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RC-positivity and scalar-flat metrics on ruled surfaces

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Abstract. Let X be a ruled surface over a curve of genus g. We prove that X has a scalar-flat Hermitian metric *if and only if* $g \geq 2$ and $m(X) > 2-2g$ where $m(X)$ is an intrinsic number depends on the complex structure of X.

Contents

1. Introduction

In his "Problem section", S.-T. Yau proposed the following classical problem ([Yau82, Problem 41]), which is investigated intensively in the last forty years.

Problem 1.1. Classify all compact Kähler surfaces with zero scalar curvature.

By the celebrated Calabi-Yau Theorem ([Yau78]), all Kähler surfaces with vanishing first Chern class (e.g. $K3$ surfaces) admit Kähler metrics with zero scalar curvature. Such metrics are usually called *scalar-flat* Kähler metrics and it is a special class of constant scalar curvature Kähler (cscK) metrics or extremal metrics. Obstructions to the existence of such metrics have been known since the pioneering works of S.-T. Yau [Yau74] and E. Calabi [Cal85]. For comprehensive discussions on this rich topic, we refer to [Yau74, Yau78, Fut83, BD88, Tian90, Sim91, Fuj92, LS93, LS94, Tian97, Don01, RS05, AP06, AT06, RT06, Ross06, CT08, Sto08, AP09, ACGT11, Sze14, Sze17] and the references therein.

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In this paper, we study the geometry of compact complex manifolds with scalar-flat Hermitian metrics (with respect to the Chern connection), which is a generalization of Problem 1.1. We begin with a characterization of compact complex manifolds with scalar-flat Hermitian metrics, which can be regarded as a Hermitian analogue of Kazdan-Warner-Bourguignon's classical work in Riemannian geometry, and we refer to [Bes86] and [Fut93] for more details.

Theorem 1.2. *A compact complex manifold X admits a scalar-flat Hermitian metric if and only if X is Chern Ricci-flat, or both K_X and K_X^{-1} are RC-positive.*

Recall that, a line bundle $\mathscr{L}$ is called *RC-positive* if it has a smooth Hermitian metric h such that its curvature $-\sqrt{-1}\partial\overline{\partial}\log h$ has at least one positive eigenvalue everywhere. By using a remarkable theorem in [TW10] established by Tosatti-Weinkove (which is a Hermitian analogue of Yau's theorem [Yau78]), the anti-canonical bundle K_X^{-1} is RC-positive if and only if X has a smooth Hermitian metric ω such that its Ricci curvature $\text{Ric}(\omega)$ has at least one positive eigenvalue everywhere. A complex manifold X is called *Chern Ricci-flat* if there exists a smooth Hermitian metric ω such that the Chern-Ricci curvature $\text{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial}\log\omega^n = 0$. On the other hand, we proved in [Yang17, Theorem 1.4] that a line bundle $\mathscr{L}$ is RC-positive if and only if its dual line bundle $\mathscr{L}^*$ is not pseudo-effective. By taking this advantage, we can verify the RC-positivity of K_X or K_X^{-1} by adapting methods in differential geometry as well as algebraic geometry.

As a straightforward application of Theorem 1.2, we obtain

Corollary 1.3. *Let X be a compact Kähler manifold. If X has a scalar-flat Kähler metric ω, then either X is a Calabi-Yau manifold, or both K_X and K_X^{-1} are RC-positive.*

For instance, if X is the blowing-up of $\mathbb{P}^2$ along m-points ($m \leq 9$), it is well-known that the anti-canonical bundle K_X^{-1} is effective (e.g. [Fri98, p. 125-p. 129]) and so it is pseudo-effective. In this case, K_X can not be RC-positive and X has no scalar-flat Hermitian (or Kähler) metrics.

Corollary 1.4. *Let $\mathbb{P}^2\#m\overline{\mathbb{P}^2}$ be the blowing-up of $\mathbb{P}^2$ along m points. If X admits a scalar-flat Hermitian metric, then $m \geq 10$.*

Indeed, it is proved by Rollin-Singer in [RS05, Theorem 1] (see also [Leb86, Leb91, LS93]) that: a complex surface X obtained by blowing-up $\mathbb{P}^2$ at 10 suitably chosen points admits a scalar-flat Kähler metric and any further blowing-up of X also admits a scalar-flat Käler metric.

A compact complex surface X is called a *ruled surface* if it is a holomorphic $\mathbb{P}^1$-bundle over a compact Riemann surface C. It is well known that any ruled surface X can be written as a projective bundle $\mathbb{P}(\mathscr{E})$ where $\mathscr{E}$ is a rank two vector bundle over C. Moreover, two ruled surfaces $\mathbb{P}(\mathscr{E})$ and $\mathbb{P}(\mathscr{E}')$ are isomorphic if and only if $\mathscr{E} \cong \mathscr{E}' \otimes \mathscr{L}$ for some line bundle $\mathscr{L}$ over C. The existence of cscK metrics on ruled surfaces are extensively studied, and we refer to [Yau74, BD88, Tian90, Sim91, Fuj92, LS93, LS94, RS05, AP06, AT06, RT06, Ross06, Sto08, ACGT11, Sze14] and the references therein. A remarkable result (e.g. [AT06, BD88, ACGT11]) asserts that: *A ruled surface $\mathbb{P}(\mathscr{E})$ admits a cscK metric if and only if $\mathscr{E}$ is poly-stable.*

In the following, we aim to classify ruled surfaces with scalar-flat Hermitian metrics. Let $\mathscr{E}$ be a rank two vector bundle over a smooth curve C. One can define a number $m(\mathscr{E})$ (e.g. [Fri98, p. 122]) which is equal to the *minimal degree* of $\mathscr{E} \otimes \mathscr{L}$ if there exists a **sheaf extension** of $\mathscr{E} \otimes \mathscr{L}$:

$$0 \to \mathcal{O}_C \to \mathscr{E} \otimes \mathscr{L} \to \mathscr{F} \to 0$$

for some line bundle $\mathscr{L}$. It is obvious that $m(\mathscr{E}) = m(\mathscr{E} \otimes \widetilde{\mathscr{L}})$ for any line bundle $\widetilde{\mathscr{L}}$. Hence, we can define an intrinsic number $m(X)$ for a ruled surface X: $m(X) = m(\mathscr{E})$ if X can be written as $\mathbb{P}(\mathscr{E})$. It is obvious that $m(X)$ is independent of the choices of $\mathscr{E}$. Let's explain the geometric meaning of $m(X)$ by the example $X = \mathbb{P}(\mathscr{L} \oplus \mathcal{O}_C) \to C$ where $\mathscr{L}$ is a line bundle. In this case, $m(X) = -|\deg(\mathscr{L})| \leq 0$. As another application of Theorem 1.2, we obtain

Theorem 1.5. *Let X be a ruled surface over a smooth curve C of genus g. Then X has a scalar-flat Hermitian metric if and only if $g \geq 2$ and $m(X) > 2 - 2g$.*

In particular, we have

Corollary 1.6. *Let $\mathscr{L} \to C$ be a line bundle over a smooth curve of genus g and $X = \mathbb{P}(\mathscr{L} \oplus \mathcal{O}_C)$. Then X has a scalar-flat Hermitian metric if and only if $g \geq 2$ and $|\deg(\mathscr{L})| < 2g - 2$.*

For instance, if C is a smooth curve of degree $d > 4$ in $\mathbb{P}^2$, then the genus of C is $g = \frac{1}{2}(d-1)(d-2)$ and the degree of $\mathcal{O}_C(1)$ is $d < 2g - 2$. Hence, $X = \mathbb{P}(\mathcal{O}_C(1) \oplus \mathcal{O}_C)$ has scalar-flat Hermitian metrics. Note also that, in Corollary 1.6, if $\deg(\mathscr{L}) = 0$, the vector bundle $\mathscr{L} \oplus \mathcal{O}_C$ is poly-stable and $X = \mathbb{P}(\mathscr{L} \oplus \mathcal{O}_C)$ admits scalar-flat Kähler metrics; however, when $0 < |\deg(\mathscr{L})| < 2g - 2$, it has no scalar-flat Kähler metrics. Moreover, we construct such examples in higher dimensional ruled manifolds.

