Abstract QFTs, Realizations, and Recursion Relations

Zhiyuan Wang

School of Mathematical Sciences, Peking University

(Joint work with Jian Zhou)

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Outline

- Backgrounds: Feynman graphs in mathematics
- Construction of abstract QFTs
  - Abstract QFT for stable graphs and recursions
  - Abstract QFT for fat graphs and recursions
- Realization of the abstract QFTs by formal integrals
- Applications:
  - A toy model: Topological 1D gravity
  - Computation of Orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$

§1.1. Backgrounds: Feynman graphs

Roughly speaking, Feynman graphs describe interactions and propagations of particles (point particles or strings).

- Path integrals;
- Summation over Feynman graphs.

However, Feynman's path integrals are infinite-dimensional in general, thus are not well-defined in mathematics.

§1.1. Backgrounds: Feynman graphs in mathematics

In mathematics, we know about finite-dimensional integrals (regarded as formal power series in the coupling constants):

- Formal Gaussian integral = Sum over thin graphs:

$$\text{eg.} \quad \int_{\mathbb{R}} \exp\left(\sum \lambda^{2g-2} f_{g,n} \cdot \frac{x^n}{n!} - \frac{1}{2\lambda^2\kappa} \cdot x^2\right) dx.$$

- Hermitian matrix models (at finite N) = Sum over fat graphs:

$$\text{eg.} \quad \int_{\mathcal{H}_N} \exp\left(\operatorname{tr} \sum_{n=1}^{\infty} \frac{g_n - \delta_{n,2}}{n g_s} M^n\right) dM \Big/ \int_{\mathcal{H}_N} \exp\left(-\frac{\operatorname{tr}(M^2)}{2g_s}\right) dM,$$

where $\mathcal{H}_N$ = space of all Hermitian matrices of size $N \times N$.

§1.1. Backgrounds: Feynman graphs in mathematics

Question : How to compute these summations over graphs or formal integrals (partition function), and their logarithms (free energy)?

- Expanding integral and its logarithm directly are hard (since it involves sum over partitions of integers);
- Writing down all possible graphs without missing and repeating and computing $|\mathrm{Aut}(\Gamma)|$ are also hard .

Other strategies:

- Derive recursion relations ;
- Relate to integrable systems (KP, BPK, KdV, $\cdots$);
- $\cdots\cdots$.

§1.1. Backgrounds: Feynman graphs in mathematics

In physics and mathematics literatures, solutions to some quadratic recursions are known to be summations over graphs:

- BCOV holomorphic anomaly equation (HAE) for quintic threefold:
  - Summations over stable graphs (at $g = 2, 3$).
  - Bershadsky, Cecotti, Ooguri, Vafa.
- Eynard-Orantin topological recursion :
  - Summations over trivalent graphs or stable graphs.
  - Eynard, Orantin; Eynard; Dunin-Barkowski, Orantin, Shadrin, Spitz.
- Virasoro constraints for Hermitian matrix models:
  - Summation over fat graphs.

§1.1. Backgrounds: Feynman graphs in mathematics

Thus it is natural to expect :

- Summation over graphs always satisfies some quadratic recursions?
- This property is determined by combinatorial properties of graphs (and independent of the specific Feynman rules)?

To formulate these recursions in terms of graphs, we are supposed to introduce some operators on graphs (linear maps on the space spanned by graphs) first.

§1.1. Backgrounds: Feynman graphs in mathematics

Another application of Feynman graphs in mathematics: describe moduli spaces of curves :

- Penner; Harer; Mumford; Thurston; Strebel; etc.

  Fat graphs describe cell decomposition of $\mathcal{M}_{g,n}^{\text{comb}}$ (combinatorial moduli space of smooth stable curves, where $2g - 2 + n > 0$).

  Structures: Whitehead collapse.

- Deligne-Mumford; Knudsen.

  Stable graphs describe stratification of $\overline{\mathcal{M}}_{g,n}$ (Deligne-Mumford moduli space of stable nodal curves, where $2g - 2 + n > 0$).

  Structures: forgetful and gluing maps.

§1.3. Backgrounds: Motivation of our works

Inspired by the above works, we introduce a formalism called:

Abstract QFTs and their realizations.

- Abstract : Summation over graphs $\rightarrow$ Summation of graphs.
- Realization : Assign Feynman rules, and obtain specific theories.

We want to derive recursions which are independent of Feynman rules .

These recursions will lead to some recursions for the specific theories pbtained by assigning Feynman rules.

§2. Abstract QFTs: Overview

We introduce two types of abstract QFTs:

- Abstract QFT for stable graphs ;
- Abstract QFT for fat graphs (ribbon graphs);

Then we introduce some operators on these graphs according to the following philosophy:

Operators on graphs are indicated by structures
of the corresponding moduli spaces.

§2. Abstract QFTs: Overview

- Stable graphs :
  - Operators: Edge-cutting/edge-adding;
  - Quadratic recursion of HAE-type;
  - Linear recursion for fixed g.
- Fat graphs :
  - Operators: Edge-contraction/vertex-splitting;
  - Abstract Virasoro constraints;
  - Quadratic recursion for abstract n-point functions.

§2.1. Abstract QFT for stable graphs

A stable graph consists of:

- Vertices $v \in V(\Gamma)$, and a genus $g_v \in \mathbb{Z}_{\geq 0}$ associated to every v;
- Internal edges $e \in E(\Gamma)$, connecting these vertices;
- External edges $e \in E^{ext}(\Gamma)$.

Stability condition:

- For a vertex v of genus 0, we have $\mathrm{val}(v) \geq 3$;
- For a vertex v of genus 1, we have $\mathrm{val}(v) \geq 1$.

Genus of a stable graph Γ:

$$g(\Gamma) := h^1(\Gamma) + \sum_{v \in V(\Gamma)} g_v,$$

where $h^1(\Gamma)$ is the number of independent loops in Γ.

§2.1. Abstract QFT for stable graphs: Geometric backgrounds

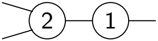
Stable graphs are dual graphs of stable curves:

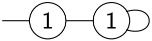
irreducible components	$\leftrightarrow$	vertices
nodal points	$\leftrightarrow$	internal edges
marked points	$\leftrightarrow$	external edges

Example

Stable curves:

Dual graphs:





§2.1. Abstract QFT for stable graphs: Geometric backgrounds

Stratification of $\overline{\mathcal{M}}_{g,n}$ is described by stable graphs:

$$\overline{\mathcal{M}}_{g,n}/S_n = \bigsqcup_{\Gamma \in \mathcal{G}^c_{g,n}} \mathcal{M}_\Gamma,$$

- $\mathcal{G}^c_{g,n} := \{$connected stable graphs of genus g with n external edges$\}$;
- $\mathcal{M}_\Gamma$: the moduli space of stable curves whose dual graph is Γ;
- Modulo S_n means we do not distinguish the n marked points here.

