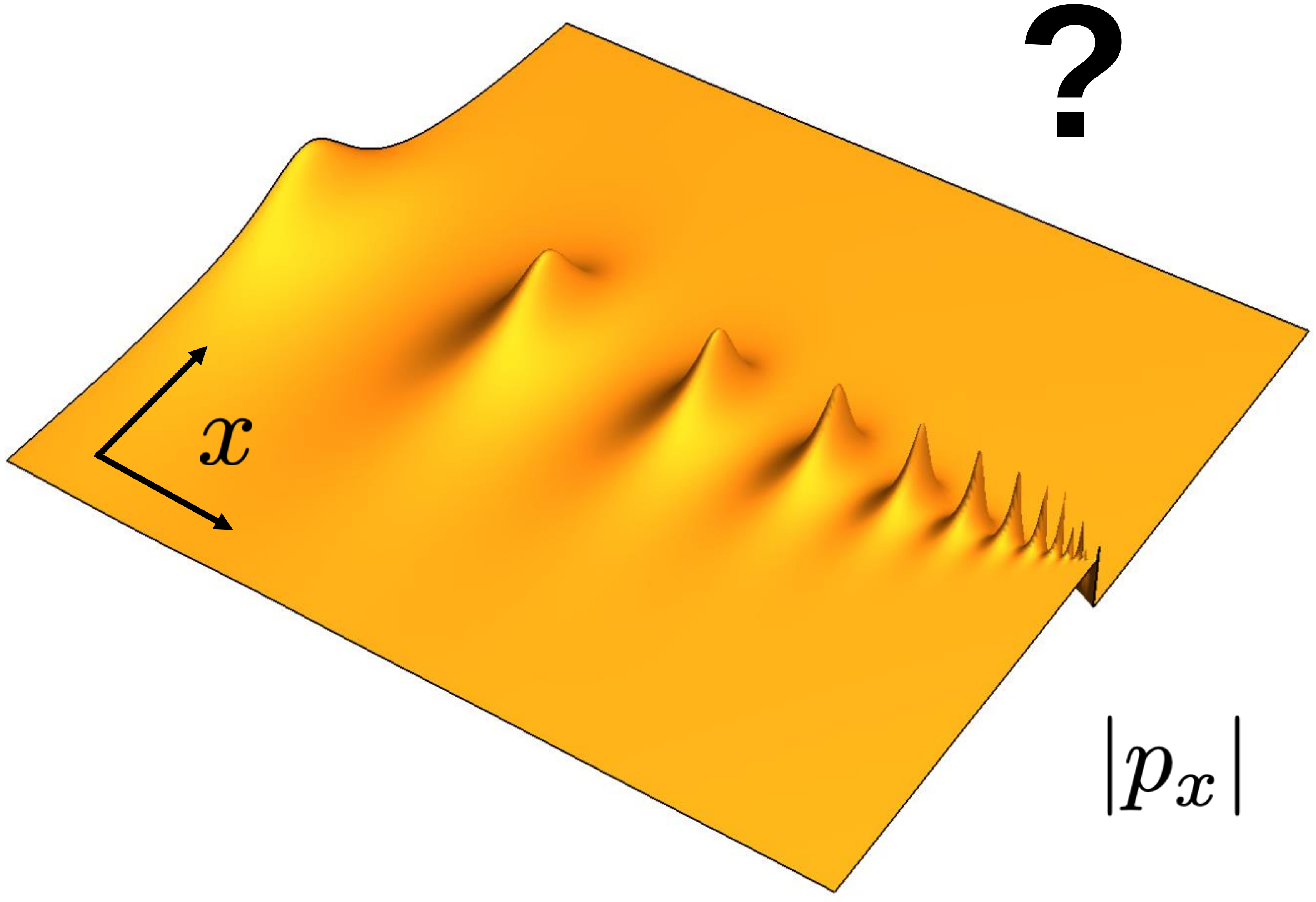
Resolving "forbidden singularities"

ETH at leading order in $c \to \infty, \;\; h_L \ll c$

ETH at leading order in $c \to \infty, \quad h_L \ll c$



ETH $+\frac{h_L}{c}+\frac{h_L^2}{c^2}+\frac{1}{c}+\frac{1}{c^2}+\frac{h_L}{c}\frac{1}{c}+\ldots$



Resolving "forbidden singularities"

Focus on vacuum block, resolution within block

L. Fitzpatrick, et al'16

Resolving "forbidden singularities"

away from the "probe limit": $\epsilon_L \propto \frac{h_L}{c}$ **finite**

Resolving “forbidden singularities”

away from the “probe limit”: $\epsilon_L \propto \dfrac{h_L}{c}$ **finite**

“two-step” resolution:

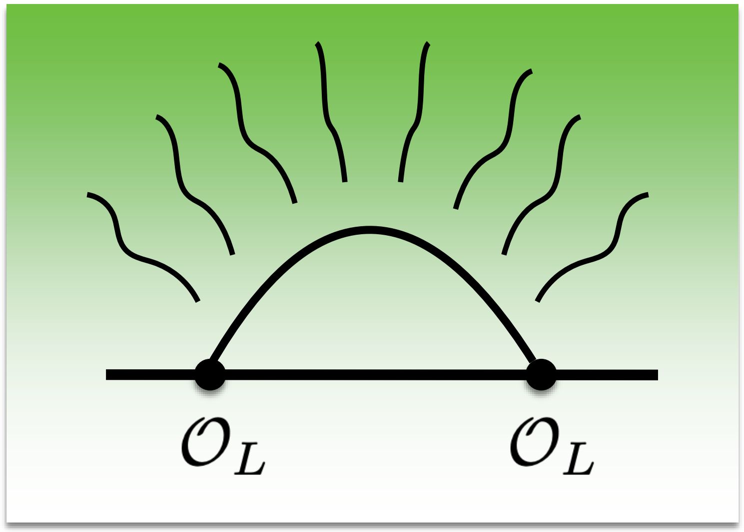
Resolving "forbidden singularities"

away from the "probe limit": $\epsilon_L \propto \frac{h_L}{c}$ **finite**

"two-step" resolution:

$$\textbf{ETH} + \mathcal{O}(\epsilon_L) + \mathcal{O}(\epsilon_L^2) + \ldots$$

"back-reacting to geometry"



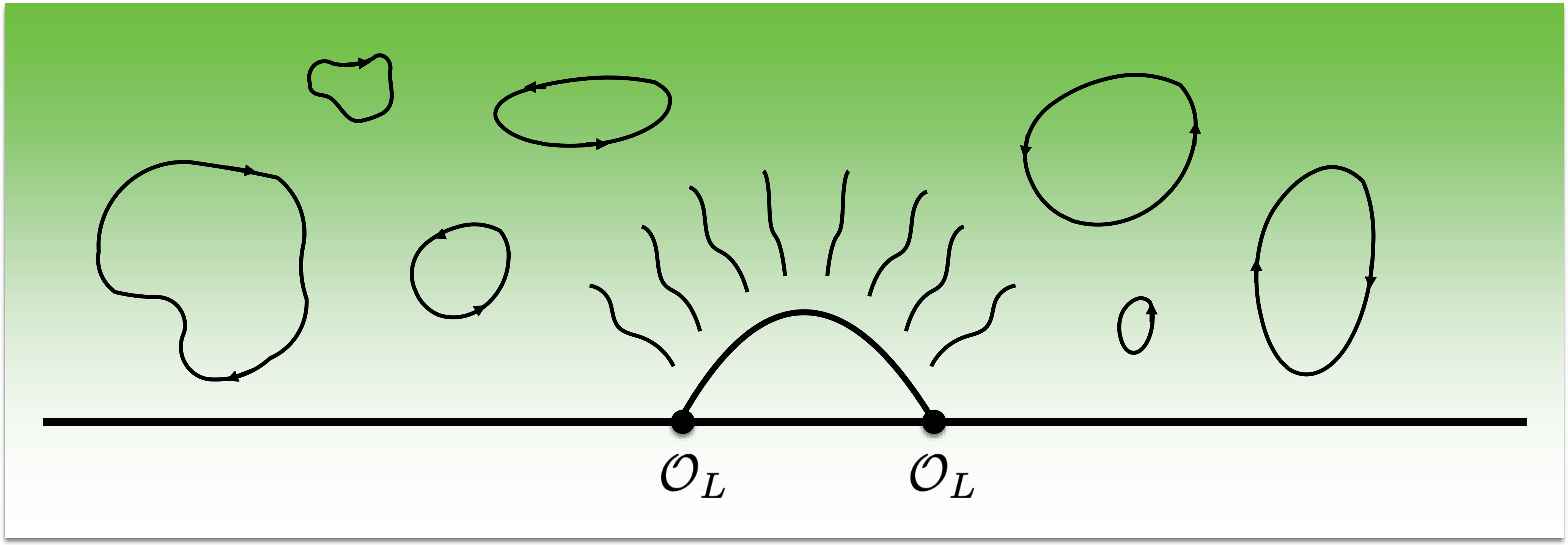
Resolving "forbidden singularities"

away from the "probe limit": $\epsilon_L \propto \frac{h_L}{c}$ **finite**

"two-step" resolution:

$$\Big[\ \textbf{ETH} + \mathcal{O}(\epsilon_L) + \mathcal{O}(\epsilon_L^2) + \ldots \Big] + \mathcal{O}(c^{-1}) + \mathcal{O}(c^{-2}) + \ldots$$

"quantum loop corrections"



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- **Resolution by “probe” corrections**
- Resolution by finite c corrections
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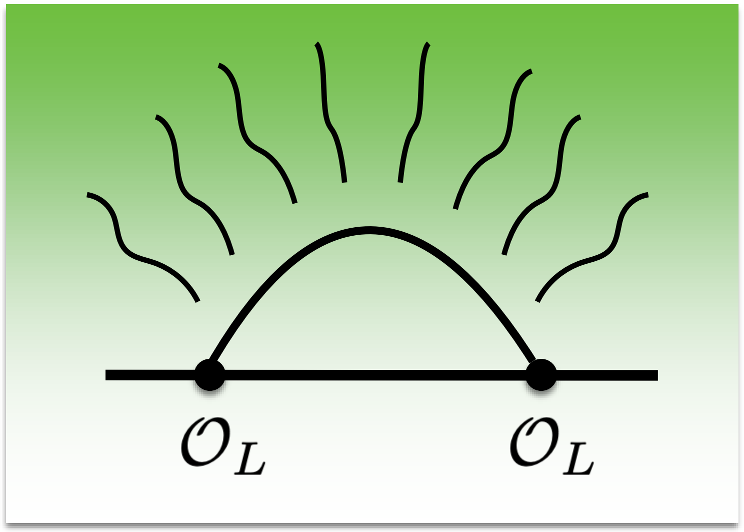
- Conclusions/Future directions



ETH $+ \mathcal{O}(\epsilon_L) + \mathcal{O}(\epsilon_L^2) + \ldots$

“back-reaction from probe”

solving the monodromy equation exactly

Resolution by "probe" corrections

Monodromy equation: transcendental equation of p_x

Resolution by "probe" corrections

$p_x = \# \epsilon_L + \# \epsilon_L^2 + \ldots$, treat p_x as another small parameter

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double series expansion of the monodromy equation:

$$\#\epsilon_L + \#p_x + \#\epsilon_L^2 + \#\epsilon_L p_x + \#p_x^2 + \ldots = 0$$

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away from "forbidden singularities"

$$2\epsilon_L - (x - x_n)p_x + \ldots = 0\ , \quad p_x \approx \frac{\epsilon_L}{x - x_n} + \mathcal{O}\left(\epsilon_L^2\right)$$

Resolution by "probe" corrections

$p_x = \#\epsilon_L + \#\epsilon_L^2 + \ldots$, treat p_x as another small parameter

double series expansion of the monodromy equation:

$$\#\epsilon_L + \#p_x + \#\epsilon_L^2 + \#\epsilon_L p_x + \#p_x^2 + \ldots = 0$$

near "forbidden singularities"

$$2\epsilon_L - (x - x_n)p_x + \ldots = 0 \; , \quad p_x \approx \frac{\epsilon_L}{x - x_n} + \frac{\#\epsilon_L^2}{(x - x_n)^2} + \ldots$$

$\wr$

0 "degenerate" need re-summation

re-summation near forbidden singularities: $x \approx x_n$

Monodromy equation:

$$2\epsilon_L - \underbrace{(x - x_n)}_{\approx 0} p_x + ... = 0$$

✗ "degenerate"

Resolution by "probe" corrections

re-summation near forbidden singularities: $x \approx x_n$

Monodromy equation:

$$2\epsilon_L - \underbrace{(x - x_n)}_{\approx 0} p_x + ... = 0$$

✗ "degenerate"

⇓ supply the next order term

$$2\epsilon_L - (x - x_n)p_x - b_n p_x^2 + ... = 0$$

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$$p_x \approx \frac{-(x - x_n) \pm \sqrt{(x - x_n)^2 + 8\epsilon_L b_n}}{2b_n}$$

"quadratic resolution"

Resolution by "probe" corrections

"quadratic resolution"

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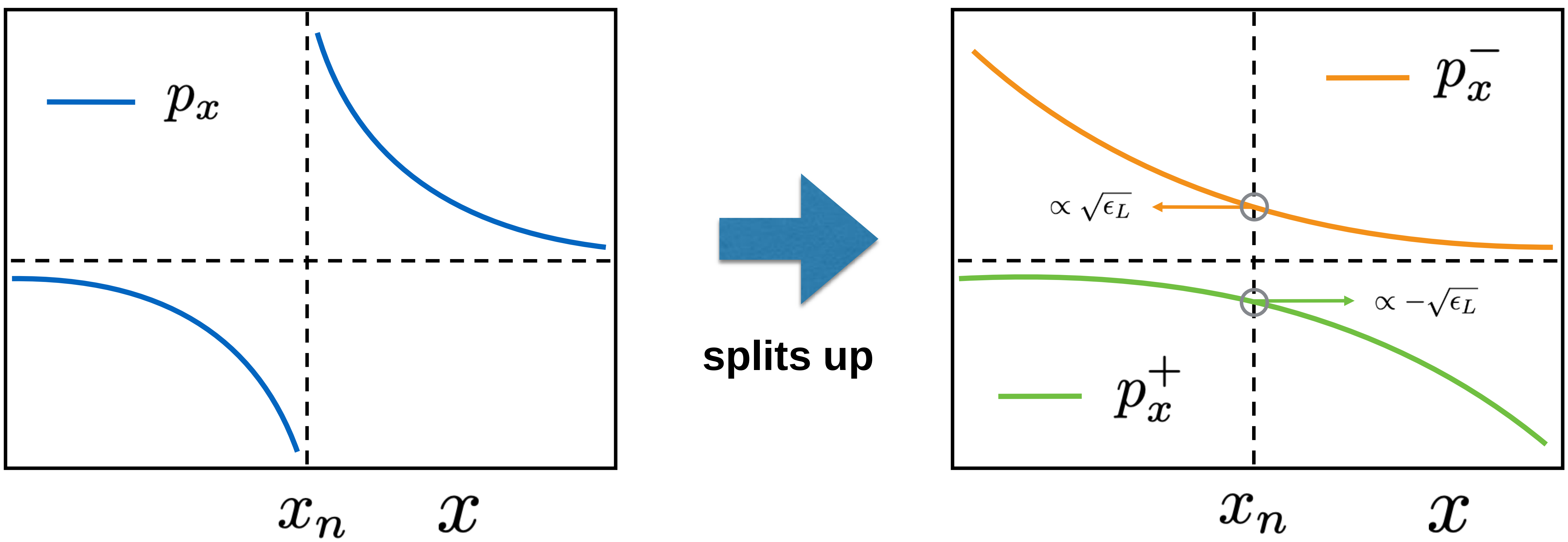

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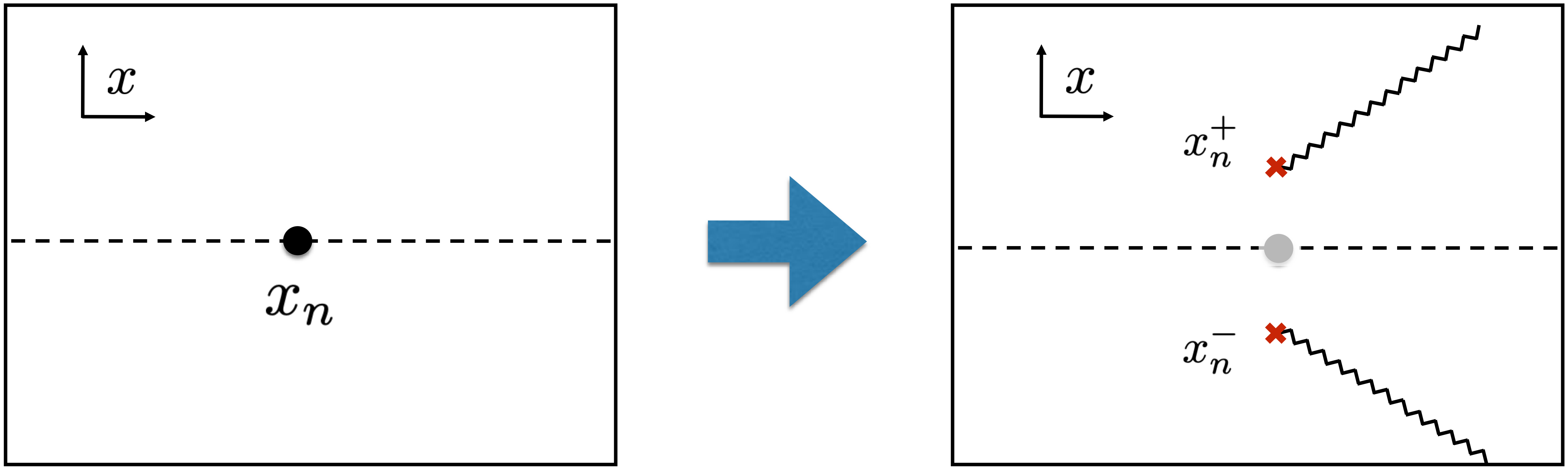

"forbidden pole"

"forbidden branch-cuts"

Resolution by "probe" corrections

What are the other branches?

Resolution by "probe" corrections

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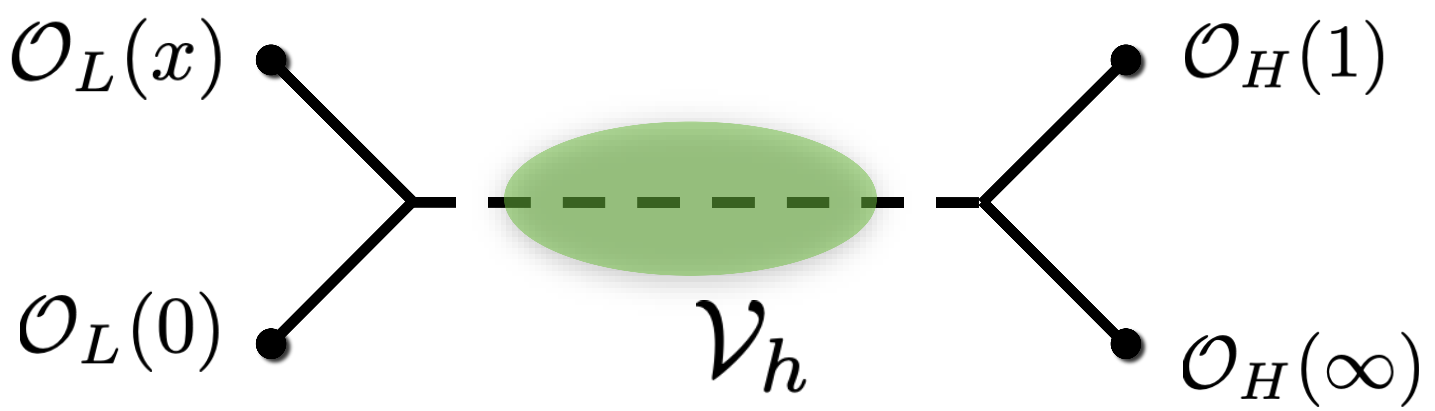
monodromy equation: $\mathrm{tr}\hat{M}_{0x} = -2\cos(\pi\Lambda_h) \qquad h = \frac{c}{24}\left(1-\Lambda_h^2\right)$



Resolution by "probe" corrections

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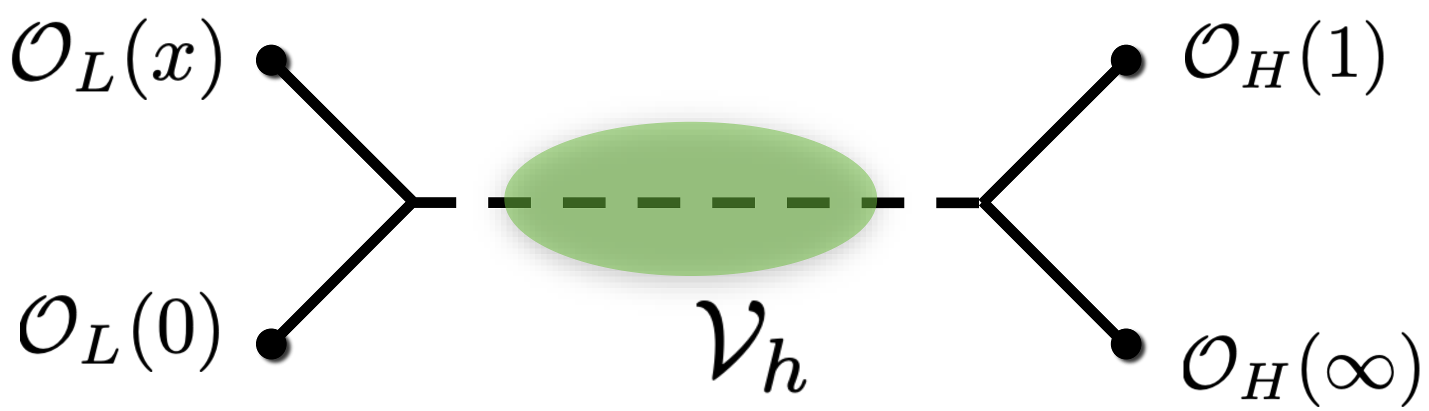


