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On the stability of recovering two sources and initial status in a stochastic hyperbolic-parabolic system

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Abstract

Consider an inverse problem of determining two stochastic source functions and the initial status simultaneously in a stochastic thermoelastic system, which is constituted of two stochastic equations of different types, namely a parabolic equation and a hyperbolic equation. To establish the conditional stability for such a coupling system in terms of some suitable norms revealing the stochastic property of the governed system, we first establish two Carleman estimates with regular weight function and two large parameters for stochastic parabolic equation and stochastic hyperbolic equation, respectively. By means of these two Carleman estimates, we finally prove the conditional stability for our inverse problem, provided the source in the elastic equation be known near the boundary and the solution be in an *a priori* bounded set. Due to the lack of information about the time derivative of wave field at the final time, the stability index with respect to the wave field at final time is found to be halved, which reveals the special characteristic of our inverse problem for the coupling system.

Keywords: inverse problem, Carleman estimate, stochastic hyperbolic-parabolic system, conditional stability

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1. Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, P)$ be a complete filtered probability space, where a one-dimensional standard Brownian motion $\{B(t)\}_{t\geqslant 0}$ is defined. Let $G \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial G =: \Gamma$ and η the outward unit normal vector on Γ. Further we set $G_T := G \times (0, T)$, $\Gamma_T := \Gamma \times (0, T)$. We consider an inverse problem for the following stochastic hyperbolic-parabolic system

$$\begin{cases} \mathrm{d}\mathbf{u}_t - \nu\Delta\mathbf{u}\,\mathrm{d}t - (\nu+\mu)\nabla\mathrm{div}\,\mathbf{u}\,\mathrm{d}t + \rho\nabla v\,\mathrm{d}t = \mathbf{f}(x,t)\mathrm{d}B(t), & (x,t)\in G_T, \\ \mathrm{d}v - \kappa\Delta v\,\mathrm{d}t + \rho\,\mathrm{div}\,\mathbf{u}_t\,\mathrm{d}t = g(x,t)\mathrm{d}B(t), & (x,t)\in G_T, \\ \mathbf{u}(x,t) = 0, \quad v(x,t) = 0, & (x,t)\in \Gamma_T, \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \quad \mathbf{u}_t(x,0) = \mathbf{u}_1(x), \quad v(x,0) = 0, & x \in G. \end{cases} \tag{1.1}$$

Here ν, μ, κ and ρ are positive constants, the vector function $\mathbf{u} = (u_1, u_2, u_3)^T$ and the scalar function v are the displacement and the temperature, respectively.

The stochastic hyperbolic-parabolic system (1.1) is used to describe temperature dependent or heat generating wave phenomena in a continuum random medium. The first stochastically perturbed hyperbolic equation in (1.1) models wave propagation in random media [32]. When the heating effect is involved in the wave phenomena, the hyperbolic equation is coupled with a stochastic parabolic equation by the second equation in (1.1) [11]. The existence of a mild solution of the stochastic thermoelastic system (1.1) in a suitable function space has been established in [10].

Physically, the source functions $\mathbf{f}(x,t)$ and $g(x,t)$ stand for the intensity of random force of the white noise type. We further assume that $g(x,t)$ has the decomposition

$$g(x,t) = h(t)R(x,t), \quad (x,t)\in G_T \tag{1.2}$$

with known function $R(x,t)$. The functions $\mathbf{u}_0$ and $\mathbf{u}_1$ represent the initial displacement and velocity, respectively. In this paper, we deal with an inverse problem of determining the sources $(\mathbf{f}(x,t), h(t))$ and the initial data $(\mathbf{u}_0, \mathbf{u}_1)$ simultaneously from extra input data

$$\mathbf{u}(x,t)|_{\varpi_T} \quad \text{and} \quad (\mathbf{u}(x,T), v(x,T))|_G,$$

where ϖ is a neighborhood of ∂G inside G satisfying $\partial\varpi \supset \partial G$.

Remark 1.1. We do not need the observation of v on ϖ_T. Indeed we can use the coupling relation between $\mathbf{u}$ and v to eliminate the local observation of v on ϖ_T.

Remark 1.2. Analogously to the treatments in [8] or [34] for the deterministic case, here we impose some restriction on the measurement domain ϖ, i.e. $\partial\varpi \supset \partial G$. The reason is that, to overcome the coupling effect, we need to consider the equations of div $\mathbf{u}$ and curl $\mathbf{u}$ simultaneously. However we have no any information of div $\mathbf{u}$ and curl $\mathbf{u}$ on ∂G. So we have to use $\mathbf{u}$ in a neighborhood ϖ of boundary ∂G, i.e. the measurement data, to cover the boundary values of div $\mathbf{u}$ and curl $\mathbf{u}$.

The Carleman estimate is crucial to solve our inverse problem. The applications of Carleman estimate to study the stability of coefficient inverse problems have a long history, for example, see [9]. Also this method is widely applied in many aspects such as unique continuation problems [30, 31, 41], control theory [1, 17, 19], coefficient inverse problems [5–7, 18, 20–22, 29, 36] and globally convergent convexification numerical methods [23, 24]. As for Carleman estimates for the deterministic thermoelastic system, we refer the readers to [8, 34, 35, 37]. Compared to [8, 34] considering the deterministic inverse problems, this paper dealing with the stochastic inverse problems, despite some similarities of the framework of proof, is still of many differences. Firstly, an unavoidable step in proving the stability of deterministic inverse problems is to differentiate with respect to time. However it is impossible for stochastic PDEs driven by time white noise. Secondly, the inverse problems discussed in [8, 34] do not touch the recovery of initial status. The reason that we can determine the initial status in this paper is that the part of time t in weight function of Carleman estimate is modified by using $(t-T)^2$ to replace $(t-t_0)^2$ in [8, 34], where $0 < t_0 < T$. This is also the key ingredient to put the term of random source $\mathbf{f}$ on the left-hand side of Carleman estimate. Finally, Itô's formula gives some new phenomena in Carleman estimates for stochastic PDEs, which allow us to discuss the inverse random source problem as the one in this paper. We would like to emphasize that as early as in 2015 [28], pointed out that both the formulation of stochastic inverse problems and the tools to solve them differ considerably from their deterministic counterpart.

Stochastic models are more reasonable compared with the deterministic ones in many real situations. Carleman estimates have been applied successfully to control problems related to stochastic PDEs, such as stochastic complex Ginzburg–Landau equations [13], stochastic Kuramoto–Sivashinsky equation [14, 15], stochastic Schrödinger equation [27], stochastic parabolic equation [4, 25, 33, 38], stochastic hyperbolic equation [40] and so on. However, Carleman estimates with singular weight function in control theory could not be applied to stochastic inverse problems. There are few work on the inverse problems for stochastic PDEs. As a pioneering work, Lü and Zhang [28] used Carleman estimate with a special weight function to obtain the global uniqueness of an inverse problem for the stochastic hyperbolic equation. Then Yuan extended this method to an inverse problem for stochastic dynamic Euler–Bernoulli beam equation [39]. Lü [26] derived the uniqueness of an inverse source problem by using a time-like Carleman estimate for the stochastic parabolic equation. As for the numerical methods for stochastic inverse problems, Bao and Xu [2] applied two kinds of regularization techniques to recover a random source function in quantifying the elastic modulus of nanomaterials, while a numerical method for an inverse medium scattering problem with a stochastic source is proposed by Bao *et al* in [3].

However, to the authors' knowledge, the inverse problems for the stochastic hyperbolic-parabolic system have not been studied thoroughly yet. Unlike the deterministic counterparts, the solution of a stochastic differential equation is not differentiable with respect to time variable. Therefore, we could not directly apply the method of proving stability of coefficient inverse problems for deterministic PDEs, such as [8, 34], to the stochastic cases. The proof of our main result is motivated by the method proposed by Lü and Zhang [28]. In comparison with the existing results such as [28, 39], the system (1.1) we discussed here includes two different type stochastic differential equations, namely, a parabolic equation and a hyperbolic equation. This means that we have to use the same weight function in the Carleman estimates for two different type stochastic equations, which is quite technical. The other difficulty arises from the strongly coupling involved in system (1.1), for which we have to introduce two large parameters for establishing our Carleman estimate.

Throughout this paper, we use t and $x=(x_1,x_2,x_3)$ to denote the time variable and the spatial variable, respectively. We also use the notations $\partial_j = \partial_{x_j}$ and $\partial_x^r = \partial_{x_1}^{r_1}\partial_{x_2}^{r_2}\partial_{x_3}^{r_3}$ for

$|r| = r_1 + r_2 + r_3$. For vector function $\mathbf{v} = (v_1, v_2, v_3)^T$, we set

$$\nabla \mathbf{v} = \left(\frac{\partial v_i}{\partial x_j}\right)_{1\leqslant i,j\leqslant 3}, \quad |\mathbf{v}| = \left(\sum_{i=1}^{3} |v_i|^2\right)^{\frac{1}{2}}, \quad |\nabla \mathbf{v}| = \left(\sum_{i,j=1}^{3} \left|\frac{\partial v_i}{\partial x_j}\right|^2\right)^{\frac{1}{2}}.$$

We denote by $L^2_{\mathcal{F}}(0,T)$ the space of all progressively measurable stochastic process X such that $\mathbb{E}(\int_0^T |X|^2\,\mathrm{d}t) < \infty$. For a Banach space H, we denote by $L^2_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted processes $X(\cdot)$ such that $\mathbb{E}(\|X(\cdot)\|_{L^2(0,T;H)}) < \infty$, with the canonical norm; by $L^\infty_{\mathcal{F}}(0,T;H)$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted bounded processes; and by $L^2_{\mathcal{F}}(\Omega;C([0,T];H))$ the Banach space consisting of all H-valued $\{\mathcal{F}_t\}_{t\geqslant 0}$-adapted continuous processes $X(\cdot)$ such that $\mathbb{E}(\|X(\cdot)\|^2_{C([0,T];H)}) < \infty$, with the canonical norm. We set

$$H_T^{(0)} = L^2_{\mathcal{F}}(0,T), \qquad H_{G_T}^{(i)} = L^2_{\mathcal{F}}(0,T;H^i(G)), \quad i = 0,1,2,3,$$
$$H_{G(0)}^{(i)} = L^2(\Omega,\mathcal{F}_0,P;H^i(G)), \qquad H_{G(T)}^{(i)} = L^2(\Omega,\mathcal{F}_T,P;H^i(G)), \quad i = 0,1,2.$$

Further, we introduce

$$X_{G_T} = \left\{\xi \in H_{G_T}^{(3)} :\ \xi_t \in H_{G_T}^{(1)}\right\} \tag{1.3}$$

endowed with the norm

$$\|\xi\|^2_{X_{G_T}} = \mathbb{E}\int_{G_T} \sum_{|\alpha|\leqslant 1} |\partial_x^\alpha \xi_t|^2\,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_{G_T} \sum_{|\alpha|\leqslant 3} |\partial_x^\alpha \xi|^2\,\mathrm{d}x\,\mathrm{d}t.$$

Moreover, we introduce cut-off functions $\chi_i \in C^\infty(G)$ $(i = 1,2)$ satisfying $0 \leqslant \chi_i \leqslant 1$ and

$$\begin{cases} \chi_i(x) = 1, & x \in \varpi^{(i-1)}, \\ \chi_i(x) = 0, & x \in G \setminus \varpi^{(i)}, \end{cases}$$

where $\varpi^{(i)}$ are chosen to satisfy

$$\varpi^{(0)} \subset \varpi^{(1)} \subset \varpi^{(2)} = \varpi, \quad \text{and} \quad \partial G \subset \partial \varpi^{(i)}, \quad i = 0,1,2.$$

Introduce $Q := -\Delta, S := -\nu\Delta - (\nu+\mu)\nabla\mathrm{div}$. Obviously, for any two functions $\mathbf{u}_1, \mathbf{u}_2 \in L^2_{\mathcal{F}}(\Omega;C^1([0,T];L^2(G)))$, there exists a constant M_ρ such that

$$\|Q^{-\frac{1}{2}}\left(\rho\,\mathrm{div}\,\mathbf{u}_{1,t} - \rho\,\mathrm{div}\,\mathbf{u}_{2,t}\right)\|_{L^2(G)} \leqslant M_\rho \|\mathbf{u}_{1,t} - \mathbf{u}_{2,t}\|_{L^2(G)}, \quad P-\text{a.s.} \tag{1.4}$$

To establish our results, we need the following assumptions.

(A1) T satisfies

$$T > \frac{\sqrt{2\nu+\mu}\,\sup_{x\in\overline{G}} |x - x_0|}{\beta}, \tag{1.5}$$

where $x_0 \in \mathbb{R}^3 \setminus \overline{G}$ is a fixed point, $\beta \in (0,\nu)$ is a constant in weight function (2.2);

(A2) The known function $R \in L^\infty_{\mathcal{F}}(0,T;W^{1,\infty}(G))$ satisfies

$$|R(x,t)| \geqslant r_0 > 0, \quad (x,t) \in G_T,\ P-\text{a.s;} \tag{1.6}$$

(A3) The constant $\kappa > M_\rho$ and $Q\mathbf{f} \in L^2_{\mathcal{F}}(0,T;L^2(G)), Qg \in L^2_{\mathcal{F}}\left(0,T;D(Q^{\frac{1}{2}})\right)$, $Q\mathbf{u}_0 \in L^2\left(\Omega,\mathcal{F}_0,P;D(S^{\frac{1}{2}})\right), Qu_1 \in L^2(\Omega,\mathcal{F}_0,P;L^2(G))$.

Remark 1.3. (A1) means the observation time T cannot be too small. Such a requirement means, to capture the unknown information, the observation time should be large enough. Since the speed of wave propagation is finite, such a requirement is physically reasonable.

Remark 1.4. According to [10], in order to guarantee the well-posed of the direct problem (1.1) we need $\kappa > M_\rho$. The assumptions of $\mathbf{f}, g, \mathbf{u}_0, \mathbf{u}_1$ are used to obtain a sufficient regularity of $\mathbf{u}$ and v for our inverse problem. However in our proof of the stability result, we do not need to apply these conditions directly.