Remark

$\mathcal{G}^c_{g,n}$ is a finite set for every (g, n) with $2g - 2 + n > 0$.

§2.1. Abstract QFT for stable graphs: Construction

Construct an abstract QFT for stable graphs by defining:

- Abstract free energy $(g \geq 2)$: $\widehat{\mathcal{F}}_g = \sum_{\Gamma \in \mathcal{G}^c_{g,0}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \Gamma.$
- Abstract n-point functions $(2g - 2 + n > 0)$: $\widehat{\mathcal{F}}_{g,n} = \sum_{\Gamma \in \mathcal{G}^c_{g,n}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \Gamma.$

They are elements in the linear space spanned by all stable graphs.

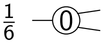
Remark

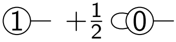
They are finite summations due to the stability condition.

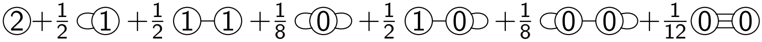
§2.1. Abstract QFT for stable graphs: Construction

Example

Here are some examples:

$\widehat{\mathcal{F}}_{0,3} =$ 

$\widehat{\mathcal{F}}_{1,1} =$ 

$\widehat{\mathcal{F}}_{2} =$ 

§2.1. Abstract QFT for stable graphs: Operators

Two types of natural maps on $\overline{\mathcal{M}}_{g,n}$:

- The forgetful map :

$$\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

forgets a marked point; contract the unstable component.

- The gluing maps :

$$\xi_1 : \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2},$$
$$\xi_2 : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n},$$

glue two marked points together to get a new nodal point.

§2.1. Abstract QFT for stable graphs: Operators

We construct two linear maps on the space spanned by stable graphs as the inverses of the above maps:

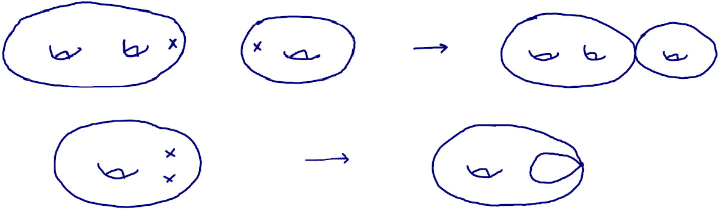
- Edge-cutting operator K: inverse of the gluing maps.
- Edge-adding operator $\mathcal{D} = \partial + \gamma$: inverse of the forgetful map.

Remark

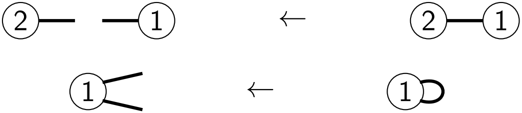
In the definitions of ∂ and γ, we need to take all possible unstable contractions into consideration.

§2.1. Abstract QFT for stable graphs: Operators

- The Gluing maps :

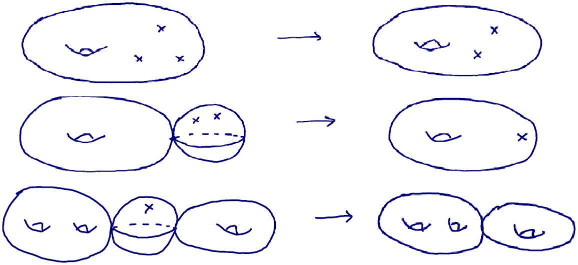


- Edge-cutting on stable graphs:

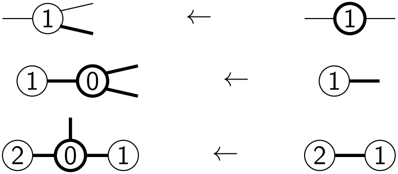


§2.1. Abstract QFT for stable graphs: Operators

- The Forgetful map :

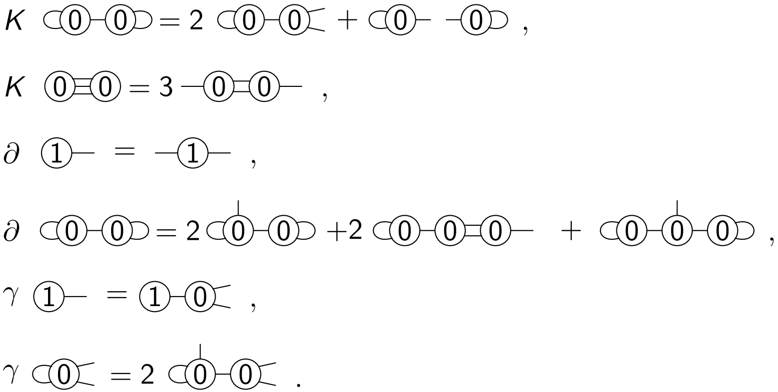


- Edge-adding on stable graphs:



§2.1. Abstract QFT for stable graphs: Operators

Example



§2.1. Abstract QFT for stable graphs: Recursions

Theorem (W-Zhou, 2019)

For $2g-2+n>0$, we have the following two types of recursions:

1) *A linear recursion with fixed g:* $\quad \mathcal{D}\widehat{\mathcal{F}}_{g,n} = (n+1)\widehat{\mathcal{F}}_{g,n+1}.$
2) *A quadratic recursion:*

$$K\widehat{\mathcal{F}}_{g,n} = \frac{1}{2}\Big(\mathcal{D}\mathcal{D}\widehat{\mathcal{F}}_{g-1,n} + \sum_{\substack{g_1+g_2=g,\\ n_1+n_2=n}} \mathcal{D}\widehat{\mathcal{F}}_{g_1,n_1}\cdot\mathcal{D}\widehat{\mathcal{F}}_{g_2,n_2}\Big), \quad 2g-2+n>0;$$

$$K\widehat{\mathcal{F}}_{g} = \frac{1}{2}\Big(\mathcal{D}\partial\widehat{\mathcal{F}}_{g-1} + \sum_{r=1}^{g-1}\partial\widehat{\mathcal{F}}_{r}\cdot\partial\widehat{\mathcal{F}}_{g-r}\Big), \quad g\geq 2,$$

where we use the convention:

$$\partial\widehat{\mathcal{F}}_1 = \mathcal{D}\widehat{\mathcal{F}}_1 := \widehat{\mathcal{F}}_{1,1}, \quad \mathcal{D}\widehat{\mathcal{F}}_{0,2} := 3\widehat{\mathcal{F}}_{0,3}, \quad \mathcal{D}\mathcal{D}\widehat{\mathcal{F}}_{0,1} := 6\widehat{\mathcal{F}}_{0,3}.$$