Proposition 1.7. *Let C be a smooth curve with genus $g \geq 2$ and $\mathscr{L}$ be a line bundle over C. Suppose $\mathscr{E} = \mathscr{L} \oplus \mathcal{O}_C^{\oplus(n-1)}$ and $X = \mathbb{P}(\mathscr{E}^*) \to C$ is the projective bundle. If $0 < \deg(\mathscr{L}) < \frac{2g-2}{n-1}$, then $\mathbb{P}(\mathscr{E}^*)$ can not support scalar-flat Kähler metrics, but it does admit scalar-flat Hermitian metrics.*

As motivated by previous results, we propose the following question.

Question 1.8. Let X be a compact Kähler manifold. Suppose X has a scalar-flat Hermitian metric. Are there any geometric conditions on X which can guarantee the existence of scalar-flat Kähler metrics?

Finally, we classify minimal compact complex surfaces with scalar-flat Hermitian metrics.

Theorem 1.9. *Let X be a minimal compact complex surface. If X admits a scalar-flat Hermitian metric, then X must be one of the following*

(1) *an Enriques surface;*
(2) *a bi-elliptic surface;*
(3) *a $K3$ surface;*
(4) *a 2-torus;*
(5) *a Kodaira surface;*
(6) *a ruled surface X over a curve C of genus $g \geq 2$ and $m(X) > 2 - 2g$;*
(7) *a class VII_0 surface with $b_2 > 0$.*

Remark 1.10. It is proved that surfaces in (1) to (6) all have scalar-flat Hermitian metrics. On the other hand, since class VII_0 surfaces with $b_2 > 0$ are not completely classified, we do not prove each class VII_0 surface with $b_2 > 0$ can support scalar-flat Hermitian metrics. Non-minimal surfaces with scalar-flat Hermitian metrics will also be studied in the sequel.

The rest of the paper is organized as follows. In Section 3, we give a characterization of compact complex manifolds with scalar-flat Hermitian metrics and prove Theorem 1.2. In Section 5, we classify ruled surfaces with scalar-flat Hermitian metrics and establish Theorem 1.5. In Section 6, we classify minimal complex surfaces with scalar-flat Hermitian metrics and obtain Theorem 1.9. In Section 7, we give some precise examples with scalar-flat Hermitian metrics (Proposition 1.7).

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2. Background materials

2.1. **Scalar curvature and total scalar curvature on complex manifolds.** Let $(\mathscr{E}, h)$ be a Hermitian holomorphic vector bundle over a complex manifold X with Chern connection ∇. Let $\{z^i\}_{i=1}^n$ be the local holomorphic coordinates on X and $\{e_\alpha\}_{\alpha=1}^r$ be a local frame of $\mathscr{E}$. The curvature tensor $R^{\mathscr{E}} \in \Gamma(X, \Lambda^{1,1}T_X^* \otimes \mathrm{End}(\mathscr{E}))$ has components

$$R^{\mathscr{E}}_{i\bar{j}\alpha\bar{\beta}} = -\frac{\partial^2 h_{\alpha\bar{\beta}}}{\partial z^i \partial \bar{z}^j} + h^{\gamma\bar{\delta}} \frac{\partial h_{\alpha\bar{\delta}}}{\partial z^i} \frac{\partial h_{\gamma\bar{\beta}}}{\partial \bar{z}^j}. \tag{2.1}$$

(Here and henceforth we sometimes adopt the Einstein convention for summation.) If (X, ω_g) is a Hermitian manifold, then (T_X, g) has Chern curvature components

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z^i \partial \bar{z}^j} + g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z^i} \frac{\partial g_{p\bar{\ell}}}{\partial \bar{z}^j}. \tag{2.2}$$

The Chern-Ricci curvature $\mathrm{Ric}(\omega_g)$ of (X, ω_g) is represented by $R_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}$. The *(Chern) scalar curvature* s of (X, ω_g) is given by

$$s = \mathrm{tr}_{\omega_g} \mathrm{Ric}(\omega_g) = g^{i\bar{j}} R_{i\bar{j}}. \tag{2.3}$$

The *total (Chern) scalar curvature* of ω_g is

$$\int_X s\omega_g^n = n \int \mathrm{Ric}(\omega_g) \wedge \omega_g^{n-1}, \tag{2.4}$$

where n is the complex dimension of X.

(1) A Hermitian metric ω_g is called a Gauduchon metric if $\partial\bar{\partial}\omega_g^{n-1} = 0$. It is proved by Gauduchon ([Gau77]) that, in the conformal class of each Hermitian metric, there exists a unique Gauduchon metric (up to constant scaling).

(2) A projective manifold X is called uniruled if it is covered by rational curves.

2.2. **Positivity of line bundles.** Let (X, ω_g) be a compact Hermitian manifold, and $\mathscr{L} \to X$ be a holomorphic line bundle.

(1) $\mathscr{L}$ is said to be *positive* (resp. *semi-positive*) if there exists a smooth Hermitian metric h on $\mathscr{L}$ such that the curvature form $R^{\mathscr{L}} = -\sqrt{-1}\partial\bar{\partial}\log h$ is a positive (resp. semi-positive) $(1,1)$-form.

(2) $\mathscr{L}$ is said to be *nef*, if for any $\varepsilon > 0$, there exists a smooth Hermitian metric h_ε on $\mathscr{L}$ such that $-\sqrt{-1}\partial\overline{\partial}\log h_\varepsilon \geq -\varepsilon\omega_g$.

(3) $\mathscr{L}$ is said to be *pseudo-effective*, if there exists a (possibly) singular Hermitian metric h on $\mathscr{L}$ such that $-\sqrt{-1}\partial\overline{\partial}\log h \geq 0$ in the sense of distributions. (See [Dem] for more details.)

(4) $\mathscr{L}$ is said to be $\mathbb{Q}$-*effective*, if there exists some positive integer m such that $H^0(X, \mathscr{L}^{\otimes m}) \neq 0$.

(5) $\mathscr{L}$ is called *unitary flat* if there exists a smooth Hermitian metric h on $\mathscr{L}$ such that the curvature of $(\mathscr{L}, h)$ is zero, i.e. $-\sqrt{-1}\partial\overline{\partial}\log h = 0$.

(6) The Kodaira dimension $\kappa(\mathscr{L})$ of $\mathscr{L}$ is defined to be

$$\kappa(\mathscr{L}) := \limsup_{m\to+\infty} \frac{\log\dim_{\mathbb{C}} H^0(X, \mathscr{L}^{\otimes m})}{\log m}$$

and the *Kodaira dimension* $\kappa(X)$ of X is defined as $\kappa(X) := \kappa(K_X)$ where the logarithm of zero is defined to be $-\infty$.