$\mathcal{V}_h$ **solve the same monodromy equation for**

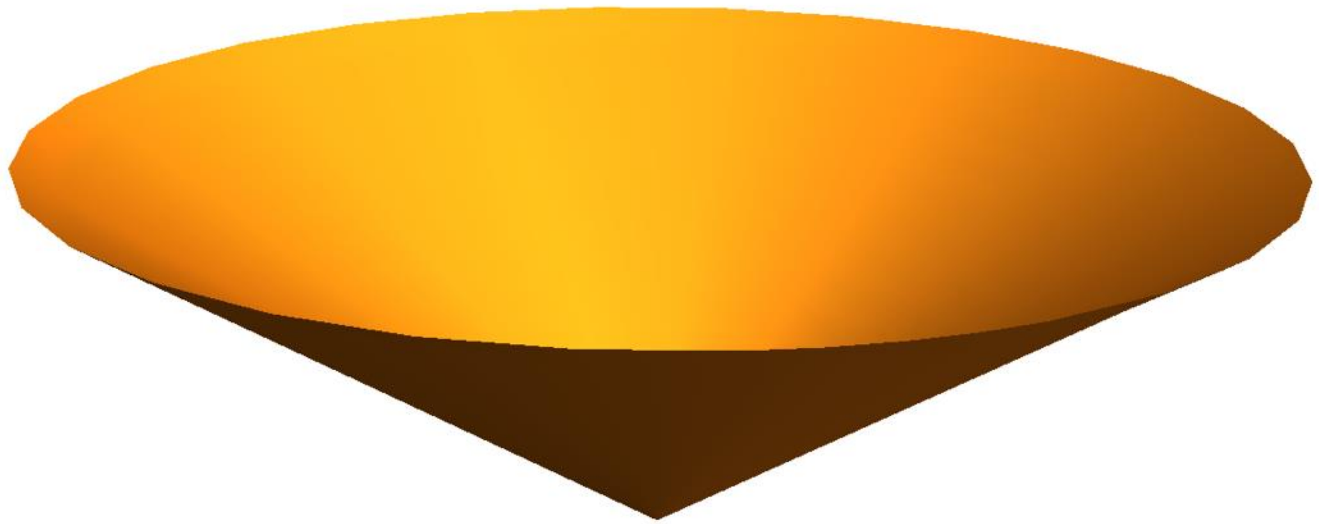
$$h = 0 \qquad\qquad h_n = -\frac{c}{6}n(n+1),\ n \in \mathbb{N}$$

vacuum block **"additional saddle"** Liam et al' 16

Resolution by "probe" corrections

"additional saddle"

$$\mathcal{V}_n : \quad h_n = -\frac{c}{6} n(n+1), \; n \in \mathbb{N}$$

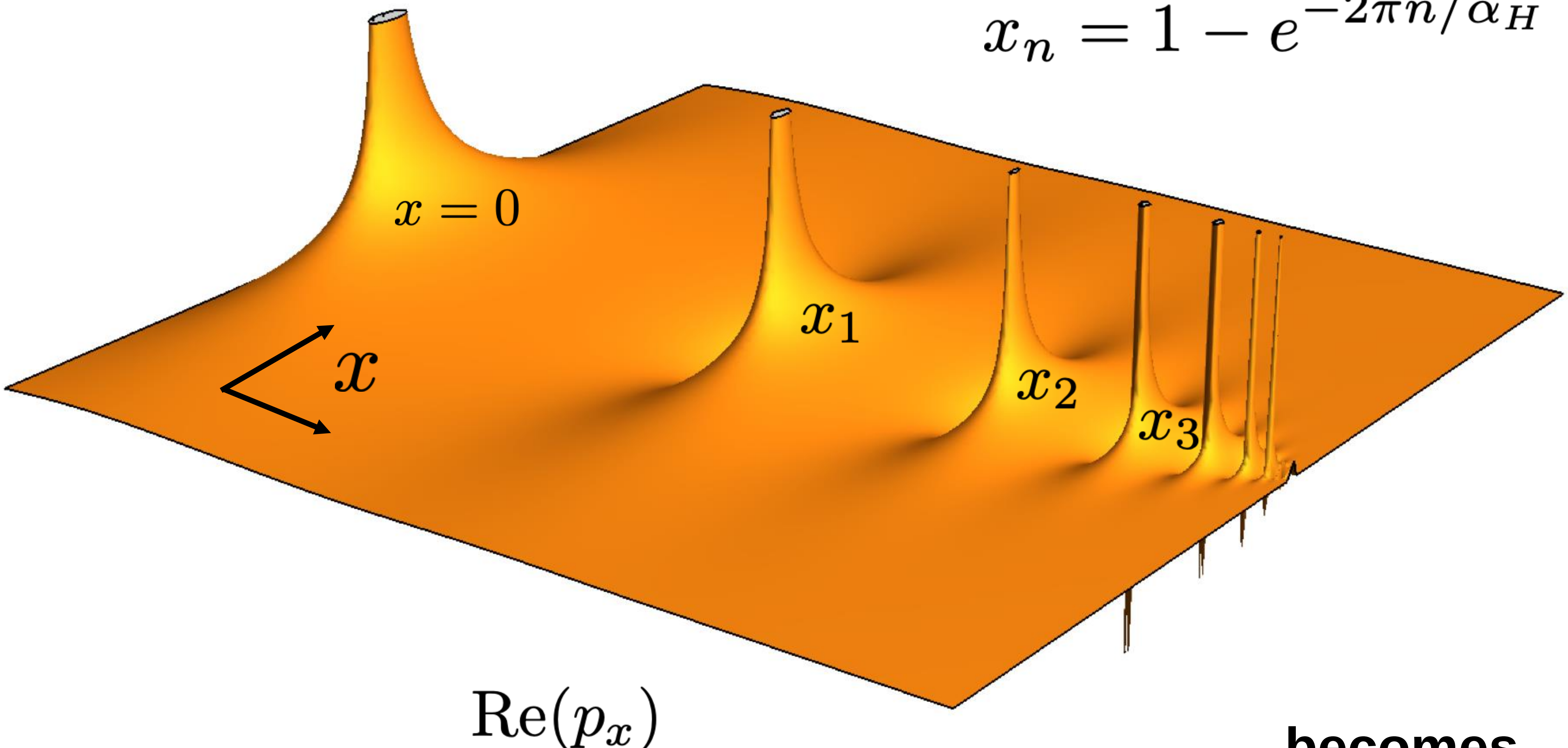


$h_n < 0$ → **"surplus angle" geometry**

Resolution by “probe” corrections

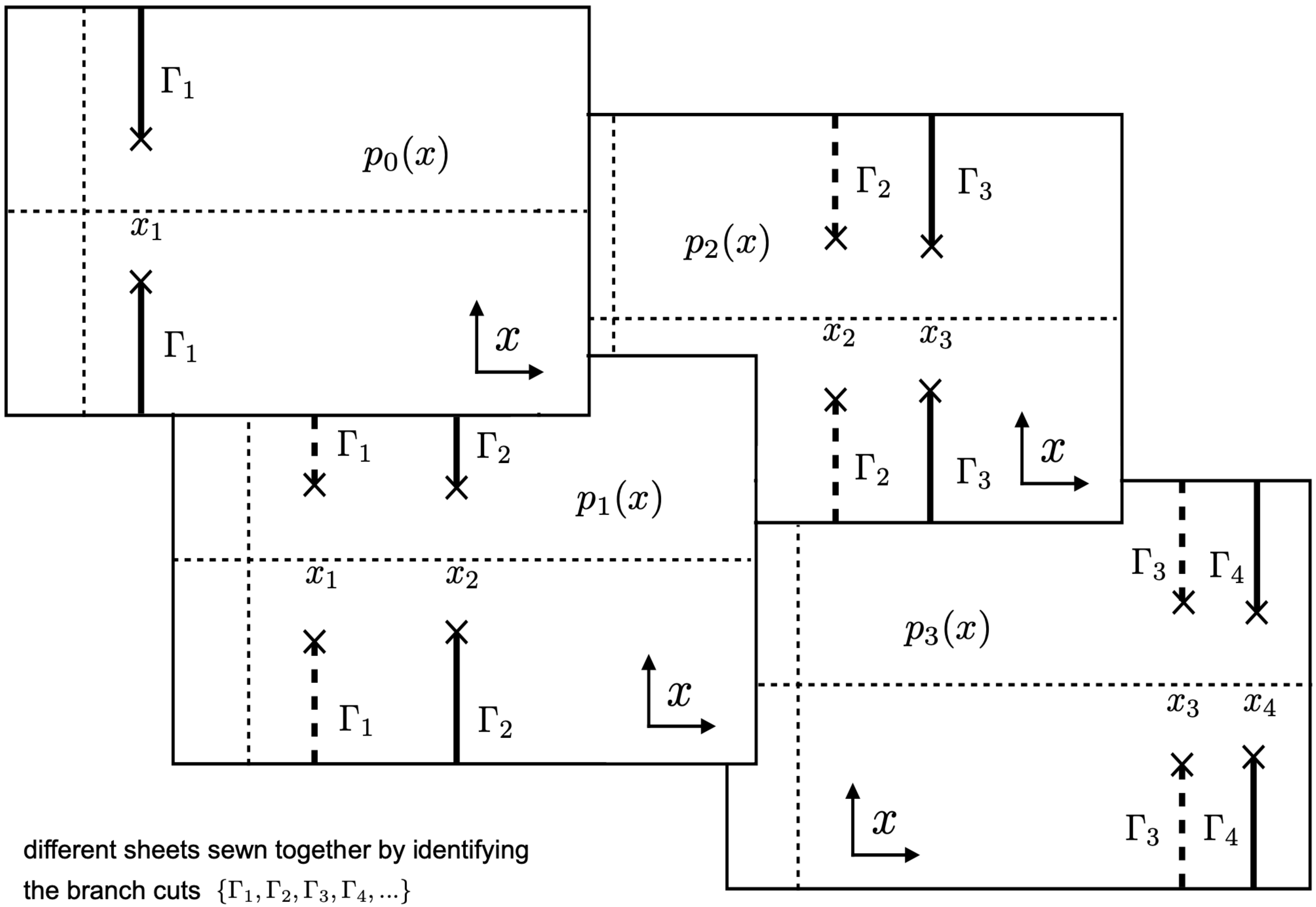
ETH $+ \mathcal{O}(\epsilon_L) + \mathcal{O}(\epsilon_L^2) + \dots$

$$x_n = 1 - e^{-2\pi n/\alpha_H}$$



becomes

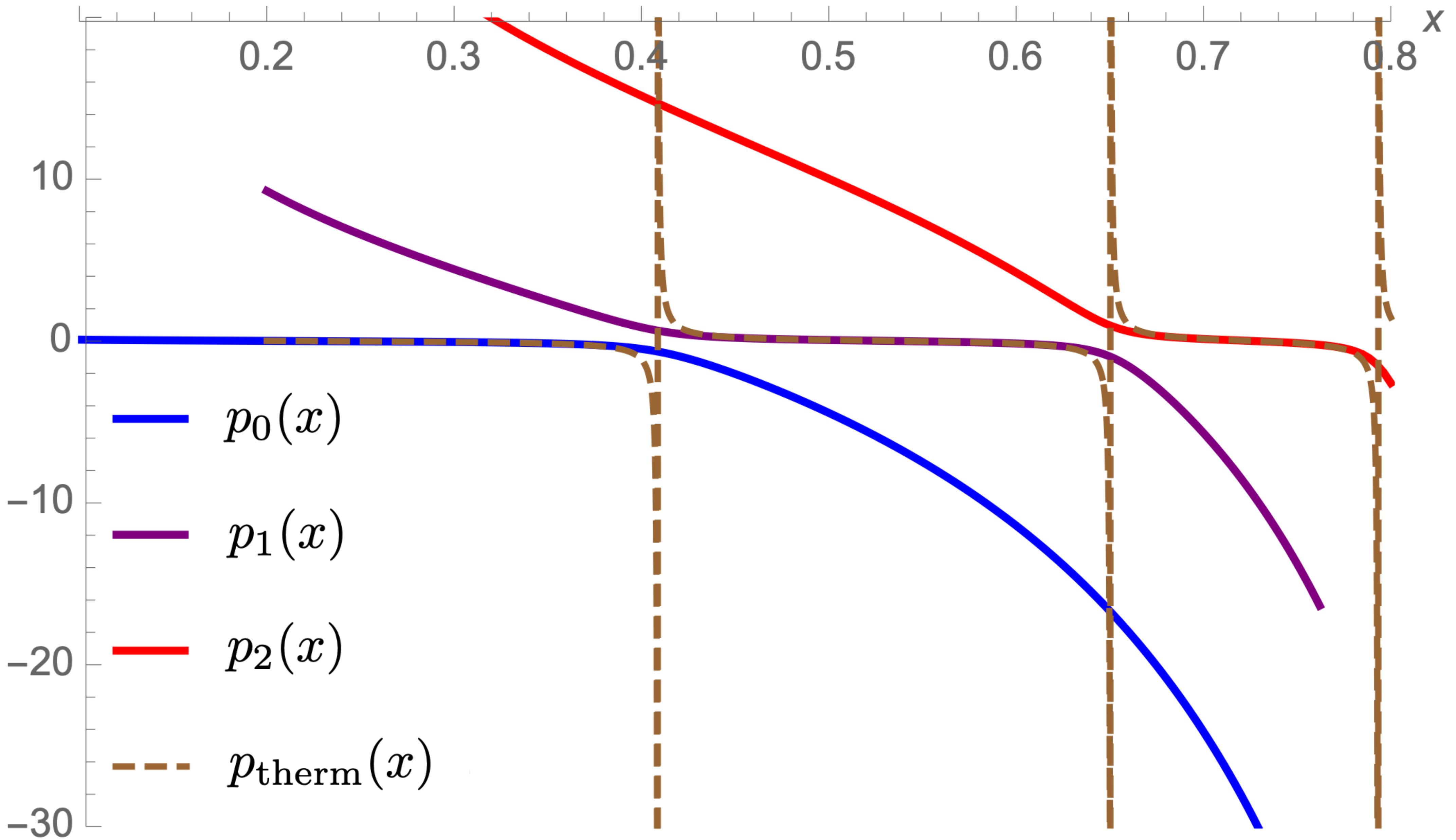
analytic structure:



different sheets sewn together by identifying the branch cuts $\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \ldots\}$

Resolution by “probe” corrections

Numerical results: $\epsilon_H = 36, \ \ \epsilon_L = 0.005$



Resolution by "probe" corrections

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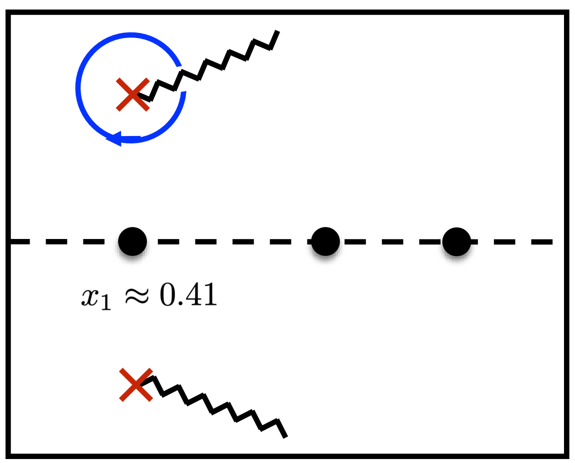
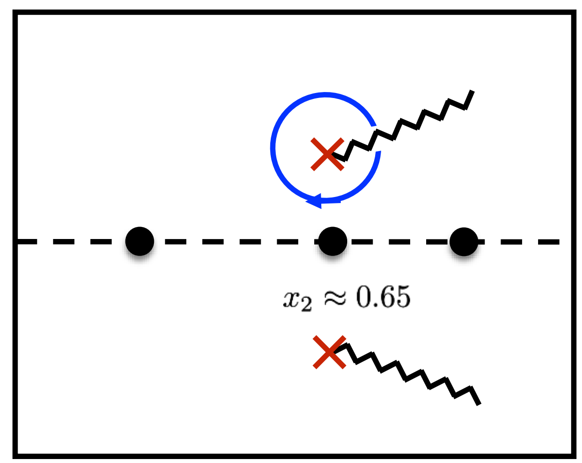
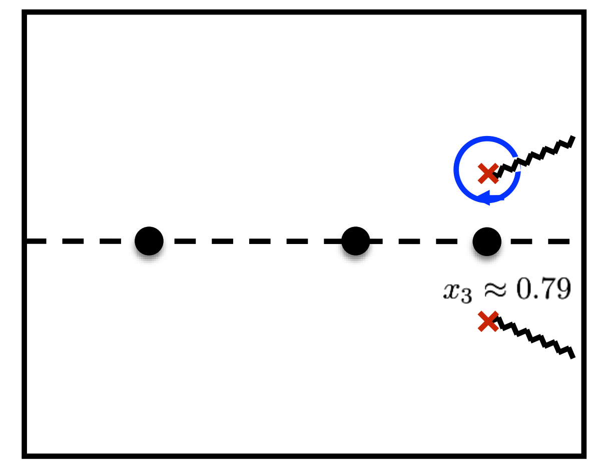
Monodromies around branch-points



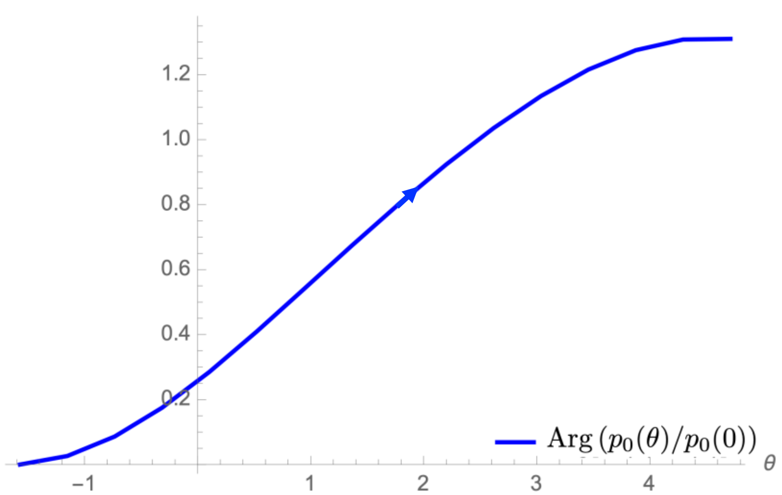
$x_1^\pm \approx 0.41 \pm 0.39i$

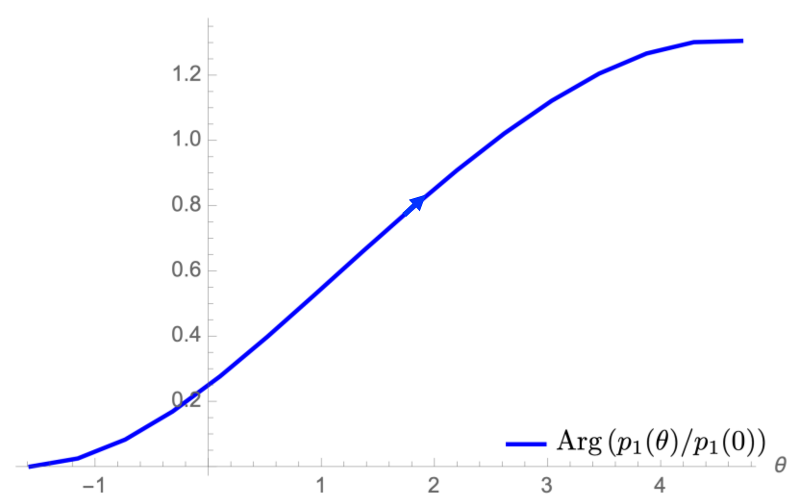


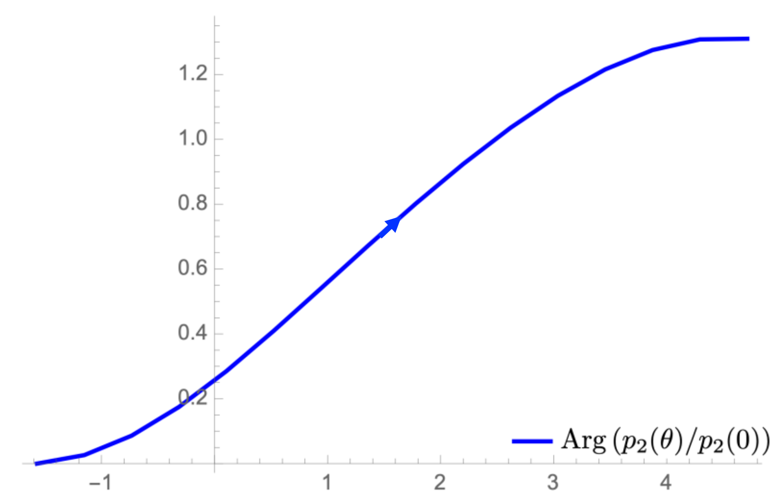
$x_2^\pm \approx 0.65 \pm 0.13i$



$x_3^\pm \approx 0.79 \pm 0.05i$





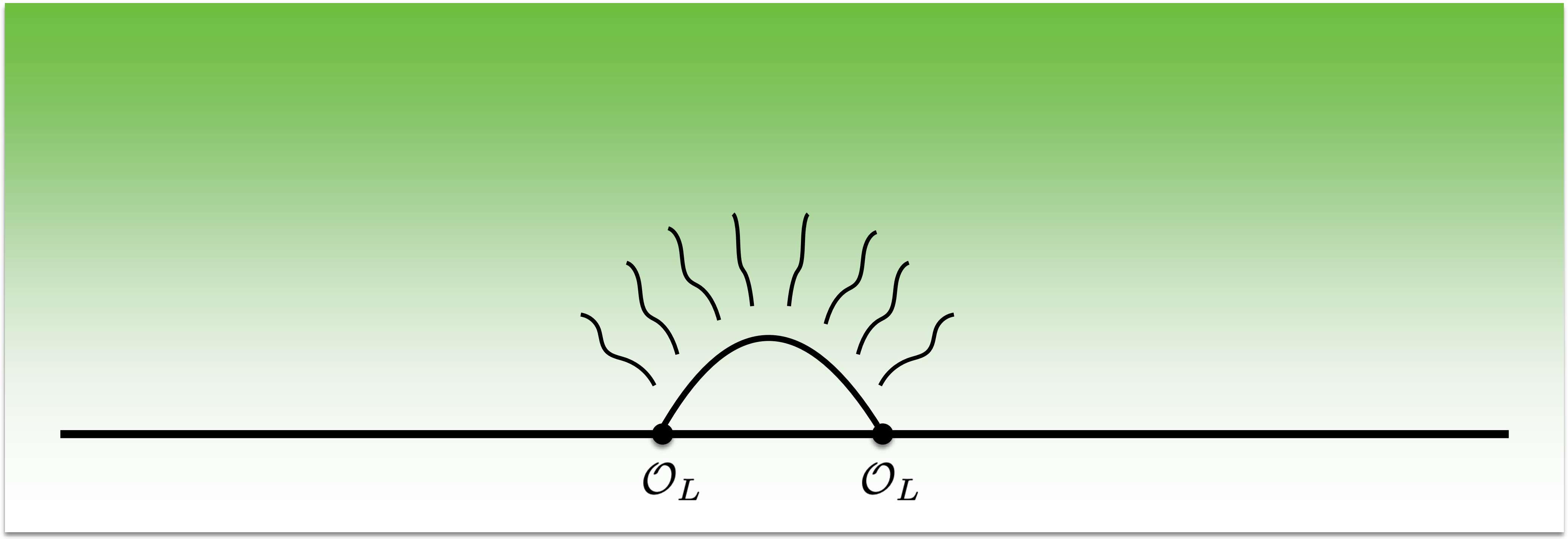


Outline

- ETH at leading order — “forbidden singularities”
- Resolution by “probe” corrections
- **Resolution by finite c corrections**
- Real time dynamics
- Conclusions/Future directions