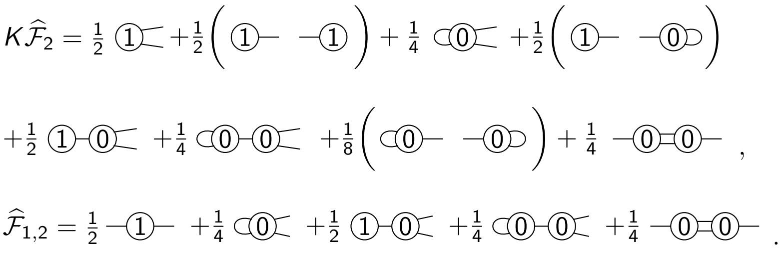
Remark

The ‘multiplication’ of two graphs means disjoint union.

§2.1. Abstract QFT for stable graphs: Recursions

Example

We have the following expressions for $K\widehat{\mathcal{F}}_2$ and $\widehat{\mathcal{F}}_{1,2}$:



One can check:

$$K\widehat{\mathcal{F}}_2 = \widehat{\mathcal{F}}_{1,2} + \frac{1}{2}\widehat{\mathcal{F}}_{1,1}\widehat{\mathcal{F}}_{1,1} = \frac{1}{2}\left(\mathcal{D}\mathcal{D}\widehat{\mathcal{F}}_1 + \mathcal{D}\widehat{\mathcal{F}}_1\mathcal{D}\widehat{\mathcal{F}}_1\right).$$

§2.2. Abstract QFT for fat graphs

Fat graphs :

- Graphs (maps) on oriented surfaces;
- 1-skeleton of a cell-decomposition of the surface (up to equivalence).

A (not necessarily stable) fat graph Γ consists of:

- Vertices (0-cells) and internal edges (1-cells);
- A cyclic order of half-edges on each vertex (induced by the orientation of the surface).

Genus of a fat graph $g(\Gamma) :=$ genus of the surface.

Euler's formula : $2 - 2g(\Gamma) = |V(\Gamma)| - |E(\Gamma)| + |F(\Gamma)|$, where:

$$V(\Gamma) = \{\text{vertices}\}, \quad E(\Gamma) = \{\text{edges}\}, \quad F(\Gamma) = \{\text{faces}\}.$$

§2.2. Abstract QFT for fat graphs: Construction

Definition (Abstract correlators)

The abstract correlator of genus g and type $\mu = (\mu_1, \cdots, \mu_n) \in \mathbb{Z}^n_{>0}$:

$$\mathcal{F}^{\mu}_g := \sum_{\Gamma \in \mathfrak{Fat}^{\mu,c}_g} \frac{1}{|\operatorname{Aut}(\Gamma)|} \Gamma.$$

It is an element in the vector space $\mathcal{V}^{\mu,c}_g := \bigoplus_{\Gamma \in \Gamma^{\mu,c}_g} \mathbb{Q}\Gamma$.

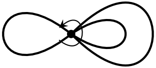
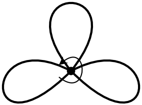
We also formally denote:

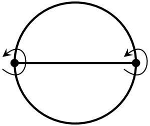
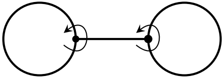
$$\mathcal{F}^{(0)}_0 := \quad \bullet \; v_1 \quad .$$

- $v_1, \cdots, v_n$: labels on vertices;
- $\mathfrak{Fat}^{\mu,c}_g = \{$ connected fat graphs of genus g, s.t. $\operatorname{val}(v_i) = \mu_i$ $\}$.

§2.2. Abstract QFT for fat graphs: Construction

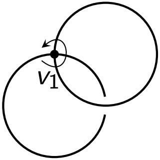
Example

$\mathcal{F}_0^{(6)} = \frac{1}{2}$  $+\frac{1}{3}$  ,

$\mathcal{F}_0^{(3,3)} = \frac{1}{3}$  $+$  ,

§2.2. Abstract QFT for fat graphs: Construction

Example

$\mathcal{F}_1^{(4)} = \frac{1}{4}$ 

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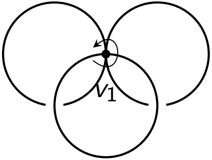
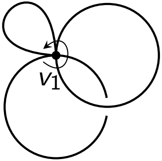
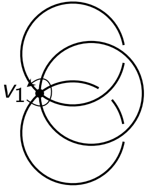
$\mathcal{F}_1^{(6)} = \frac{1}{2}$ 

$+$ 

$+\frac{1}{6}$ 

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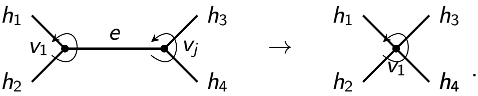
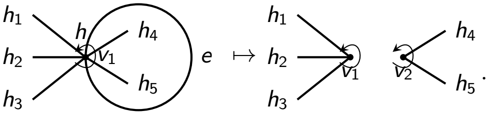
§2.2. Abstract QFT for fat graphs: Operators

Roughly speaking, define the edge-contraction operator by:

- If $e \in E(\Gamma)$ is not a loop (the 'Whitehead collapse'):



- If $e \in E(\Gamma)$ is not a loop (degeneration of the surface):

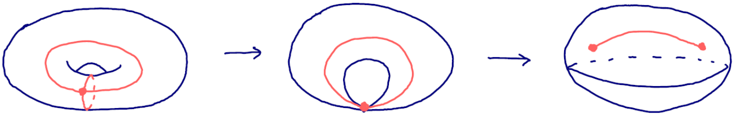


- Suitably relabel the vertices.

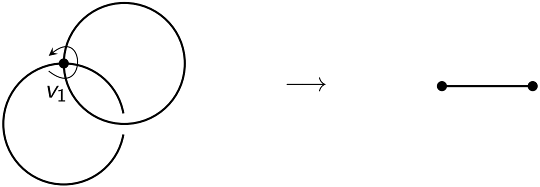
We formulate this procedure as a linear map K_1 on the vector space spanned by all fat graphs.