Moreover, for any sufficiently smooth known function $\bar{\mathbf{f}}$ in ϖ_T and some constant $M > 0$, introduce the admissible set

$$\mathscr{W} = \Big\{(\mathbf{f}, h, \mathbf{u}_0, \mathbf{u}_1) \in H^{(1)}_{G(T)} \times H^{(0)}_T \times H^{(2)}_{G(0)} \times H^{(1)}_{G(0)} :\ \mathbf{f} = \bar{\mathbf{f}} \text{ in } \varpi_T,\ P-\text{a.s.}, \\ \|\mathbf{f}\|_{H^{(1)}_{G_T}} + \|h\|_{H^{(0)}_T} + \|\mathbf{u}_0\|_{H^{(2)}_{G(0)}} + \|\mathbf{u}_1\|_{H^{(1)}_{G(0)}} \leqslant M\Big\}.$$

Remark 1.5. $\mathbf{f} = \bar{\mathbf{f}}$ in ϖ_T means that we already know some information near the boundary ∂G about the unknown $\mathbf{f}$. From a technical point of view, this condition is used to deal with the lack of boundary condition about $\operatorname{div}\mathbf{u}$ and $\operatorname{curl}\mathbf{u}$. More precisely, to apply theorem 2.2 in next section to $\operatorname{div}\mathbf{u}$ and $\operatorname{curl}\mathbf{u}$, we have to introduce a cut-off function to truncate $\operatorname{div}\mathbf{u}$ and $\operatorname{curl}\mathbf{u}$ to zero on ∂G, so that we can only identify $\operatorname{div}\mathbf{f}$ and $\operatorname{curl}\mathbf{f}$ in any interior of G_T. So we have to append the information of $\mathbf{f}$ near the boundary ∂G. Moreover, it is also a technical condition for getting rid of the local observation of v.

Our main result in this paper is the following conditional stability.

Theorem 1.1. *Let $(\mathbf{f}^{(i)}, h^{(i)}, \mathbf{u}_0^{(i)}, u_1^{(i)}) \in \mathscr{W}$ for $i = 1, 2$, and let (A1)–(A3) hold. Then there exists a constant $C = C(x_0, G, T, r_0, M, \nu, \mu, \rho, \kappa) > 0$ such that*

$$\begin{aligned} &\left\|\mathbf{u}_0^{(1)} - \mathbf{u}_0^{(2)}\right\|_{H^{(1)}_{G(0)}} + \left\|\mathbf{u}_1^{(1)} - \mathbf{u}_1^{(2)}\right\|_{H^{(0)}_{G(0)}} + \|h^{(1)} - h^{(2)}\|_{H^{(0)}_T} + \|\sqrt{T-t}(\mathbf{f}^{(1)} - \mathbf{f}^{(2)})\|_{H^{(0)}_{G_T}} \\ &\quad + \|\sqrt{T-t}(\operatorname{div}\mathbf{f}^{(1)} - \operatorname{div}\mathbf{f}^{(2)})\|_{H^{(0)}_{G_T}} + \|\sqrt{T-t}(\operatorname{curl}\mathbf{f}^{(1)} - \operatorname{curl}\mathbf{f}^{(2)})\|_{H^{(0)}_{G_T}} \\ &\leqslant C\left(\|\mathbf{u}^{(1)}(\cdot,T) - \mathbf{u}^{(2)}(\cdot,T)\|^{\frac{1}{2}}_{H^{(2)}_{G(T)}} + \|v^{(1)}(\cdot,T) - v^{(2)}(\cdot,T)\|^{\frac{1}{2}}_{H^{(1)}_{G(T)}} + \|\mathbf{u}^{(1)} - \mathbf{u}^{(2)}\|_{X_{\varpi_T}}\right), \end{aligned} \tag{1.7}$$

where $(\mathbf{u}^{(i)}, v^{(i)})$ is the solution to (1.1) corresponding to $(\mathbf{f}^{(i)}, g^{(i)}, \mathbf{u}_0^{(i)}, \mathbf{u}_1^{(i)})$ for $i = 1, 2$, respectively.

Remark 1.6. Obviously, we can immediately obtain the uniqueness of our inverse problem from theorem 1.1. More precisely, under the same assumptions as in theorem 1.1 and if

$$(\mathbf{u}^{(1)}(x,T), v^{(1)}(x,T)) = (\mathbf{u}^{(2)}(x,T), v^{(2)}(x,T)), \quad x \in G, \quad P-\text{a.s.},$$

and

$$\mathbf{u}^{(1)}(x,t)=\mathbf{u}^{(2)}(x,t),\quad (x,t)\in\varpi\times(0,T),\quad P-\text{a.s.},$$

then $\mathbf{f}^{(1)}=\mathbf{f}^{(2)}$, $g^{(1)}=g^{(2)}$ in G_T and $\mathbf{u}_0^{(1)}=\mathbf{u}_0^{(2)}$, $\mathbf{u}_1^{(1)}=\mathbf{u}_1^{(2)}$ in G, P-a.s.

Remark 1.7. In order to study the inverse random source problem, we need to choose a regular weight function to put the term of random source on the left-hand side of Carleman estimate. When the weight function is singular, we could directly drop the terms at the time T. In our case we have to deal with these terms when proving our Carleman estimate, see (2.51) below. More precisely, the lack of the information about $\mathbf{u}_t$ at $t=T$ leads to the power $\frac{1}{2}$ for $\left\|\mathbf{u}^{(1)}(\cdot,T)-\mathbf{u}^{(2)}(\cdot,T)\right\|_{H^{(2)}_{G(T)}}$ and $\left\|v^{(1)}(\cdot,T)-v^{(2)}(\cdot,T)\right\|_{H^{(1)}_{G(T)}}$ in theorem 1.1.

The rest of the paper is organized as follows. In section 2, we give two Carleman estimates which will be used in the proof of our main result. In section 3, we prove our stability result, i.e., theorem 1.1.

2. The Carleman estimates

Now we establish two Carleman estimates with a regular weight function and two large parameters for a stochastic parabolic equation and a stochastic hyperbolic equation, respectively.

In order to formulate our Carleman estimates, we introduce some notations. Let λ and s be two large parameters. We define the regular weight function φ by

$$\varphi(x,t)=\mathrm{e}^{\lambda\Phi(x,t)},\quad (x,t)\in G_T \tag{2.1}$$

with

$$\Phi(x,t)=|x-x_0|^2-\beta(t-T)^2+M_1,\quad (x,t)\in G_T, \tag{2.2}$$

where $x_0\in\mathbb{R}^3\setminus\overline{G}$, β is a positive constant. M_1 is chosen sufficiently large such that $\Phi(x,t)>0$ for all $(x,t)\in\overline{G_T}$. Furthermore we set

$$\varphi(x,0)=\varphi_0(x),\quad x\in G.$$

We also use the notation $O(\gamma)$, which satisfies $|O(\gamma)|\leqslant C\gamma$ with a constant C independent of s and λ.

There are two results in this section. In the first one we apply the regular weight function to obtain a new Carleman estimate for the stochastic parabolic equation, in which the source H_1 is put on the left-hand side of (2.3) below. The second one is another Carleman estimate for the stochastic hyperbolic equation. In these two Carleman estimates we introduce two large parameters, which is necessary to deal with the couplings when proving the stability of our inverse problem.

Theorem 2.1. *Let $F_1\in H^{(0)}_{G_T}$, $H_1\in H^{(1)}_{G_T}$. Then for any $\beta>0$ there exist positive constants $\lambda_1=\lambda_1(x_0,G,T,\kappa)$, $s_1=s_1(\lambda)$ and $C_1=C_1(x_0,G,T,\kappa)$, $C_2=C_2(\lambda)$ such that*

$$\mathbb{E}\int_{G_T}\frac{1}{s}\sum_{i,j=1}^{3}|\partial_i\partial_j y|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}s\lambda^2\varphi^2|\nabla y|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\mathbb{E}\int_{G_T}s^3\lambda^4\varphi^4|y|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}s\lambda^2\varphi^2|H_1|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C_1\mathbb{E}\int_{G_T}\varphi|F_1|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+C_1\mathbb{E}\int_{G_T}s\varphi^2|\nabla H_1|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$+C_1\mathbb{E}\int_{\varpi_T^{(1)}}s^3\lambda^2\varphi^4|\nabla y|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+C_2(\lambda)s^2\,\mathrm{e}^{C_2(\lambda)s}\|y(\cdot,T)\|^2_{H^{(1)}_{G(T)}} \tag{2.3}$$

for all $\lambda\geqslant\lambda_1$, $s\geqslant s_1$ *and all* $y\in L^2_{\mathcal{F}}(\Omega;C([0,T];H^1_0(G)))\cap L^2_{\mathcal{F}}(0,T;H^2(G))$ *satisfying*

$$\begin{cases}\mathrm{d}y-\kappa\Delta y\,\mathrm{d}t=F_1\,\mathrm{d}t+H_1\,\mathrm{d}B(t), & (x,t)\in G_T,\\ y(x,t)=0, & (x,t)\in\Gamma_T,\\ y(x,0)=0, & x\in G.\end{cases} \tag{2.4}$$

Remark 2.1. Unlike the stochastic hyperbolic equation, we could not get rid of the term containing of ∇H_1 on the right-hand side of (2.3). This is the reason why we need to assume that the unknown part in the source function g is spatial independent, i.e., $h(t)$, rather than the whole $g(x,t)$.

Remark 2.2. In order to overcome the difficulty arising from the couplings, we have to eliminate φ^{-1} in the term $\int_{G_T}\frac{1}{s\varphi}\sum_{i,j=1}^{3}|\partial_i\partial_j y|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$ in usual Carleman estimates for the stochastic parabolic equation, e.g. lemma 1 in [38]. To do this, we introduce $\tilde{y}=\varphi^{\frac{1}{2}}y$. Unlike the deterministic case, we apply this change of variables in the beginning of proving the Carleman estimate, rather than after deriving the Carleman estimate of y as done in [8]. In this case, we could not absorb the term of H_1 appearing on the right-hand side of the resulting inequality by applying variable transforms. However, in order to determine the unknown h in the stochastic thermoelastic model we need to put the term of H_1 on the left-hand side of our Carleman estimate.

Remark 2.3. In [25], Liu extended the Carleman estimate for stochastic parabolic equation proposed by Tang and Zhang [33], where the author eliminated one extra gradient term and relaxed some assumptions on the regularity for coefficients. Lü [26] established an $x-$independent Carleman estimate for stochastic parabolic equation to study two kinds of inverse source problems. In these two papers, the random source terms are on the right-hand side of Carleman estimates, so these estimates could not be applied to our inverse random source problem. In order to study our inverse problem including the recovery of the random source h, we have to use a different weight function to put the random source term on the left-hand side of the Carleman estimate.

Proof. Let $\tilde{y}=\varphi^{\frac{1}{2}}y$. Then $\tilde{y}$ satisfies

$$\begin{cases}\mathrm{d}\tilde{y}-\kappa\Delta\tilde{y}\,\mathrm{d}t=\tilde{F}_1\,\mathrm{d}t+\varphi^{\frac{1}{2}}H_1\,\mathrm{d}B(t), & (x,t)\in G_T,\\ \tilde{y}(x,t)=0, & (x,t)\in\Gamma_T,\\ \tilde{y}(x,0)=0, & x\in G,\end{cases} \tag{2.5}$$

where

$$\tilde{F}_1 = -\kappa\lambda\nabla\Phi\cdot\nabla\tilde{y} + \left(\frac{1}{4}\kappa\lambda^2|\nabla\Phi|^2 - \frac{1}{2}\kappa\lambda\Delta\Phi + \frac{1}{2}\lambda\Phi_t\right)\tilde{y} + \varphi^{\frac{1}{2}}F_1.$$

We set $l = s\varphi$, $\theta = \mathrm{e}^l$ and $Y = \theta\tilde{y}$. A direct calculation gives

$$\theta(\mathrm{d}\tilde{y} - \kappa\Delta\tilde{y}\,\mathrm{d}t) = I_1 + I\,\mathrm{d}t \tag{2.6}$$

with

$$I_1 = \mathrm{d}Y + 2\kappa\nabla l\cdot\nabla Y\,\mathrm{d}t + \Psi Y\,\mathrm{d}t, \quad I = -\kappa\Delta Y - \kappa|\nabla l|^2Y + (-l_t + \kappa\Delta l - \Psi)Y,$$

where Ψ will be specified below. Multiplying I in both sides of (2.6), we further have

$$\theta I(\mathrm{d}\tilde{y} - \kappa\Delta\tilde{y}\,\mathrm{d}t) = \mathrm{II}_1 + I^2\,\mathrm{d}t. \tag{2.7}$$

Now compute II_1 term by term. We first split the product II_1 into a sum of nine terms

$$\mathrm{II}_1 = \sum_{i,j=1}^{3} I_{ij}, \tag{2.8}$$

where I_{ij} is the product of the ith term of I and the jth term of I_1. Proceeding as done in [33], we apply Itô's stochastic calculus to yield

- $$\begin{aligned} I_{11} &= -\kappa\Delta Y\,\mathrm{d}Y = -\kappa\nabla\cdot(\nabla Y\,\mathrm{d}Y) + \kappa\nabla Y\cdot\nabla\,\mathrm{d}Y \\ &= -\kappa\nabla\cdot(\nabla Y\,\mathrm{d}Y) + \frac{1}{2}\kappa\,\mathrm{d}\left(|\nabla Y|^2\right) - \frac{1}{2}\kappa|\nabla\,\mathrm{d}Y|^2, \end{aligned}$$

- $$I_{12} = -\kappa|\nabla l|^2Y\,\mathrm{d}Y = -\frac{1}{2}\kappa\,\mathrm{d}\left(|\nabla l|^2|Y|^2\right) + \kappa(\nabla l\cdot\nabla l_t)|Y|^2\,\mathrm{d}t + \frac{1}{2}\kappa|\nabla l|^2(\mathrm{d}Y)^2,$$

- $$\begin{aligned} I_{13} &= (-l_t + \kappa\Delta l - \Psi)Y\,\mathrm{d}Y \\ &= \frac{1}{2}\,\mathrm{d}\left[(-l_t + \kappa\Delta l - \Psi)|Y|^2\right] - \frac{1}{2}\left(-l_{tt} + \kappa\Delta l_t - \Psi_t\right)|Y|^2\,\mathrm{d}t \\ &\quad - \frac{1}{2}(-l_t + \kappa\Delta l - \Psi)(\mathrm{d}Y)^2, \end{aligned}$$

- $$\begin{aligned} I_{21} &= -2\kappa^2\left(\nabla l\cdot\nabla Y\right)\Delta Y\,\mathrm{d}t \\ &= -2\kappa^2\sum_{i,j=1}^{3}\left(l_{x_i}Y_{x_i}Y_{x_j}\right)_{x_j}\mathrm{d}t + 2\kappa^2\sum_{i,j=1}^{3} l_{x_ix_j}Y_{x_i}Y_{x_j}\,\mathrm{d}t + \kappa^2\sum_{i,j=1}^{3}\left(l_{x_j}|Y_{x_i}|^2\right)_{x_j}\mathrm{d}t \\ &\quad - \kappa^2\Delta l|\nabla Y|^2\mathrm{d}t, \end{aligned}$$