2.3. **Positivity of vector bundles.** The points of the projective bundle $\mathbb{P}(\mathscr{E}^*)$ of $\mathscr{E} \to X$ can be identified with the hyperplanes of $\mathscr{E}$. Note that every hyperplane $\mathscr{V}$ in $\mathscr{E}_z$ corresponds bijectively to the line of linear forms in $\mathscr{E}_z^*$ which vanish on $\mathscr{V}$. Let $\pi : \mathbb{P}(\mathscr{E}^*) \to X$ be the natural projection. There is a tautological hyperplane subbundle $\mathscr{S}$ of $\pi^*\mathscr{E}$ such that $\mathscr{S}_{[\xi]} = \xi^{-1}(0) \subset \mathscr{E}_z$ for all $\xi \in \mathscr{E}_z^* \setminus \{0\}$. The quotient line bundle $\pi^*\mathscr{E}/\mathscr{S}$ is denoted $\mathcal{O}_{\mathscr{E}}(1)$ and is called the *tautological line bundle* associated to $\mathscr{E} \to X$. Hence there is an exact sequence of vector bundles over $\mathbb{P}(\mathscr{E}^*)$, $0 \to \mathscr{S} \to \pi^*\mathscr{E} \to \mathcal{O}_{\mathscr{E}}(1) \to 0$. A holomorphic vector bundle $\mathscr{E} \to X$ is called *ample* (resp. *nef*) if the line bundle $\mathcal{O}_{\mathscr{E}}(1)$ is ample (resp. nef) over $\mathbb{P}(\mathscr{E}^*)$. (**Caution**: In general, $\mathbb{P}(\mathscr{E})$ and $\mathbb{P}(\mathscr{E}^*)$ are not isomorphic! $\mathcal{O}_{\mathscr{E}}(1)$ is the tautological line bundle of $\mathbb{P}(\mathscr{E}^*)$, and $\mathcal{O}_{\mathscr{E}^*}(1)$ is the tautological line bundle of $\mathbb{P}(\mathscr{E})$.) A Hermitian holomorphic vector bundle $(\mathscr{E}, h)$ over a complex manifold X is called *Griffiths positive* if at each point $q \in X$ and for any nonzero vector $v \in \mathscr{E}_q$, and any nonzero vector $u \in T_qX$, $R^{\mathscr{E}}(u, \overline{u}, v, \overline{v}) > 0$.

2.4. **RC-positive line bundles.** Let's recall that

Definition 2.1. A line bundle $\mathscr{L}$ is called *RC-positive* if it has a smooth Hermitian metric h such that its curvature $R^{(\mathscr{L},h)} = -\sqrt{-1}\partial\overline{\partial}\log h$ has at least one positive eigenvalue everywhere.

In [Yang17, Theorem 1.4], we obtained an equivalent characterization for RC-positive line bundles.

Theorem 2.2. *Let $\mathscr{L}$ be a holomorphic line bundle over a compact complex manifold X. The following statements are equivalent.*

(1) *$\mathscr{L}$ is RC-positive;*
(2) *the dual line bundle $\mathscr{L}^*$ is not pseudo-effective.*

Hence, we obtain

Corollary 2.3. *A line bundle $\mathscr{L}$ is unitary flat if and only if neither $\mathscr{L}$ nor $\mathscr{L}^*$ is RC-positive.*

Proof. It is easy to see that $\mathscr{L}$ is unitary flat if and only if both $\mathscr{L}$ and $\mathscr{L}^*$ are pseudo-effective (e.g. [Yang17a, Theorem 3.4]). Hence, Corollary 2.3 follows from Theorem 2.2. ☐

By using Theorem 2.2, the classical result of [BDPP13, Theorem] and Yau's theorem [Yau78], we obtain in [Yang17, Corollary 1.9] that

Theorem 2.4. *A projective manifold X is uniruled if and only if K_X^{-1} is RC-positive, i.e. X has a smooth Hermitian metric ω such that the Ricci curvature* $\mathrm{Ric}(\omega)$ *has at least one positive eigenvalue everywhere.*

3. Characterizations of complex manifolds with scalar-flat metrics

In this section, we shall prove Theorem 1.2. Let ω be a smooth Hermitian metric on a compact complex manifold X. For simplicity, we denote by $\mathscr{F}(\omega)$ the total (Chern) scalar curvature of ω, i.e.

$$\mathscr{F}(\omega) = \int_X s\omega^n = n\int_X \mathrm{Ric}(\omega)\wedge\omega^{n-1}.$$

Note that, when X is not Kähler, the total scalar curvature differs from the total scalar curvature of the Levi-Civita connection of the underlying Riemannian metric (e.g. [LY17]). Let $\mathscr{W}$ be the space of smooth Gauduchon metrics on X. We obtained in [Yang17a, Theorem 1.1] a complete characterization on the image of the total scalar curvature function $\mathscr{F}:\mathscr{W}\to\mathbb{R}$ following [Gau77, Mi82, La99] (see also some special cases in [Tel06, Gau77, HW12]). By Theorem 2.2, we obtain the following result.

Theorem 3.1. *The image of the total scalar function $\mathscr{F}:\mathscr{W}\to\mathbb{R}$ has exactly four different cases:*

(1) *$\mathscr{F}(\mathscr{W})=\mathbb{R}$ if and only if both K_X and K_X^{-1} are RC-positive;*

(2) $\mathscr{F}(\mathscr{W}) = \mathbb{R}^{>0}$ *if and only if* K_X^{-1} *is RC-positive but* K_X *is not RC-positive;*
(3) $\mathscr{F}(\mathscr{W}) = \mathbb{R}^{<0}$ *if and only if* K_X *is RC-positive but* K_X^{-1} *is not RC-positive;*
(4) $\mathscr{F}(\mathscr{W}) = \{0\}$ *if and only if* X *is Ricci-flat; or equivalently, neither* K_X *nor* K_X^{-1} *is RC-positive.*

Proof. We obtained in [Yang17a, Theorem 1.1] that the image of the total scalar function $\mathscr{F} : \mathscr{W} \to \mathbb{R}$ has exactly four different cases:

(1) $\mathscr{F}(\mathscr{W}) = \mathbb{R}$, if and only if neither K_X nor K_X^{-1} is pseudo-effective;
(2) $\mathscr{F}(\mathscr{W}) = \mathbb{R}^{>0}$, if and only if K_X^{-1} is pseudo-effective but not unitary flat;
(3) $\mathscr{F}(\mathscr{W}) = \mathbb{R}^{<0}$, if and only if K_X is pseudo-effective but not unitary flat;
(4) $\mathscr{F}(\mathscr{W}) = \{0\}$, if and only if K_X is unitary flat.