§2.2. Abstract QFT for fat graphs: Operators

Edge-contraction indicates degeneration of surfaces. For example, the second case above (i.e., contracting a loop) means:



On fat graphs, one has:



§2.2. Abstract QFT for fat graphs: Recursion

One can recursively compute $\mathcal{F}_g^{\mu}$ with lower g or lower μ:

Theorem (W-Zhou, 2021)

The following quadratic recursion relation holds:

$$
\begin{aligned}
K_1(\mathcal{F}_g^{\mu}) =& \ \delta_{g,0}\delta_{n,1}\delta_{\mu_1,2}\mathcal{F}_{0,\{1\}}^{(0)}\mathcal{F}_{0,\{2\}}^{(0)} + \sum_{j=2}^{n}(\mu_1+\mu_j-2)\mathcal{F}_g^{(\mu_1+\mu_j-2,\mu_{[n]\backslash\{1,j\}})} \\
&+ \sum_{\substack{\alpha+\beta=\mu_1-2\\ \alpha\geq 1,\beta\geq 1}} \alpha\beta\Big(\mathcal{F}_{g-1}^{(\alpha,\beta,\mu_{[n]\backslash\{1\}})} + \sum_{\substack{g_1+g_2=g\\ I\sqcup J=[n]\backslash\{1\}}} \mathcal{F}_{g_1,\{1\}\sqcup(I+1)}^{(\alpha,\mu_I)}\mathcal{F}_{g_2,\{2\}\sqcup(J+1)}^{(\beta,\mu_J)}\Big) \\
&+ (\mu_1-2)\cdot\mathcal{F}_{0,\{1\}}^{(0)}\mathcal{F}_{g,[n+1]\backslash\{1\}}^{(\mu_1-2,\mu_{[n]\backslash\{1\}})} + (\mu_1-2)\cdot\mathcal{F}_{0,\{2\}}^{(0)}\mathcal{F}_{g,[n+1]\backslash\{2\}}^{(\mu_1-2,\mu_{[n]\backslash\{1\}})},
\end{aligned}
$$

where:

- $[n] := \{1,2,\cdots,n\}$;
- $I+1 := \{i_1+1,\cdots,i_k+1\}$, *for* $I=\{i_1,\cdots,i_k\}$;
- $\mathcal{F}_{g,I}^{\mu}$ *is obtained by relabeling the vertices using indices in* I *to* $\mathcal{F}_g^{\mu}$;
- $\mathcal{F}_g^{(\mu_1-2,\mu_{[n]\backslash\{1\}})} := 0$ *for* $\mu_1 < 2$.

§2.2. Abstract QFT for fat graphs: Recursion

Define 'generating series' of the above abstract correlators:

Definition

- Abstract free energy of genus g for fat graphs:

$$\mathcal{F}_g := \sum_{n\geq 1} \frac{1}{n!} \sum_{\mu\in\mathbb{Z}^n_{>0}} \mathcal{F}^{\mu}_g$$

(after forgetting the labels on the right-hand side).

- Abstract partition function for fat graphs:

$$\mathcal{Z} := \exp\left(\sum_{g=0}^{\infty} g_s^{2g-2}\mathcal{F}_g\right) = 1 + \sum_{g=-\infty}^{+\infty} g_s^{2g-2} \sum_{n\geq 1} \frac{1}{n!} \sum_{\mu\in\mathbb{Z}^n_{>0}} \sum_{\Gamma\in\mathfrak{Fat}^{\mu}_g} \frac{1}{|\mathrm{Aut}(\Gamma)|}\Gamma$$

(after forgetting the labels on the right-hand side), where $\mathfrak{Fat}^{\mu}_g$ is the set of all (not necessarily connected) fat graphs of genus g and type μ, and '1' is an 'empty graph'.

§2.2. Abstract QFT for fat graphs: Recursion

Remark

Genus and automorphism of a disconnected graph $\Gamma = \Gamma_1 \sqcup \cdots \sqcup \Gamma_k$:

- $g(\Gamma) := g(\Gamma_1) + \cdots + g(\Gamma_k) - k + 1$;
- $\mathrm{Aut}(\Gamma) := \mathrm{Aut}(\Gamma_1) \times \cdots \times \mathrm{Aut}(\Gamma_k)$.

§2.2. Abstract QFT for fat graphs: Recursion

The above quadratic recursion relation for abstract correlators can be reformulated in the following way:

Theorem (W-Zhou, 2021)

The following abstract Virasoro constraints holds:

$$\mathcal{L}_m(\mathcal{Z}) = 0, \qquad \forall m \geq -1,$$

where (roughly speaking) the abstract Virasoro operators $\{\mathcal{L}_m\}_{m\geq -1}$ *are constructed using some vertex-splitting operators. Moreover, one has:*

$$[\mathcal{L}_m, \mathcal{L}_n] = (m-n)\mathcal{L}_{m+n}, \quad \forall m, n \geq -1,$$

where the Lie bracket $[,]$ *is 'almost' the commutator.*

Remark

Vertex-splitting are 'inverse' to edge-contraction.

§2.2. Abstract QFT for fat graphs: Recursion

Abstract Virasoro operators are of the following form:

$$\mathcal{L}_{-1} := -\partial_1 + \sum_{n\geq 1} \mathcal{S}_{n,n} + g_s^{-2} \cdot \gamma_{-1},$$

$$\mathcal{L}_0 := -2\partial_2 + \sum_{n\geq 1} \mathcal{S}_{n,n-1} + g_s^{-2} \cdot \gamma_0,$$

$$\mathcal{L}_m := -(m+2)\partial_{m+2} + \sum_{n\geq 1} \mathcal{S}_{n+m,n-1} + g_s^2 \cdot \sum_{n=1}^{m-1} \mathcal{J}_{n,m-n} + 2\mathcal{J}_{m,0}(- \sqcup \Gamma_{dot}), \quad m \geq 1.$$

These operators describe inverse procedures of edge-contraction. We mark the vertex v that we are going to apply the edge-contraction operator to.

- ∂_k: to choose a valence k vertex to be marked;
- $\mathcal{S}_{n,k}$ (vertex-splitting): the inverse of contraction of a non-loop;
- $\mathcal{J}_{k,l}$: the inverse of contraction of a loop.

Here we omit detailed definition (which is natural but very lengthy!).