- $$I_{22} = -2\kappa^2|\nabla l|^2\left(\nabla l\cdot\nabla Y\right)Y\,\mathrm{d}t = -\kappa^2\nabla\cdot(|\nabla l|^2|Y|^2\nabla l)\mathrm{d}t + \kappa^2\nabla\cdot(|\nabla l|^2\nabla l)|Y|^2\,\mathrm{d}t,$$

- $$\begin{aligned} I_{23} &= 2\kappa(-l_t + \kappa\Delta l - \Psi)\left(\nabla l\cdot\nabla Y\right)Y\,\mathrm{d}t \\ &= \kappa\nabla\cdot\left[(-l_t + \kappa\Delta l - \Psi)|Y|^2\nabla l\right]\mathrm{d}t - \kappa\nabla\cdot\left[(-l_t + \kappa\Delta l - \Psi)\nabla l\right]|Y|^2\,\mathrm{d}t, \end{aligned}$$

- $I_{31} = -\,\kappa\Psi Y\Delta Y\,\mathrm{d}t = -\kappa\nabla\cdot(\Psi Y\nabla Y)\mathrm{d}t + \kappa\Psi|\nabla Y|^2\,\mathrm{d}t + \kappa\,(\nabla\Psi\cdot\nabla Y)\,Y\,\mathrm{d}t,$

- $I_{32} = -\,\kappa|\nabla l|^2\Psi|Y|^2\,\mathrm{d}t,$

- $I_{33} = (-l_t + \kappa\Delta l - \Psi)\Psi|Y|^2\,\mathrm{d}t.$

Therefore, we find that

$$\begin{aligned}\theta I(\mathrm{d}\tilde{y} - \kappa\Delta\tilde{y}\,\mathrm{d}t) = {}& I^2\,\mathrm{d}t + \left[\kappa^2\nabla\cdot(|\nabla l|^2\nabla l) - \kappa|\nabla l|^2\Psi\right]|Y|^2\,\mathrm{d}t\\ &+ \left(\kappa\Psi - \kappa^2\Delta l\right)|\nabla Y|^2\,\mathrm{d}t + 2\kappa^2\sum_{i,j=1}^{3} l_{x_ix_j}Y_{x_i}Y_{x_j}\,\mathrm{d}t\\ &+ \frac{1}{2}\left(\kappa|\nabla l|^2 + l_t - \kappa\Delta l + \Psi\right)(\mathrm{d}Y)^2 - \frac{1}{2}\kappa|\nabla\,\mathrm{d}Y|^2 + J_1 + J_2 + J_3,\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}J_1 = {}& \frac{1}{2}\kappa\mathrm{d}\left(|\nabla Y|^2\right) - \frac{1}{2}\kappa\mathrm{d}\left(|\nabla l|^2|Y|^2\right) + \frac{1}{2}\,\mathrm{d}\left[(-l_t + \kappa\Delta l - \Psi)|Y|^2\right],\\ J_2 = {}& -\kappa\nabla\cdot(\nabla Y\,\mathrm{d}Y) - 2\kappa^2\sum_{i,j=1}^{3}\left(l_{x_i}Y_{x_i}Y_{x_j}\right)_{x_j}\mathrm{d}t + \kappa^2\sum_{i,j=1}^{3}\left(l_{x_j}|Y_{x_i}|^2\right)_{x_j}\mathrm{d}t\\ &- \kappa^2\nabla\cdot(|\nabla l|^2|Y|^2\nabla l)\mathrm{d}t + \kappa\nabla\cdot\left[(-l_t + \kappa\Delta l - \Psi)|Y|^2\nabla l\right]\mathrm{d}t - \kappa\nabla\cdot(\Psi Y\nabla Y)\mathrm{d}t,\\ J_3 = {}& \kappa(\nabla l\cdot\nabla l_t)|Y|^2\,\mathrm{d}t - z\frac{1}{2}(-l_{tt} + \kappa\Delta l_t - \Psi_t)|Y|^2\mathrm{d}t - \kappa\nabla\cdot\left[(-l_t + \kappa\Delta l - \Psi)\nabla l\right]|Y|^2\,\mathrm{d}t\\ &+ (-l_t + \kappa\Delta l - \Psi)\Psi|Y|^2\,\mathrm{d}t + \kappa(\nabla\Psi\cdot\nabla Y)Y\,\mathrm{d}t.\end{aligned}$$

Integrating (2.9) over G_T and taking mathematical expectation, we obtain

$$\begin{aligned}\mathbb{E}\int_{G_T}\theta I(\mathrm{d}\tilde{y} - \kappa\Delta\tilde{y}\,\mathrm{d}t)\mathrm{d}x = {}& \mathbb{E}\int_{G_T} I^2\,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_{G_T}\left[\kappa^2\nabla\cdot(|\nabla l|^2\nabla l) - \kappa|\nabla l|^2\Psi\right]|Y|^2\,\mathrm{d}x\,\mathrm{d}t\\ &+ \mathbb{E}\int_{G_T}\left(\left(\kappa\Psi - \kappa^2\Delta l\right)|\nabla Y|^2 + 2\kappa^2\sum_{i,j=1}^{3} l_{x_ix_j}Y_{x_i}Y_{x_j}\right)\mathrm{d}x\,\mathrm{d}t\\ &+ \mathbb{E}\int_{G_T}\frac{1}{2}\left(\kappa|\nabla l|^2 + l_t - \kappa\Delta l + \Psi\right)(\mathrm{d}Y)^2\,\mathrm{d}x - \mathbb{E}\int_{G_T}\frac{1}{2}\kappa|\nabla\,\mathrm{d}Y|^2\,\mathrm{d}x\\ &+ \mathbb{E}\int_{G_T} J_1\,\mathrm{d}x + \mathbb{E}\int_{G_T} J_2\,\mathrm{d}x + \mathbb{E}\int_{G_T} J_3\,\mathrm{d}x.\end{aligned} \tag{2.10}$$

Taking $\Psi = \tau\kappa\Delta l$ with some positive constant $\tau \in (5/2, 3)$, we obtain

$$\begin{cases}\kappa^2\nabla\cdot(|\nabla l|^2\nabla l)-\kappa|\nabla l|^2\Psi=(3-\tau)\kappa^2|\nabla l|^2\Delta l\geqslant(3-\tau)\kappa^2\,\mathrm{d}^4s^3\lambda^4\varphi^3,\\ \kappa\Psi-\kappa^2\Delta l=(\tau-1)\kappa^2\Delta l\geqslant(\tau-1)\kappa^2\,\mathrm{d}^2s\lambda^2\varphi,\end{cases}\tag{2.11}$$

where $d=\min_{x\in\overline{G}}|x-x_0|>0$. Obviously,

$$\sum_{i,j=1}^{3}l_{x_ix_j}Y_{x_i}Y_{x_j}=s\lambda^2\varphi|\nabla\Phi\cdot\nabla Y|^2+s\lambda\varphi\sum_{i=1}^{3}\Phi_{x_ix_i}|Y_{x_i}|^2\geqslant 0.\tag{2.12}$$

From (2.11) and (2.12), it follows that

$$\begin{aligned}&\mathbb{E}\int_{G_T}\left[\kappa^2\nabla\cdot(|\nabla l|^2\nabla l)-\kappa|\nabla l|^2\Psi\right]|Y|^2\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\mathbb{E}\int_{G_T}\left(\left(\kappa\Psi-\kappa^2\Delta l\right)|\nabla Y|^2+2\kappa^2\sum_{i,j=1}^{3}l_{x_ix_j}Y_{x_i}Y_{x_j}\right)\mathrm{d}x\,\mathrm{d}t\\&\geqslant\mathbb{E}\int_{G_T}(3-\tau)\kappa^2\,\mathrm{d}^4s^3\lambda^4\varphi^3|Y|^2\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}(\tau-1)\kappa^2\,\mathrm{d}^2s\lambda^2\varphi|\nabla Y|^2\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{2.13}$$

On the other hand, noting that

$$(\mathrm{d}Y)^2=\theta^2\varphi|H_1|^2\,\mathrm{d}t\tag{2.14}$$

and

$$\begin{aligned}|\nabla\,\mathrm{d}Y|^2&=|\nabla\theta|^2(\mathrm{d}\tilde{y})^2+\theta^2|\nabla\,\mathrm{d}\tilde{y}|^2+2\theta\,\mathrm{d}\tilde{y}\,(\nabla\theta\cdot\nabla\,\mathrm{d}\tilde{y})\\&\leqslant|\nabla\theta|^2(\mathrm{d}\tilde{y})^2+\theta^2|\nabla\,\mathrm{d}\tilde{y}|^2+\frac{\Delta l}{2}\theta^2(\mathrm{d}\tilde{y})^2+\frac{2}{\Delta l}|\nabla\theta|^2|\nabla\,\mathrm{d}\tilde{y}|^2\\&\leqslant\left(|\nabla l|^2+\frac{1}{2}\Delta l\right)\theta^2\varphi|H_1|^2\,\mathrm{d}t+\left(1+2\frac{|\nabla l|^2}{\Delta l}\right)\theta^2\left|\nabla\left(\varphi^{\frac{1}{2}}H_1\right)\right|^2\mathrm{d}t,\end{aligned}\tag{2.15}$$

we have

$$\begin{aligned}&\mathbb{E}\int_{G_T}\frac{1}{2}\left(\kappa|\nabla l|^2+l_t-\kappa\Delta l+\Psi\right)(\mathrm{d}Y)^2\,\mathrm{d}x-\mathbb{E}\int_{G_T}\frac{1}{2}\kappa|\nabla\,\mathrm{d}Y|^2\,\mathrm{d}x\\&\quad\geqslant\mathbb{E}\int_{G_T}\left[\left(\frac{\tau}{2}-\frac{3}{4}\right)\kappa\Delta l+\frac{1}{2}l_t-\frac{1}{4}\kappa\lambda^2|\nabla\Phi|^2\left(1+2\frac{|\nabla l|^2}{\Delta l}\right)\right]\theta^2\varphi|H_1|^2\,\mathrm{d}x\,\mathrm{d}t\\&\qquad-\mathbb{E}\int_{G_T}\kappa\left(1+2\frac{|\nabla l|^2}{\Delta l}\right)\theta^2\varphi|\nabla H_1|^2\,\mathrm{d}x\,\mathrm{d}t\\&\quad\geqslant\mathbb{E}\int_{G_T}\left(\frac{\tau}{2}-\frac{5}{4}\right)\kappa\,\mathrm{d}^2s\lambda^2\varphi^2\theta^2|H_1|^2\,\mathrm{d}x\,\mathrm{d}t-C_1\mathbb{E}\int_{G_T}s\varphi^2\theta^2|\nabla H_1|^2\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{2.16}$$

where we have used

$$\left(\frac{\tau}{2}-\frac{3}{4}\right)\kappa\Delta l+\frac{1}{2}l_t-\frac{1}{4}\kappa\lambda^2|\nabla\Phi|^2\left(1+2\frac{|\nabla l|^2}{\Delta l}\right)$$

$$=\left(\frac{\tau}{2}-\frac{3}{4}\right)\kappa(s\lambda^2\varphi|\nabla\Phi|^2+s\lambda\varphi\Delta\Phi)+\beta s\lambda\varphi\,(T-t)$$

$$-\frac{1}{4}\kappa\lambda^2|\nabla\Phi|^2\left(1+2\frac{s^2\lambda^2\varphi^2|\nabla\Phi|^2}{s\lambda^2\varphi|\nabla\Phi|^2+s\lambda\varphi\Delta\Phi}\right)$$

$$\geqslant\left(\frac{\tau}{2}-\frac{5}{4}\right)\kappa s\lambda^2\varphi|\nabla\Phi|^2+\left(\frac{\tau}{2}-\frac{3}{4}\right)\kappa s\lambda\varphi\Delta\Phi-|O(\lambda^2)|\geqslant\left(\frac{\tau}{2}-\frac{5}{4}\right)\kappa \mathrm{d}^2 s\lambda^2\varphi$$

for s sufficiently large such that $\left(\frac{\tau}{2}-\frac{3}{4}\right)\kappa s\lambda\varphi\Delta\Phi>|O(\lambda^2)|$. Substituting (2.13) and (2.16) into (2.10), we obtain that

$$\begin{aligned}
&C_1\mathbb{E}\int_{G_T}\theta I(\mathrm{d}\tilde{y}-\kappa\Delta\tilde{y}\,\mathrm{d}t)\mathrm{d}x\\
&\quad\geqslant C_1\mathbb{E}\int_{G_T}I^2\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}s^3\lambda^4\varphi^3|Y|^2\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}s\lambda^2\varphi|\nabla Y|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\mathbb{E}\int_{G_T}s\lambda^2\varphi^2\theta^2|H_1|^2\,\mathrm{d}x\,\mathrm{d}t-C_1\mathbb{E}\int_{G_T}s\varphi^2\theta^2|\nabla H_1|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad+C_1\mathbb{E}\int_{G_T}J_1\,\mathrm{d}x+C_1\mathbb{E}\int_{G_T}J_2\,\mathrm{d}x+C_1\mathbb{E}\int_{G_T}J_3\,\mathrm{d}x.
\end{aligned}\tag{2.17}$$

Now we estimate the terms of J_1, J_2 and J_3. Since $\tilde{y}=0$ on Γ_T due to (2.5), we have $Y=0$ and $\nabla Y=\frac{\partial Y}{\partial\eta}\eta$ on Γ_T. Together with $Y(x,0)=0$, P-a.s. in G, we then use integration by parts to yield

$$\begin{aligned}
&\mathbb{E}\int_{G_T}J_1\,\mathrm{d}x+\mathbb{E}\int_{G_T}J_2\,\mathrm{d}x\\
&\quad=\int_G\left[\frac{1}{2}\kappa\left(|\nabla Y(x,T)|^2\right)-\frac{1}{2}\kappa\left(|\nabla l|^2|Y(x,T)|^2\right)+\frac{1}{2}(-l_t+\kappa\Delta l-\Psi)|Y(x,T)|^2\right]\mathrm{d}x\\
&\qquad-\mathbb{E}\int_{\Gamma_T}2\kappa^2(\nabla l\cdot\nabla Y)\frac{\partial Y}{\partial\eta}\,\mathrm{d}S\,\mathrm{d}t+\mathbb{E}\int_{\Gamma_T}\kappa^2|\nabla Y|^2\frac{\partial l}{\partial\eta}\,\mathrm{d}S\,\mathrm{d}t\\
&\quad\geqslant -C_2(\lambda)s^2\,\mathrm{e}^{C_2(\lambda)s}\|y(\cdot,T)\|^2_{H^{(1)}_{G(T)}}-C_1\mathbb{E}\int_{\Gamma_T}|\nabla l|\left|\frac{\partial Y}{\partial\eta}\right|^2\mathrm{d}S\,\mathrm{d}t.
\end{aligned}\tag{2.18}$$