By [TW10, Corollary 2], K_X is unitary flat if and only if X is Ricci-flat, i.e. there exists a Hermitian metric ω on X such that $\mathrm{Ric}(\omega) = 0$. Hence Theorem 3.1 follows from Theorem 2.2 and Corollary 2.3. □

Remark 3.2. It is easy to see that Theorem 3.1 also holds for Bott-Chern classes ([Yang17a, Theorem 3.4])

As an application of Theorem 3.1, we establish Theorem 1.2, that is,

Theorem 3.3. *Let* X *be a compact complex manifold. Then* X *admits a scalar-flat Hermitian metric if and only if* X *is Ricci-flat, or both* K_X *and* K_X^{-1} *are RC-positive.*

Proof. If X has a scalar-flat Hermitian metric ω, in the conformal class of ω, there exists a Gauduchon metric $\omega_f = e^f\omega$. Then the total scalar curvature s_f of the Gauduchon metric ω_f is

$$s_f = n\int_X \mathrm{Ric}(\omega_f) \wedge \omega_f^{n-1} = n\int_X \left(\mathrm{Ric}(\omega) - n\sqrt{-1}\partial\overline{\partial} f\right) \wedge \omega_f^{n-1}. \tag{3.1}$$

Since ω_f is Gauduchon, i.e. $\partial\overline{\partial}\omega_f^{n-1} = 0$, an integration by part yields

$$\begin{aligned} s_f &= n\int_X \mathrm{Ric}(\omega) \wedge \omega_f^{n-1} \\ &= n\int_X \mathrm{Ric}(\omega) \wedge e^{(n-1)f}\omega^{n-1} \\ &= \int_X e^{(n-1)f} \cdot \mathrm{tr}_\omega \mathrm{Ric}(\omega) \cdot \omega^n. \end{aligned}$$

Since ω has zero scalar curvature, i.e. $\mathrm{tr}_\omega \mathrm{Ric}(\omega) = 0$, we deduce that the total scalar curvature s_f of the Gauduchon metric ω_f is zero. By Theorem 3.1, we conclude that either X is Ricci-flat, or both K_X and K_X^{-1} are RC-positive.

On the other hand, suppose either X is Ricci-flat, or both K_X and K_X^{-1} are RC-positive, by Theorem 3.1 again, we know X has a Gauduchon metric ω_G with zero total scalar curvature. By a conformal perturbation method, it is easy to see that there exists a Hermitian metric ω with zero scalar curvature (e.g. [Yang17a, Lemma 3.2]). Indeed, let s_G be the scalar curvature of ω_G. It is well-known (e.g.[Gau77] or [CTW16, Theorem 2.2]) that the following equation

$$s_G - \mathrm{tr}_{\omega_G} \sqrt{-1}\partial\bar{\partial} f = 0 \tag{3.2}$$

has a solution $f \in C^\infty(X)$ since ω_G is Gauduchon and its total scalar curvature $\int_X s_G \omega_G^n$ is zero. Let $\omega = e^{\frac{f}{n}} \omega_G$. Then the scalar curvature s of ω is,

$$\begin{aligned} s &= \mathrm{tr}_\omega \mathrm{Ric}(\omega) = -\mathrm{tr}_\omega \sqrt{-1}\partial\bar{\partial} \log(\omega^n) \\ &= -e^{-\frac{f}{n}} \mathrm{tr}_{\omega_G} \sqrt{-1}\partial\bar{\partial} \log(e^f \omega_G^n) \\ &= -e^{-\frac{f}{n}} \left(s_G - \mathrm{tr}_{\omega_G} \sqrt{-1}\partial\bar{\partial} f \right) \\ &= 0. \end{aligned}$$

The proof of Theorem 1.2 is completed. □

The proof of Corollary 1.3. It is a special case of Theorem 1.2 since Kähler manifolds with unitary flat K_X are Kähler Calabi-Yau. □

Corollary 3.4. *Let X be a compact Kähler manifold. Suppose X has a scalar-flat Hermitian metric, or a Gauduchon metric with zero total scalar curvature. If K_X or K_X^{-1} is pseudo-effective, then X is a Kähler Calabi-Yau manifold.*

4. Projective bundles with scalar-flat metrics

In this section, we prove the following result.

Theorem 4.1. *Let $\mathscr{E}$ be a nef vector bundle of rank $r \geq 2$ over a smooth curve C with genus $g \geq 2$ and $X = \mathbb{P}(\mathscr{E})$. If $0 \leq \deg(\mathscr{E}) < 2g - 2$, then both K_X and K_X^{-1} are RC-positive. In particular, X has scalar-flat Hermitian metrics.*

Let's recall some elementary settings. Suppose $\dim_{\mathbb{C}} Y = n$ and $r = \mathrm{rank}(\mathscr{E})$. Let π be the projection $\mathbb{P}(\mathscr{E}^*) \to Y$ and $\mathscr{L} = \mathcal{O}_{\mathscr{E}}(1)$. Let $(e_1, \cdots, e_r)$ be the local holomorphic frame on $\mathscr{E}$ and the dual frame on $\mathscr{E}^*$ is denoted by $(e^1, \cdots, e^r)$. The corresponding holomorphic coordinates on $\mathscr{E}^*$ are denoted by

$(W_1, \cdots, W_r)$. If $(h_{\alpha\overline{\beta}})$ is the matrix representation of a smooth metric $h^{\mathscr{E}}$ on $\mathscr{E}$ with respect to the basis $\{e_\alpha\}_{\alpha=1}^r$, then the induced Hermitian metric $h^{\mathscr{L}}$ on $\mathscr{L}$ can be written as $h^{\mathscr{L}} = \frac{1}{\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta}$. The curvature of $(\mathscr{L}, h^{\mathscr{L}})$ is

$$R^{\mathscr{L}} = \sqrt{-1}\partial\overline{\partial}\log\left(\sum h^{\alpha\overline{\beta}} W_\alpha \overline{W}_\beta\right) \tag{4.1}$$

where ∂ and $\overline{\partial}$ are operators on the total space $\mathbb{P}(\mathscr{E}^*)$. We fix a point $p \in \mathbb{P}(\mathscr{E}^*)$, then there exist local holomorphic coordinates $(z^1, \cdots, z^n)$ centered at point $q = \pi(p) \in Y$ and local holomorphic basis $\{e_1, \cdots, e_r\}$ of $\mathscr{E}$ around q such that

$$h_{\alpha\overline{\beta}} = \delta_{\alpha\overline{\beta}} - R^{\mathscr{E}}_{i\overline{j}\alpha\overline{\beta}} z^i \overline{z}^j + O(|z|^3) \tag{4.2}$$

Without loss of generality, we assume p is the point $(0, \cdots, 0, [a_1, \cdots, a_r])$ with $a_r = 1$. On the chart $U = \{W_r = 1\}$ of the fiber $\mathbb{P}^{r-1}$, we set $w^A = W_A$ for $A = 1, \cdots, r-1$. By formula (4.1) and (4.2)

$$R^{\mathscr{L}}(p) = \sqrt{-1}\sum R^{\mathscr{E}}_{i\overline{j}\alpha\overline{\beta}} \frac{a_\beta \overline{a}_\alpha}{|a|^2} dz^i \wedge d\overline{z}^j + \omega_{\mathrm{FS}} \tag{4.3}$$

where $|a|^2 = \sum\limits_{\alpha=1}^{r} |a_\alpha|^2$ and $\omega_{\mathrm{FS}} = \sqrt{-1} \sum\limits_{A,B=1}^{r-1} \left(\frac{\delta_{AB}}{|a|^2} - \frac{a_B \overline{a}_A}{|a|^4}\right) dw^A \wedge d\overline{w}^B$ is the Fubini-Study metric on the fiber $\mathbb{P}^{r-1}$.