Resolution by finite c corrections

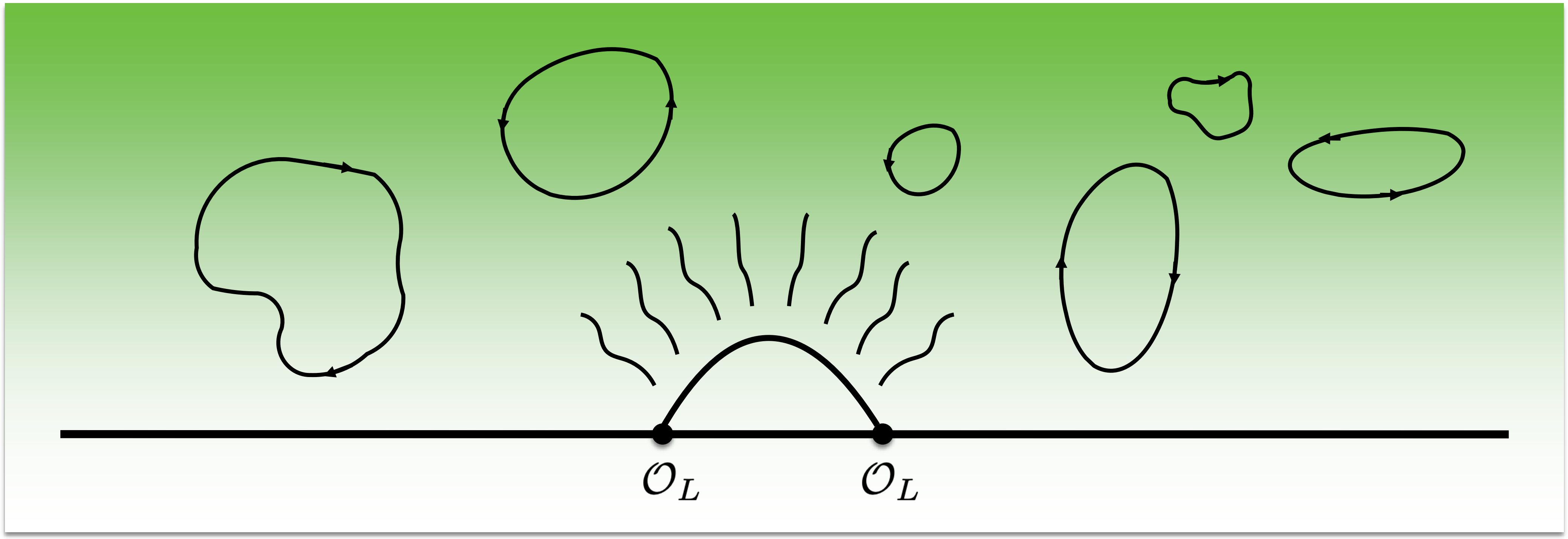
ETH $+\,\mathcal{O}(\epsilon_L) + \mathcal{O}(\epsilon_L^2) + \ldots$



“forbidden branch-cuts”

Resolution by finite c corrections

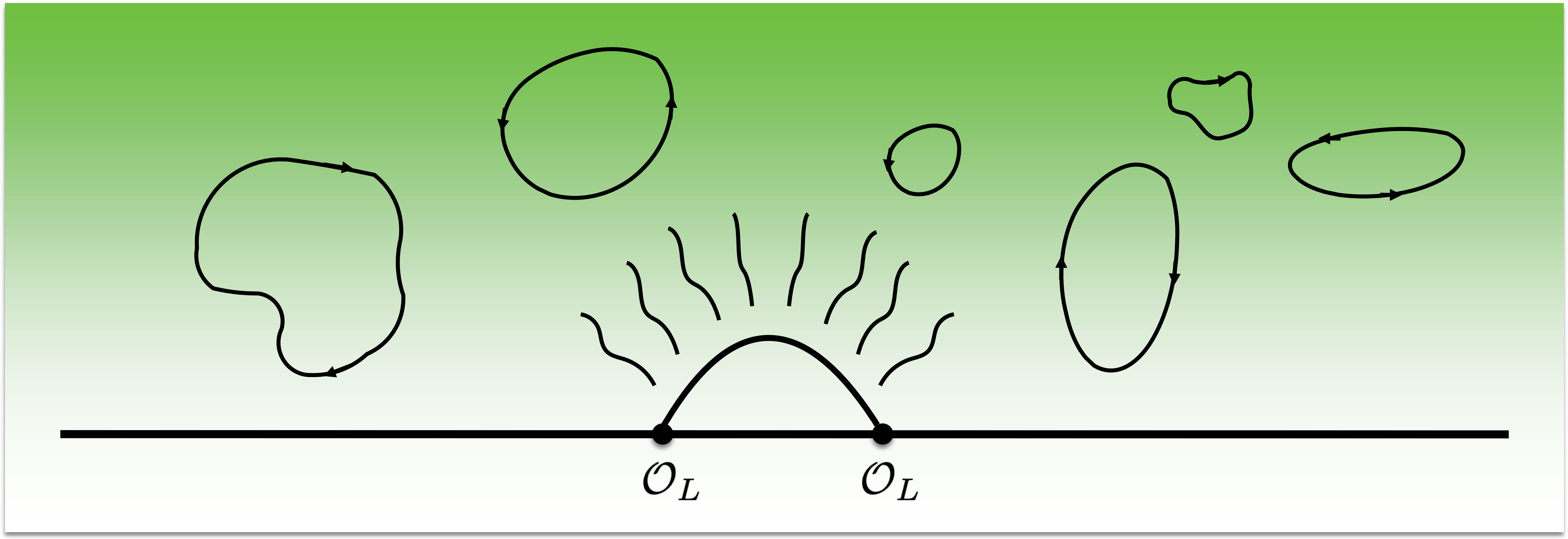
$$\left[\textbf{ETH} + \mathcal{O}(\epsilon_L) + \mathcal{O}(\epsilon_L^2) + \ldots \right] + \mathcal{O}(c^{-1}) + \mathcal{O}(c^{-2}) + \ldots$$



~~**“forbidden branch-cuts”**~~

Resolution by finite c corrections

$$\Big[\textbf{ETH} + \mathcal{O}(\epsilon_L) + \mathcal{O}(\epsilon_L^2) + \ldots \Big] + \mathcal{O}(c^{-1}) + \mathcal{O}(c^{-2}) + \ldots$$



~~"forbidden branch-cuts"~~

HOW?

Resolution by finite c corrections

opposite question:

For analytic $\mathcal{V}(c,x)$**, how do branch-cuts emerge in** $p(x) \propto \partial_x \mathcal{V}(c,x)/\mathcal{V}(c,x)$ **in the limit** $c \to \infty$ **?**

Resolution by finite c corrections

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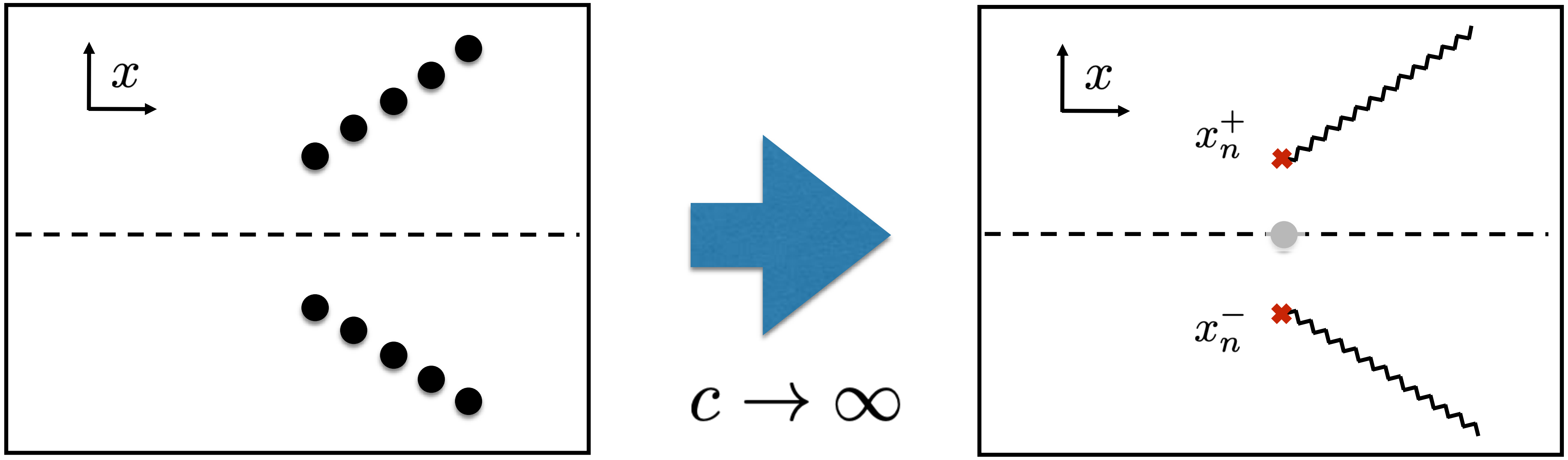
one common scenario: condensation of poles



Resolution by finite c corrections

opposite question:

> **For analytic** $\mathcal{V}(c,x)$**, how do branch-cuts emerge in** $p(x) \propto \partial_x \mathcal{V}(c,x)/\mathcal{V}(c,x)$ **in the limit** $c \to \infty$ **?**

one common scenario: condensation of poles

zeros of $\mathcal{V}(c,x)$

poles of $p(x) \propto \partial_x \mathcal{V}(c,x)/\mathcal{V}(c,x)$

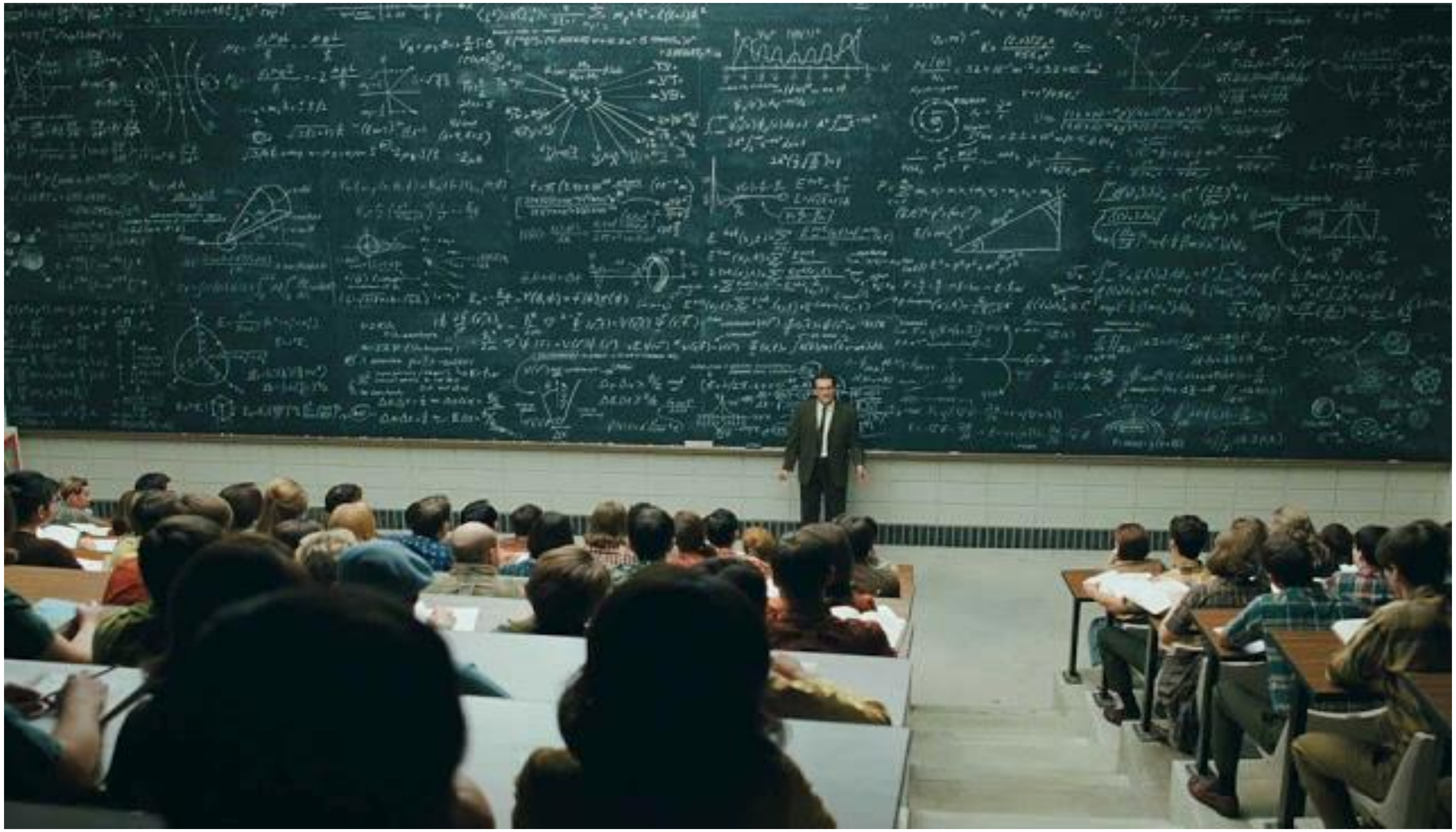
Resolution by finite c corrections

a natural guess: condensation of zeros for $\lim_{c \to \infty} \mathcal{V}(c, x)$

Resolution by finite c corrections

a natural guess: condensation of zeros for $\lim_{c\to\infty} \mathcal{V}(c,x)$

Analytic checks: very difficult!



Numerical checks:

Zamolodchikov's recursive relation to 1000th order…

Resolution by finite c corrections

Zamolodchikov's recursive relation:

generates a convergent series expansion in $q(x)$ for $\mathcal{V}_h(x)$ at finite c

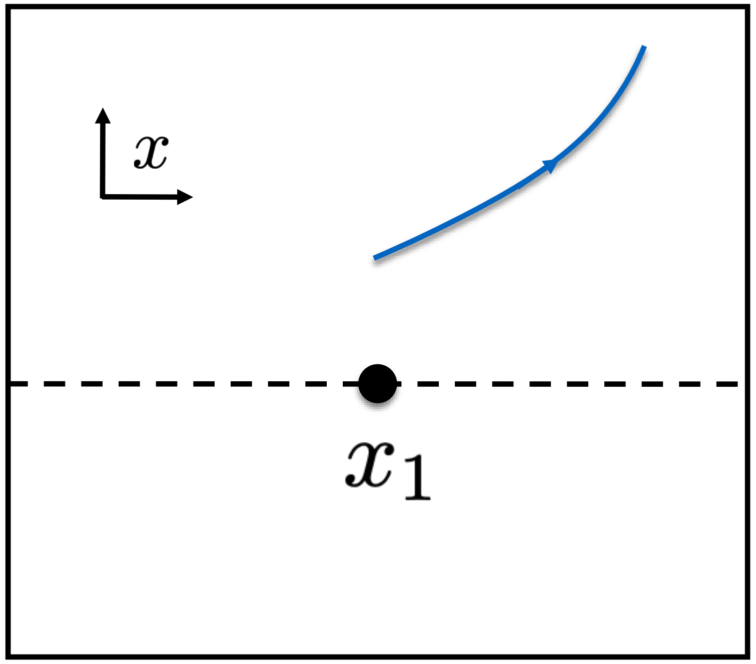
$$\mathcal{V}_h(x) \propto 1 + \#q + \#q^2 + \#q^3 + \ldots$$

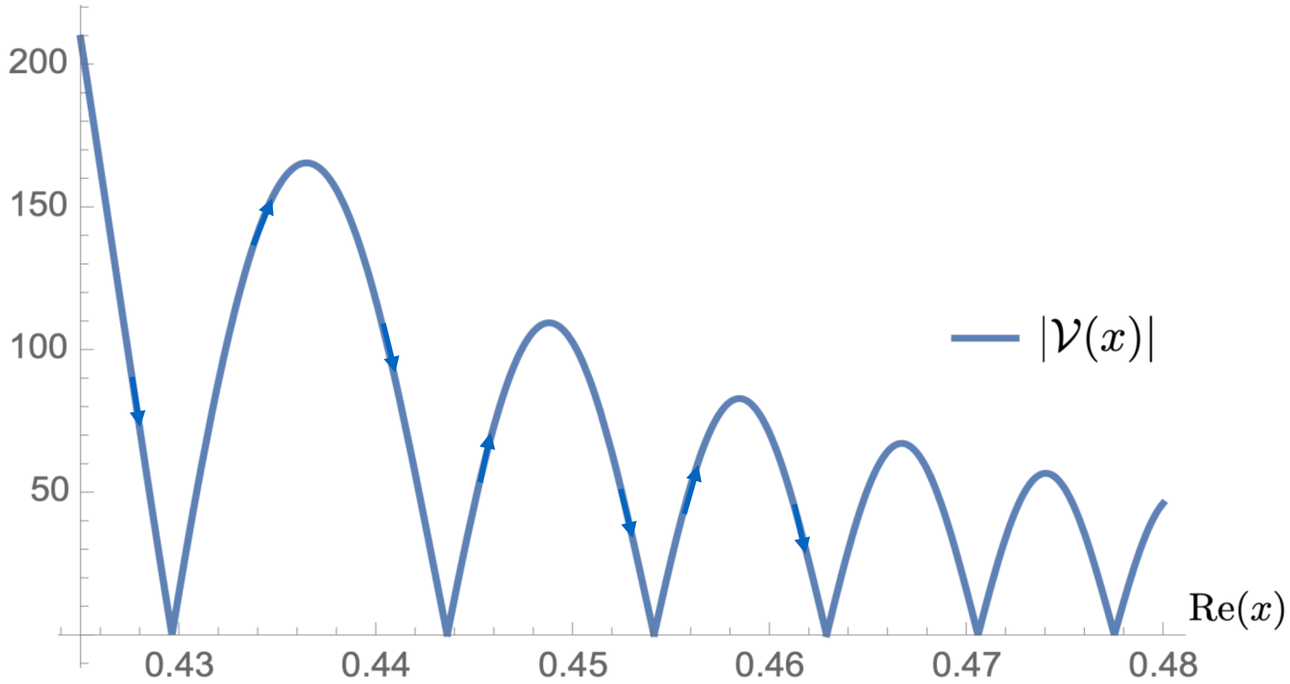
$$q = e^{i\pi\tau}, \ \tau = i\frac{K(1-x)}{K(x)}$$