By using

$$\begin{cases}|\nabla l|\leqslant C_1s\lambda\varphi,\quad |\nabla l_t|+|l_{tt}|+|\Delta l|\leqslant C_1s\lambda^2\varphi,\quad |\Delta l_t|+|\nabla\Psi|+|\Psi_t|\leqslant C_1s\lambda^3\varphi,\\ |\nabla\cdot[(-l_t+\kappa\Delta l-\Psi)\nabla l]\,|\leqslant C_1s^2\lambda^4\varphi^2,\quad |(-l_t+\kappa\Delta l-\Psi)\Psi|\leqslant C_1s^2\lambda^4\varphi^2\end{cases}$$

and Hölder inequality, we obtain

$$\mathbb{E}\int_{G_T} J_3\,\mathrm{d}x \geqslant -C_1\mathbb{E}\int_{G_T} s^2\lambda^4\varphi^2|Y|^2\,\mathrm{d}x\,\mathrm{d}t - C_1\mathbb{E}\int_{G_T} s\lambda^3\varphi|\nabla Y||Y|\,\mathrm{d}x\,\mathrm{d}t$$
$$\geqslant -C_1\mathbb{E}\int_{G_T} s^2\lambda^4\varphi^2|Y|^2\,\mathrm{d}x\,\mathrm{d}t - C_1\mathbb{E}\int_{G_T} \lambda^2|\nabla Y|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.19}$$

Then from (2.17)–(2.19) we deduce that there exist positive constants $\lambda_1^{(2)}$ and $s_1^{(2)}$ such that for all $\lambda \geqslant \lambda_1^{(2)}$ and $s \geqslant s_1^{(2)}$, it holds that

$$C_1\mathbb{E}\int_{G_T} \theta I(\mathrm{d}\tilde{y} - \kappa\Delta\tilde{y}\mathrm{d}t)\mathrm{d}x$$
$$\geqslant C_1\mathbb{E}\int_{G_T} I^2\,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_{G_T} s^3\lambda^4\varphi^3|Y|^2\,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_{G_T} s\lambda^2\varphi|\nabla Y|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \mathbb{E}\int_{G_T} s\lambda\varphi^2\theta^2|H_1|^2\,\mathrm{d}x\,\mathrm{d}t - C_1\mathbb{E}\int_{G_T} s\varphi^2\theta^2|\nabla H_1|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$- C_1\mathbb{E}\int_{\Gamma_T} |\nabla l|\left|\frac{\partial Y}{\partial\eta}\right|^2\mathrm{d}S\,\mathrm{d}t - C_2(\lambda)s^2\,\mathrm{e}^{C_2(\lambda)s}\|y(\cdot,T)\|^2_{H^{(1)}_{G(T)}}. \tag{2.20}$$

Noting that $\mathbb{E}\int_{G_T}\theta I\varphi^{\frac{1}{2}}H_1\,\mathrm{d}B(t) = 0$, we then deduce from (2.5) that

$$\mathbb{E}\int_{G_T} \theta I(\mathrm{d}\tilde{y} - \kappa\Delta\tilde{y}\,\mathrm{d}t)\mathrm{d}x = \mathbb{E}\int_{G_T} \theta I\left(\tilde{F}_1\,\mathrm{d}t + \varphi^{\frac{1}{2}}H_1\,\mathrm{d}B(t)\right)\mathrm{d}x$$
$$= \mathbb{E}\int_{G_T} \theta I\tilde{F}_1\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \frac{1}{2}\mathbb{E}\int_{G_T} I^2\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{2}\mathbb{E}\int_{G_T} \theta^2|\tilde{F}_1|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.21}$$

Therefore, it follows from (2.20) and (2.21) that

$$C_1\mathbb{E}\int_{G_T} \theta^2|\tilde{F}_1|^2\,\mathrm{d}x\,\mathrm{d}t \geqslant C_1\mathbb{E}\int_{G_T} I^2\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_{G_T} s^3\lambda^4\varphi^3|Y|^2\,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_{G_T} s\lambda^2\varphi|\nabla Y|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \mathbb{E}\int_{G_T} s\lambda\varphi^2\theta^2|H_1|^2\,\mathrm{d}x\,\mathrm{d}t - C_1\mathbb{E}\int_{G_T} s\varphi^2\theta^2|\nabla H_1|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$- C_1\mathbb{E}\int_{\Gamma_T} s\lambda\varphi\left|\frac{\partial Y}{\partial\eta}\right|^2\mathrm{d}S\,\mathrm{d}t - C_2(\lambda)s^2\,\mathrm{e}^{C_2(\lambda)s}\|y(\cdot,T)\|^2_{H^{(1)}_{G(T)}}. \tag{2.22}$$

By the definition of I we obtain

$$\mathbb{E}\int_{G_T} \frac{1}{s\varphi}|\Delta Y|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant \mathbb{E}\int_{G_T} I^2\,\mathrm{d}x\,\mathrm{d}t + C_1\mathbb{E}\int_{G_T} s^3\lambda^4\varphi^3|Y|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.23}$$

Consequently, substituting (2.23) into (2.22) yields

$$\mathbb{E}\int_{G_T}\frac{1}{s\varphi}|\Delta Y|^2\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}s\lambda^2\varphi|\nabla Y|^2\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}s^3\lambda^4\varphi^3|Y|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+\mathbb{E}\int_{G_T}s\lambda\varphi^2\theta^2|H_1|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C_1\mathbb{E}\int_{G_T}\theta^2|\tilde{F}_1|^2\,\mathrm{d}x\,\mathrm{d}t+C_1\mathbb{E}\int_{G_T}s\varphi^2\theta^2|\nabla H_1|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$+C_1\mathbb{E}\int_{\Gamma_T}s\lambda\varphi\left|\frac{\partial Y}{\partial\eta}\right|^2\mathrm{d}S\,\mathrm{d}t-C_2(\lambda)s^2\,\mathrm{e}^{C_2(\lambda)s}\|y(\cdot,T)\|^2_{H^{(1)}_{G(T)}}. \tag{2.24}$$

Moreover, for a.s. $\omega\in\Omega$ it holds that

$$\begin{cases}\Delta\left(\dfrac{Y}{\sqrt{\varphi}}\right)=\dfrac{\Delta Y}{\sqrt{\varphi}}-\dfrac{\lambda}{\sqrt{\varphi}}\nabla\Phi\cdot\nabla Y+\left(\dfrac{1}{4}\dfrac{\lambda^2}{\sqrt{\varphi}}|\nabla\Phi|^2-\dfrac{1}{2}\lambda\dfrac{1}{\sqrt{\varphi}}\Delta\Phi\right)Y, & (x,t)\in G_T,\\ \dfrac{Y}{\sqrt{\varphi}}(x,t)=0, & (x,t)\in\Gamma_T.\end{cases} \tag{2.25}$$

Then by H^2-estimate for the solution of the linear elliptic equation [16], we obtain

$$\int_G\sum_{i,j=1}^{3}\left|\partial_i\partial_j\left(\frac{Y}{\sqrt{\varphi}}\right)\right|^2\mathrm{d}x\leqslant C_1\left(\int_G\frac{1}{\varphi}|\Delta Y|^2+\frac{\lambda^2}{\varphi}|\nabla Y|^2+\frac{\lambda^4}{\varphi}Y^2\right)\mathrm{d}x, \tag{2.26}$$

which implies that

$$\mathbb{E}\int_{G_T}\frac{1}{\varphi}\sum_{i,j=1}^{3}|\partial_i\partial_j\,Y|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C_1\mathbb{E}\int_{G_T}\sum_{i,j=1}^{3}\left|\partial_i\partial_j\left(\frac{Y}{\sqrt{\varphi}}\right)\right|^2\mathrm{d}x\,\mathrm{d}t+C_1\mathbb{E}\int_{G_T}\frac{\lambda^2}{\varphi}|\nabla Y|^2\,\mathrm{d}x\,\mathrm{d}t+C_1\mathbb{E}\int_{G_T}\frac{\lambda^4}{\varphi}|Y|^2\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C_1\mathbb{E}\int_{G_T}\frac{1}{\varphi}|\Delta Y|^2\,\mathrm{d}x\,\mathrm{d}t+C_1\mathbb{E}\int_{G_T}\frac{\lambda^2}{\varphi}|\nabla Y|^2\,\mathrm{d}x\,\mathrm{d}t+C_1\mathbb{E}\int_{G_T}\frac{\lambda^4}{\varphi}|Y|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.27}$$

Hence, substituting (2.27) into (2.24) and going back to $\tilde{y}$ we find that

$$\mathbb{E}\int_{G_T}\frac{1}{s\varphi}\sum_{i,j=1}^{3}|\partial_i\partial_j\tilde{y}|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}s\lambda^2\varphi|\nabla\tilde{y}|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$+\mathbb{E}\int_{G_T}s^3\lambda^4\varphi^3|\tilde{y}|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}s\lambda\varphi^2|H_1|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C_1\mathbb{E}\int_{G_T}|\tilde{F}_1|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+C_1\mathbb{E}\int_{G_T}s\varphi^2|\nabla H_1|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$+C_1\mathbb{E}\int_{\Gamma_T}s\lambda\varphi\left|\frac{\partial\tilde{y}}{\partial\eta}\right|^2\mathrm{e}^{2s\varphi}\,\mathrm{d}S\,\mathrm{d}t+C_2(\lambda)s^2\,\mathrm{e}^{C_2(\lambda)s}\|y(\cdot,T)\|^2_{H^{(1)}_{G(T)}}. \tag{2.28}$$

On the other hand, we have

$$\mathbb{E}\int_{G_T} |\tilde{F}_1|^2 \, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t$$
$$\leqslant C_1\mathbb{E}\int_{G_T} \left(\lambda^2|\nabla\tilde{y}|^2 + \lambda^4|\tilde{y}|^2\right) \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t + C_1\int_{G_T} \varphi|F_1|^2\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t. \tag{2.29}$$

By using (2.28), (2.29) and $\tilde{y} = \varphi^{\frac{1}{2}}y$, we obtain

$$\mathbb{E}\int_{G_T} \frac{1}{s}\sum_{i,j=1}^{3} |\partial_i\partial_j y|^2\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t + \mathbb{E}\int_{G_T} s\lambda^2\varphi^2|\nabla y|^2\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t$$
$$+ \mathbb{E}\int_{G_T} s^3\lambda^4\varphi^4|y|^2\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t + \mathbb{E}\int_{G_T} s\lambda\varphi^2|H_1|^2\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t$$
$$\leqslant C_1\mathbb{E}\int_{G_T} \varphi|F_1|^2\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t + C_1\mathbb{E}\int_{G_T} s\varphi^2|\nabla H_1|^2\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t$$
$$+ C_1\mathbb{E}\int_{\Gamma_T} s\lambda\varphi^2\left|\frac{\partial y}{\partial\eta}\right|^2 \mathrm{e}^{2s\varphi}\, \mathrm{d}S\, \mathrm{d}t + C_2(\lambda)s^2\, \mathrm{e}^{C_2(\lambda)s}\|y(\cdot, T)\|^2_{H^{(1)}_{G(T)}}. \tag{2.30}$$

Finally, we eliminate the boundary term in (2.30) by the local observation of y on $\varpi^{(1)}$. To do this, we choose a vector function $\mathbf{g}_0 \in C^1(\overline{G}; \mathbb{R}^3)$ such that $\mathbf{g}_0 = \eta$ on Γ. Then by integration by parts and Young's inequality with ϵ, we obtain

$$\mathbb{E}\int_{\Gamma_T} s\lambda\varphi^2\left|\frac{\partial y}{\partial\eta}\right|^2 \mathrm{e}^{2s\varphi}\, \mathrm{d}S\, \mathrm{d}t$$
$$= \mathbb{E}\int_{G_T} s\lambda\varphi^2\chi_1(\nabla y\cdot\mathbf{g}_0)\Delta y\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t + \mathbb{E}\int_{G_T} s\lambda\nabla\left[\varphi^2\chi_1(\nabla y\cdot\mathbf{g}_0)\mathrm{e}^{2s\varphi}\right]\cdot\nabla y\, \mathrm{d}x\, \mathrm{d}t$$
$$= \mathbb{E}\int_{\varpi^{(1)}_T} s\lambda\varphi^2\chi_1(\nabla y\cdot\mathbf{g}_0)\Delta y\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t + \mathbb{E}\int_{\varpi^{(1)}_T} s\lambda\varphi^2\chi_1\left[\nabla(\nabla y\cdot\mathbf{g}_0)\cdot\nabla y\right]\mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t$$
$$+ \mathbb{E}\int_{\varpi^{(1)}_T} s\lambda(\nabla y\cdot\mathbf{g}_0)\left(2\lambda\varphi^2\chi_1\nabla\Phi + \varphi^2\nabla\chi_1 + 2s\lambda\varphi^3\chi_1\nabla\Phi\right)\cdot\nabla y\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t$$
$$\leqslant \epsilon\mathbb{E}\int_{\varpi^{(1)}_T} \frac{1}{s}\sum_{i,j=1}^{3} |\partial_i\partial_j y|^2\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t + C_1(\epsilon)\mathbb{E}\int_{\varpi^{(1)}_T} s^3\lambda^2\varphi^4|\nabla y|^2\, \mathrm{e}^{2s\varphi}\, \mathrm{d}x\, \mathrm{d}t. \tag{2.31}$$

By taking ϵ sufficiently small, we can absorb the first term on the right-hand side of (2.31) by the term of $\partial_i\partial_j y$ on the left-hand side of (2.30), and then obtain the desired estimate (2.3). This completes the proof of theorem 2.1. □

Our next result is a Carleman estimate for the stochastic hyperbolic equation with two large parameters.