§2.2. Abstract QFT for fat graphs: Recursion

One can also collect the abstract correlators in the following way:

$$\mathcal{W}_{g,n} := \delta_{g,0}\delta_{n,1}\mathcal{F}_0^{(0)} + \sum_{\mu\in\mathbb{Z}^n_{>0}} \mu_1\mu_2\cdots\mu_n\mathcal{F}_g^{\mu}.$$

Theorem (W-Zhou, 2021)

The abstract Virasoro constraints can be reformulated in the following way (which resembles the Eynard-Orantin topological recursion):

$$(1-2\mathcal{T}\sigma_1^{-1})\mathcal{W}_{g,n} = \sum_{j=2}^{n} \sigma_j\mathcal{S}_{\{1;j\}}\sigma_1^{-1}\mathcal{W}_{g,n-1} + \mathcal{J}_{\{1,2\}}\sigma_1^{-1}\sigma_2^{-1}\Big(\mathcal{W}_{g-1,n+1} + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[n+1]\setminus\{1,2\}}}^{s} \mathcal{W}_{g_1,\{1\}\sqcup I}\cdot\mathcal{W}_{g_2,\{2\}\sqcup J}\Big),$$

- $\mathcal{W}_{g,I}$: *obtained from* $\mathcal{W}_{g,|I|}$ *by relabelling the vertices using indices in* I;
- $\mathcal{S}_{1;j}$, $\mathcal{J}_{1,2}$, $\mathcal{T}$, σ_j: *certain operators on fat graphs.*

§2.2. Abstract QFT for fat graphs: Recursion

Summary: Three (equivalent) quadratic recursions for fat graphs:

- For the abstract correlators $\mathcal{F}_g^\mu$:
  - Quadratic recursion using edge-contraction operator K_1;
- For the abstract partition function $\mathcal{Z}$:
  - Abstract Virasoro constraints;
- For the abstract n-point functions $\mathcal{W}_{g,n}$:
  - Quadratic recursion which resembles the E-O recursion.

§2.3. Abstract QFT for ordinary graphs

We have also constructed an abstract QFT for ordinary graphs (W-Zhou, in preparation).

Roughly speaking, an ordinary graphs is a (not necessarily stable) thin graph whose vertices have no genus.

Here we omit the details. In this case we have:

- Abstract flow equations and abstract polymer equation;
- Abstract Virasoro constraints;
- Abstract bilinear relation (of Hirota type).

§3. Realizations of Abstract QFTs

We construct realizations of an Abstract QFT:

- Assign a suitable Feynman rule $\Gamma \mapsto w_\Gamma$ to each Feynman graph, where w_Γ is a formal variable (or a function, a formal power series, etc.).
- The abstract correlators, abstract free energy, and abstract partition function will be realized by some functions or formal power series:

$$\sum \frac{1}{|\mathrm{Aut}(\Gamma)|}\Gamma \quad \mapsto \quad \sum \frac{1}{|\mathrm{Aut}(\Gamma)|} w_\Gamma .$$

- An operator $\mathcal{O}$ (edge-cutting/adding/contraction, abstract Virasoro, etc.) acting on graphs is realized by an operator O, if:

$$w_{\mathcal{O}(\Gamma)} = O(w_\Gamma), \quad \forall \Gamma.$$

- If we know the realizations of the edge-cutting/edge-adding or edge-contraction/vertex-splitting operators, then we obtain recursions for realizations of abstract correlators, abstract free energies, or abstract partition function automatically.

§3. Realizations of Abstract QFTs: Formal integrals

Examples of realizations: Formal integrals.

- Formal Gaussian integrals: for stable graphs :
  - $\int_{\mathbb{R}} \exp\Big(\sum_{2g-2+n>0} \lambda^{2g-2} f_{g,n} \cdot \frac{x^n}{n!} - \frac{1}{2\lambda^2\kappa} \cdot x^2 \Big) dx.$
  - $w_\Gamma = \prod_{v \in V(\Gamma)} f_{g(v),\mathrm{val}(v)} \cdot \prod_{e \in E(\Gamma)} \kappa.$
- Hermitian one-matrix model (at finite N): for fat graphs :
  - $\int_{\mathcal{H}_N} \exp\Big(\mathrm{tr} \sum_{n=1}^{\infty} \frac{g_n - \delta_{n,2}}{n g_s} M^n \Big) dM \Big/ \int_{\mathcal{H}_N} \exp\Big(-\frac{\mathrm{tr}(M^2)}{2 g_s} \Big) dM.$
  - $w_\Gamma = t^{|F(\Gamma)|} \prod_{v \in V(\Gamma)} g_{\mathrm{val}(v)}$, where $t = N g_s$ is the 't Hooft coupling constant.

§3. Realizations of Abstract QFTs: Formal integrals

Example (Stable graphs and HAE type recursion)

Let $\{F_g(t)\}_{g\geq 0}$ be a sequence of holomorphic functions. We set

$$F_{g,0} := F_g(t); \qquad F_{g,n} := F_g^{(n)}(t) = (\frac{\partial}{\partial t})^n F_g(t), \quad n > 0,$$

and let $\widehat{F}_g$ be the summation over stable graphs:

$$\widehat{F}_g := \sum_{\Gamma \in \mathcal{G}^c_{g,0}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} F_{g(v),\mathrm{val}(v)} \cdot \prod_{e \in E(\Gamma)} \kappa.$$

If $\kappa = \frac{1}{C - F_0''(t)}$, where C is either a constant or anti-holomorphic in t, then:

$$\partial_\kappa \widehat{F}_g = \frac{1}{2}\left(D_t \partial_t \widehat{F}_{g-1} + \sum_{r=1}^{g-1} \partial_t \widehat{F}_r \cdot \partial_t \widehat{F}_{g-r} \right), \quad g \geq 2,$$

where $D_t = \partial_t + \kappa F_0'''$ is a covariant derivative.

- Edge-cutting operator $K \quad \to \quad$ partial derivative ∂_κ;
- Edge-adding operators $\mathcal{D}$ and $\partial \quad \to \quad D_t$ and ∂_t.

§3. Realizations of Abstract QFTs: Formal integrals

Example (Fat graphs and Hermitian one-matrix models)

- The abstract Virasoro operators $\{\mathcal{L}_m\}_{m\geq -1}$ can be realized by the fat Virasoro operators for Hermitian one-matrix models:

$$L_{-1,t} = -\frac{\partial}{\partial g_1} + \sum_{n\geq 1} n g_{n+1}\frac{\partial}{\partial g_n} + t g_1 g_s^{-2},$$

$$L_{0,t} = -2\frac{\partial}{\partial g_2} + \sum_{n\geq 1} n g_n \frac{\partial}{\partial g_n} + t^2 g_s^{-2},$$

$$L_{m,t} = \sum_{k\geq 1}(k+m)(g_k - \delta_{k,2})\frac{\partial}{\partial g_{k+m}} + g_s^2 \sum_{k=1}^{m-1} k(m-k)\frac{\partial}{\partial g_k}\frac{\partial}{\partial g_{m-k}} + 2tm\frac{\partial}{\partial g_m}, \quad m\geq 1.$$