Lemma 4.2. *If $\mathscr{E}$ is Griffiths-positive, then $\mathcal{O}_{\mathscr{E}^*}(-1)$ is RC-positive.*

Proof. It follows from formula (4.3). Indeed, by (4.3), the induced metric on $\mathcal{O}_{\mathscr{E}^*}(-1)$ over $\mathbb{P}(\mathscr{E})$ has curvature form

$$R^{\mathcal{O}_{\mathscr{E}^*}(-1)} = -\left(\sqrt{-1}\sum R^{\mathscr{E}^*}_{i\overline{j}\alpha\overline{\beta}} \frac{a_\beta \overline{a}_\alpha}{|a|^2} dz^i \wedge d\overline{z}^j + \omega_{\mathrm{FS}}\right).$$

On the other hand, $R^{\mathscr{E}^*} = -\left(R^{\mathscr{E}}\right)^t$ and so

$$R^{\mathcal{O}_{\mathscr{E}^*}(-1)} = \sqrt{-1}\sum R^{\mathscr{E}}_{i\overline{j}\alpha\overline{\beta}} \frac{a_\alpha \overline{a}_\beta}{|a|^2} dz^i \wedge d\overline{z}^j - \omega_{\mathrm{FS}}.$$

Hence, $\mathcal{O}_{\mathscr{E}^*}(-1)$ is RC-positive if $(\mathscr{E}, h^{\mathscr{E}})$ is Griffiths-positive. □

Lemma 4.3. *If $\mathscr{E}$ is a nef vector bundle over a smooth curve C. Then for any ample line bundle $\mathscr{A}$ over C and any $k \geq 0$, $\mathcal{O}_{\mathscr{E}^*}(-k) \otimes \pi^*(\mathscr{A})$ is RC-positive.*

Proof. It is easy to see that $\mathrm{Sym}^{\otimes k}\mathscr{E} \otimes \mathscr{A}$ is an ample vector bundle over C. By [CF90], $\mathrm{Sym}^{\otimes k}\mathscr{E} \otimes \mathscr{A}$ has a smooth Griffiths-positive metric. In particular, by Lemma 4.2, the dual tautological line bundle

$$\mathcal{O}_{\mathrm{Sym}^{\otimes k}\mathscr{E}^* \otimes \mathscr{A}^*}(-1) \tag{4.4}$$

is RC-positive. More precisely, the base curve C direction is a positive direction of the curvature tensor of $\mathcal{O}_{\mathrm{Sym}^{\otimes k}\mathscr{E}^*\otimes\mathscr{A}^*}(-1)$. On the other hand, we have the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{O}_{\mathscr{E}^*}(-k)\otimes\pi^*(\mathscr{A}) & \longrightarrow & \mathcal{O}_{\mathrm{Sym}^{\otimes k}\mathscr{E}^*}(-1)\otimes\pi_k^*(\mathscr{A}) & \longrightarrow & \mathcal{O}_{\mathrm{Sym}^{\otimes k}\mathscr{E}^*\otimes\mathscr{A}^*}(-1) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{P}(\mathscr{E}) & \xrightarrow{\nu_k} & \mathbb{P}(\mathrm{Sym}^{\otimes k}\mathscr{E}) & \xrightarrow{i} & \mathbb{P}(\mathrm{Sym}^{\otimes k}\mathscr{E}\otimes\mathscr{A}) \\
{\scriptstyle\pi}\downarrow & & {\scriptstyle\pi_k}\downarrow & & \downarrow \\
C & \xrightarrow{f} & C & \xrightarrow{f} & C,
\end{array}$$

where $\nu_k : \mathscr{E} \to \mathrm{Sym}^{\otimes \mathrm{k}}\mathscr{E}$ is the k-th Veronese map, $f = \mathrm{Identity}$ and i is an isomorphism. It is easy to see that $\mathcal{O}_{\mathscr{E}^*}(-k)\otimes\pi^*(\mathscr{A})$ is RC-positive, i.e., the induced curvature has a positive direction along the base C direction. □

The proof of Theorem 4.1. By using the projection formula on $X = \mathbb{P}(\mathscr{E})$,

$$K_X = \mathcal{O}_{\mathscr{E}^*}(-n)\otimes\pi^*(K_C\otimes\det\mathscr{E}^*),$$

where $\pi : X \to C$ is the projection. If $\deg(\mathscr{E}) < 2g-2 = \deg(K_C)$, then $\deg(K_C\otimes\det\mathscr{E}^*) > 0$ and so $K_C\otimes\det\mathscr{E}^*$ is ample. By Lemma 4.3, K_X is RC-positive. On the other hand, by Theorem 2.4, it is easy to see that K_X^{-1} is RC-positive. Hence, by Theorem 1.2, X has scalar-flat Hermitian metrics. □

5. Classification of ruled surfaces with scalar-flat Hermitian metrics

In this section, we classify ruled surfaces with scalar-flat Hermitian metrics and prove Theorem 1.5. It is well-known that any ruled surface X can be written as a projective bundle $\mathbb{P}(\mathscr{E})$ where $\mathscr{E}$ is a rank two vector bundle over a smooth curve C with genus g. Moreover, two ruled surfaces $\mathbb{P}(\mathscr{E})$ and $\mathbb{P}(\mathscr{E}')$ are isomorphic if and only if $\mathscr{E}\cong\mathscr{E}'\otimes\mathscr{L}$ for some line bundle $\mathscr{L}$ over C. Since $\mathscr{E}$ has rank two and $X\cong\mathbb{P}(\mathscr{E})\cong\mathbb{P}(\mathscr{E}^*)$, we shall use projection formulas

$$K_X = \mathcal{O}_{\mathscr{E}}(-2)\otimes\pi^*(K_C\otimes\det\mathscr{E}),\quad \pi : \mathbb{P}(\mathscr{E}^*)\to C$$

and

$$K_X = \mathcal{O}_{\mathscr{E}^*}(-2)\otimes\pi^*(K_C\otimes\det\mathscr{E}^*),\quad \pi : \mathbb{P}(\mathscr{E})\to C$$

alternatively.

When $g = 0$, $C\cong\mathbb{P}^1$ and each rank two vector bundle can be written as $\mathscr{E} = \mathcal{O}_{\mathbb{P}^1}(a)\oplus\mathcal{O}_{\mathbb{P}^1}(b)$. We can write a ruled surface over $\mathbb{P}^1$ as $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-k)\oplus\mathcal{O}_{\mathbb{P}^1})$.

Proposition 5.1. *Let $X = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-k) \oplus \mathcal{O}_{\mathbb{P}^1})$ be a Hirzebruch surface. Then the anti-canonical line bundle K_X^{-1} is effective and X has no scalar-flat Hermitian metrics.*

Proof. Let $\mathscr{E} = \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}$ and $X = \mathbb{P}(\mathscr{E}^*)$. We have $K_X^{-1} = \mathcal{O}_{\mathscr{E}}(2) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(2-k))$. By the direct image formula (e.g. [Laz04, p.90]), we have

$$\begin{aligned} H^0(X, K_X^{-1}) &= H^0(X, \mathcal{O}_{\mathscr{E}}(2) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(2-k)) \\ &= H^0(\mathbb{P}^1, \mathrm{Sym}^{\otimes 2}\mathscr{E} \otimes \mathcal{O}_{\mathbb{P}^1}(2-k)) \\ &= H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(k+2) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(2-k)) \\ &\neq 0 \end{aligned}$$

for any k. Therefore, K_X^{-1} is effective and K_X is not RC-positive. By Theorem 1.2, X has no scalar-flat Hermitian metrics. □

Theorem 5.2. *Let $X = \mathbb{P}(\mathscr{E}^*) \to C$ be a projective bundle over an elliptic curve C where $\mathscr{E} \to C$ is a rank two vector bundle. Then the K_X is not RC-positive and X has no scalar-flat Hermitian metrics.*

Proof. We divide the proof into three different cases.