Theorem 2.2. *Let* $F_2 \in H^{(0)}_{G_T}$, $H_2 \in H^{(0)}_{G_T}$, $z_0 \in H^{(1)}_{G(0)}$, $z_1 \in H^{(0)}_{G(0)}$ *and* $\alpha = \{\nu, 2\nu + \mu\}$, *and let* T *satisfy*

$$T > \frac{\sqrt{\alpha}\,\sup_{x\in\overline{G}} |x - x_0|}{\beta}. \tag{2.32}$$

Then for any $\beta \in (0, \alpha)$ *there exist positive constants* $\lambda_2 = \lambda_2(x_0, G, T, \alpha)$, $s_2 = s_2(\lambda)$ *and*

$$C_3 = C_3(x_0, G, T, \alpha), \quad C_4 = C_4\left(\lambda, \|z_0\|_{H^{(1)}_{G(0)}}, \|z_1\|_{H^{(0)}_{G(0)}}, \|F_2\|_{H^{(0)}_{G_T}}, \|H_2\|_{H^{(0)}_{G_T}}\right)$$

such that

$$\begin{aligned}
&\mathbb{E}\int_{G_T} s\lambda\varphi|z_t|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_{G_T} s\lambda\varphi|\nabla z|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \mathbb{E}\int_{G_T} s^3\lambda^3\varphi^3|z|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_{G_T} s\lambda\varphi(T-t)|H_2|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \mathbb{E}\int_{G}\left[s\lambda\varphi_0\left(|\nabla z_0|^2 + |z_1|^2\right) + s^3\lambda^3\varphi_0^3|z_0|^2\right]\mathrm{e}^{2s\varphi_0}\,\mathrm{d}x\\
&\leqslant C_3\mathbb{E}\int_{G_T}|F_2|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + C_4(\lambda)s^3\,\mathrm{e}^{C_4(\lambda)s}\left(\|z\|^2_{H^{(2)}_{\varpi^{(1)}_T}} + \|z(\cdot, T)\|_{H^{(1)}_{G(T)}}\right)
\end{aligned} \tag{2.33}$$

for all $\lambda \geqslant \lambda_2$, $s \geqslant s_2$ *and all* $z \in L^2_{\mathcal{F}}(\Omega; C([0, T]; H^1_0(G))) \cap L^2_{\mathcal{F}}(0, T; H^2(G))$ *satisfying*

$$\begin{cases}
\mathrm{d}z_t - \alpha\Delta z\,\mathrm{d}t = F_2\,\mathrm{d}t + H_2\,\mathrm{d}B(t), & (x, t) \in G_T,\\
z(x, t) = 0, & (x, t) \in \Gamma_T,\\
z(x, 0) = z_0(x),\ z_t(x, 0) = z_1(x), & x \in G.
\end{cases} \tag{2.34}$$

Remark 2.4. The proof follows from the ideas in [12, 28, 40]. However, since we introduce the second parameter λ to overcome the difficulty arising from the strongly coupling in the stochastic thermoelastic system (1.1), we have to choose a new function Ψ different from the one in [28].

Proof. We split the proof into three steps.

Step 1. Apply a fundamental identity for the stochastic hyperbolic operator.

Let $l = s\varphi$, $\theta = \mathrm{e}^l$ and $Z = \theta z$. Applying the identity for the stochastic hyperbolic operator proposed by Zhang, i.e., with $(p^{ij})_{3\times 3} = \alpha I$ in [40], we obtain

$$\theta\,[-2l_tZ_t+2\alpha\,(\nabla l\cdot\nabla Z)+\Psi Z]\,(\mathrm{d}z_t-\alpha\Delta z\,\mathrm{d}t)$$
$$+\,\mathrm{div}\Bigg[2\alpha^2\,(\nabla l\cdot\nabla Z)\,\nabla Z-\alpha^2|\nabla Z|^2\nabla l-2\alpha l_tZ_t\nabla Z$$
$$+\,\alpha|Z_t|^2\nabla l+\alpha\Psi Z\nabla Z-\alpha\left(A\nabla l+\frac{\nabla\Psi}{2}\right)|Z|^2\Bigg]\,\mathrm{d}t$$
$$+\,\mathrm{d}\left[\alpha l_t|\nabla Z|^2-2\alpha(\nabla l\cdot\nabla Z)Z_t+l_t|Z_t|^2-\Psi Z_tZ+\left(Al_t+\frac{\Psi_t}{2}\right)|Z|^2\right]$$
$$=\Bigg\{(l_{tt}+\alpha\Delta l-\Psi)\,|Z_t|^2-4\alpha\,(\nabla l_t\cdot\nabla Z)\,Z_t+\left(\alpha l_{tt}-\alpha^2\Delta l+\alpha\Psi\right)|\nabla Z|^2$$
$$+\,2\alpha^2\sum_{i,j=1}^{3}l_{x_ix_j}Z_{x_i}Z_{x_j}+B|Z|^2+[-2l_tZ_t+2\alpha\,(\nabla l\cdot\nabla Z)+\Psi Z]^2\Bigg\}\,\mathrm{d}t$$
$$+\,\theta^2l_t(\mathrm{d}z_t)^2, \tag{2.35}$$

where $\Psi\in C^2(\overline{G}_T)$ will be specified below, A and B have the expressions

$$\begin{cases}A=\left(l_t^2-l_{tt}\right)-\left(\alpha|\nabla l|^2-\alpha\Delta l\right)-\Psi,\\ B=A\Psi+(Al_t)_t-\alpha\displaystyle\sum_{i=1}^{3}(Al_{x_i})_{x_i}+\frac{1}{2}\left(\Psi_{tt}-\alpha\Delta\Psi\right).\end{cases}$$

Step 2. Find the positive lower bound of the terms on the right-hand side of (2.35).

We take Ψ as

$$\Psi=-\gamma s\lambda\varphi(\Phi_{tt}-\alpha\Delta\Phi)-s\lambda^2\varphi(|\Phi_t|^2-\alpha|\nabla\Phi|^2)$$

with a suitable constant γ independent of λ and s, which will be chosen in (2.43).

Firstly we estimate $B|Z|^2$. Obviously,

$$A=s^2\lambda^2\varphi^2\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)+O(s\lambda^2\varphi). \tag{2.36}$$

By the definitions of A and Ψ, we have

$$A\Psi=\left[s^2\lambda^2\varphi^2\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)+O(s\lambda^2\varphi)\right]\left[-\gamma s\lambda\varphi(\Phi_{tt}-\alpha\Delta\Phi)-s\lambda^2\varphi(|\Phi_t|^2-\alpha|\nabla\Phi|^2)\right]$$
$$=-s^3\lambda^4\varphi^3\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)^2-\gamma s^3\lambda^3\varphi^3(|\Phi_t|^2-\alpha|\nabla\Phi|^2)(\Phi_{tt}-\alpha\Delta\Phi)+O(s^2\lambda^4\varphi^2). \tag{2.37}$$

Similarly, we can prove that

$$(Al_t)_t=\left[s^2\lambda^2\varphi^2\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)+O(s\lambda^2\varphi)\right](s\lambda^2\varphi|\Phi_t|^2+s\lambda\varphi\Phi_{tt})$$
$$+\left[s^2\lambda^2\varphi^2\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)_t+2s^2\lambda^3\varphi^2\Phi_t\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)+O(s\lambda^3\varphi)\right]s\lambda\varphi\Phi_t$$
$$=3s^3\lambda^4\varphi^3|\Phi_t|^2\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)$$
$$+s^3\lambda^3\varphi^3\left[\Phi_{tt}\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)+\Phi_t\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)_t\right]+O(s^2\lambda^4\varphi^2) \tag{2.38}$$

and

$$
\begin{aligned}
&-\alpha\sum_{i=1}^{3}(Al_{x_i})_{x_i}\\
&=\alpha\left[s^2\lambda^2\varphi^2\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)+O(s\lambda^2\varphi)\right](s\lambda^2\varphi|\nabla\Phi|^2+s\lambda\varphi\Delta\Phi)\\
&-\alpha\left[s^2\lambda^2\varphi^2\nabla\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)+2s^2\lambda^3\varphi^2\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)\nabla\Phi+O(s\lambda^3\varphi)\right]\cdot(s\lambda\varphi\nabla\Phi)\\
&=3\alpha s^3\lambda^4\varphi^3|\nabla\Phi|^2\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)-\alpha s^3\lambda^3\varphi^3\left[\Delta\Phi\left(|\Phi_t|^2\right.\right.\\
&\quad\left.\left.-\alpha|\nabla\Phi|^2\right)+\nabla\Phi\cdot\nabla\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)\right]+O(s^2\lambda^4\varphi^2). \qquad (2.39)
\end{aligned}
$$

Therefore, from (2.37)–(2.39) it follows that

$$B=C^*s^3\lambda^3\varphi^3+O(s^2\lambda^4\varphi^2) \qquad (2.40)$$

with

$$
\begin{aligned}
C^*&=2\lambda\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)^2-\gamma(\Phi_{tt}-\alpha\Delta\Phi)(|\Phi_t|^2-\alpha|\nabla\Phi|^2)\\
&\quad+\Phi_{tt}\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)-\alpha\Delta\Phi\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)+2|\Phi_t|^2\Phi_{tt}+2\alpha^2|\nabla\Phi|^2\Delta\Phi.
\end{aligned}
$$

For some suitable small $\epsilon_0>0$, we set

$$G_T^{(1)}=\left\{(x,t)\in G_T \mid \left||\Phi_t(x,t)|^2-\alpha|\nabla\Phi(x,t)|^2\right|\geqslant\epsilon_0\right\},\quad G_T^{(2)}=G_T\setminus G_T^{(1)}.$$

Then for $(x,t)\in G_T^{(1)}$, we obtain that there exists $\lambda_2^{(1)}>0$ such that

$$C^*=2\lambda\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)^2+O(1)\geqslant 2\epsilon_0^2\lambda+O(1)\geqslant 2\epsilon_0$$

for all $\lambda>\lambda_2^{(1)}$. On the other hand, for $(x,t)\in G_T^{(2)}$, we also have

$$
\begin{aligned}
C^*&\geqslant-\gamma(\Phi_{tt}-\alpha\Delta\Phi)(|\Phi_t|^2-\alpha|\nabla\Phi|^2)+\Phi_{tt}\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)\\
&\quad-\alpha\Delta\Phi\left(|\Phi_t|^2-\alpha|\nabla\Phi|^2\right)+(12\alpha^2-4\alpha\beta)|\nabla\Phi|^2-4\beta\epsilon_0\\
&\geqslant(12\alpha^2-4\alpha\beta)|\nabla\Phi|^2-\epsilon_0|O(1)|\geqslant 2\epsilon_0
\end{aligned}
$$

for

$$\beta\in(0,\alpha)\quad \epsilon_0=\frac{(12\alpha^2-4\alpha\beta)\min\limits_{x\in\overline{G}}4|x-x_0|^2}{2+|O(1)|}>0.$$

In conclusion, for any $\beta\in(0,\alpha)$ we obtain

$$B|Z|^2\geqslant\epsilon_0 s^3\lambda^3\varphi^3|Z|^2, \qquad (2.41)$$

if we take $\lambda\geqslant\lambda_2^{(1)}$ and $s\geqslant s_2^{(1)}:=\frac{|O(\lambda)|}{\epsilon_0}$.

Next we deal with the first order terms of Z

$$(l_{tt}+\alpha\Delta l-\Psi)\,|Z_t|^2-4\alpha\,(\nabla l_t\cdot\nabla Z)\,Z_t+\left(\alpha l_{tt}-\alpha^2\Delta l+\alpha\Psi\right)|\nabla Z|^2+2\alpha^2\sum_{i,j=1}^{3}l_{x_ix_j}Z_{x_i}Z_{x_j}$$
$$=\left\{2s\lambda^2\varphi|\Phi_t|^2+s\lambda\varphi\,[(1+\gamma)\Phi_{tt}+\alpha(1-\gamma)\Delta\Phi]\right\}|Z_t|^2$$
$$-4\alpha s\lambda^2\varphi\Phi_t\,(\nabla\Phi\cdot\nabla Z)\,Z_t+s\lambda\varphi\left[\alpha(1-\gamma)\Phi_{tt}+\alpha^2(-1+\gamma)\Delta\Phi\right]|\nabla Z|^2$$
$$+2\alpha^2\left[2s\lambda\varphi|\nabla Z|^2+s\lambda^2\varphi(\nabla\Phi\cdot\nabla Z)^2\right]$$
$$=2s\lambda^2\varphi[\Phi_tZ_t-\alpha(\nabla\Phi\cdot\nabla Z)]^2+s\lambda\varphi\,(-2\beta\gamma-2\beta+6\alpha-6\alpha\gamma)\,|Z_t|^2$$
$$+s\lambda\varphi\left(-2\alpha\beta+2\alpha\beta\gamma-2\alpha^2+6\alpha^2\gamma\right)|\nabla Z|^2. \tag{2.42}$$

We choose γ such that

$$\frac{\alpha+\beta}{3\alpha+\beta}<\gamma<\frac{3\alpha-\beta}{3\alpha+\beta}, \tag{2.43}$$

which is nonempty for all $\beta\in(0,\alpha)$. Then from (2.42) and (2.43) we deduce

$$(l_{tt}+\alpha\Delta l-\Psi)\,|Z_t|^2-4\alpha\,(\nabla l_t\cdot\nabla Z)\,Z_t+\left(\alpha l_{tt}-\alpha^2\Delta l+\alpha\Psi\right)|\nabla Z|^2$$
$$+2\alpha^2\sum_{i,j=1}^{N}l_{x_ix_j}Z_{x_i}Z_{x_j}\geqslant\epsilon_1 s\lambda\varphi|Z_t|^2+\epsilon_1 s\lambda\varphi|\nabla Z|^2 \tag{2.44}$$

with $\epsilon_1=\min\{-2\beta\gamma-2\beta+6\alpha-6\alpha\gamma,-2\alpha\beta+2\alpha\beta\gamma-2\alpha^2+6\alpha^2\gamma\}>0$ due to (2.43).

Thus, substituting (2.41) and (2.44) into (2.35), we obtain for any $\beta\in(0,\alpha)$ that

$$\theta\,[-2l_tZ_t+2\alpha\,(\nabla l\cdot\nabla Z)+\Psi Z]\,(\mathrm{d}z_t-\alpha\Delta z\,\mathrm{d}t)$$
$$+\,\mathrm{div}\Bigg[2\alpha^2\,(\nabla l\cdot\nabla Z)\,\nabla Z-\alpha^2|\nabla Z|^2\nabla l-2\alpha l_tZ_t\nabla Z$$
$$+\alpha|Z_t|^2\nabla l+\alpha\Psi Z\nabla Z-\alpha\left(A\nabla l+\frac{\nabla\Psi}{2}\right)|Z|^2\Bigg]\mathrm{d}t$$
$$+\,\mathrm{d}\left[\alpha l_t|\nabla Z|^2-2\alpha(\nabla l\cdot\nabla Z)Z_t+l_tZ_t^2-\Psi Z_tZ+\left(Al_t+\frac{\Psi_t}{2}\right)|Z|^2\right]$$
$$\geqslant\epsilon_2\left(s\lambda\varphi|Z_t|^2+s\lambda\varphi|\nabla Z|^2+s^3\lambda^3\varphi^3|Z|^2\right)\mathrm{d}t$$
$$+[-2l_tZ_t+2\alpha\,(\nabla l\cdot\nabla Z)+\Psi Z]^2\mathrm{d}t+\theta^2l_t(\mathrm{d}z_t)^2, \tag{2.45}$$

if we take $\lambda\geqslant\lambda_2^{(1)}$ and $s\geqslant s_2^{(1)}$. Here $\epsilon_2=\min\{\epsilon_0,\epsilon_1\}$.