Numerical results: $c = 1000, \ \epsilon_H = 36, \ \ \epsilon_L = 0.05$

compute $\mathcal{V}_{\rm vac}(x)$ **to 1000th order**

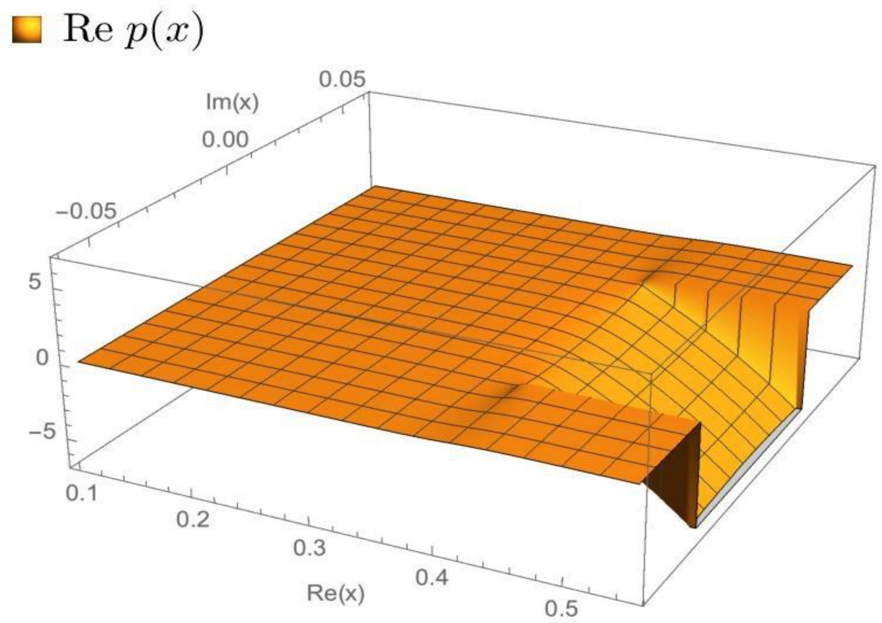
near the first forbidden singularity $x_1 \approx 0.41$

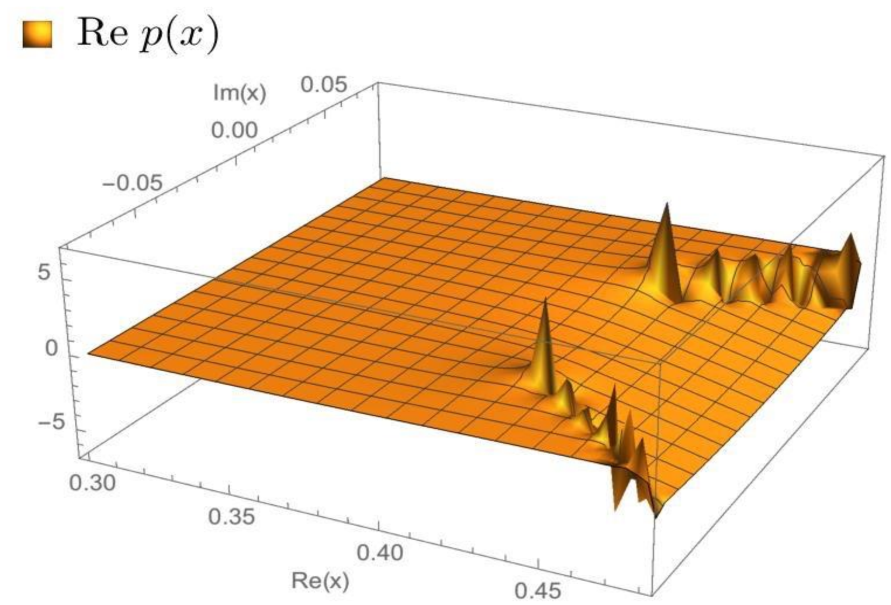




Resolution by finite c corrections

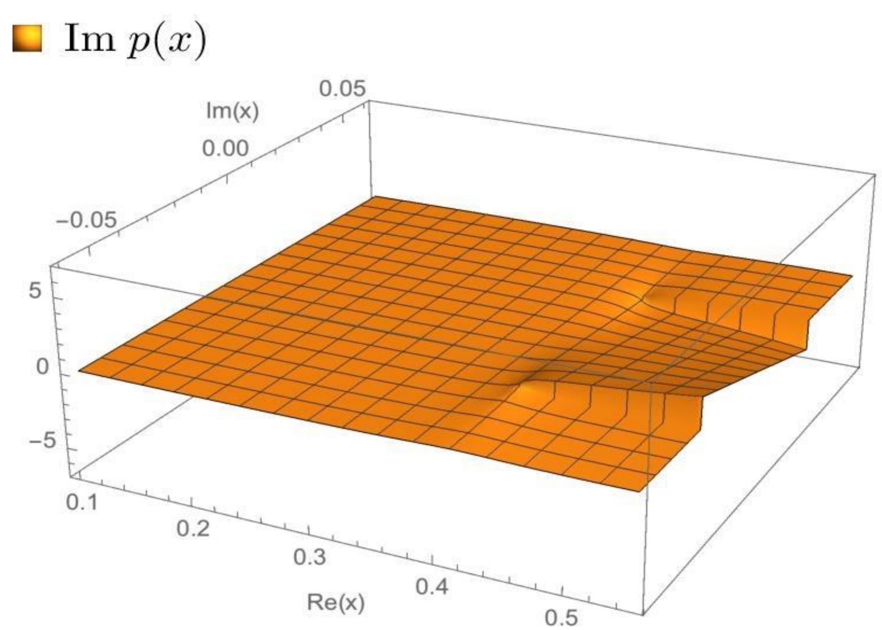
resolution of “forbidden branch-cuts”

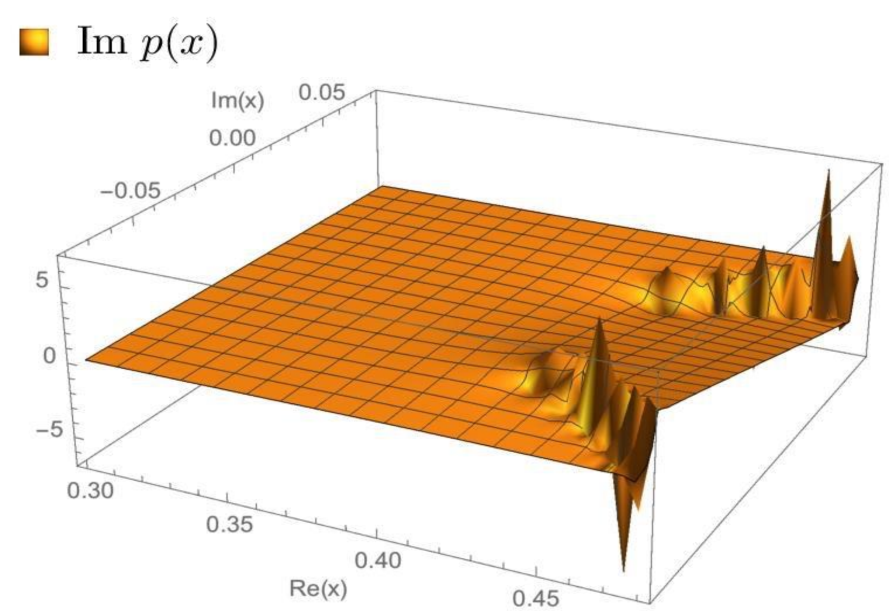




$c \to \infty$

$c = 1000$





Resolution by finite c corrections

Comments:

- re-summing probe corrections important intermediate step for revealing the final picture
- strong evidence for Stoke's phenomena
- resolved branch-cuts emerge as anti-Stoke's curves

Outline

- ETH at leading order — “forbidden singularities”
- Resolution by “probe” corrections
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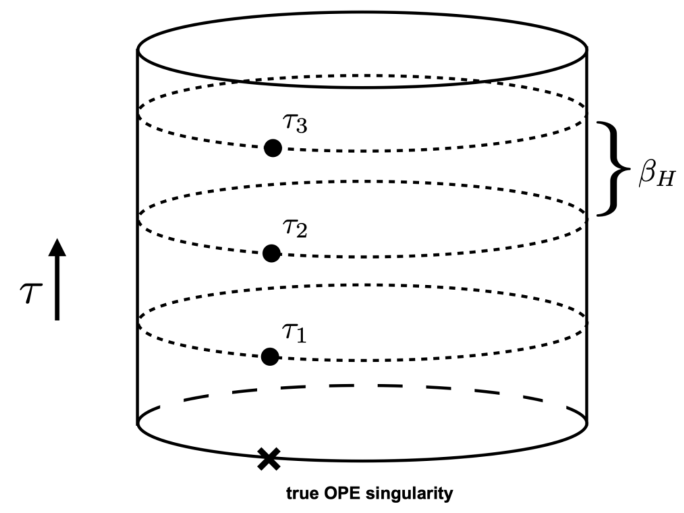
- **Real time dynamics**
- Conclusions/Future directions

ETH in different regimes

Euclidean time:

$$\langle H|\mathcal{O}_L(\tau)\mathcal{O}_L(0)|H\rangle$$



"forbidden singularities"

Lorentzian time:

$$\langle H|\mathcal{O}_L(t)\mathcal{O}_L(0)|H\rangle \propto \exp\left[-2\pi T_H h_L t\right]$$

exponential time decay

Corrections from finite c effects:

Euclidean time:

$$\langle H|\mathcal{O}_L(\tau)\mathcal{O}_L(0)|H\rangle$$

resolving "forbidden singularities"

Lorentzian time:

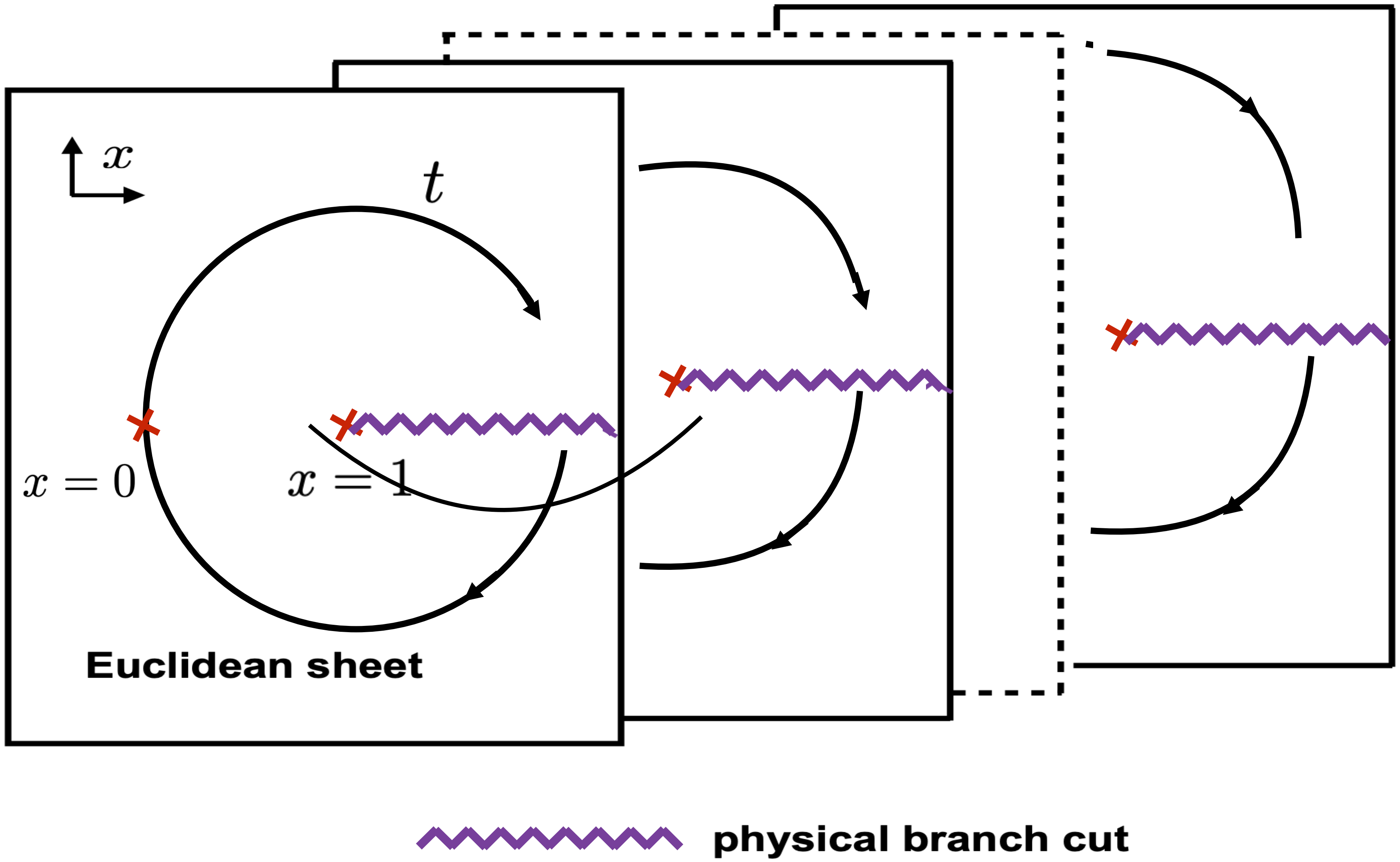
$$\langle H|\mathcal{O}_L(t)\mathcal{O}_L(0)|H\rangle \propto \exp\left[-2\pi T_H h_L t\right]$$

exit from exponential decay at later time

Are they related?

Are they related? Naively no.

$$x = 1 - e^{-it}$$



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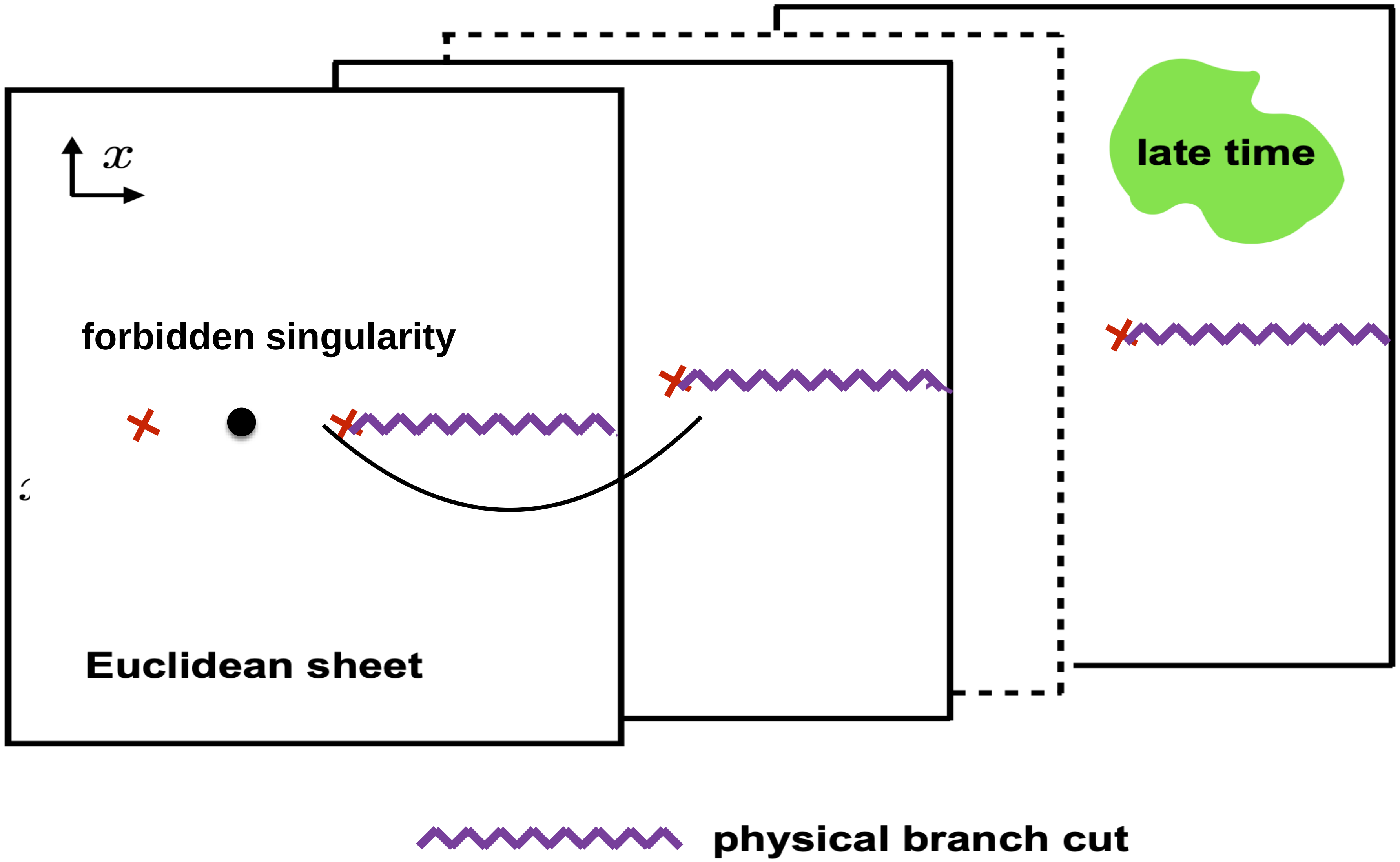