- The abstract Virasoro constraints are realized by fat Virasoro constraints:

$$L_m Z_N = 0, \quad \forall m\geq -1;$$
$$[L_m, L_n] = (m-n)L_{m+n}, \quad \forall m,n\geq -1.$$

§3. Realizations of Abstract QFTs: Formal integrals

Example (Fat graphs and Hermitian one-matrix models)

A direct consequence of the Virasoro constraints is the following cut-and-join type representation for Z_N:

$$Z_N = \exp(M)(1),$$

where:

$$M = \frac{1}{2} \sum_{i+j+k=-2} :\alpha_i \alpha_j \alpha_k: + \frac{t}{2} \sum_{i+j=-2} :\alpha_i \alpha_j: + \frac{t^2}{2} \alpha_{-2},$$

where $\{\alpha_n\}$ are the bosons (here we denote $p_n := g_n$):

$$\alpha_n = \begin{cases} p_{-n}, & n < 0; \\ 0, & n = 0; \\ n \frac{\partial}{\partial p_n}, & n > 0. \end{cases}$$

Since $M \in \widehat{\mathfrak{gl}(\infty)}$, one obtains a new proof of (Shaw-Tu-Yen 1992; etc.):

Corollary

The partition function Z_N is a tau-function of the KP hierarchy.

§3. Realizations of Abstract QFTs: Formal integrals

Example (Fat graphs and Hermitian one-matrix models)

- The abstract n-point functions $\mathcal{W}_{g,n}$ are realized by:

$$W^{\mathrm{H}}_{g,n}(x_1,\cdots,x_n) := \delta_{g,0}\delta_{n,1}\cdot t x_1^{-1} + \sum_{\mu\in\mathbb{Z}^n_{>0}} \langle p_{\mu_1}\cdots p_{\mu_n}\rangle^c_g \cdot x_1^{-(\mu_1+1)}\cdots x_n^{-(\mu_n+1)},$$

where $\langle p_{\mu_1}\cdots p_{\mu_n}\rangle^c_g$ are fat correlators of the Hermitian 1MM.

- The quadratic recursion for $\mathcal{W}_{g,n}$ is realized by:

$$\begin{aligned}&W^{\mathrm{H}}_{g,n}(x_1,x_2,\cdots,x_n)\\ =&\sum_{j=2}^{n}\widetilde{D}^{\mathrm{H}}_{x_1,x_j}W^{\mathrm{H}}_{g,n-1}(x_1,\cdots,\hat{x}_j,\cdots,x_n)+\widetilde{E}^{\mathrm{H}}_{x_1,u,v}W^{\mathrm{H}}_{g-1,n+1}(u,v,x_2,\cdots,x_n)\\ &+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=[n]\backslash\{1\}}}^{s}\widetilde{E}^{\mathrm{H}}_{x_1,u,v}\Big(W^{\mathrm{H}}_{g_1,|I|+1}(u,x_I)\cdot W^{\mathrm{H}}_{g_2,|J|+1}(v,x_J)\Big)\end{aligned}$$

for $(g,n)\neq(0,1)$, where $\widetilde{D}$, $\widetilde{E}$ are some differential operators.

§3. Realizations for fat graphs: A conjecture towards EO

Recall the recursion for abstract n-point functions $\mathcal{W}_{g,n}$ for fat graphs:

$$(1-2\mathcal{T}\sigma_1^{-1})\mathcal{W}_{g,n}=\sum_{j=2}^{n}\sigma_j\mathcal{S}_{\{1;j\}}\sigma_1^{-1}\mathcal{W}_{g,n-1}+\mathcal{J}_{\{1,2\}}\sigma_1^{-1}\sigma_2^{-1}\Big(\mathcal{W}_{g-1,n+1}$$
$$+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=[n+1]\backslash\{1,2\}}}^{s}\mathcal{W}_{g_1,\{1\}\sqcup I}\cdot\mathcal{W}_{g_2,\{2\}\sqcup J}\Big),$$

Conjecture (W-Zhou, 2021)

Assigning suitable Feynman rules to fat graphs, then the realization of above recursion is equivalent to the Eynard-Orantin topological recursion. Here:

- *The spectral curve is the realization of the spectral curve for the abstract QFT (which emerges from the abstract Virasoro constraints);*
- *The Bergman kernel is the realization of $\mathcal{W}_{0,2}$.*

Known to be true for Hermitian 1MM at finite N, by [Zhou, in preparation].

§4. Applications

Now let us see two applications:

- Topological 1D gravity;
- Orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$.

We explain how to see recursion relations and integrable systems from their (various) graph expansions.

§4.1. Application: Topological 1D gravity

The partition function Z^{1D} of the topological 1D gravity is the following 1-dimensional formal Gaussian integral :

$$Z^{1D} := \frac{1}{(2\pi\lambda^2)^{\frac{1}{2}}} \int dx \exp\left[\frac{1}{\lambda^2}\left(-\frac{1}{2}x^2 + \sum_{n\geq 1} t_{n-1}\frac{x^n}{n!}\right)\right],$$

where $\{t_n\}_{n\geq 0}$ are the coupling constants .

Question : Computing the free energy $F^{1D} = \log Z^{1D}$.

Remark

Z^{1D} is the special case $N = 1$ of the Hermitian one-matrix models, thus results for Z_N applies to Z^{1D} by taking $N = 1$.

§4.1. Application: Topological 1D gravity

Z^{1D} can be expanded using both fat and thin graphs.

(1) Fat graph expansion :

Topological 1D gravity is the special case $N = 1$ of Hermitian 1MM, and this gives a fat graph expansion of Z^{1D}. Thus:

- One has the fat Virasoro constraints :

$$L_m(Z^{1D}) = 0, \quad \forall m \geq -1;$$
$$[L_m, L_n] = (m-n)L_{m+n}, \quad \forall m, n \geq -1,$$

  where $\{L_m\}$ are the Virasoro operators of the Hermitian one-matrix models evaluated at $N = 1$.

- Z^{1D} is a tau-function of KP hierarchy .