Step 3. Take mathematical expectation and complete the proof.

Integrating both sides of (2.45) over G_T and taking mathematical expectation in Ω, we find that

$$
\begin{aligned}
&\mathbb{E}\int_{G_T}\theta\,[-2l_tZ_t+2\alpha\,(\nabla l\cdot\nabla Z)+\Psi Z]\,(\mathrm{d}z_t-\alpha\Delta z\,\mathrm{d}t)\,\mathrm{d}x\\
&\qquad+\mathbb{E}\int_{\Gamma_T}\Big[2\alpha^2\,(\nabla l\cdot\nabla Z)\,\nabla Z-\alpha^2|\nabla Z|^2\nabla l-2\alpha l_tZ_t\nabla Z\\
&\qquad+\alpha|Z_t|^2\nabla l+\alpha\Psi Z\nabla Z-\alpha\left(A\nabla l+\frac{\nabla\Psi}{2}\right)|Z|^2\Big]\cdot\eta\,\mathrm{d}S\,\mathrm{d}t\\
&\qquad+\mathbb{E}\int_{G}\left[\alpha l_t|\nabla Z|^2-2\alpha(\nabla l\cdot\nabla Z)Z_t+l_t|Z_t|^2-\Psi Z_tZ+\left(Al_t+\frac{\Psi_t}{2}\right)|Z|^2\right]\mathrm{d}x\Big|_{t=0}^{t=T}\\
&\quad\geqslant\epsilon_2\mathbb{E}\int_{G_T}\left(s\lambda\varphi|Z_t|^2+s\lambda\varphi|\nabla Z|^2+s^3\lambda^3\varphi^3|Z|^2\right)\mathrm{d}x\,\mathrm{d}t\\
&\qquad+\mathbb{E}\int_{G_T}[-2l_tZ_t+2\alpha\,(\nabla l\cdot\nabla Z)+\Psi Z]^2\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}\theta^2l_t(\mathrm{d}z_t)^2\,\mathrm{d}x.
\end{aligned}
\tag{2.46}
$$

Now we estimate each term on the left-hand side of (2.46). For the first term, by the equation for z in (2.34) we have

$$
\begin{aligned}
&\mathbb{E}\int_{G_T}\theta\,[-2l_tZ_t+2\alpha\,(\nabla l\cdot\nabla Z)+\Psi Z]\,(\mathrm{d}z_t-\alpha\Delta z\,\mathrm{d}t)\,\mathrm{d}x\\
&\quad=\mathbb{E}\int_{G_T}\theta\,[-2l_tZ_t+2\alpha\,(\nabla l\cdot\nabla Z)+\Psi Z]\,(F_2\,\mathrm{d}t+H_2\,\mathrm{d}B(t))\,\mathrm{d}x\\
&\quad\leqslant\frac{1}{2}\mathbb{E}\int_{G_T}[-2l_tZ_t+2\alpha\,(\nabla l\cdot\nabla Z)+\Psi Z]^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{2}\mathbb{E}\int_{G_T}\theta^2|F_2|^2\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{2.47}
$$

Noting that $z=0$ on Γ_T, we have $Z=0$ and $\nabla Z=\frac{\partial Z}{\partial\eta}\eta$ on Γ_T. Consequently, we have

$$
\begin{aligned}
&\mathbb{E}\int_{\Gamma_T}\Big[2\alpha^2\,(\nabla l\cdot\nabla Z)\,\nabla Z-\alpha^2|\nabla Z|^2\nabla l-2\alpha l_tZ_t\nabla Z\\
&\qquad+\alpha|Z_t|^2\nabla l+\alpha\Psi Z\nabla Z-\alpha\left(A\nabla l+\frac{\nabla\Psi}{2}\right)|Z|^2\Big]\cdot\eta\,\mathrm{d}S\,\mathrm{d}t\\
&\quad=\mathbb{E}\int_{\Gamma_T}\alpha^2\frac{\partial l}{\partial\eta}\left|\frac{\partial Z}{\partial\eta}\right|^2\mathrm{d}S\,\mathrm{d}t\leqslant C_3\mathbb{E}\int_{\Gamma_T}s\lambda\varphi\left|\frac{\partial z}{\partial\eta}\right|^2\mathrm{e}^{2s\varphi}\,\mathrm{d}S\,\mathrm{d}t.
\end{aligned}
\tag{2.48}
$$

Moreover, by a similar argument to (2.31), we obtain

$$
\begin{aligned}
&\mathbb{E}\int_{\Gamma_T}s\lambda\varphi\left|\frac{\partial z}{\partial\eta}\right|^2\mathrm{e}^{2s\varphi}\,\mathrm{d}S\,\mathrm{d}t\\
&\quad\leqslant C_3\mathbb{E}\int_{\varpi_T^{(1)}}\frac{1}{s\varphi}\sum_{i,j=1}^{3}|\partial_i\partial_jz|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+C_3\mathbb{E}\int_{\varpi_T^{(1)}}s^3\lambda^2\varphi^3|\nabla z|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{2.49}
$$

Therefore, from (2.48) and (2.49) we deduce that

$$\mathbb{E}\int_{\Gamma_T}\left[2\alpha^2\left(\nabla l\cdot\nabla Z\right)\nabla Z-\alpha^2|\nabla Z|^2\nabla l-2\alpha l_t Z_t\nabla Z\right.$$
$$\left.+\alpha|Z_t|^2\nabla l+\alpha\Psi Z\nabla Z-\alpha\left(A\nabla l+\frac{\nabla\Psi}{2}\right)|Z|^2\right]\cdot\eta\,\mathrm{d}S\,\mathrm{d}t$$
$$\leqslant C_4(\lambda)s^3\,\mathrm{e}^{C_4(\lambda)s}\|z\|^2_{H^{(2)}_{\varpi^{(1)}_T}}. \tag{2.50}$$

Next we analyze the terms corresponding to $t=0$ and $t=T$. By $l_t(x,T)=0$ in G, together with the following standard estimate for stochastic hyperbolic equation [28]:

$$\|z\|_{L^2_{\mathcal{F}}(\Omega;C^1([0,T];L^2(G)))}+\|z\|_{L^2_{\mathcal{F}}(\Omega;C([0,T];H^1_0(G)))}$$
$$\leqslant C_3\left(\|z_0\|_{H^{(1)}_{G(0)}}+\|z_1\|_{H^{(0)}_{G(0)}}+\|F_2\|_{H^{(0)}_{G_T}}+\|H_2\|_{H^{(0)}_{G_T}}\right),$$

we have for $t=T$ that

$$\mathbb{E}\int_G\left[\alpha l_t|\nabla Z|^2-2\alpha(\nabla l\cdot\nabla Z)Z_t+l_t|Z_t|^2-\Psi Z_t Z+\left(Al_t+\frac{\Psi_t}{2}\right)|Z|^2\right]\mathrm{d}x$$
$$\leqslant C_4(\lambda)s^3\,\mathrm{e}^{C_4(\lambda)s}\|z(\cdot,T)\|^2_{H^{(1)}_{G(T)}}+C_4(\lambda)s^2\,\mathrm{e}^{C_4(\lambda)s}\|z(\cdot,T)\|_{H^{(1)}_{G(T)}}\|z_t(\cdot,T)\|_{H^{(0)}_{G(T)}}$$
$$\leqslant C_4(\lambda)s^3\,\mathrm{e}^{C_4(\lambda)s}\left(\|z(\cdot,T)\|_{H^{(1)}_{G(T)}}+\|z_t(\cdot,T)\|_{H^{(0)}_{G(T)}}\right)\|z(\cdot,T)\|_{H^{(1)}_{G(T)}}$$
$$\leqslant C_4(\lambda)s^3\,\mathrm{e}^{C_4(\lambda)s}\|z(\cdot,T)\|_{H^{(1)}_{G(T)}}. \tag{2.51}$$

On the other hand, by $l_t(x,0)=2\beta Ts\lambda\varphi_0(x)$, we obtain for $t=0$ that

$$\mathbb{E}\int_G\left[\alpha l_t|\nabla Z|^2-2\alpha(\nabla l\cdot\nabla Z)Z_t+l_t|Z_t|^2-\Psi Z_t Z+\left(Al_t+\frac{\Psi_t}{2}\right)|Z|^2\right]\mathrm{d}x$$
$$=\mathbb{E}\int_G 2\alpha\beta Ts\lambda\varphi_0|\nabla Z|^2\,\mathrm{d}x-\mathbb{E}\int_G 4\alpha s\lambda\varphi_0\left[(x-x_0)\cdot\nabla Z\right]Z_t\,\mathrm{d}x$$
$$+\mathbb{E}\int_G 2\beta Ts\lambda\varphi_0|Z_t|^2\,\mathrm{d}x+\mathbb{E}\int_G O(s\lambda^2\varphi_0)ZZ_t\,\mathrm{d}x$$
$$+\mathbb{E}\int_G\left[8\beta T\left(\beta^2T^2-\alpha|x-x_0|^2\right)s^3\lambda^3\varphi_0^3+O(s^2\lambda^3\varphi_0^2)\right]|Z|^2\,\mathrm{d}x. \tag{2.52}$$

Together with

$$\mathbb{E}\int_G 4\alpha s\lambda\varphi_0\left[(x-x_0)\cdot\nabla Z\right]Z_t\,\mathrm{d}x$$
$$\leqslant\mathbb{E}\int_G 2\alpha\sqrt{\alpha}s\lambda\varphi_0|x-x_0||\nabla Z|^2\,\mathrm{d}x+\mathbb{E}\int_G 2\sqrt{\alpha}s\lambda\varphi_0|x-x_0||Z_t|^2\,\mathrm{d}x \tag{2.53}$$

and

$$\mathbb{E}\int_G O(s\lambda^2\varphi_0)ZZ_t\,\mathrm{d}x \leqslant \epsilon_3\mathbb{E}\int_G 2s\lambda\varphi_0|Z_t|^2\,\mathrm{d}x + C(\epsilon_3)\mathbb{E}\int_G |O(s\lambda^3\varphi_0)||Z|^2\,\mathrm{d}x, \tag{2.54}$$

we further have for $t = 0$ that

$$\begin{aligned}&\mathbb{E}\int_G \left[\alpha l_t|\nabla Z|^2 - 2\alpha(\nabla l\cdot\nabla Z)Z_t + l_t|Z_t|^2 - \Psi Z_t Z + \left(Al_t + \frac{\Psi_t}{2}\right)|Z|^2\right]\mathrm{d}x\\ &\quad\geqslant \mathbb{E}\int_G \left[8\beta T(\beta^2T^2 - \alpha|x-x_0|^2)s^3\lambda^3\varphi_0^3 - |O(s^2\lambda^3\varphi_0^2)| - C(\epsilon_3)|O(s\lambda^3\varphi_0)|\right]|Z|^2\,\mathrm{d}x\\ &\qquad + \mathbb{E}\int_G 2\alpha s\lambda\varphi_0(\beta T - \sqrt{\alpha}|x-x_0|)|\nabla Z|^2\,\mathrm{d}x\\ &\qquad + \mathbb{E}\int_G 2s\lambda\varphi_0(\beta T - \sqrt{\alpha}|x-x_0| - \epsilon_3)|Z_t|^2\,\mathrm{d}x.\end{aligned} \tag{2.55}$$

By (2.32) we can choose $\epsilon_3 > 0$ sufficiently small to satisfy $\beta T - \sqrt{\alpha}|x - x_0| > 2\epsilon_3$ for all $x \in G$. Therefore for $t = 0$, there exists $s_2^{(2)} > 0$ such that for any $s \geqslant s_2^{(2)}$, it holds that

$$\begin{aligned}&\mathbb{E}\int_G \left[\alpha l_t|\nabla Z|^2 - 2\alpha(\nabla l\cdot\nabla Z)Z_t + l_t|Z_t|^2 - \Psi Z_t Z + \left(Al_t + \frac{\Psi_t}{2}\right)|Z|^2\right]\mathrm{d}x\\ &\quad\geqslant \epsilon_4\mathbb{E}\int_G \left[s\lambda\varphi_0\left(|\nabla Z(x,0)|^2 + |Z_t(x,0)|^2\right) + s^3\lambda^3\varphi_0^3|Z(x,0)|^2\right]\mathrm{d}x\end{aligned} \tag{2.56}$$

with $\epsilon_4 = \min\{4\alpha\epsilon_3, 2\epsilon_3, 32\epsilon_3^3\} > 0$. Hence,

$$\begin{aligned}&\mathbb{E}\int_G \left[\alpha l_t|\nabla Z|^2 - 2\alpha(\nabla l\cdot\nabla Z)Z_t + l_t|Z_t|^2 - \Psi Z_t Z + \left(Al_t + \frac{\Psi_t}{2}\right)|Z|^2\right]\mathrm{d}x\Big|_{t=0}^{t=T}\\ &\quad\leqslant -\epsilon_4\mathbb{E}\int_G \left[s\lambda\varphi_0\left(|\nabla Z(x,0)|^2 + |Z_t(x,0)|^2\right) + s^3\lambda^3\varphi_0^3|Z(x,0)|^2\right]\mathrm{d}x\\ &\qquad + C_4(\lambda)s^3\,\mathrm{e}^{C_4(\lambda)s}\|z(\cdot,T)\|_{H^{(1)}_{G(T)}}.\end{aligned} \tag{2.57}$$

Moreover,

$$\mathbb{E}\int_{G_T} \theta^2 l_t(\mathrm{d}z_t)^2\,\mathrm{d}x = \mathbb{E}\int_{G_T} 2\beta s\lambda\varphi\,(T-t)\,\theta^2|H_2|^2\,\mathrm{d}x\,\mathrm{d}t. \tag{2.58}$$

Substituting (2.47), (2.50), (2.57) and (2.58) into (2.46) and going back to y, we obtain the desired estimate (2.33). This completes the proof of theorem 2.2. □

3. Proof of theorem 1.1

We are now in a position to prove theorem 1.1 for our inverse problem by using theorems 2.1 and 2.2. First we fix $\beta \in (0, \nu)$ to satisfy the conditions in theorem 2.2. We also use C to denote a generic positive constant depending on $x_0, G, T, r_0, M, \nu, \mu, \rho$ and κ, but independent of λ and s.