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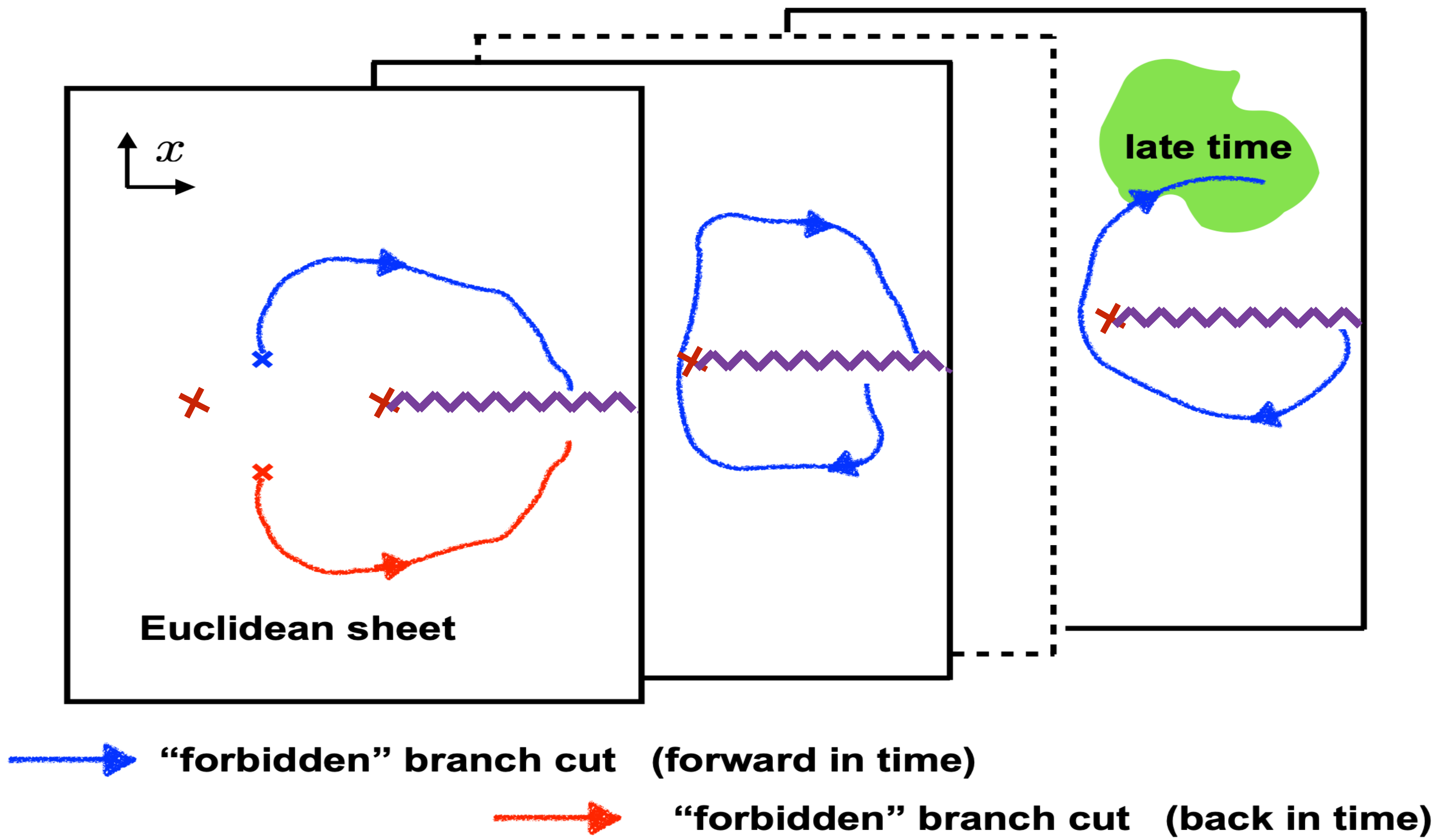
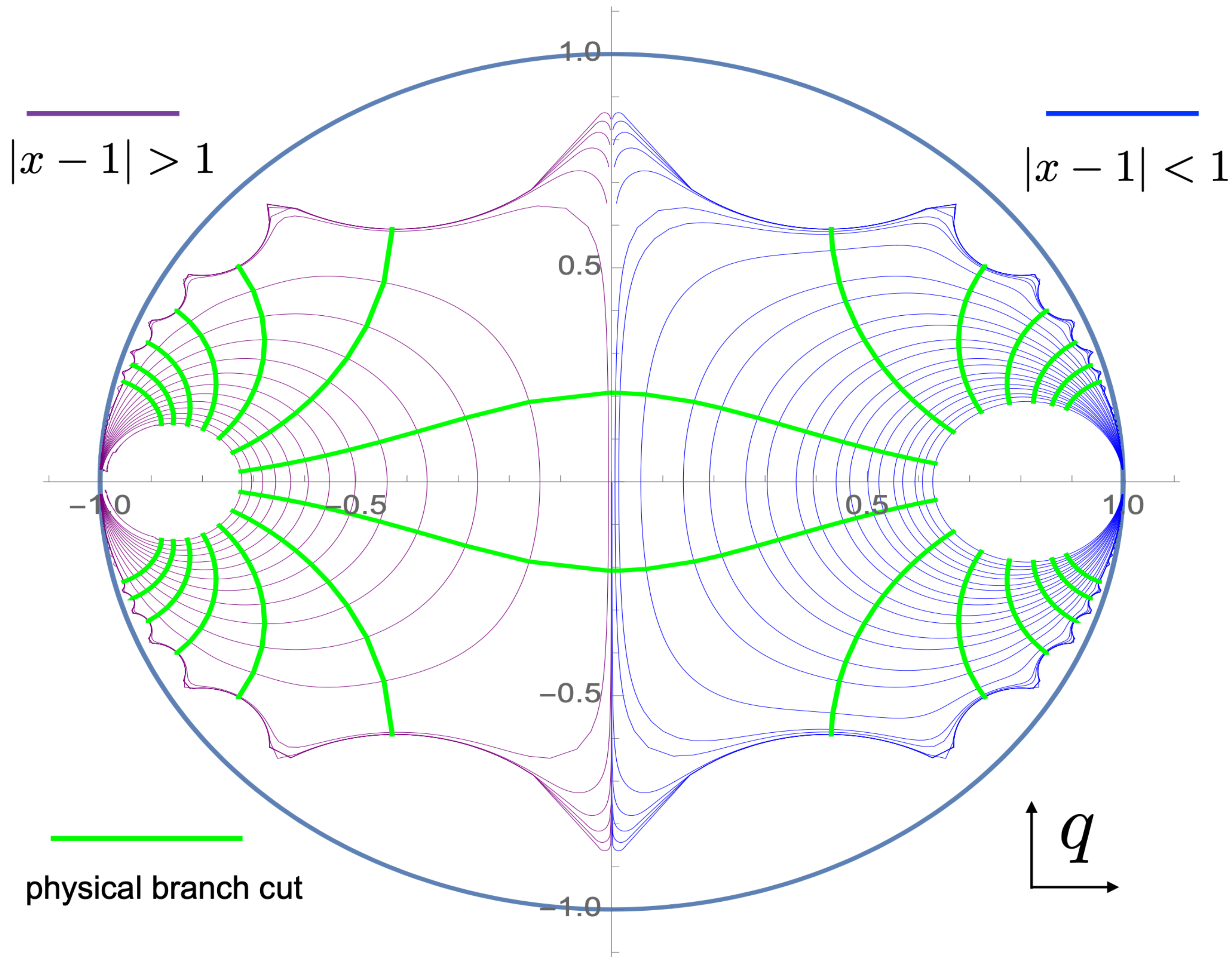
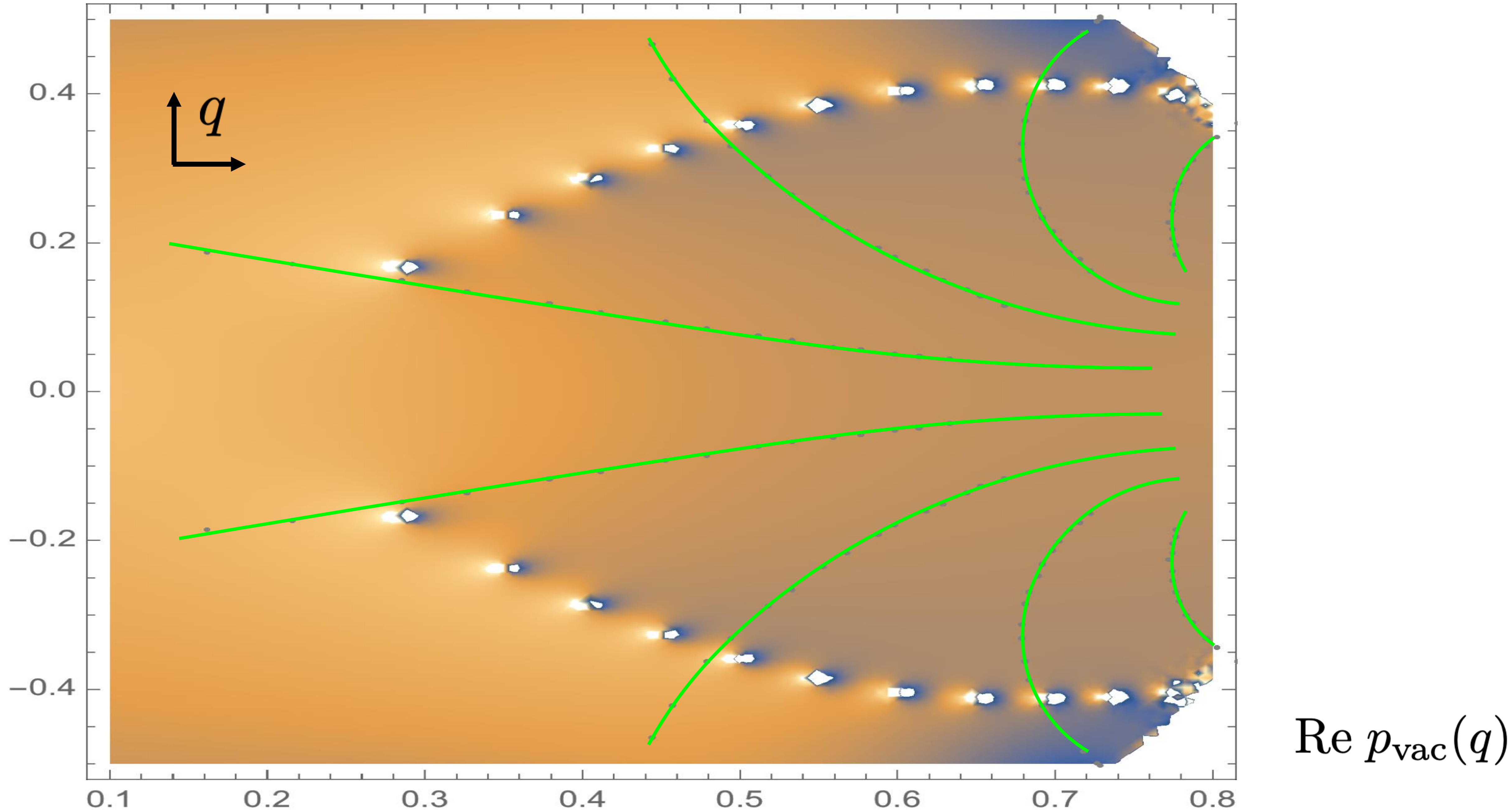


Numerical results: (on q-plane)



Numerical results: $c = 30, h_H = 5, h_L = 0.5$



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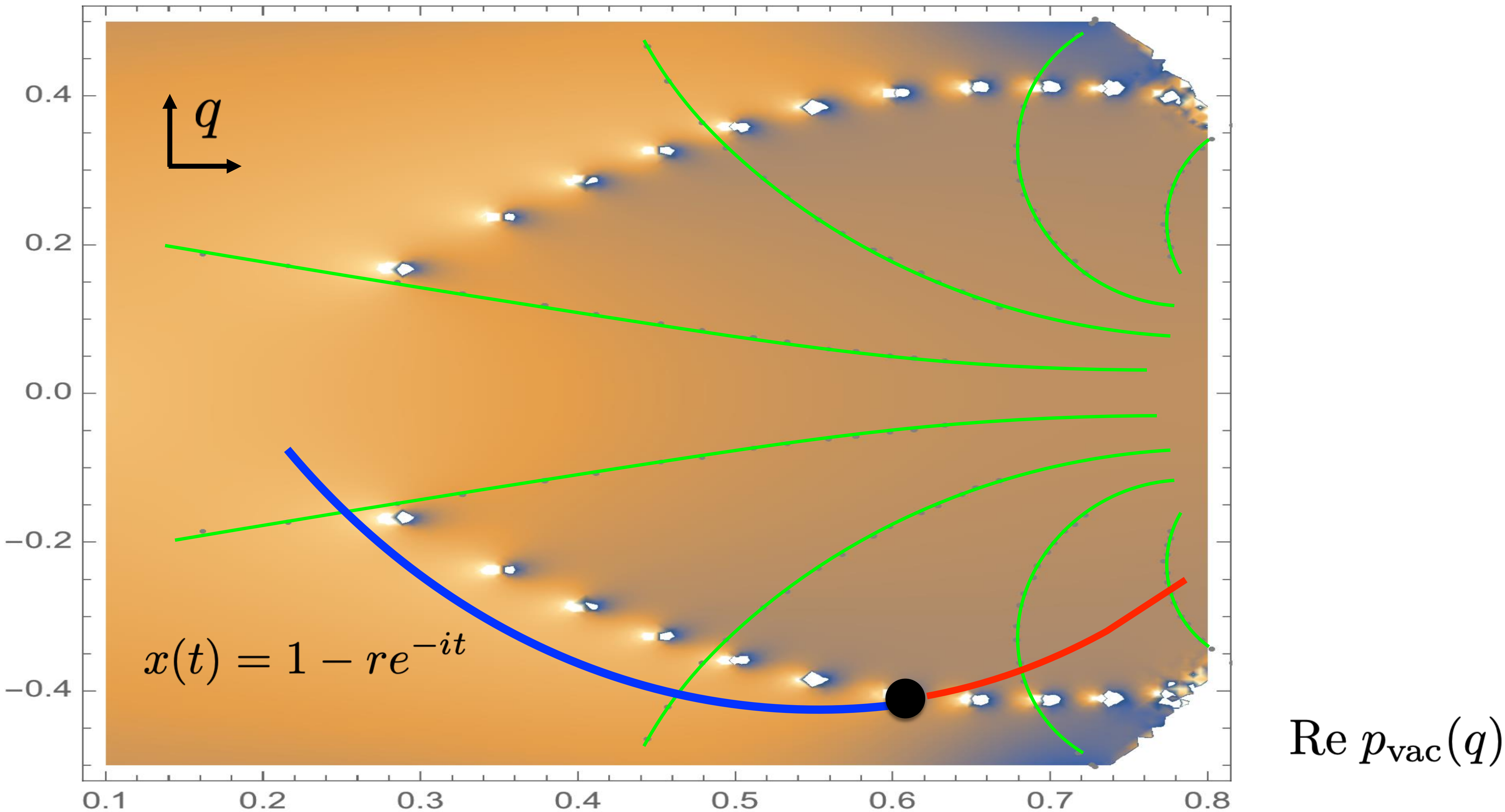

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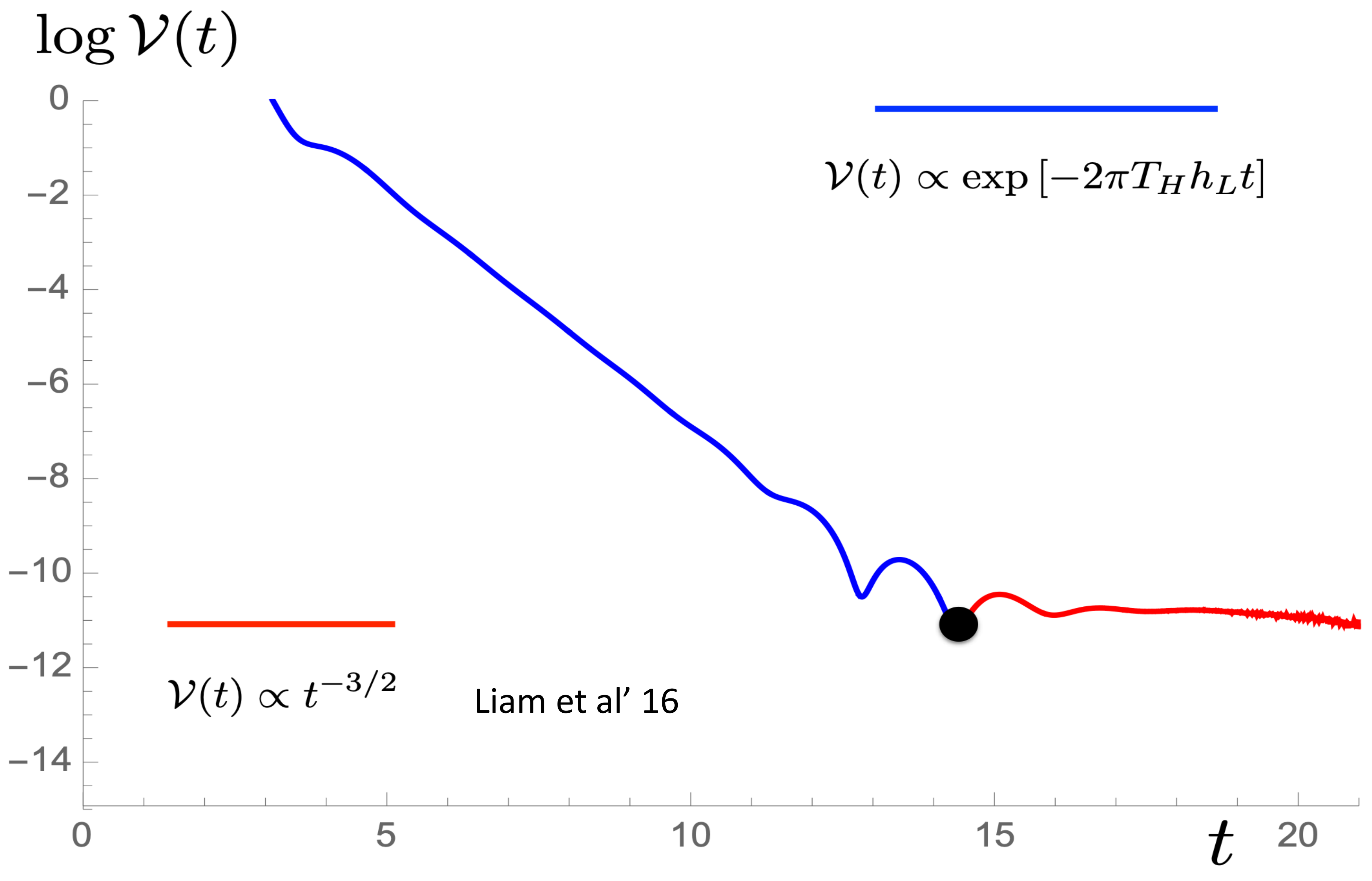

Comments:

- Similar transitions also observed in spectral form factors $|Z(\beta+it)|$ for the SYK model (Cotler, et al'17) and the BTZ black holes (Dyer et al'17)
- For the SYK model, the transition arises from one-loop effect in the Schwarzian effective action.
- For the BTZ black holes, the transition arises from an infinite number of saddle-switchings (non-perturbative effects).
- For the Virasoro vacuum block, the transition arises from a single Stoke's phenomena (non-perturbative effect).

Outline

- ETH at leading order — “forbidden singularities”
- Resolution by “probe” corrections
- Resolution by finite c corrections
- Real time dynamics
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Conclusions

- ETH for $\mathcal{O}_{obs} \propto \mathcal{O}_L(\tau)\mathcal{O}_L(0)$: "forbidden singularities"
- re-summing "probe" corrections: "forbidden branch-cuts"

  —— "additional saddles"
- similar change for the micro-canonical ensemble
- finite c: condensation of series of zeros, Stoke's phenomena
- related to real-time dynamics, exits from exponential decay

Future directions

- understand explicitly the underlying Stoke’s phenomena
- “universal behavior” among blocks?
- relating to spectral form factor?
- bulk (gravitational) interpretation (additional saddle, etc)

Thank you!

Supplementary Slides

The method of monodromy

consider light degenerate operator Ψ , $\Delta \equiv -\frac{1}{2} - \frac{3b^2}{4}$

$$c \equiv 1 + 6(b + 1/b)^2 \qquad b \ll 1$$

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \mathcal{O}_2(z_2, \bar{z}_2) \Psi(z, \bar{z}) \mathcal{O}_3(z_3, \bar{z}_3) \mathcal{O}_4(z_4, \bar{z}_4) \rangle$$

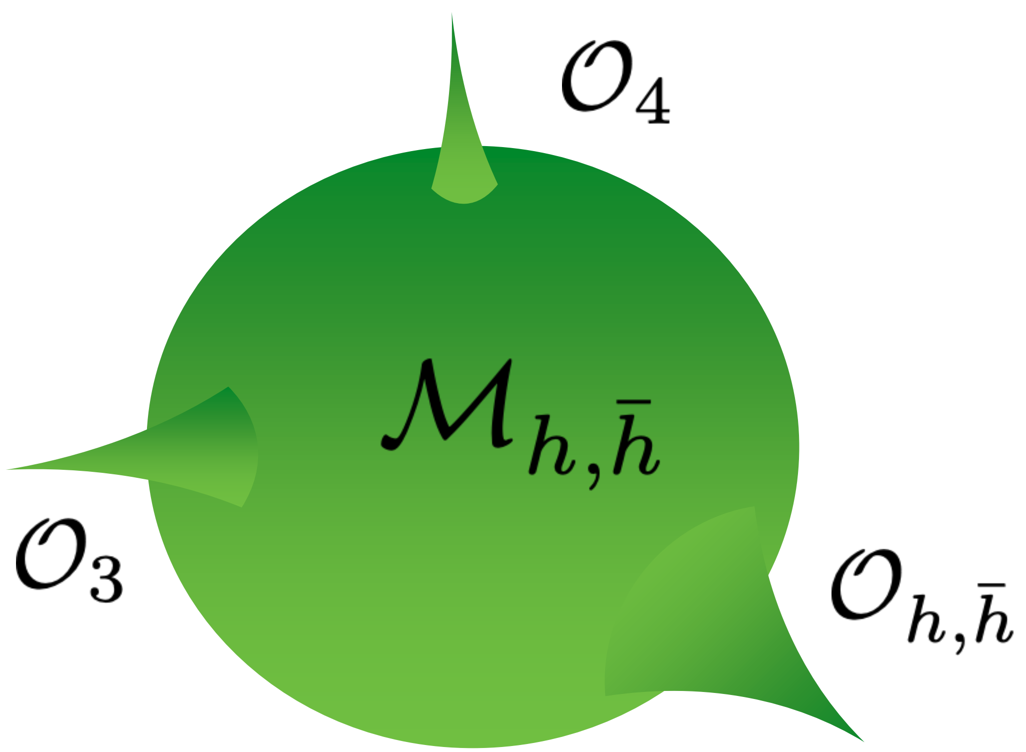
satisfies:

$$\left[\frac{1}{b^2} \partial_z^2 + \sum_i \left(\frac{\Delta_i}{(z - z_i)^2} + \frac{1}{z - z_i} \partial_i \right) \right] \langle \mathcal{O}_1 \mathcal{O}_2 \Psi \mathcal{O}_3 \mathcal{O}_4 \rangle = 0$$

The method of monodromy

$$\langle \mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2(z_2,\bar{z}_2)\Psi(z,\bar{z})\mathcal{O}_3(z_3,\bar{z}_3)\mathcal{O}_4(z_4,\bar{z}_4)\rangle$$

$$\approx \sum_{h,\bar{h}} C^{12}_{h,\bar{h}} C^{34}_{h,\bar{h}} \Psi_{h,\bar{h}}(z_i,\bar{z}_i,z,\bar{z}) \mathcal{V}_h(z_i) \bar{\mathcal{V}}_{\bar{h}}(\bar{z}_i)$$



$$\Psi_{h,\bar{h}}(z_i,\bar{z}_i,z,\bar{z}) \sim \langle \Psi(z,\bar{z}) \rangle_{\mathcal{M}_{h,\bar{h}}}$$

Ψ **just probes** $\mathcal{M}_{h,\bar{h}}$

no “back-reaction” on $\mathcal{M}_{h,\bar{h}}$

The method of monodromy

$$\mathcal{V}_{h,\bar{h}}(z_i) \sim e^{-\frac{c}{6}S_{cl}(z_i)}, \ \Psi_{h,\bar{h}}(z,\bar{z}) \sim \mathcal{O}(c^0)$$

leading order in $c \to \infty$

$$\Psi_h''(z) + T(z)\Psi_h(z) = 0, \ T(z) = \sum_i \left\{ \frac{\epsilon_i}{(z-z_i)^2} - \frac{6}{c}\frac{\partial_i S_{cl}}{z-z_i} \right\}$$

conformal symmetry $z_1 \to 0, z_2 \to x, z_3 \to 1, z_4 \to \infty$

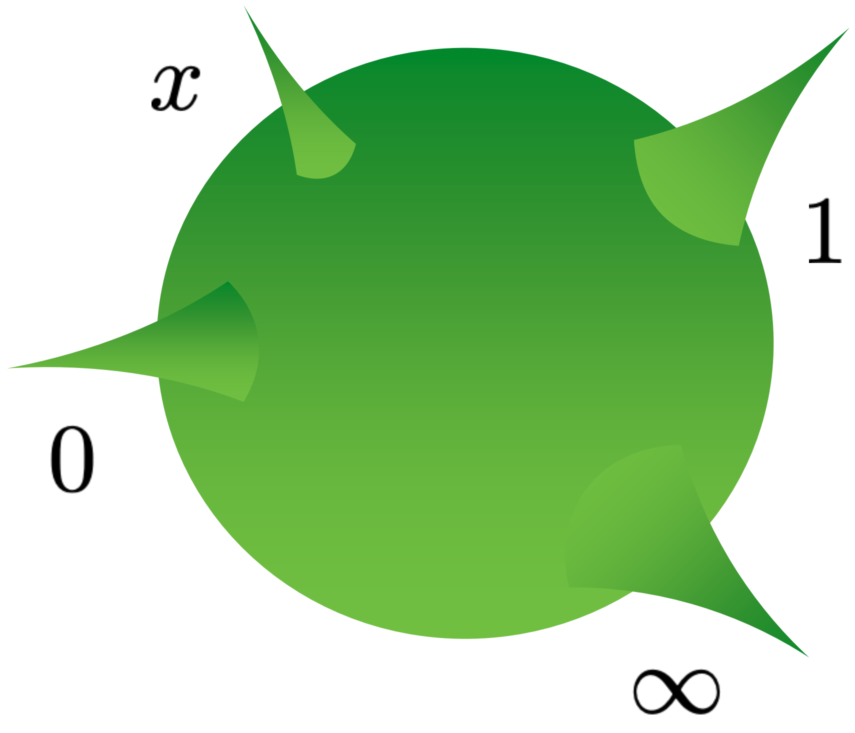
$$T(z) = \frac{\epsilon_1}{z^2} + \frac{\epsilon_2}{(z-x)^2} + \frac{\epsilon_3}{(1-z)^2} + \frac{\sum_i \epsilon_i - 2\epsilon_4}{z(1-z)} - \frac{p_x x(1-x)}{z(z-x)(1-z)}$$

$$p_x = -\frac{6}{c}\partial_x \ln \mathcal{V}_h(x)$$ **“accessory parameter”**

How do the block info $h, \bar{h}$ enter?