§4.1. Application: Topological 1D gravity

(2) Stable graph expansion :

Z^{1D} can be represented as a summation over stable graphs (whose vertices are all of genus zero):

Theorem (Zhou, 2014)

For every $g \geq 2$, F_g^{1D} is a finite summation:

$$F_g^{1D} = \sum_{\Gamma \in \mathcal{G}_g^{st,c}} \frac{1}{|\mathrm{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} I_{\mathrm{val}(v)-1} \cdot \prod_{e \in E(\Gamma)} \frac{1}{1 - I_1},$$

where $\mathcal{G}_g^{st,c}$ is the set of all connected ordinary stable graphs of genus g. And for $g = 0, 1$, one has:

$$F_0^{1D} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)!} (I_k + \delta_{k,1}) I_0^{k+1},$$

$$F_1^{1D} = \frac{1}{2} \log \frac{1}{1 - I_1}.$$

§4.1. Application: Topological 1D gravity

The renormalized coupling constants $\{I_k\}$ are defined by (Itzykson-Zuber for 2D gravity; Zhou for 1D gravity):

$$I_0 = \sum_{k=1}^{\infty} \frac{1}{k} \sum_{p_1+\cdots+p_k=k-1} \frac{t_{p_1}}{p_1!} \cdots \frac{t_{p_k}}{p_k!},$$
$$I_k = \sum_{n\geq 0} t_{n+k} \frac{I_0^n}{n!}, \quad k \geq 1,$$

and

$$t_k = \sum_{n=0}^{\infty} \frac{(-1)^n I_0^n}{n!} I_{n+k}, \quad k \geq 0.$$

Remark

Here I_0 is the critical point of $S(x) := -\frac{1}{2}x^2 + \sum_{n\geq 1} t_{n-1} \frac{x^n}{n!}$, and $\{I_k\}_{k\geq 1}$ are the Taylor coefficients of $S(x)$ expanded at $x = I_0$.

§4.1. Application: Topological 1D gravity

Consider the realizations of Edge-cutting/edge-adding operators on stable graphs:

Theorem (W-Zhou, 2019)

For every $g \geq 2$, we have:

$$\partial_\kappa F_g^{1D} = \frac{1}{2}\left((d_X + \kappa I_2) d_X F_{g-1}^{1D} + \sum_{r=1}^{g-1} d_X F_r^{1D} \cdot d_X F_{g-r}^{1D}\right),$$

where d_X is the operator $d_X := \sum_{k\geq 1} I_{k+1} \frac{\partial}{\partial I_k}$.

The above Theorem enables us to solve $F_g (g \geq 2)$ recursively using the initial value $F_1^{1D} = \frac{1}{2} \log \frac{1}{1-I_1}$.

§4.1. Application: Topological 1D gravity

For example:

$$F_2^{1D} = \frac{1}{8}\frac{I_3}{(1-I_1)^2} + \frac{5}{24}\frac{I_2^2}{(1-I_1)^3},$$

$$F_3^{1D} = \frac{1}{48}\frac{I_5}{(1-I_1)^3} + \frac{1}{12}\frac{I_3^2}{(1-I_1)^4} + \frac{7}{48}\frac{I_2 I_4}{(1-I_1)^4} + \frac{25}{48}\frac{I_2^2 I_3}{(1-I_1)^5} + \frac{5}{16}\frac{I_2^4}{(1-I_1)^6},$$

$$F_4^{1D} = \frac{1}{384}\frac{I_7}{(1-I_1)^4} + \frac{1}{32}\frac{I_2 I_6}{(1-I_1)^5} + \frac{5}{96}\frac{I_3 I_5}{(1-I_1)^5} + \frac{21}{640}\frac{I_4^2}{(1-I_1)^5}$$
$$+ \frac{113}{576}\frac{I_2^2 I_5}{(1-I_1)^6} + \frac{11}{96}\frac{I_3^3}{(1-I_1)^6} + \frac{7}{12}\frac{I_2 I_3 I_4}{(1-I_1)^6} + \frac{445}{288}\frac{I_2^2 I_3^2}{(1-I_1)^7}$$
$$+ \frac{161}{192}\frac{I_2^3 I_4}{(1-I_1)^7} + \frac{985}{384}\frac{I_2^4 I_3}{(1-I_1)^8} + \frac{1105}{1152}\frac{I_2^6}{(1-I_1)^9}.$$

For every $g \geq 2$, F_g^{1D} is a polynomial in $\frac{1}{1-I_1}$ and $I_2, I_3, \cdots, I_{2g-1}$.

Remark

This is the $N = 1$ case of the Hermitian one-matrix models. For polynomial structure (Itzykson-Zuber ansatz) for general N (and fat genus expansion), see [Zhang-Zhou 2019].

§4.2. Application: Orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$

We also present an application in algebraic geometry .

Recall that $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$ are complex orbifolds of dimension $3g - 3 + n$.

- Manifold: locally U, $U \subset \mathbb{C}^k$ open subset;
- Orbifold: locally U/G.

In general, many topological notions for manifolds can be generalized to the orbifold case:

- Vector bundles $\to$ orbi-bundles;
- Cohomology $\to$ orbifold cohomology;
- Euler characteristics $\to$ orbifold Euler characteristics;
- $\cdots\cdots$.

§4.2. Application: Orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$

The orbifold Euler characteristics of $\mathcal{M}_{g,n}$ are given by:

Theorem (Harer-Zagier; Penner)

The orbifold Euler characteristics of $\mathcal{M}_{g,n}$ are given by the following Harer-Zagier formula :

$$\chi(\mathcal{M}_{g,n}) = (-1)^n \cdot \frac{(2g-1)B_{2g}}{(2g)!}(2g+n-3)!,$$

for $2g-2+n>0$.

Recall: $\mathcal{M}_{g,n} \times \mathbb{R}_+^n$ has a cell-decomposition indexed by fat graphs.

- Harer-Zagier: prove by computing sum over fat graphs (with $n=1$);
- Penner: prove by matrix integration (Penner model).

§4.2. Application: Orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$

Question : We want to compute orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$.

Recall: $\overline{\mathcal{M}}_{g,n}$ has a stratification indexed by stable graphs of type (g, n). Thus the orbifold Euler characteristics are given by (Bini-Harer):

$$\chi(\overline{\mathcal{M}}_{g,n}/S_n) = \sum_{\Gamma \in \mathcal{G}^c_{g,n}} \frac{1}{|\operatorname{Aut}(\Gamma)|} \prod_{v \in V(\Gamma)} \chi(\mathcal{M}_{g(v),\mathrm{val}(v)}).$$

where the weights of vertices are given by the Harer-Zagier formula.