Firstly, we show a regularity result for the solution $(\mathbf{u}, v)$ of the direct problem (1.1), i.e. $\mathbf{u} \in X_{G_T} \cap L^2_{\mathcal{F}}(\Omega; C([0,T]; H^2(G)))$, $v \in H^{(2)}_{G_T} \times L^2_{\mathcal{F}}(\Omega; C([0,T]; H^1(G)))$, under the condition (A4). This regularity is sufficient for our inverse problem. Let

$$\mathcal{A} = \begin{pmatrix} 0 & -1 & 0 \\ S & 0 & 0 \\ 0 & 0 & \kappa Q \end{pmatrix},$$

where $Q = -\Delta, S = -\nu\Delta - (\nu + \mu)\nabla\mathrm{div}$. We introduce

$$\mathbf{W} = (\mathbf{u}, \mathbf{u}_t, v)^T, \qquad \mathbf{D} = (0, \mathbf{f}, g)^T, \qquad \mathbf{B}(\mathbf{W}) = (0, -\rho\nabla v, -\rho\,\mathrm{div}\,\mathbf{u}_t)^T. \tag{3.1}$$

Then the problem (1.1) is equivalent to

$$\mathrm{d}\mathbf{W} + \mathcal{A}\mathbf{W}\,\mathrm{d}t = \mathbf{B}(\mathbf{W})\mathrm{d}t + \mathbf{D}\,\mathrm{d}B(t), \quad \mathbf{W}|_{t=0} = \mathbf{W}_0 := (\mathbf{u}_0, \mathbf{u}_1, 0)^T. \tag{3.2}$$

We consider problem (3.2) in space $\mathcal{H} = D(S^{\frac{1}{2}}) \times L^2(G) \times D(Q^{\frac{1}{2}})$.

As done in [10], we know that there exists a mild solution $\mathbf{W} \in L^2_{\mathcal{F}}(0,T;\mathcal{H})$ of problem (1.1), denoted by

$$\mathbf{W}(t) = \mathrm{e}^{-\mathcal{A}t}\mathbf{W}_0 + \int_0^t \mathrm{e}^{-\mathcal{A}(t-\tau)}\mathbf{B}(\mathbf{W}(\tau))\mathrm{d}\tau + \int_0^t \mathrm{e}^{-\mathcal{A}(t-\tau)}\mathbf{D}(\tau)\mathrm{d}B(\tau). \tag{3.3}$$

Therefore

$$\|Q\mathbf{W}\|^2_{L^2_{\mathcal{F}}(0,T;\mathcal{H})} \leqslant C\|Q\mathbf{W}_0\|^2_{L^2(\Omega,\mathcal{F}_0,P;\mathcal{H})} + \left\|\int_0^t \mathrm{e}^{-\mathcal{A}(t-\tau)}\mathbf{B}(Q\mathbf{W}(\tau))\mathrm{d}\tau\right\|^2_{L^2_{\mathcal{F}}(0,T;\mathcal{H})} + \left\|\int_0^t \mathrm{e}^{-\mathcal{A}(t-\tau)}Q\mathbf{D}(\tau)\mathrm{d}B(\tau)\right\|^2_{L^2_{\mathcal{F}}(0,T;\mathcal{H})}. \tag{3.4}$$

Similar to (37) and (38) in [10], we obtain

$$\left\|\int_0^t \mathrm{e}^{-\mathcal{A}(t-\tau)}\mathbf{B}(Q\mathbf{W}(\tau))\mathrm{d}\tau\right\|^2_{L^2_{\mathcal{F}}(0,T;\mathcal{H})} \leqslant \left(CT^{\frac{1}{2}} + \frac{M_\rho}{\kappa}\right)\|Q\mathbf{W}\|^2_{L^2_{\mathcal{F}}(0,T;\mathcal{H})}. \tag{3.5}$$

Additionally,

$$\begin{aligned} &\left\|\int_0^t \mathrm{e}^{-\mathcal{A}(t-\tau)}Q\mathbf{D}(\tau)\mathrm{d}B(\tau)\right\|^2_{L^2_{\mathcal{F}}(0,T;\mathcal{H})} \\ &\quad = \mathbb{E}\int_0^T \left|\int_0^t \mathrm{e}^{-\mathcal{A}(t-\tau)}\|Q\mathbf{D}(\tau)\|_{\mathcal{H}}\,\mathrm{d}B(\tau)\right|^2 \mathrm{d}t \\ &\quad = \mathbb{E}\int_0^T\int_0^t \left(e^{-\mathcal{A}(t-\tau)}\|Q\mathbf{D}(\tau)\|_{\mathcal{H}}\right)^2 \mathrm{d}\tau\,\mathrm{d}t \leqslant CT\|Q\mathbf{D}\|^2_{L^2_{\mathcal{F}}(0,T;\mathcal{H})}. \end{aligned} \tag{3.6}$$

Therefore, together with $\kappa > M_\rho$, it follows from (3.4)–(3.6) that

$$\|Q\mathbf{W}\|^2_{L^2_{\mathcal{F}}(0,T_0;\mathcal{H})} \leqslant C\left(\|Q\mathbf{W}_0\|^2_{L^2(\Omega,\mathcal{F}_0,P;\mathcal{H})} + \|Q\mathbf{D}\|^2_{L^2_{\mathcal{F}}(0,T_0;\mathcal{H})}\right) \tag{3.7}$$

for sufficiently small T_0. Since T_0 does not depend on the initial data, we can extend $[0, T_0]$ to $[0, T]$ by repeating the same process. Thus (3.7) holds for any T, which means $(\mathbf{u}, v) \in X_{G_T} \times H^{(2)}_{G_T}$. By a similar argument to step 2 of the proof of theorem 3.3 in [10], we further obtain $Q\mathbf{u} \in L^2_{\mathcal{F}}\left(\Omega; C([0,T]; D(S^{\frac{1}{2}})\right)$ and $Qv \in L^2_{\mathcal{F}}(\Omega; C([0,T]; D(Q^{\frac{1}{2}})))$, which implies $\mathbf{u} \in L^2_{\mathcal{F}}(\Omega; C([0,T]; H^2(G)))$ and $v \in L^2_{\mathcal{F}}(\Omega; C([0,T]; H^1(G)))$.

Now we prove our stability result. For simplicity, we set

$$\begin{aligned}
&\mathbf{p}(x,t) = \mathbf{u}^{(1)}(x,t) - \mathbf{u}^{(2)}(x,t), \qquad q(x,t) = v^{(1)}(x,t) - v^{(2)}(x,t),\\
&\mathbf{p}_0(x) = \mathbf{u}_0^{(1)}(x) - \mathbf{u}_0^{(2)}(x), \qquad \mathbf{p}_1(x) = \mathbf{u}_1^{(1)}(x) - \mathbf{u}_1^{(2)}(x),\\
&\mathbf{F}(x,t) = \mathbf{f}^{(1)}(x,t) - \mathbf{f}^{(2)}(x,t), \qquad \vartheta(t) = h^{(1)}(t) - h^{(2)}(t).
\end{aligned}$$

Then by (1.1), we easily see that

$$\begin{cases}
\mathrm{d}\mathbf{p}_t - \nu\Delta\mathbf{p}\,\mathrm{d}t - (\nu+\mu)\nabla\,\mathrm{div}\,\mathbf{p}\,\mathrm{d}t + \rho\nabla q\,\mathrm{d}t = \mathbf{F}(x,t)\mathrm{d}B(t), & (x,t) \in G_T,\\
\mathrm{d}q - \kappa\Delta q\,\mathrm{d}t + \rho\,\mathrm{div}\,\mathbf{p}_t\,\mathrm{d}t = \vartheta(t)R(x,t)\mathrm{d}B(t), & (x,t) \in G_T,\\
\mathbf{p}(x,t) = 0, \quad q(x,t) = 0, & (x,t) \in \Gamma_T,\\
\mathbf{p}(x,0) = \mathbf{p}_0(x), \quad \mathbf{p}_t(x,0) = \mathbf{p}_1(x), \quad q(x,0) = 0, & (x,t) \in \Omega.
\end{cases} \tag{3.8}$$

By $\mathbf{f}^{(1)} = \mathbf{f}^{(2)} = \bar{\mathbf{f}}$ in ϖ_T, P-a.s., we have

$$\mathbf{F}(x,t) = 0, \quad (x,t) \in \varpi_T, \quad P-\text{a.s.} \tag{3.9}$$

Set $\pi = \mathrm{div}\,\mathbf{p}$ and $\zeta = \mathrm{curl}\,\mathbf{p}$. Then we have

$$\begin{cases}
\mathrm{d}\pi_t - (2\nu+\mu)\Delta\pi\,\mathrm{d}t = -\rho\Delta q\,\mathrm{d}t + \mathrm{div}\,\mathbf{F}(x,t)\mathrm{d}B(t), & (x,t) \in G_T,\\
\mathrm{d}\zeta_t - \nu\Delta\zeta\,\mathrm{d}t = \mathrm{curl}\,\mathbf{F}(x,t)\mathrm{d}B(t), & (x,t) \in G_T.
\end{cases} \tag{3.10}$$

In order to apply theorem 2.2, we introduce a cut-off function $\chi \in C^\infty(G)$ satisfying $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ in $G\backslash\varpi^{(1)}$ and $\mathrm{Supp}(\chi) \subset G$. Then $(\tilde{\pi}, \tilde{\zeta}) := (\chi\pi, \chi\zeta)$ satisfies

$$\begin{cases}
\mathrm{d}\tilde{\pi}_t - (2\nu+\mu)\Delta\tilde{\pi}\,\mathrm{d}t = \xi_1\,\mathrm{d}t + \chi\,\mathrm{div}\,\mathbf{F}(x,t)\mathrm{d}B(t), & (x,t) \in Q_T,\\
\mathrm{d}\tilde{\zeta}_t - \nu\Delta\tilde{\zeta}\,\mathrm{d}t = \xi_2\,\mathrm{d}t + \chi\,\mathrm{curl}\,\mathbf{F}(x,t)\mathrm{d}B(t), & (x,t) \in Q_T,\\
\tilde{\pi}(x,t) = 0, \quad \tilde{\zeta}(x,t) = 0, & (x,t) \in \Gamma_T,\\
\tilde{\pi}(x,0) = \chi\,\mathrm{div}\,\mathbf{p}_0(x), \quad \tilde{\pi}_t(x,0) = \chi\,\mathrm{div}\,\mathbf{p}_1(x), & x \in G,\\
\tilde{\zeta}(x,0) = \chi\,\mathrm{curl}\,\mathbf{p}_0(x), \quad \tilde{\zeta}_t(x,0) = \chi\,\mathrm{curl}\,\mathbf{p}_1(x), & x \in G,
\end{cases} \tag{3.11}$$

where

$$\xi_1 = -(2\nu + \mu)(2\nabla\chi\cdot\nabla\pi + \Delta\chi\pi) - \rho\chi\Delta q,\ \xi_2 = -\nu(2\nabla\chi\cdot\nabla\zeta + \Delta\chi\zeta).$$

Applying theorem 2.2 to $\tilde{\pi}$ and $\tilde{\zeta}$, we obtain

$$\begin{aligned}
&\mathbb{E}\int_{G_T\setminus\varpi_T^{(1)}} \left[s\lambda\varphi\left(|\pi_t|^2 + |\zeta_t|^2\right) + s\lambda\varphi\left(|\nabla\pi|^2 + |\nabla\zeta|^2\right) + s^3\lambda^3\varphi^3\left(|\pi|^2 + |\zeta|^2\right)\right]\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \mathbb{E}\int_{G_T\setminus\varpi_T^{(1)}} s\lambda\varphi(T-t)\left(|\mathrm{div}\,\mathbf{F}|^2 + |\mathrm{curl}\ \mathbf{F}|^2\right)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C\mathbb{E}\int_{G_T}\left(|\nabla\chi|^2 + |\Delta\chi|^2\right)\left(|\nabla\pi|^2 + |\pi|^2 + |\nabla\zeta|^2 + |\zeta|^2\right)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + C\mathbb{E}\int_{G_T}|\Delta q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + C(\lambda)s^3\,\mathrm{e}^{C(\lambda)s}\left(\|\pi\|^2_{H^{(2)}_{\varpi_T^{(1)}}} + \|\pi(\cdot,T)\|_{H^{(1)}_{G(T)}}\right.\\
&\quad \left. + \|\zeta\|^2_{H^{(2)}_{\varpi_T^{(1)}}} + \|\zeta(\cdot,T)\|_{H^{(1)}_{G(T)}}\right)
\end{aligned}\tag{3.12}$$

for all $\lambda \geqslant \lambda_2$ and $s \geqslant s_2$. Together with $\mathrm{div}\,\mathbf{F} = \mathrm{curl}\,\mathbf{F} = 0$ in $\varpi_T^{(1)}$ by (3.9) and $\mathrm{Supp}(\nabla\chi)$, $\mathrm{Supp}(\Delta\chi) \subset \varpi^{(1)} \subset \varpi$, we further obtain

$$\begin{aligned}
&\mathbb{E}\int_{G_T}\left[s^2\lambda\varphi\left(|\pi_t|^2 + |\zeta_t|^2\right) + s^2\lambda\varphi\left(|\nabla\pi|^2 + |\nabla\zeta|^2\right) + s^4\lambda^3\varphi^3\left(|\pi|^2 + |\zeta|^2\right)\right]\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \mathbb{E}\int_{G_T} s^2\lambda\varphi(T-t)\left(|\mathrm{div}\,\mathbf{F}|^2 + |\ \mathrm{curl}\ \mathbf{F}|^2\right)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C\mathbb{E}\int_{G_T} s|\Delta q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + C(\lambda)s^4\mathrm{e}^{C(\lambda)s}\left(\|\mathbf{p}\|^2_{X_{\varpi_T}} + \|\mathbf{p}(\cdot,T)\|_{H^{(2)}_{G(T)}}\right).
\end{aligned}\tag{3.13}$$

On the other hand, applying theorem 2.1 to q and using (A2), we find that

$$\begin{aligned}
&\mathbb{E}\int_{G_T}\left(\frac{1}{s}\sum_{i,j=1}^{3}|\partial_i\partial_j q|^2 + s\lambda^2\varphi^2|\nabla q|^2 + s^3\lambda^4\varphi^4|q|^2\right)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_{G_T}s\lambda^2\varphi^2|\vartheta|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C\mathbb{E}\int_{G_T}\varphi|\pi_t|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + C\mathbb{E}\int_{G_T}s\varphi^2|\vartheta|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad + C\mathbb{E}\int_{\varpi_T^{(1)}}s^3\lambda^2\varphi^4|\nabla q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + C(\lambda)s^2\,\mathrm{e}^{C(\lambda)s}\|q(\cdot,T)\|^2_{H^{(1)}_{G(T)}}
\end{aligned}\tag{3.14}$$

for all $\lambda \geqslant \lambda_1$ and $s \geqslant s_1$.