$$\Psi_h''(z) + T(z)\Psi_h(z) = 0$$, **two solutions** $\{\Psi_h^+(z), \Psi_h^-(z)\}$

regular singularities at $z = \{0, x, 1, \infty\}$



$$\begin{pmatrix} \Psi_h^+ \\ \Psi_h^- \end{pmatrix}_{\circlearrowright} = \hat{M} \begin{pmatrix} \Psi_h^+ \\ \Psi_h^- \end{pmatrix}$$

The method of monodromy

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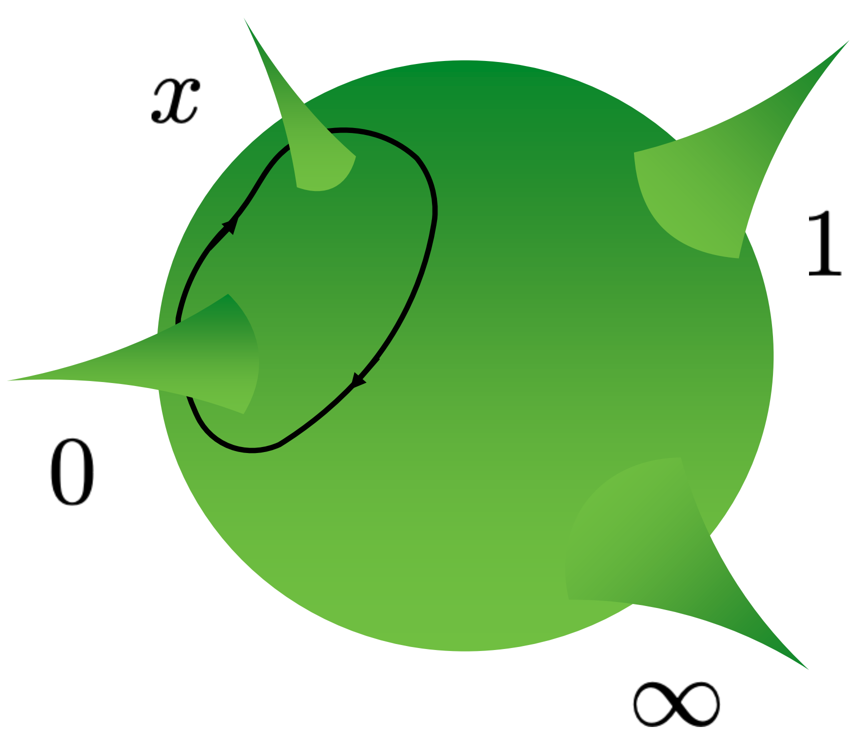

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$$\mathrm{tr}\hat{M}_{0x} = -2\cos\left(\pi\Lambda_h\right)$$

$$h = \frac{c}{24}\left(1 - \Lambda_h^2\right)$$

The method of monodromy

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$$\Psi_h''(z) + T(z)\Psi_h(z) = 0$$ **,** **two solutions** $\{\Psi_h^+(z), \Psi_h^-(z)\}$

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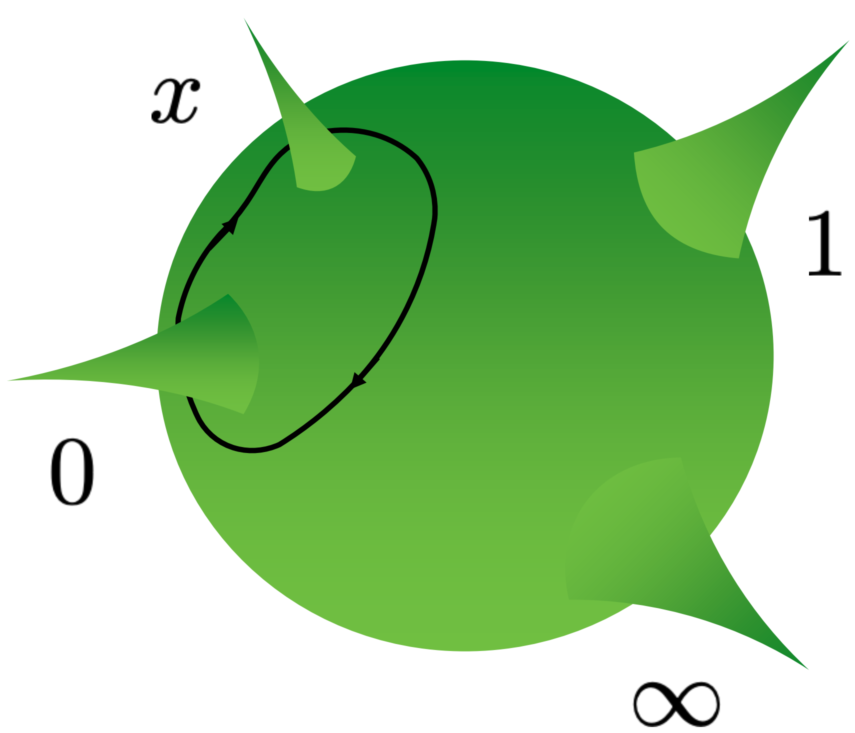

$$\begin{pmatrix} \Psi_h^+ \\ \Psi_h^- \end{pmatrix}_{\circlearrowright} = \hat{M} \begin{pmatrix} \Psi_h^+ \\ \Psi_h^- \end{pmatrix}$$

vacuum block: $h = 0$

$$\hat{M}_{0x} = \mathbb{1}$$

Zamolodchikov's recursive relation:

$$\mathcal{V}(c,h_i,h_p,x) = (16q)^{h_p-\frac{c-1}{24}} x^{\frac{c-1}{24}} (1-x)^{\frac{c-1}{24}-h_2-h_3} \theta_3(q)^{\frac{c-1}{2}-4\sum_i h_i} H(c,h_i,h_p,q)$$

$$q = e^{i\pi\tau},\ \tau = i\frac{K(1-x)}{K(x)},\ \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

$$H(c,h_i,h_p,q) = 1 + \sum_{m\geq 1, n\geq 1}^{\infty} \frac{(16q)^{mn}\,\hat{R}_{mn}(c,h_i)}{h_p - h_{p,mn}(c)} H(c,h_i,h_{p,mn}+mn,q)$$

$$\hat{R}_{mn}(c,h_i) = \quad -\frac{1}{2}\frac{\prod_{j,k}\left(\lambda_2+\lambda_1-\frac{\lambda_{jk}}{2}\right)\left(\lambda_2-\lambda_1-\frac{\lambda_{jk}}{2}\right)\left(\lambda_3+\lambda_4-\frac{\lambda_{jk}}{2}\right)\left(\lambda_3-\lambda_4-\frac{\lambda_{jk}}{2}\right)}{\prod_{a,b}\lambda_{ab}}$$

$$h_{p,mn}(c) = \frac{1}{4}(n^2-1)t(c) + \frac{1}{4}(m^2-1)\frac{1}{t(c)} - \frac{1}{2}(mn-1)$$

$$t(c) = 1 + \frac{1}{12}\left(1-c\pm\sqrt{(1-c)(25-c)}\right)$$

$$i = -m+1, -m+3, \ldots, m-3, m-1;\quad j = -n+1, -n+3, \ldots, n-3, n-1$$

$$-m+1 \leq a \leq m,\ -n+1 \leq b \leq n,\ (a,b) \neq (0,0), (a,b) \neq (m,n)$$

$$\lambda_i = \quad \sqrt{h_i + \frac{1-c}{24}},\ \ \lambda_{pq} = \frac{1}{\sqrt{24}}\left\{(p+q)\sqrt{1-c} + (p-q)\sqrt{25-c}\right\}$$

How are the "additional saddles" related at finite c?

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One possible structure (speculative!): **infinite-order ODE for** $\mathcal{V}(x)$

$$\sum_{k=0}^{\infty} h_k(c,x)\partial_x^k \mathcal{V}(x) = 0, \quad \lim_{c\to\infty} h_k(c,x) \sim \left(\frac{c}{6}\right)^{-k} g_k(x)$$

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WKB solutions: $\mathcal{V}(x) \propto e^{-\frac{c}{6}\int^x p(x')}$

$$c \to \infty \qquad \sum_{n=0}^{\infty} g_n(x) p(x)^n = 0$$

"monodromy equation"

roots of the equation: $p_0(x), p_1(x), p_2(x), \ldots$

infinitely many WKB solutions:

$$e^{-\frac{c}{6}\int^x p_0(x')}, \; e^{-\frac{c}{6}\int^x p_1(x')}, \; e^{-\frac{c}{6}\int^x p_2(x')}, \; \cdots$$

Stoke's phenomena

"branch point" ⟷ **"turning point"**

$$p_m(x^*) = p_n(x^*)$$

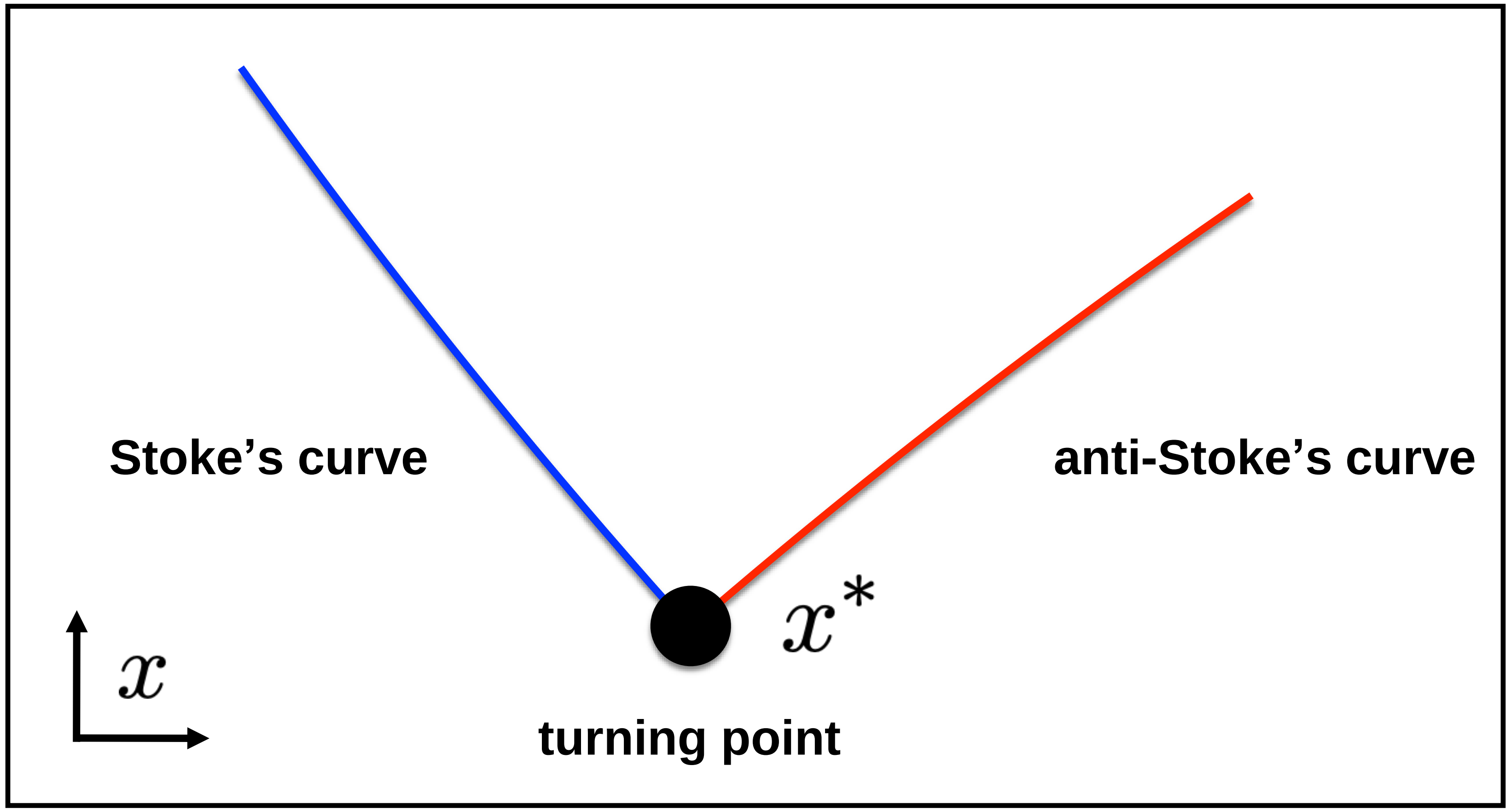
infinitely many WKB solutions:

$$e^{-\frac{c}{6}\int^{x} p_0(x')}, \; e^{-\frac{c}{6}\int^{x} p_1(x')}, \; e^{-\frac{c}{6}\int^{x} p_2(x')}, \; \cdots$$

Stoke's phenomena

$$\mathrm{Re}\int^{x} p_n(x') = \mathrm{Re}\int^{x} p_m(x')$$

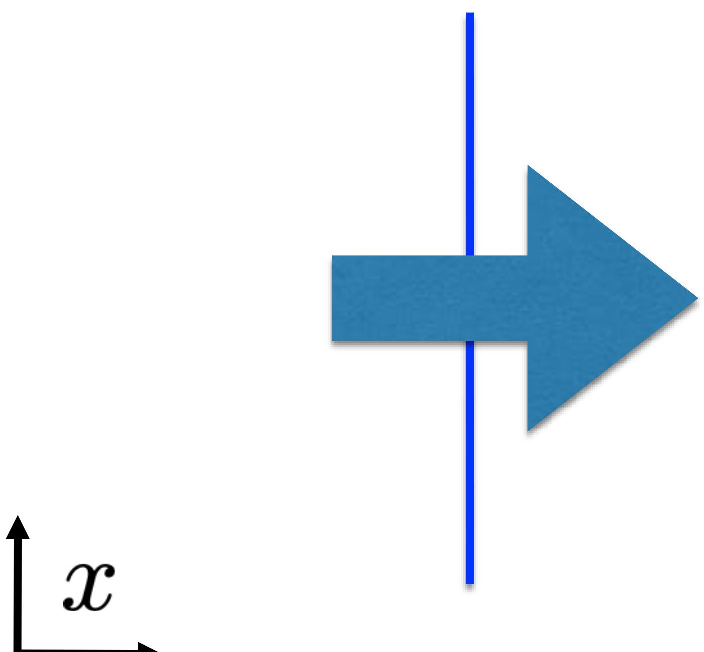
$$\mathrm{Im}\int^{x} p_n(x') = \mathrm{Im}\int^{x} p_m(x')$$



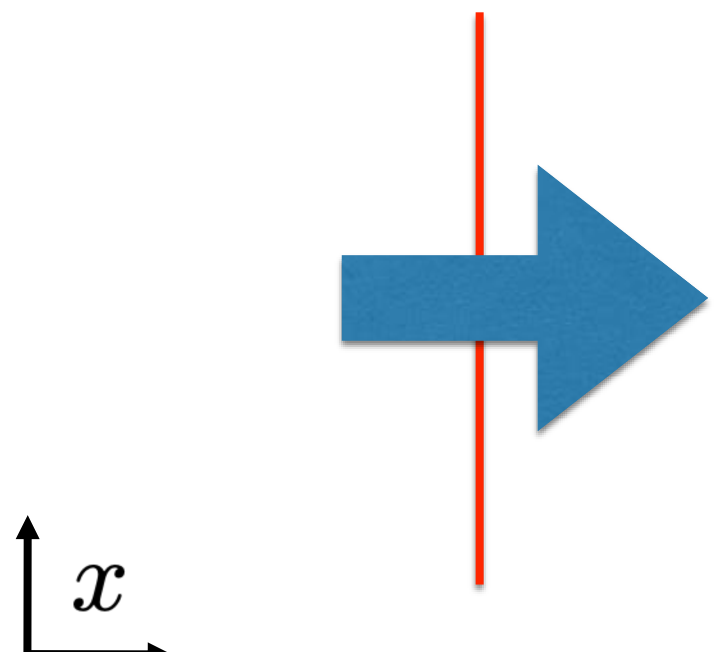
Stokes’ phenomena

Stoke’s phenomena:

$$\mathcal{V}^d(x) \propto e^{-\frac{c}{6}\int^x p_+(x')} \gg \mathcal{V}^s(x) \propto e^{-\frac{c}{6}\int^x p_-(x')}$$



Stoke’s curve



anti-Stoke’s curve

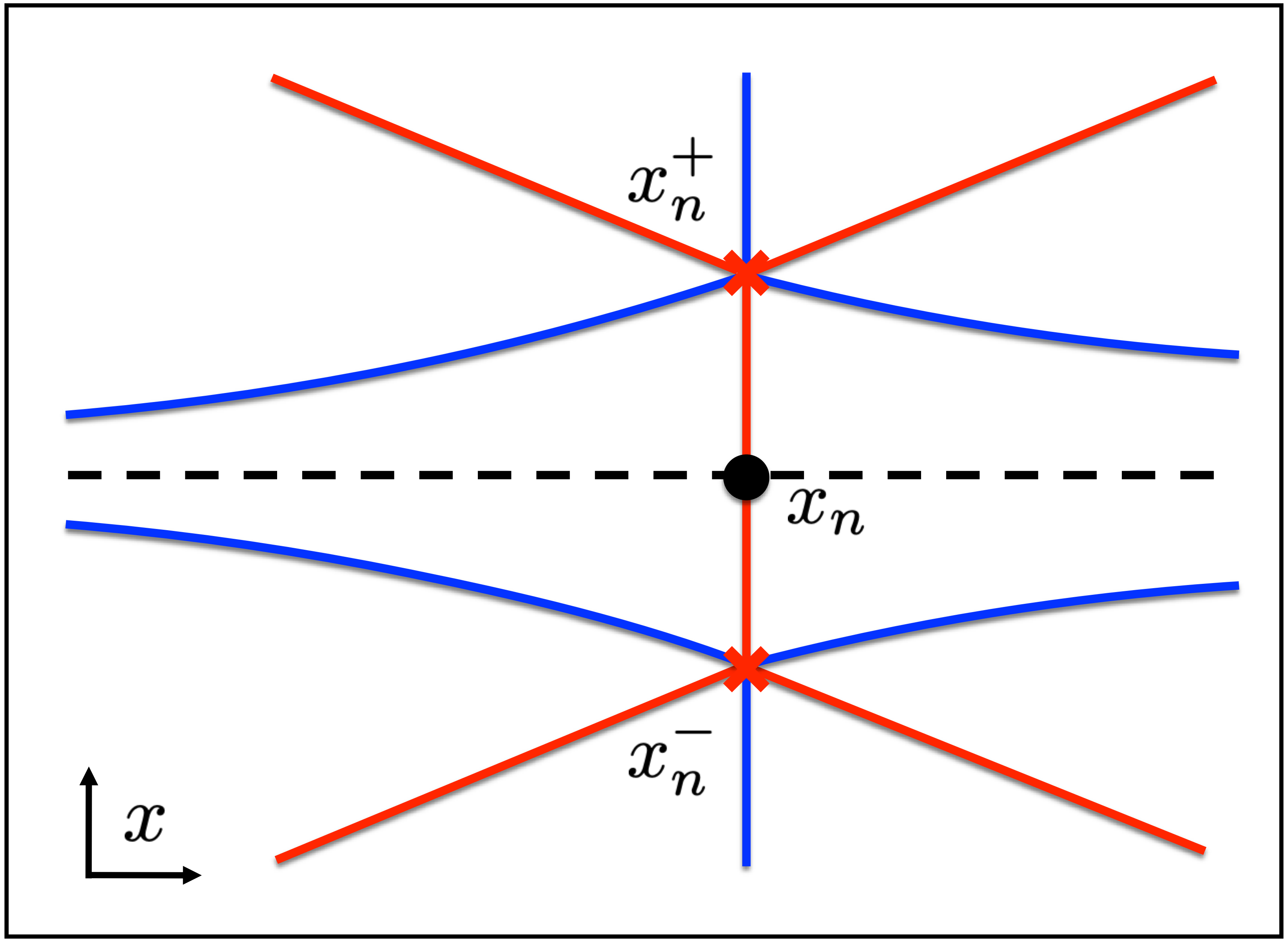
$$\mathcal{V}^d(x) \to \mathcal{V}^d(x) + T\mathcal{V}^s(x)$$