Define the refined orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}/S_n$ to be:

$$\chi_{g,n}(t,\kappa) := \sum_{\Gamma \in \mathcal{G}^c_{g,n}} \frac{\kappa^{|E(\Gamma)|}}{|\operatorname{Aut}(\Gamma)|} \cdot \prod_{v \in V(\Gamma)} \left(\chi(\mathcal{M}_{g_v,n_v}) \cdot t^{2-2g_v-n_v} \right).$$

Then the orbifold Euler characteristic of $\overline{\mathcal{M}}_{g,n}$ can be recovered by:

$$\chi(\overline{\mathcal{M}}_{g,n}) = n! \cdot \chi_{g,n}(1,1).$$

§4.2. Application: Orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$

Then the edge-cutting/edge-adding operators are realized by some differential operators in variables t and κ, and we have:

Theorem (W-Zhou, 2018)

For $2g-2+n>0$, we have

$$D\chi_{g,n} = (n+1)\chi_{g,n+1}, \quad (1)$$

$$\frac{\partial}{\partial\kappa}\chi_{g,n} = \frac{1}{2}\Big(DD\chi_{g-1,n} + \sum_{\substack{g_1+g_2=g,\\ n_1+n_2=n}} D\chi_{g_1,n_1} D\chi_{g_2,n_2}\Big), \quad (2)$$

where $D\chi_{g,n} := \big(\frac{\partial}{\partial t} + \kappa^2 t^{-1}\cdot\frac{\partial}{\partial\kappa} + n\cdot\kappa t^{-1}\big)\chi_{g,n}$.

The above recursions enable us to compute $\chi_{g,n}$ for $2g-2+n>0$:

- The quadratic recursion (2) enables us to compute $\chi_{g,n}$ using $\{\chi_{r,h}\}$ with $r<g$ or $(r=g, h<h)$;
- The linear recursion (1) enables us to compute $\chi_{g,n}$ using $\chi_{g,0}$ for $g\geq 2$ (and $\chi_{0,3}$, $\chi_{1,1}$ for $g=0,1$ respectively.)

§4.2. Application: Orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$

We can derive some formulas using these recursions. E.g., for $g = 0$:

Theorem (W-Zhou, 2018)

For $n \geq 3$, we have $\chi(\overline{\mathcal{M}}_{0,n}) = n! \cdot \big[\sum_{k=0}^{n-3} A_k(x)\big]_n$ where $[\cdot]_n$ means the coefficient of x^n. The functions $A_k(x)$ are given by:

$$A_0 = \frac{1}{2}(1+x)^2\log(1+x) - \frac{1}{2}x - \frac{3}{4}x^2,$$

$$A_1 = \frac{1}{2} + (1+x)(\log(1+x)-1) + \frac{1}{2}(1+x)^2(\log(1+x)-1)^2,$$

$$A_k = \sum_{m=2-k}^{2} \frac{(x+1)^m}{(m+2)!} \sum_{l=0}^{k-m-1} \frac{\big(\log(x+1)-1\big)^l}{l!} e_{l+m}(1-m,\cdots,k-m-1)$$

for $k \geq 2$, where e_l are the elementary symmetric polynomials.

For higher genera, $\chi(\overline{\mathcal{M}}_{g,n}) = n! \cdot \big[\sum_{k=0}^{3g-3+n}\sum_{p=0}^{3g-3} a^p_{g,0} A^p_{g,k}(x)\big]_n$, where $A^p_{g,k}(x)$ is given by an explicit expression similar to A_k.

§4.2. Application: Orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$

Moreover, the graph sum formula/Gaussian integral formula for $\chi(\overline{\mathcal{M}}_{g,n})$ leads to the relation to topological 1D gravity:

Theorem (W-Zhou, 2019)

Let y, z be two formal variable and denote by $\chi(y,z)$ the following generating series of the orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$:

$$\chi(y,z) := \sum_{2g-2+n>0} \frac{y^n z^{2-2g}}{n!} \cdot \chi(\overline{\mathcal{M}}_{g,n}) - \widetilde{V}_0(y,z),$$

then we have:

$$\chi(y,z) = F^{1D}(t)|_{t_n = \widetilde{V}_{n+1}(y,z), n\geq 0},$$

where $\widetilde{V}_n(y,z)$ are the generating series of $\chi(\mathcal{M}_{g,m})$:

$$\widetilde{V}_n(y,z) := -\frac{1}{2}y^2 z^2 \cdot \delta_{n,0} + yz \cdot \delta_{n,1} + \sum_{g\geq 0,\ g>\frac{1}{2}(2-n)} \chi(\mathcal{M}_{g,n})^{2-2g-n},$$

given by the Harer-Zagier formula.

§4.2. Application: Orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$

Since Z^{1D} is a tau-function of the KP hierarchy, one can use the techniques of integrable systems to derive a formula for the generating series $\chi(y,z)$:

Theorem (W-Zhou, 2021)

The generating series $\chi(y,z)$ is given by:

$$\chi(y,z) = \widetilde{\Psi}\left(\sum_{n\geq 1} (-1)^{n-1} \sum_{n\text{-cycles } \sigma} \prod_{i=1}^{n} \widehat{A}^{1D}(z_{\sigma(i)}, z_{\sigma(i+1)}) - \frac{1}{(z_1 - z_2)^2} \right),$$

where:

$$\widehat{A}^{1D}(\xi,\eta) = \sum_{k=0}^{\infty} (2k+1)!! \cdot \xi^{-1}\eta^{-2k-2} + \frac{1}{\xi - \eta},$$

and $\widetilde{\Psi}$ is a particular evaluation (at times $\{\widetilde{V}_n\}$).

§4.2. Application: Orbifold Euler characteristics of $\overline{\mathcal{M}}_{g,n}$

Example

$g \backslash n$	0	1	2	3
0				1
1		$\frac{5}{12}$	$\frac{1}{2}$	$\frac{17}{12}$
2	$\frac{119}{1440}$	$\frac{247}{1440}$	$\frac{413}{720}$	$\frac{89}{32}$
3	$\frac{8027}{181440}$	$\frac{13159}{72576}$	$\frac{179651}{181440}$	$\frac{495611}{72576}$
4	$\frac{2097827}{43545600}$	$\frac{5160601}{17418240}$	$\frac{97471547}{43545600}$	$\frac{1747463783}{87091200}$
5	$\frac{150427667}{1916006400}$	$\frac{1060344499}{1642291200}$	$\frac{35763130021}{5748019200}$	$\frac{157928041517}{2299207680}$
6	$\frac{31966432414753}{188305108992000}$	$\frac{43927799939987}{25107347865600}$	$\frac{350875518979697}{17118646272000}$	$\frac{14466239894532961}{53801459712000}$
7	$\frac{21067150021261}{46115536896000}$	$\frac{2557845805129901}{4519322615808000}$	$\frac{5346168720992921}{68474585088000}$	$\frac{7660506498435083339}{645617516544000}$
8	$\frac{27108194937436478387}{18438836272496640000}$	$\frac{71323310082487963309}{3352515685908480000}$	$\frac{622747665915354051 6409}{18438836272496640000}$	$\frac{40987641590826353281 7}{70243185799987200}$

Thank you!