Case 1. Suppose $\mathscr{E}$ is indecomposable and $\deg \mathscr{E} = 0$. A well-known result of Atiyah asserts that an indecomposable vector bundle over an elliptic curve is semi-stable and so $\mathscr{E}$ is semi-stable (e.g. [Tu93, Appendix A]). On the other hand, a semi-stable vector bundle over a curve is nef if $\deg(\mathscr{E}) \geq 0$ (e.g. [Laz04, Theorem 6.4.15]). Hence $\mathscr{E}$ is nef. By using the projection formula,

$$K_X^{-1} = \mathcal{O}_{\mathscr{E}}(2) \otimes \pi^*(K_C^{-1} \otimes \det \mathscr{E}^*) = \mathcal{O}_{\mathscr{E}}(2) \otimes \pi^*(\det \mathscr{E}^*) \tag{5.1}$$

we deduce K_X^{-1} is nef.

Case 2. Suppose $\mathscr{E}$ is indecomposable and $\deg(\mathscr{E}) \neq 0$. There exists an étale base change $f : C' \to C$ of degree k where k is an integer such that $2|k$, and C' is also an elliptic curve. Suppose $X' = \mathbb{P}(f^*\mathscr{E}^*)$, then we have the commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{f'} & X \\ {\scriptstyle \pi'}\big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ C' & \xrightarrow{f} & C. \end{array}$$

Let ℓ be an integer defined as

$$\ell = \frac{\deg(f^*\mathscr{E})}{2} = \frac{k \deg(\mathscr{E})}{2}, \tag{5.2}$$

and $\mathscr{F}$ be a line bundle over Y such that $\deg(\mathscr{F}) = -\ell$. Now we set

$$\widetilde{\mathscr{E}} = f^*\mathscr{E} \otimes \mathscr{F},$$

then $\deg(\widetilde{\mathscr{E}}) = 0$. Since $\mathscr{E}$ is indecomposable, it is semi-stable. Therefore $f^*\mathscr{E}$ is semi-stable (e.g. [Laz04, Lemma 6.4.12]) and so $\widetilde{\mathscr{E}}$ is semi-stable. Therefore, $\widetilde{\mathscr{E}}$ is nef since $\deg(\widetilde{\mathscr{E}}) = 0$. By projection formula again, we have

$$K_{X'}^{-1} = \mathcal{O}_{\widetilde{\mathscr{E}}}(2) \otimes \pi^*(\det \widetilde{\mathscr{E}}).$$

We deduce $K_{X'}^{-1}$ is nef. Hence K_X^{-1} is nef.

Case 3. If $\mathscr{E}$ is decomposable, then there exits a line bundle $\mathscr{L}$ such that

$$\mathscr{E} = \mathscr{L} \oplus (\mathscr{L}^{-1} \otimes \det \mathscr{E}).$$

By the projection formula (5.1) again, we have

$$\begin{aligned} H^0(X, K_X^{-1}) &= H^0(X, \mathcal{O}_{\mathscr{E}}(2) \otimes \pi^*(\det \mathscr{E}^*)) \cong H^0(C, \mathrm{Sym}^{\otimes 2}\mathscr{E} \otimes \det \mathscr{E}^*)) \\ &= H^0\left(C, \left(\mathscr{L}^2 \otimes \det \mathscr{E}^*\right) \oplus \mathcal{O}_C \oplus \left(\mathscr{L}^{-2} \otimes \det \mathscr{E}\right)\right) \\ &\neq 0 \end{aligned}$$

So K_X^{-1} is effective.

In summary, we conclude that the anti-canonical line bundle K_X^{-1} is pseudo-effective, i.e. K_X is not RC-positive. By Theorem 1.2, X has no scalar-flat Hermitian metrics. □

Finally, we deal with ruled surfaces over curves of genus $g \geq 2$. For a rank two vector bundle $\mathscr{E}$ over a curve C, in general, it is not clear whether $\mathscr{E}$ has an extension by $\mathcal{O}_C$:

$$0 \to \mathcal{O}_C \to \mathscr{E} \to \mathscr{F} \to 0 \tag{5.3}$$

where $\mathscr{F}$ is a coherent sheaf over C. However, one can obtain such an extension for $\mathscr{E} \otimes \mathscr{L}$ where $\mathscr{L}$ is some suitable line bundle. This enables us to make the following definition (see [Fri98, p.121-p.124] for more details).

Definition 5.3. Let $\mathscr{E}$ be a rank two vector bundle over a smooth curve C. The number $m(\mathscr{E})$ is defined to be the minimal degree of $\mathscr{E} \otimes \mathscr{L}$ where there exists a sheaf extension of $\mathscr{E} \otimes \mathscr{L}$:

$$0 \to \mathcal{O}_C \to \mathscr{E} \otimes \mathscr{L} \to \mathscr{F} \to 0 \tag{5.4}$$

for some line bundle $\mathscr{L}$ over C.

It is easy to see that for a sufficiently ample line bundle $\mathscr{L}$, $H^0(C, \mathscr{E} \otimes \mathscr{L}) \neq 0$ and a global section of $\mathscr{E} \otimes \mathscr{L}$ gives an extension (5.4). Hence, $m(\mathscr{E})$ is well-defined. It is obvious that $m(\mathscr{E}) = m(\mathscr{E} \otimes \widetilde{\mathscr{L}})$ for any line bundle $\widetilde{\mathscr{L}}$. Nagata proved in [Nag70, Theorem 1] (see also [Fri98, p. 123]) that

Theorem 5.4. *$m(\mathscr{E}) \leq g$.*

(Note that, in [Fri98, p. 123], the notion $e(\mathscr{E})$ is exactly $-m(\mathscr{E})$.)

As we pointed out before, any ruled surface X can be written as a projective bundle $\mathbb{P}(\mathscr{E})$ and two ruled surfaces $\mathbb{P}(\mathscr{E})$ and $\mathbb{P}(\mathscr{E}')$ are isomorphic if and only if $\mathscr{E} \cong \mathscr{E}' \otimes \mathscr{L}$ for some line bundle $\mathscr{L}$, then we can define $m(X)$ by $m(\mathscr{E})$ for any ruled surface $X = \mathbb{P}(\mathscr{E})$.

One can see that the definition of $m(\mathscr{E})$ is related to stability of coherent sheaves. If $m(\mathscr{E}) > 0$, then $\mathscr{E}$ is stable. Indeed, for any rank one sub-sheaf $\mathscr{L}$ of $\mathscr{E}$, we have the short exact sequence:

$$0 \to \mathscr{L} \to \mathscr{E} \to \mathscr{F} \to 0.$$

Since $\mathscr{E}$ is torsion free, $\mathscr{L}$ is torsion free and we know $\mathscr{L}$ is a line bundle. Therefore,

$$0 \to \mathcal{O}_C \to \mathscr{E} \otimes \mathscr{L}^{-1} \to \mathscr{F} \otimes \mathscr{L}^{-1} \to 0.$$

By the definition of $m(\mathscr{E})$, we have $\deg(\mathscr{E} \otimes \mathscr{L}^{-1}) \geq m(\mathscr{E}) > 0$ which is equivalent to $\deg \mathscr{L} < \frac{\deg \mathscr{E}}{2}$. This implies $\mathscr{E}$ is stable. Conversely, if $\mathscr{E}$ is stable, by a similar argument, we can conclude $m(\mathscr{E}) > 0$. Hence, we obtain a fact pointed out in [Fri98, Proposition 12, p. 123].