$$\mathcal{V}^s(x) \to \mathcal{V}^s(x)$$

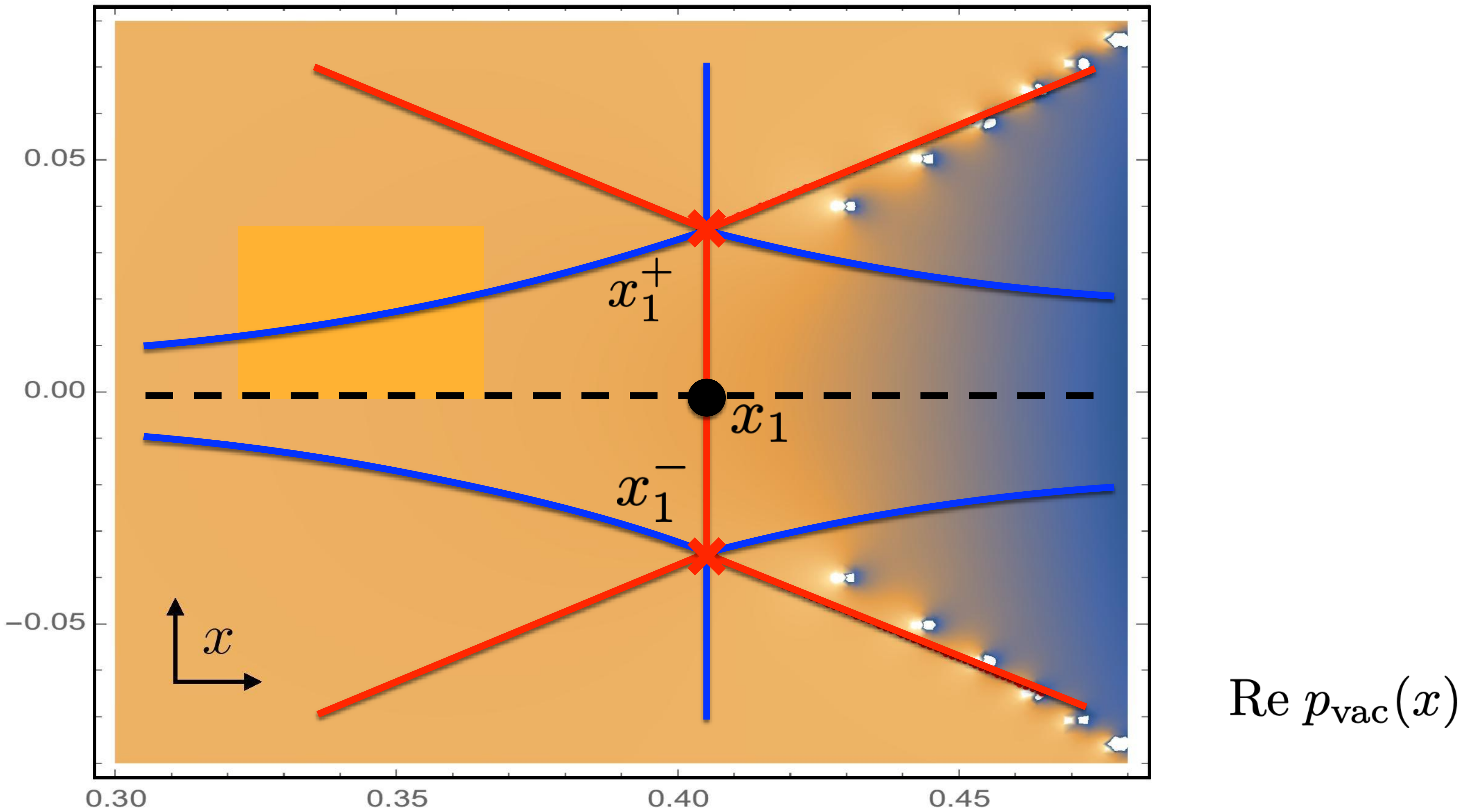
$$\mathcal{V}^d(x) \leftrightarrow \mathcal{V}^s(x)$$

near "forbidden branch-points"

$$p_{\pm}(x) = \frac{-(x - x_n) \mp \sqrt{(x - x_n^+)(x - x_n^-)}}{2b_n}$$



near "forbidden branch-points": consistency with numerics



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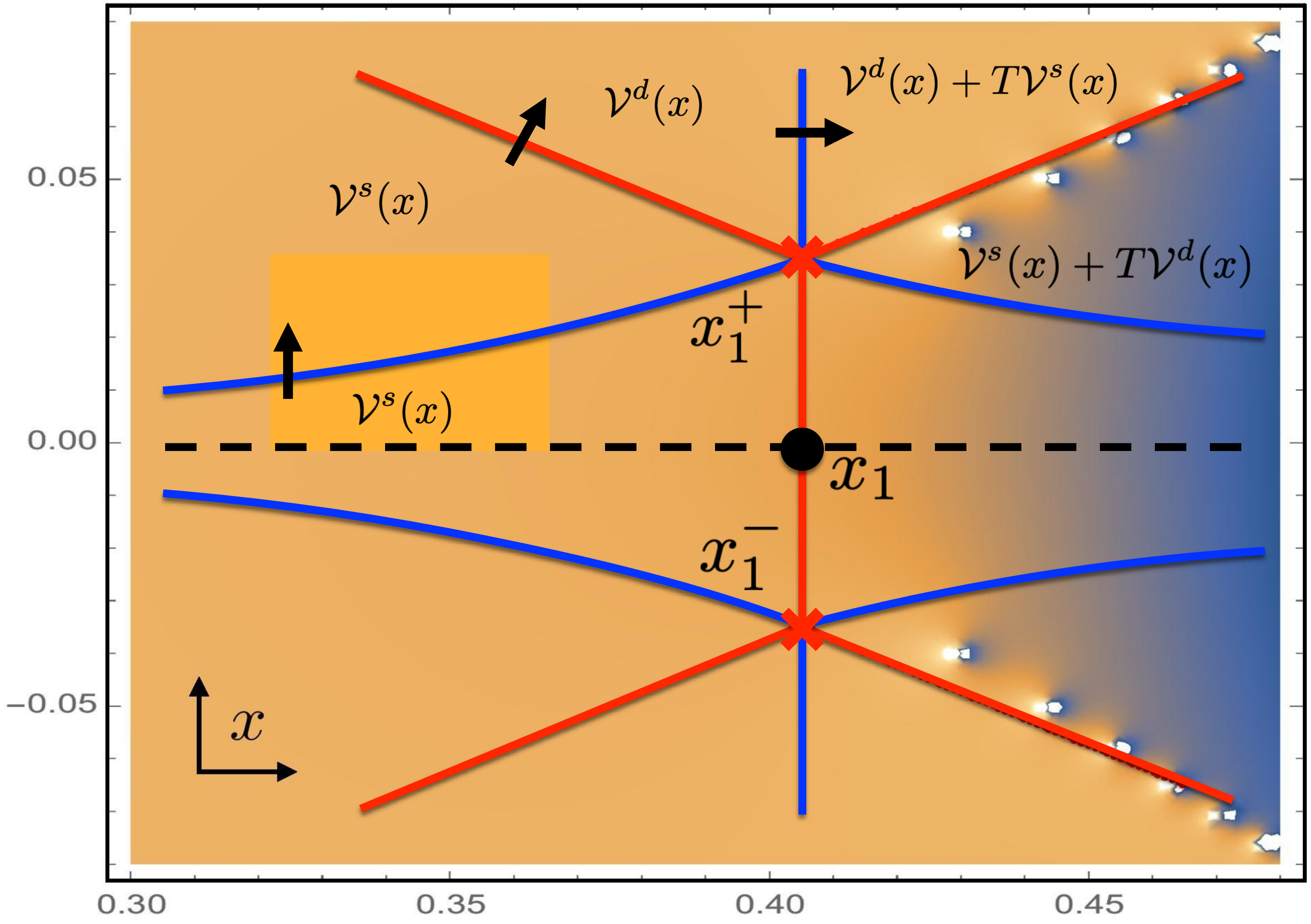
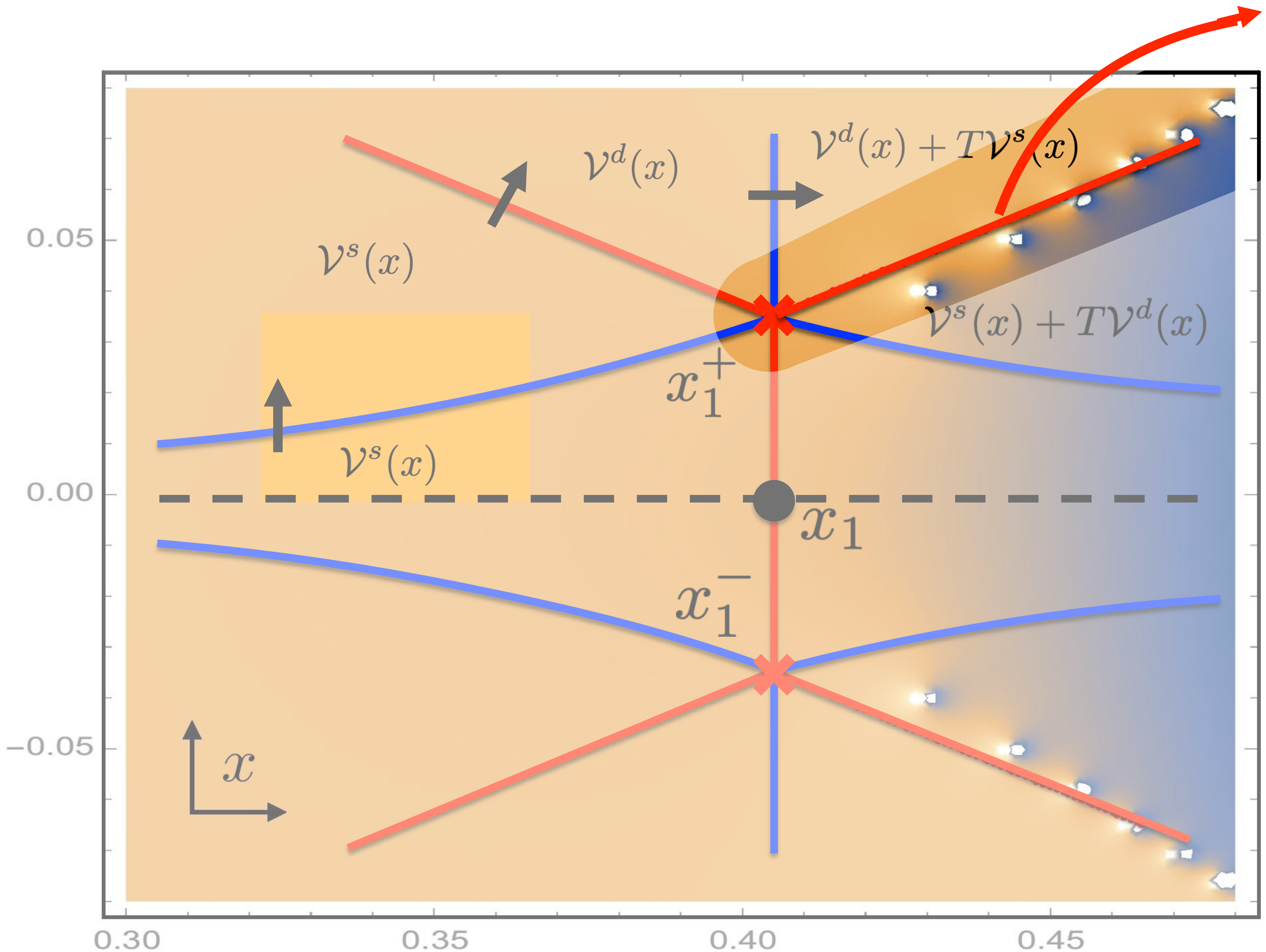


Re $p_{\mathrm{vac}}(x)$

near "forbidden branch-points": consistency with numerics



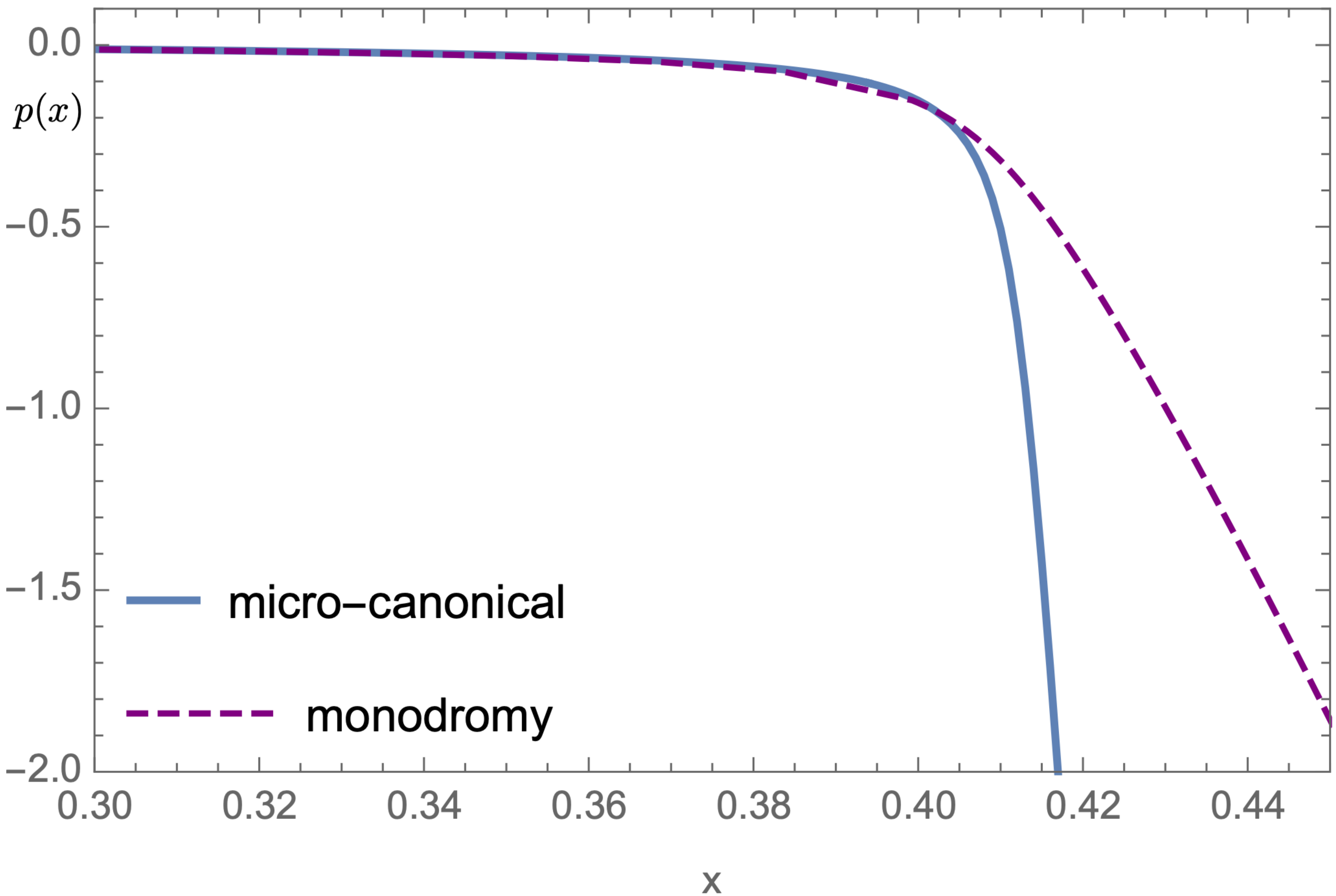
$\mathcal{V}^d(x) \sim \mathcal{V}^s(x)$