Multiplying (3.14) by $s^2(C+1)$ and adding up (3.13), we have

$$
\begin{aligned}
&\mathbb{E}\int_{G_T}\left[s^2\lambda\varphi\left(|\pi_t|^2+|\zeta_t|^2\right)+s^2\lambda\varphi\left(|\nabla\pi|^2+|\nabla\zeta|^2\right)+s^4\lambda^3\varphi^3\left(|\pi|^2+|\zeta|^2\right)\right]\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+(C+1)\mathbb{E}\int_{G_T}\left(s\sum_{i,j=1}^{3}|\partial_i\partial_j q|^2+s^3\lambda^2\varphi^2|\nabla q|^2+s^5\lambda^4\varphi^4|q|^2\right)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\mathbb{E}\int_{G_T}s^2\lambda\varphi(T-t)\left(|\mathrm{div}\,\mathbf{F}|^2+|\,\mathrm{curl}\,\mathbf{F}|^2\right)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+(C+1)\mathbb{E}\int_{G_T}s^3\lambda^2\varphi^2|\vartheta|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C\mathbb{E}\int_{G_T}s|\Delta q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+C(C+1)\mathbb{E}\int_{G_T}s^2\varphi|\pi_t|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+C(C+1)\mathbb{E}\int_{G_T}s^3\varphi^2|\vartheta|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+C(C+1)\mathbb{E}\int_{\varpi_T^{(1)}}s^5\lambda^2\varphi^4|\nabla q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+C(\lambda)s^4\,\mathrm{e}^{C(\lambda)s}\left(\|\mathbf{p}\|^2_{X_{\varpi_T}}+\|\mathbf{p}(\cdot,T)\|_{H^{(2)}_{G(T)}}\right)+(C+1)C(\lambda)s^2\,\mathrm{e}^{C(\lambda)s}\|q(\cdot,T)\|^2_{H^{(1)}_{G(T)}}.
\end{aligned}
\tag{3.15}
$$

Obviously, for λ sufficiently large such that $\lambda\geqslant C(C+1)$, we can absorb the terms of π_t and ϑ on the right-hand side of (3.15). Then it holds

$$
\begin{aligned}
&\mathbb{E}\int_{G_T}\left[s^2\lambda\varphi|\pi_t|^2+s^2\lambda\varphi|\nabla\pi|^2+s^4\lambda^3\varphi^3|\pi|^2\right]\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\mathbb{E}\int_{G_T}\left(s\sum_{i,j=1}^{3}|\partial_i\partial_j q|^2+s^3\lambda^2\varphi^2|\nabla q|^2\right)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\mathbb{E}\int_{G_T}s^2\lambda\varphi(T-t)\left(|\mathrm{div}\,\mathbf{F}|^2+|\,\mathrm{curl}\,\mathbf{F}|^2\right)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}s^3\lambda^2\varphi^2|\vartheta|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\leqslant C(\lambda)s^4\,\mathrm{e}^{C(\lambda)s}\left(\|\mathbf{p}\|^2_{X_{\varpi_T}}+\|\mathbf{p}(\cdot,T)\|_{H^{(2)}_{G(T)}}+\|q(\cdot,T)\|^2_{H^{(1)}_{G(T)}}\right)\\
&\quad+C\mathbb{E}\int_{\varpi_T^{(1)}}s^5\lambda^2\varphi^4|\nabla q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{3.16}
$$

Moreover, applying theorem 2.2 to $\mathbf{p}$ yields that

$$
\begin{aligned}
&\mathbb{E}\int_{G_T}\left(s\lambda\varphi|\mathbf{p}_t|^2+s\lambda\varphi|\nabla\mathbf{p}|^2+s^3\lambda^3\varphi^3|\mathbf{p}|^2\right)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+\mathbb{E}\int_{G_T}s\lambda\varphi(T-t)|\mathbf{F}|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\mathbb{E}\int_{G}\left[s\lambda\varphi_0\left(|\nabla\mathbf{p}_0|^2+|\mathbf{p}_1|^2\right)+s^3\lambda^3\varphi_0^3|\mathbf{p}_0|^2\right]\mathrm{e}^{2s\varphi_0}\,\mathrm{d}x\\
&\leqslant C\mathbb{E}\int_{G_T}\left(|\nabla\pi|^2+|\nabla q|^2\right)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t+C(\lambda)s^3\,\mathrm{e}^{C(\lambda)s}\left(\|\mathbf{p}\|^2_{H^{(2)}_{\varpi_T^{(1)}}}+\|\mathbf{p}(\cdot,T)\|_{H^{(1)}_{G(T)}}\right)
\end{aligned}
\tag{3.17}
$$

for all $\lambda \geqslant \lambda_2$ and $s \geqslant s_2$. From (3.16) and (3.17), it follows that

$$\begin{aligned}
&\mathbb{E}\int_{G_T}\left(s\sum_{i,j=1}^{3}|\partial_i\partial_j q|^2 + s^3\lambda^2\varphi^2|\nabla q|^2\right)\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + \|\vartheta\|^2_{H^{(0)}_T} + \|\mathbf{p}_0\|^2_{H^{(1)}_{G(0)}} + \|\mathbf{p}_1\|^2_{H^{(0)}_{G(0)}}\\
&\quad + \left\|\sqrt{T-t}\mathbf{F}\right\|^2_{H^{(0)}_{G_T}} + \left\|\sqrt{T-t}\,\mathrm{div}\,\mathbf{F}\right\|^2_{H^{(0)}_{G_T}} + \left\|\sqrt{T-t}\ \mathrm{curl}\ \mathbf{F}\right\|^2_{H^{(0)}_{G_T}}\\
&\leqslant C(\lambda)s^4\,\mathrm{e}^{C(\lambda)s}\left(\|\mathbf{p}\|^2_{X_{\varpi_T}} + \|\mathbf{p}(\cdot,T)\|_{H^{(2)}_{G(T)}} + \|q(\cdot,T)\|^2_{H^{(1)}_{G(T)}}\right)\\
&\quad + C\mathbb{E}\int_{\varpi^{(1)}_T} s^5\lambda^2\varphi^4|\nabla q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{3.18}$$

Next, we estimate the local integral of ∇q on $\varpi^{(1)}$. By Ito's formula, we know

$$\begin{aligned}
&\mathrm{d}\left[s^5\lambda^2\chi_2\varphi^4\left(\nabla q\cdot\mathbf{p}_t\right)\mathrm{e}^{2s\varphi}\right]\\
&\quad = s^5\lambda^2\chi_2\left(\varphi^4\,\mathrm{e}^{2s\varphi}\right)_t\left(\nabla q\cdot\mathbf{p}_t\right)\mathrm{d}t + s^5\lambda^2\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\left(\nabla q\cdot\,\mathrm{d}\mathbf{p}_t\right)\\
&\qquad + s^5\lambda^2\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\left(\mathbf{p}_t\cdot\nabla\,\mathrm{d}q\right) + s^5\lambda^2\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\left(\mathrm{d}\mathbf{p}_t\cdot\nabla\,\mathrm{d}q\right).
\end{aligned}\tag{3.19}$$

Integrating (3.19) over G_T, taking mathematical expectation in Ω and noting that $q(x,0)=0$ in G, P-a.s., we have

$$\begin{aligned}
&\mathbb{E}\int_G\left[s^5\lambda^2\chi_2\varphi^4\left(\nabla q\cdot\mathbf{p}_t\right)\mathrm{e}^{2s\varphi}\right]\big|_{t=T}\mathrm{d}x\\
&\quad = \mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\left(\varphi^4\,\mathrm{e}^{2s\varphi}\right)_t\left(\nabla q\cdot\mathbf{p}_t\right)\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\left(\nabla q\cdot\mathrm{d}\mathbf{p}_t\right)\mathrm{d}x\\
&\qquad + \mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\varphi^4\mathrm{e}^{2s\varphi}\left(\mathbf{p}_t\cdot\nabla\mathrm{d}q\right)\mathrm{d}x + \mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\left(\mathrm{d}\mathbf{p}_t\cdot\nabla\mathrm{d}q\right)\mathrm{d}x\\
&\quad = \mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\left(\varphi^4\,\mathrm{e}^{2s\varphi}\right)_t\left(\nabla q\cdot\mathbf{p}_t\right)\mathrm{d}x\,\mathrm{d}t + \mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\left(\mathbf{F}\cdot\nabla\vartheta\right)\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\nabla q\cdot\left(\nu\Delta\mathbf{p}\,\mathrm{d}t + (\nu+\mu)\nabla\pi\,\mathrm{d}t - \rho\nabla q\,\mathrm{d}t + \mathbf{F}\mathrm{d}B(t)\right)\mathrm{d}x\\
&\qquad + \mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\mathbf{p}_t\cdot\left(\kappa\nabla\Delta q\,\mathrm{d}t - \rho\nabla\pi_t\,\mathrm{d}t + \nabla\vartheta\,\mathrm{d}B(t)\right)\mathrm{d}x.
\end{aligned}\tag{3.20}$$

By (3.9), we obtain

$$\mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\left(\mathbf{F}\cdot\nabla\vartheta\right)\mathrm{d}x\,\mathrm{d}t \leqslant \mathbb{E}\int_{\varpi_T} s^5\lambda^2\varphi^4\,\mathrm{e}^{2s\varphi}\left|\mathbf{F}\cdot\nabla\vartheta\right|\,\mathrm{d}x\,\mathrm{d}t = 0.\tag{3.21}$$

Therefore, by (3.20) and (3.21) we further obtain

$$\rho\mathbb{E}\int_{\varpi_T^{(1)}} s^5\lambda^2\varphi^4|\nabla q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t \leqslant K_1+K_2+K_3, \tag{3.22}$$

where

$$K_1 = \mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\left(\varphi^4\,\mathrm{e}^{2s\varphi}\right)_t\left(\nabla q\cdot\mathbf{p}_t\right)\mathrm{d}x\,\mathrm{d}t$$
$$+\mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\nabla q\cdot(\nu\Delta\mathbf{p}\,\mathrm{d}t+(\nu+\mu)\nabla\pi\,\mathrm{d}t)\,\mathrm{d}x,$$
$$K_2 = \mathbb{E}\int_{G_T} s^5\lambda^2\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\mathbf{p}_t\cdot(\kappa\nabla\Delta q\,\mathrm{d}t-\rho\nabla\pi_t\,\mathrm{d}t)\,\mathrm{d}x,$$
$$K_3 = -\mathbb{E}\int_G \left[s^5\lambda^2\chi_2\varphi^4\left(\nabla q\cdot\mathbf{p}_t\right)\mathrm{e}^{2s\varphi}\right]\Big|_{t=T}\mathrm{d}x.$$

Now we estimate K_1, K_2 and K_3. By $\left|\left(\varphi^4\,\mathrm{e}^{2s\varphi}\right)_t\right| \leqslant Cs\lambda\varphi^5\,\mathrm{e}^{2s\varphi}$, we obtain

$$K_1 \leqslant \mathbb{E}\int_{G_T} s^2\lambda^2\varphi^2|\nabla q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + C(\lambda)s^{10}\,\mathrm{e}^{C(\lambda)s}\|\mathbf{p}\|^2_{X_{\varpi_T}}. \tag{3.23}$$

Integration by parts and Hölder inequality yields the following estimate for K_2

$$K_2 = -\mathbb{E}\int_{G_T} s^5\lambda^2\left[\nabla\left(\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\right)\cdot\mathbf{p}_t+\chi_2\varphi^4\,\mathrm{e}^{2s\varphi}\pi_t\right](\kappa\Delta q-\rho\pi_t)\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant \mathbb{E}\int_{G_T}|\Delta q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + C(\lambda)s^{12}\,\mathrm{e}^{C(\lambda)s}\|\mathbf{p}\|^2_{X_{\varpi_T}}. \tag{3.24}$$

For K_3, we have

$$K_3 \leqslant C(\lambda)s^5\,\mathrm{e}^{C(\lambda)s}\|\mathbf{p}_t(\cdot,T)\|_{H^{(0)}_{G(T)}}\|q(\cdot,T)\|_{H^{(1)}_{G(T)}} \leqslant C(\lambda)s^5\,\mathrm{e}^{C(\lambda)s}\|q(\cdot,T)\|_{H^{(1)}_{G(T)}}, \tag{3.25}$$

where we have used the energy estimate

$$\|\mathbf{p}\|_{L^2_{\mathcal{F}}(\Omega;C^1([0,T];L^2(G)))}$$
$$\leqslant C\left(\|\mathbf{p}_0\|_{H^{(1)}_{G(0)}}+\|\mathbf{p}_1\|_{H^{(0)}_{G(0)}}+\|\mathbf{F}\|_{H^{(0)}_{G_T}}+\|\nabla q\|_{H^{(0)}_{G_T}}+\|\nabla\pi\|_{H^{(0)}_{G_T}}\right) \leqslant C(M). \tag{3.26}$$

Substituting (3.23)–(3.26) into (3.22), we obtain

$$\mathbb{E}\int_{\varpi_T^{(1)}} s^5\lambda^2\varphi^4|\nabla q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$\leqslant C\mathbb{E}\int_{G_T} s^2\lambda^2\varphi^2|\nabla q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t + C\mathbb{E}\int_{G_T}|\Delta q|^2\,\mathrm{e}^{2s\varphi}\,\mathrm{d}x\,\mathrm{d}t$$
$$+C(\lambda)s^{12}\,\mathrm{e}^{C(\lambda)s}\|\mathbf{p}\|^2_{X_{\varpi_T}} + C(\lambda)s^5\,\mathrm{e}^{C(\lambda)s}\|q(\cdot,T)\|_{H^{(1)}_{G(T)}}. \tag{3.27}$$

Finally, substituting (3.27) into (3.18) and taking s sufficiently large to absorb the first two terms on the right-hand side of (3.27), we obtain (1.7). This completes the proof of theorem 1.1. □

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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