Proposition 5.5. *If $\mathscr{E}$ is a rank two vector bundle over a Riemann surface C, then $\mathscr{E}$ is stable if and only if $m(\mathscr{E}) > 0$.*

The proof of Theorem 1.5. Let X be a ruled surface which can support scalar-flat Hermitian metrics. We can write $X = \mathbb{P}(\mathscr{E}_o)$ for some rank 2 vector bundle $\mathscr{E}_o$ over a smooth curve C. Note that, since $\mathscr{E}_o$ has rank 2, $\mathscr{E}_o \cong \mathscr{E}_o^* \otimes \det \mathscr{E}_o$ and so $X \cong \mathbb{P}(\mathscr{E}_o) \cong \mathbb{P}(\mathscr{E}_o^*)$. By Proposition 5.1 and Theorem 5.2, we know the genus $g(C) \geq 2$. On the other hand, by the above discussion, we can write $X = \mathbb{P}(\mathscr{E})$ where $\deg(\mathscr{E}) = m(X)$ and $\mathscr{E}$ has an extension

$$0 \to \mathcal{O}_C \to \mathscr{E} \to \mathscr{F} \to 0. \tag{5.5}$$

Hence, $\deg(\mathscr{E}) = \deg(\mathscr{F}) = m(X)$.

(1). If $m(X) = \deg \mathscr{F} \leq 2-2g$, $X \cong \mathbb{P}(\mathscr{E}^*) \cong \mathbb{P}(\mathscr{E})$ has no scalar-flat Hermitian metrics. Indeed, we consider $X = \mathbb{P}(\mathscr{E}^*)$. By the exact sequence (5.5), we have

$$0 \to H^0(C, \mathcal{O}_C) \to H^0(C, \mathscr{E}) \to \cdots .$$

Therefore, $H^0(C, \mathscr{E}) \neq 0$. By the Le Potier isomorphism ([LeP75]), we have

$$H^0(\mathbb{P}(\mathscr{E}^*), \mathcal{O}_{\mathscr{E}}(1)) \cong H^0(C, \mathscr{E}) \neq 0.$$

Hence, $\mathcal{O}_{\mathscr{E}}(1)$ is effective and so it is pseudo-effective. On the other hand, since $\deg(\mathscr{E}) \leq 2-2g = -\deg(K_C)$, we deduce $K_C^{-1} \otimes \det \mathscr{E}^*$ is semi-positive. By the projection formula $K_X^{-1} = \mathcal{O}_{\mathscr{E}}(2) \otimes \pi^*(K_C^{-1} \otimes \det \mathscr{E}^*)$, we know K_X^{-1} is pseudo-effective. By Theorem 2.2, K_X is not RC-positive. By Theorem 1.2, X has no scalar-flat Hermitian metrics.

(2). If $2-2g < m(X) = \deg(\mathscr{E}) = \deg(\mathscr{F}) \leq 0$, we know $0 \leq \deg(\mathscr{E}^*) < 2g-2$. Since $\mathcal{O}_C$ and $\mathscr{F}^*$ are nef, by the dual exact sequence of (5.5),

$$0 \to \mathscr{F}^* \to \mathscr{E}^* \to \mathcal{O}_C \to 0,$$

we deduce $\mathscr{E}^*$ is nef with $0 \leq \deg(\mathscr{E}^*) < 2g-2$. By Theorem 4.1, $X \cong \mathbb{P}(\mathscr{E}^*)$ can support scalar-flat Hermitian metrics.

(3). If $0 < m(X) = \deg(\mathscr{E}) = \deg(\mathscr{F}) < 2g-2$, by the exact sequence (5.5), $\mathscr{E}$ is nef with $0 < \deg(\mathscr{E}) < 2g-2$. By Theorem 4.1, $X \cong \mathbb{P}(\mathscr{E})$ admits scalar-flat Hermitian metrics. Note that $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}^*)$.

(4). Suppose $m(X) \geq 2g-2$. By Theorem 5.4, $m(X) \leq g$. Hence, in this case, we have $g = 2$ and $m(X) = \deg(\mathscr{E}) = 2$. We work on $X = \mathbb{P}(\mathscr{E})$. By Proposition 5.5, $\mathscr{E}$ is a stable vector bundle and $\deg(\mathscr{E}) = 2$. By ([Laz04, Theorem 6.4.15]), we know $\mathscr{E}$ is an ample vector bundle over a smooth curve. According to [CF90], $\mathscr{E}$ has a smooth Griffiths-positive metric. By using Lemma 4.2, $\mathcal{O}_{\mathscr{E}^*}(-1)$ is RC-positive. By the projection formula again, we have

$$K_X = \mathcal{O}_{\mathscr{E}^*}(-2) \otimes \pi^*(K_C \otimes \det \mathscr{E}^*).$$

Since $\deg(K_C) = \deg(\mathscr{E}) = 2$, we know $K_C \otimes \det \mathscr{E}^*$ and $\pi^*(K_C \otimes \det \mathscr{E}^*)$ are unitary flat. Hence, we deduce K_X is RC-positive. Since X is uniruled, by Theorem 2.4, K_X^{-1} is RC-positive. Then we can apply Theorem 1.2 and assert that X has scalar-flat Hermitian metrics.

In summary, we prove that a ruled surface X over a smooth curve C admits scalar-flat Hermitian metrics if and only if $g(C) \geq 2$ and $m(X) > 2-2g$. The proof of Theorem 1.5 is completed. □

6. Classification of minimal surfaces with scalar-flat Hermitian metrics

In this section, we classify minimal surfaces with scalar-flat Hermitian metrics and prove Theorem 1.9.

Proposition 6.1. *Let X be a compact complex manifold. If X admits a scalar-flat Hermitian metric, then the Kodaira dimension $\kappa(X) = 0$ or $\kappa(X) = -\infty$.*

Proof. According to the proof of Theorem 1.2, if X admits a scalar-flat Hermitian metric, then X has a Gauduchon metric with zero total scalar curvature. By Theorem [Yang17a, Theorem 1.4], $\kappa(X) = 0$ or $\kappa(X) = -\infty$. □

If X is a minimal surface with Kodaira dimension $\kappa(X) = 0$, X is exactly one of the following (e.g. [BHPV04])

(1) an Enriques surface;
(2) a bi-elliptic surface;
(3) a K3 surface;
(4) a torus;
(5) a Kodaira surface.

In this case, it is well-known that X has torsion canonical line bundle, i.e. $K_X^{\otimes 6} = \mathcal{O}_X$ (e.g. [BHPV04, p. 244]). Hence, X admits scalar-flat Hermitian metrics.

If X is a minimal surface with Kodaira dimension $\kappa(X) = -\infty$, then X lies in one of the following classes:

(1) minimal rational surfaces;
(2) ruled surfaces of genus $g \geq 1$;
(3) minimal surfaces of class VII_0.