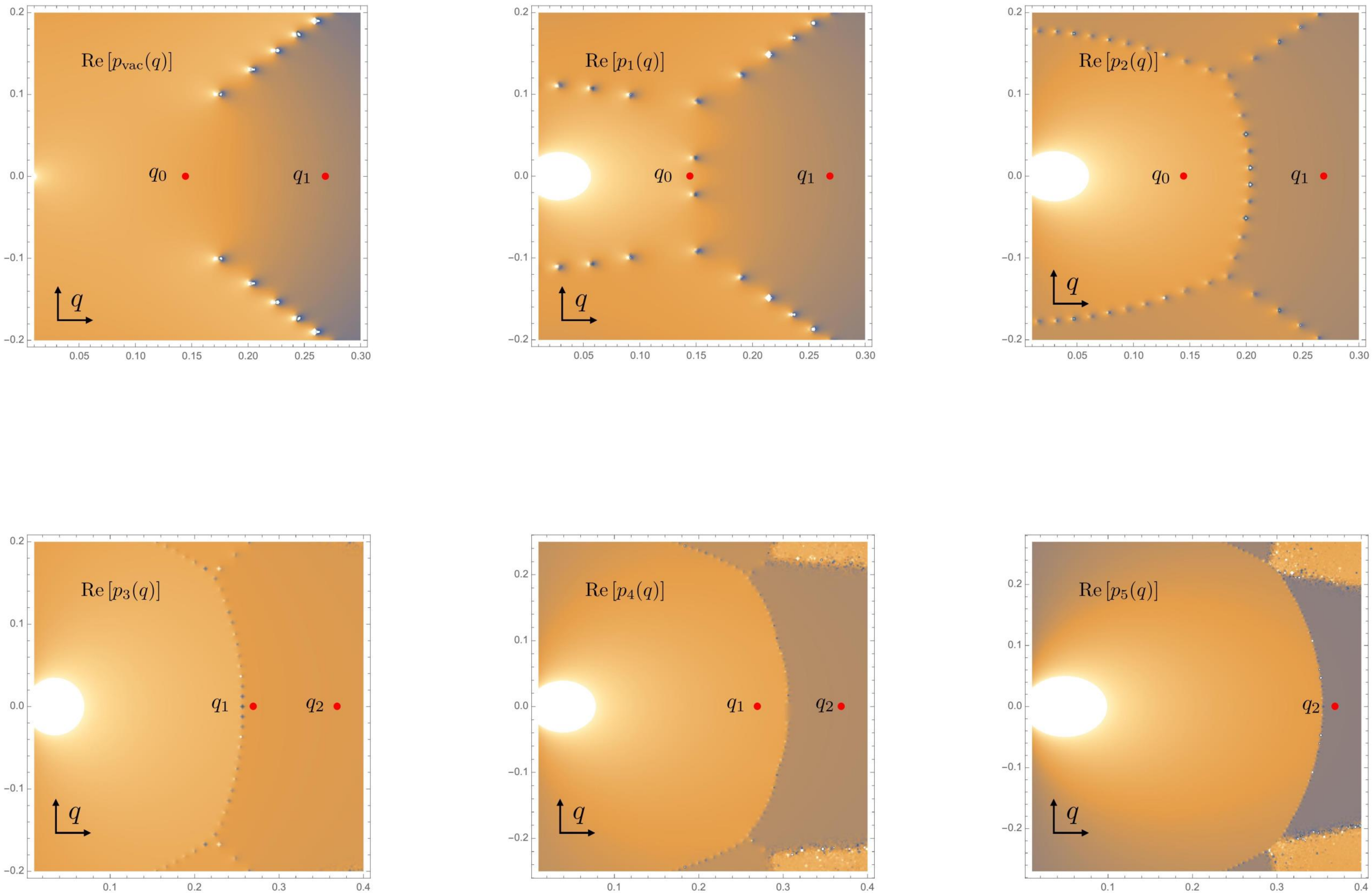
oscillatory

zeros of $\mathcal{V}_{\text{vac}}(x)$

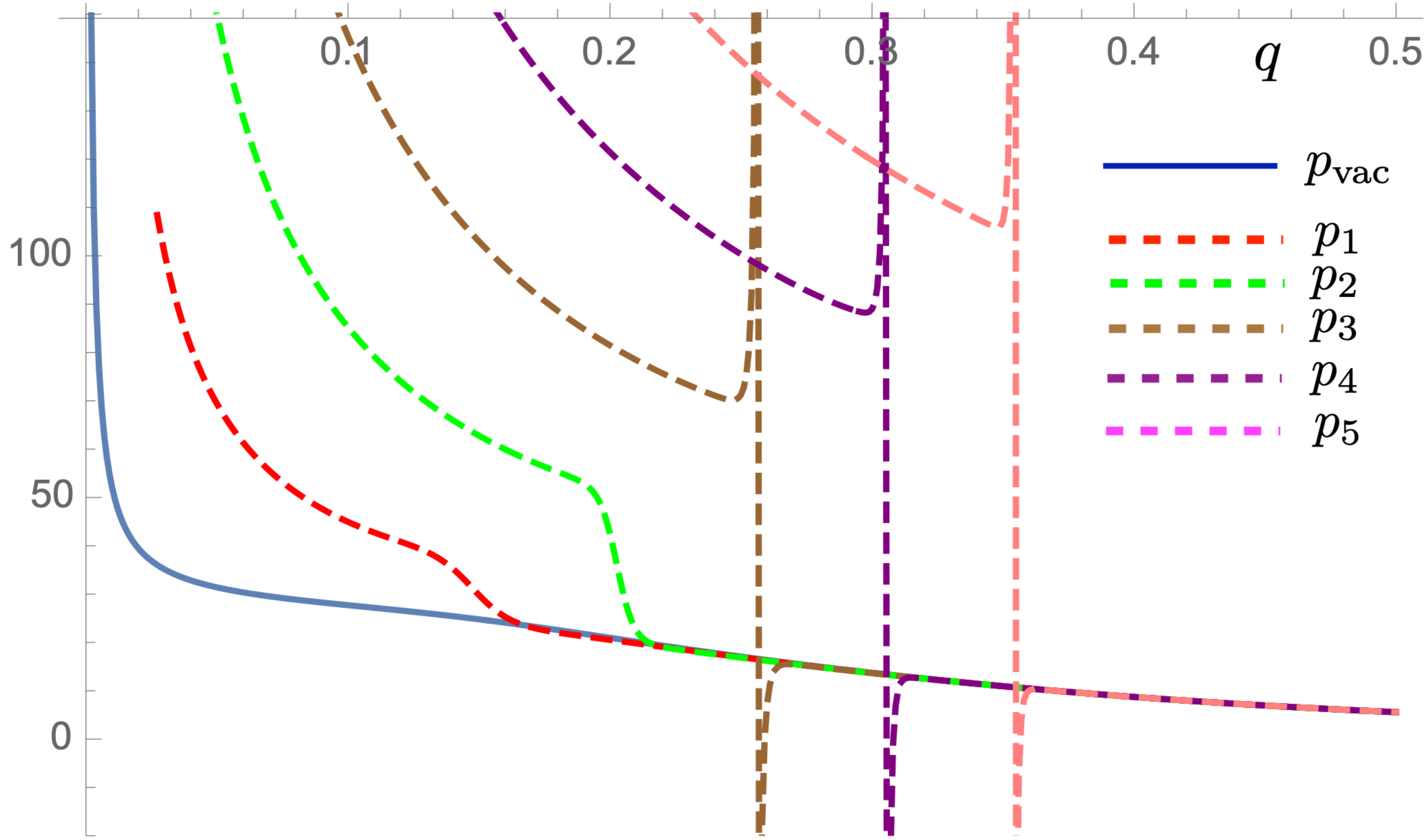
$\text{Re}\, p_{\text{vac}}(x)$

micro-canonical vs eigenstate

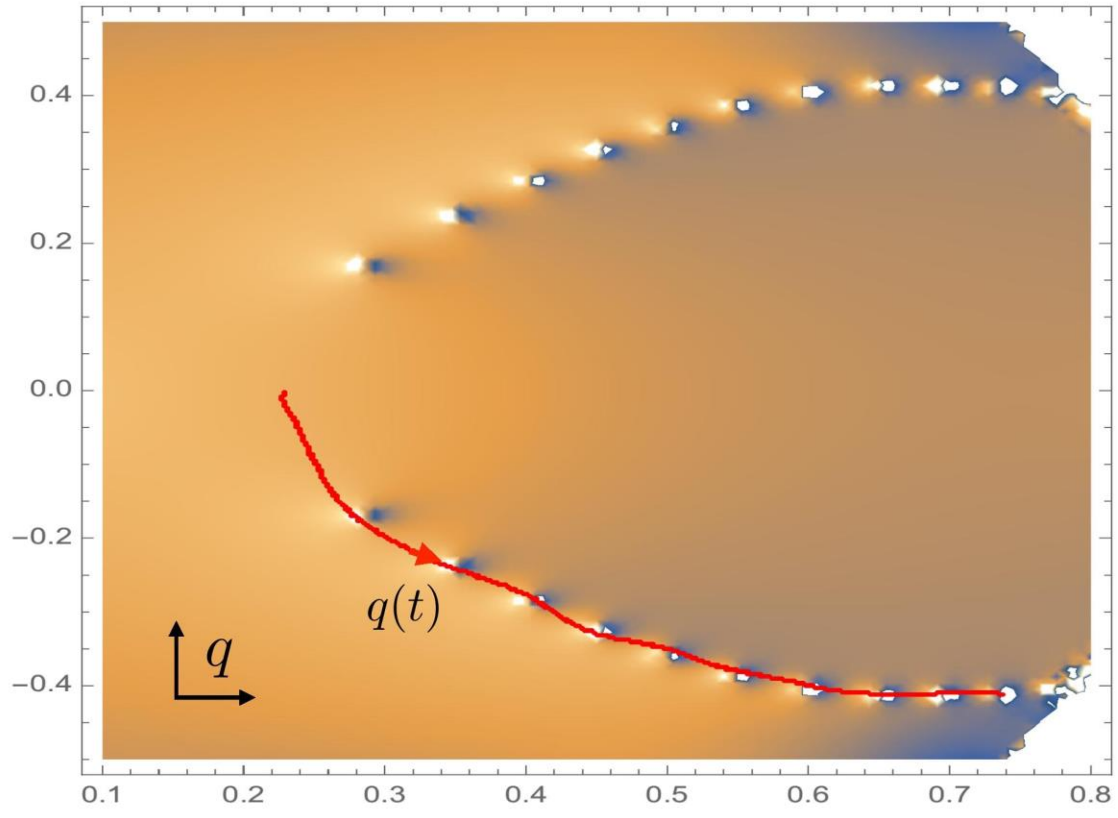




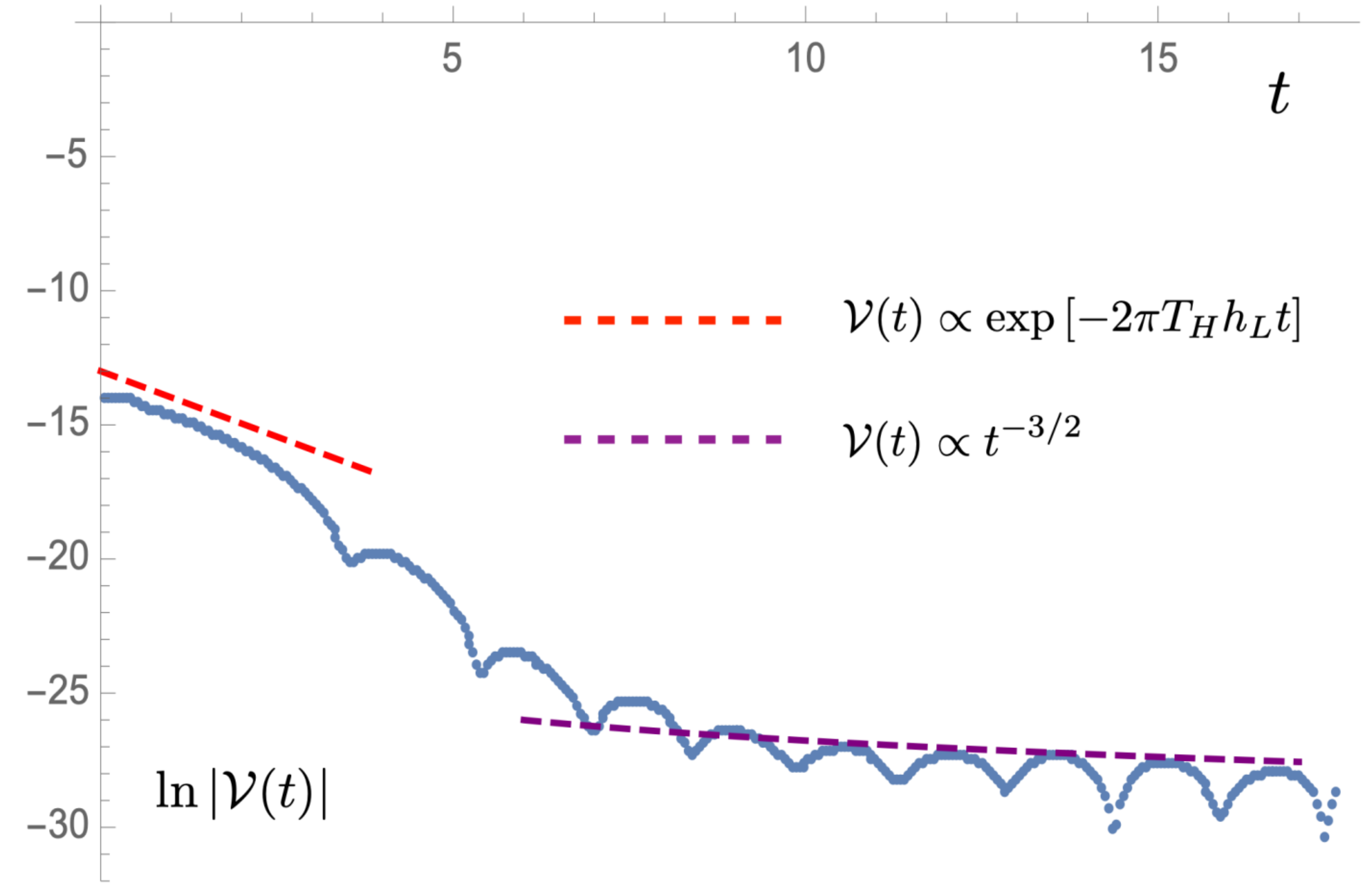
Re [p_vac(q)]
Re [p_1(q)]
Re [p_2(q)]
Re [p_3(q)]
Re [p_4(q)]
Re [p_5(q)]
q_0
q_1
q_2
q



0.1
0.2
0.3
0.4
0.5
q
100
50
0
$p_{\rm vac}$
p_1
p_2
p_3
p_4
p_5



0.4
0.2
0.0
−0.2
−0.4
0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8
$q(t)$
q



5 10 15
t
−5
−10
−15
−20
−25
−30
$\mathcal{V}(t) \propto \exp[-2\pi T_H h_L t]$
$\mathcal{V}(t) \propto t^{-3/2}$
$\ln|\mathcal{V}(t)|$

Resolution by "probe" corrections (optional)

Probe effects in Micro-canonical ensemble

ETH:

$$\langle \mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle_H \approx \langle \mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle_{\rm micro}$$

in the probe limit

$$\langle \dots \rangle_{\rm micro} \approx \langle \dots \rangle_{\rm canonical}$$

"forbidden singularities"

thermal periodicity

Resolution by "probe" corrections (optional)

Probe effects in Micro-canonical ensemble

ETH:

$$\langle \mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle_H \approx \langle \mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle_{\rm micro}$$

+ finite probe effects

"forbidden branch-cuts"

?

$\langle \ldots \rangle_{\rm micro} \not\approx \langle \ldots \rangle_{\rm canonical}$

Probe effects in Micro-canonical ensemble

$$\rho(E)\langle\mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle^E_{\text{micro}} \propto \int_\Gamma d\beta\, e^{\beta E} Z(\beta)\langle\mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle_\beta$$

Probe effects in Micro-canonical ensemble

$$\rho(E)\langle\mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle^E_{\text{micro}} \propto \int_\Gamma d\beta\, e^{\beta E} Z(\beta)\langle\mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle_\beta$$

$c \to \infty$: saddle point approximation β^*

Resolution by “probe” corrections (optional)

Probe effects in Micro-canonical ensemble

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$c \to \infty$: saddle point approximation β^*

probe limit :

$$E + Z'(\beta^*)/Z(\beta^*) = 0 \quad \Rightarrow \quad \langle\ldots\rangle_{\mathrm{micro}} \approx \langle\ldots\rangle_{\mathrm{canonical}}$$

Resolution by "probe" corrections (optional)

Probe effects in Micro-canonical ensemble

$$\rho(E)\langle \mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle^{E}_{\text{micro}} \propto \int_\Gamma d\beta\, e^{\beta E} Z(\beta)\langle \mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle_\beta$$

$c \to \infty$: saddle point approximation β^*

probe limit :

$$E + Z'(\beta^*)/Z(\beta^*) = 0$$

⇨

$$\langle \dots \rangle_{\text{micro}} \approx \langle \dots \rangle_{\text{canonical}}$$

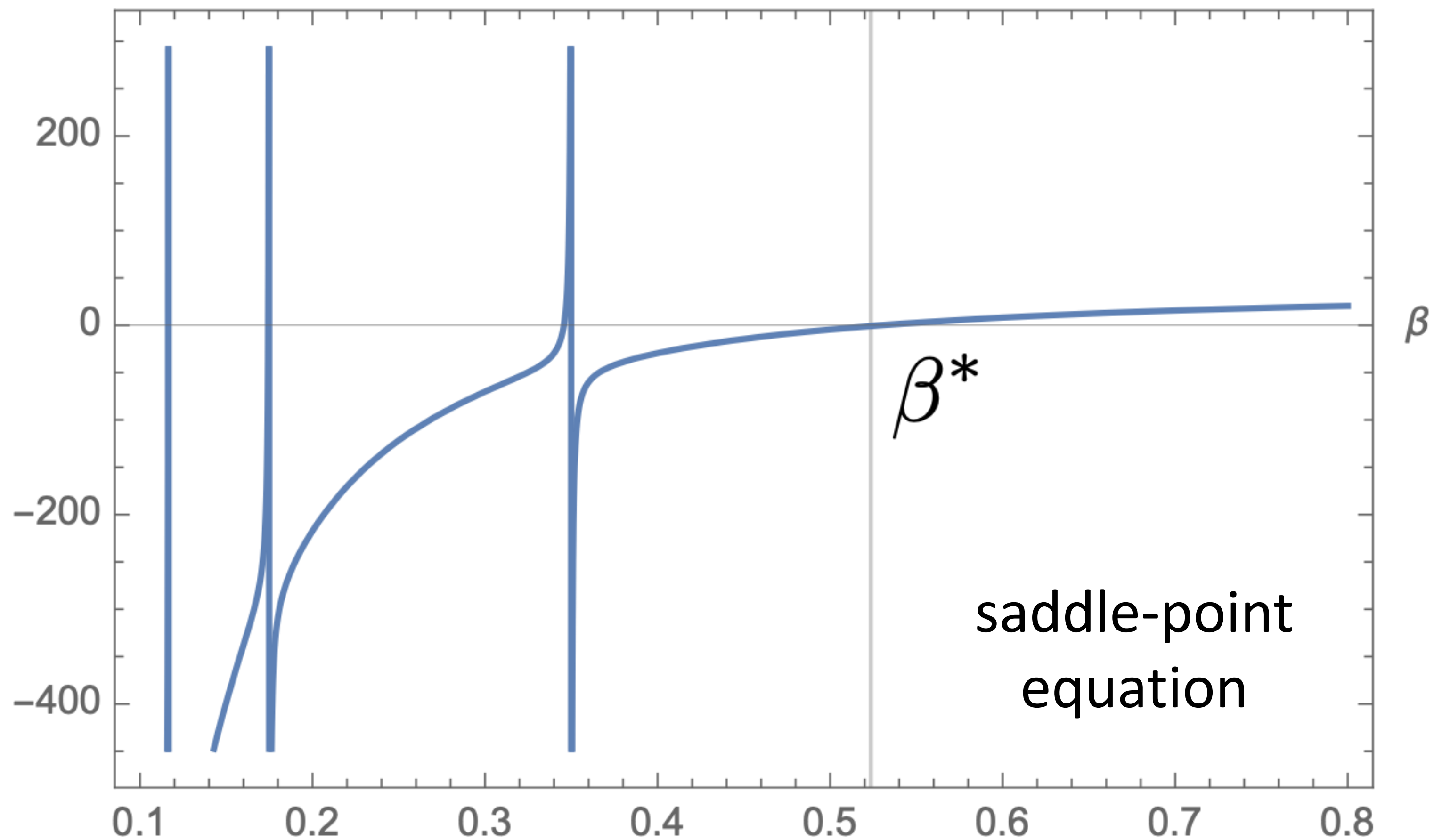
+ finite probe effects :

$$E + Z'(\beta^*)/Z(\beta^*) + \ln' \langle \mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle_{\beta^*} = 0$$

Probe effects in Micro-canonical ensemble

$$E + Z'(\beta^*)/Z(\beta^*) + \ln' \langle \mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle_{\beta^*} = 0$$

“switching of dominant saddle”



$$\tau < \beta_E$$

$$\langle \ldots \rangle_{\text{micro}} \approx \langle \ldots \rangle_{\text{canonical}}$$

Probe effects in Micro-canonical ensemble

$$E + Z'(\beta^*)/Z(\beta^*) + \ln' \langle \mathcal{O}_L(\tau)\mathcal{O}_L(0)\rangle_{\beta^*} = 0$$

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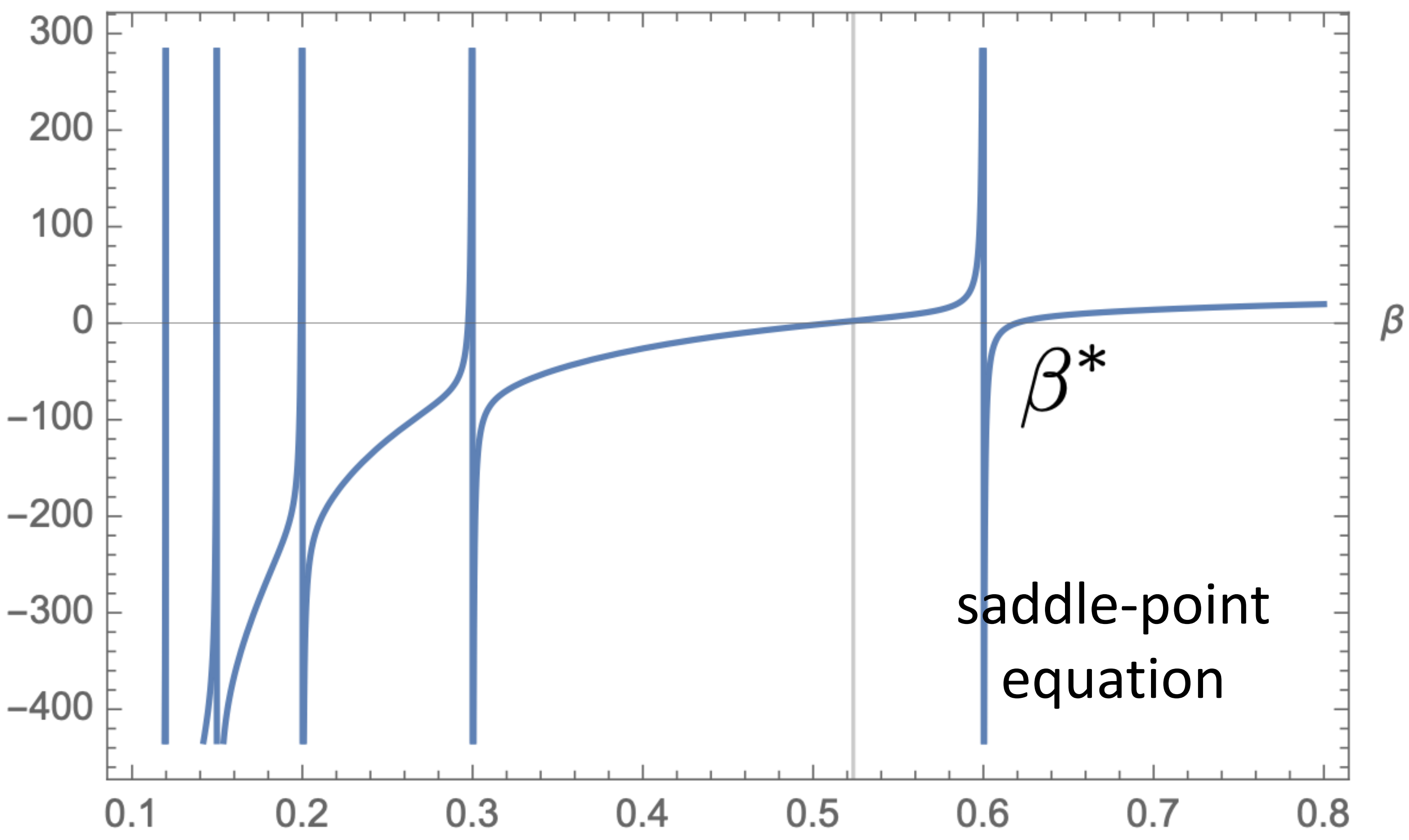

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$$\langle \cdots \rangle_{\text{micro}} \not\approx \langle \cdots \rangle_{\text{canonical}}$$

Probe effects in Micro-canonical ensemble

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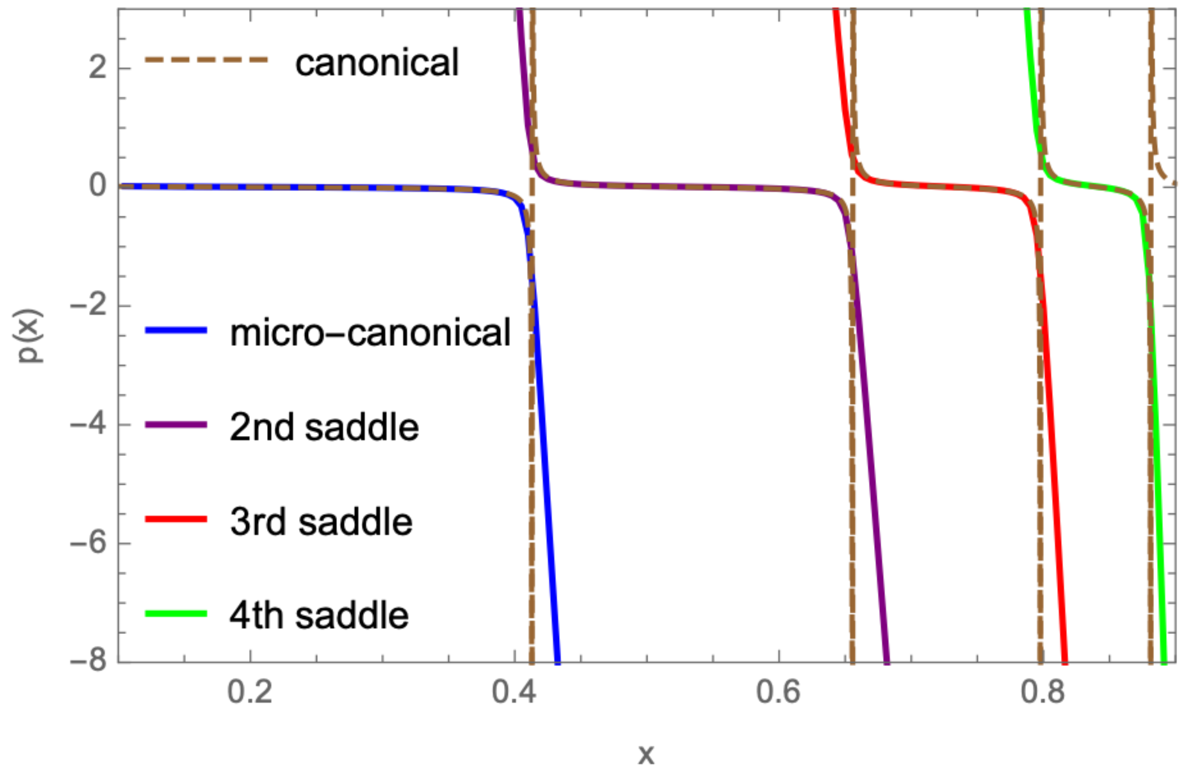
"switching of dominant saddle"

$$\langle \mathcal{O}_L(\tau)\mathcal{O}(0)\rangle_H \quad \text{v.s.} \quad \langle \mathcal{O}_L(\tau)\mathcal{O}(0)\rangle_{\rm micro}$$

+ finite probe corrections, similar change in analytic structure

Resolution by "probe" corrections (optional)



$L \to \infty$ approximation

micro-canonical

eigenstate