Minimal rational surfaces are either $\mathbb{P}^2$ or Hirzebruch surfaces. Hence, by Proposition 5.1, they can not support scalar-flat Hermitian metrics.

If X is a minimal ruled surfaces of genus $g \geq 1$, by Theorem 1.9, X has a scalar-flat Hermitian metric if and only if $g \geq 2$ and $m(X) > 2-2g$.

If X is a minimal surface of class VII_0, then X is one of the following

- class VII_0 surfaces with $b_2 > 0$;
- Inoue surfaces: a class VII_0 surface has $b_2 = 0$ and contains no curves;

- Hopf surfaces: its universal covering is $\mathbb{C}^2 - \{0\}$, or equivalently a class VII_0 surface has $b_2 = 0$ and contains a curve.

According to the proof of [Tel06, Remark 4.2] (see also [TW13] or [HLY18, Theorem 5.1]), we know Inoue surfaces all have K_X semi-positive but not unitary flat, and so it can not support scalar-flat Hermitian metrics. Similarly, it is proved in [Tel06, Remark 4.3], all Hopf surfaces have semi-positive anti-canonical bundle, and so it has no scalar-flat Hermitian metrics. For class VII_0 surfaces with $b_2 > 0$, they are not completely classified, and it is possible that some of them can support scalar-flat Hermitian metrics (see the discussion in [Tel06, p. 977-p. 979]). The proof of Theorem 1.9 is completed.

7. Examples

In this section, we exhibit several examples on ruled manifolds with scalar-flat Hermitian metrics. As a straightforward application of Theorem 1.5, we get the following result.

Corollary 7.1. *Let $\mathscr{L} \to C$ be a line bundle over a smooth curve of genus g and $X = \mathbb{P}(\mathscr{L} \oplus \mathcal{O}_C)$. Then X has a scalar-flat Hermitian metric if and only if $g \geq 2$ and $|\deg(\mathscr{L})| < 2g - 2$.*

We can also construct higher dimensional ruled manifolds with scalar-flat metrics.

Theorem 7.2. *Let C be a smooth curve with genus $g \geq 2$ and $\mathscr{L}$ be a line bundle over C. Suppose $\mathscr{E} = \mathscr{L} \oplus \mathcal{O}_C^{\oplus(n-1)}$ and $X = \mathbb{P}(\mathscr{E}^*) \to C$ is the projective bundle. If $0 \leq \deg(\mathscr{L}) < \frac{2g-2}{n-1}$, then both K_X and K_X^{-1} are RC-positive.*

Proof. By using the projection formula, we know

$$K_X = \mathcal{O}_{\mathscr{E}}(-n) \otimes \pi^*(K_C \otimes \det \mathscr{E}), \tag{7.1}$$

where $\pi : X \to C$ is the projection. Fix an arbitrary smooth Hermitian metric $h^{\mathscr{L}}$ on $\mathscr{L}$ and the trivial metric on $\mathcal{O}_C$. Let $\{z\}$ be the local holomorphic coordinate on C. The curvature form of $(\mathscr{L}, h^{\mathscr{L}})$ is

$$R^{\mathscr{L}} = -\sqrt{-1}\partial\overline{\partial} \log h^{\mathscr{L}} = \sqrt{-1}\kappa dz \wedge d\overline{z}. \tag{7.2}$$

Similarly, fix a smooth metric h^{K_C} on K_C, and its curvature form is

$$R^{K_C} = -\sqrt{-1}\partial\overline{\partial} \log h^{K_C} = \sqrt{-1}\gamma dz \wedge d\overline{z}. \tag{7.3}$$

Hence, $\mathscr{E}$ has the curvature form

$$R^{\mathscr{E}} = \sqrt{-1}\kappa dz \wedge d\overline{z} \otimes e^1 \otimes e^1 + \sum_{i=2}^{n} \sqrt{-1} \cdot 0 \cdot dz \wedge d\overline{z} \otimes e^i \otimes e^i, \tag{7.4}$$

where $e^1 = e_{\mathscr{L}}$ is the local frame of $\mathscr{L}$ and for $i \geq 2$, $e^i = e$ is the local holomorphic frame on $\mathcal{O}_C$ with the order in the direct sum $\mathscr{E} = \mathscr{L} \oplus \mathcal{O}_C^{\oplus(n-1)}$. Therefore, by (4.3), $\mathcal{O}_{\mathscr{E}}(-n)$ has the curvature form at some point

$$R^{\mathcal{O}_{\mathscr{E}}(-n)} = \sqrt{-1}\left(-n\kappa\frac{|a_1|^2}{|a|^2}dz \wedge d\overline{z}\right) - n\omega_{\mathrm{FS}}.$$

Hence, by formula (7.1), the curvature of K_X is given by

$$R^{K_X} = \sqrt{-1}\left(\left((\kappa+\gamma) - n\kappa\frac{|a_1|^2}{|a|^2}\right) dz \wedge d\overline{z}\right) - n\omega_{\mathrm{FS}}.$$

Since $\deg(\mathscr{L}) \geq 0$, we can choose the smooth metric $h^{\mathscr{L}}$ such that its curvature is semi-positive, i.e. $\kappa \geq 0$. Therefore,

$$R^{K_X} \geq \sqrt{-1}\left((\gamma - (n-1)\kappa)\, dz \wedge d\overline{z}\right) - n\omega_{\mathrm{FS}}. \tag{7.5}$$

The condition $0 \leq \deg(\mathscr{L}) < \frac{2g-2}{n-1}$ implies $\deg\left(K_C \otimes \mathscr{L}^{1-n}\right) > 0$. Therefore, we can choose the Hermitian metric h^{K_C} on K_C such that $h^{K_C} \otimes (h^{\mathscr{L}})^{1-n}$ has positive curvature, i.e.

$$\gamma - (n-1)\kappa > 0.$$

By (7.5), we know the curvature of K_X is positive along the base direction, i.e., K_X is RC-positive. The RC-positivity of K_X^{-1} follows from Theorem 2.4. □

Example 7.3. Let $n \geq 2$ be an integer. Let C be a smooth curve of degree $d \geq n+3$ in $\mathbb{P}^2$. It is easy to see that $\deg(\mathcal{O}_C(1)) = d$ and C is a curve of genus

$$g = \frac{(d-1)(d-2)}{2}. \tag{7.6}$$

Let $\mathscr{L} = \mathcal{O}_C(1)$ and $\mathscr{E} = \mathscr{L} \oplus \mathcal{O}_C^{\oplus(n-1)}$ and $X := \mathbb{P}(\mathscr{E}^*) \to C$ be the projective bundle. Note that $\dim_{\mathbb{C}} X = n$. Then

$$\frac{2g-2}{n-1} = \frac{d(d-3)}{n-1} \geq \frac{d \cdot n}{n-1} > d = \deg(\mathscr{L}) > 0.$$

Hence, the pair $(X, C, \mathscr{L}, \mathscr{E})$ satisfies the conditions in Theorem 7.2. In particular, both K_X and K_X^{-1} are RC-positive.

The proof of Proposition 1.7. By Theorem 7.2 and Theorem 1.2, X admits a scalar-flat Hermitian metric. On the other hand, by [ACGT11, Theorem 1], X has no scalar-flat Kähler metrics since $\mathscr{E} = \mathscr{L} \oplus \mathcal{O}_C^{\oplus(n-1)}$ is not polystable. □

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