Resurgence and BPS invariants

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1908.07065: Grassi, Gu, Marino
2007.10190: Garoufalidis, Gu, Marino
2012.00062: Garoufalidis, Gu, Marino
2104.07437: Gu, Marino
2111.04763: Garoufalidis, Gu, Marino, Wheeler

- Models in QM and QFT are typically studied through perturbation series in weak coupling limit

$$\varphi(g_s) = \sum_{n=0}^{\infty} a_n g_s^n, \quad a_n \sim \frac{n!}{A^n}.$$

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- In (extended) supersymmetric theories, (solitonic) BPS states are special.
  - They are annihilated by some supercharges.
  - They saturate the BPS bound $M = |Z|$.
  - In a typical gauge theory with charge lattice $\Gamma \cong \mathbb{Z}^{2r}$, the central charge is discretely valued $Z_\gamma = \gamma \cdot Z$.
  - The number of BPS states $\Omega(\gamma)$ is stable with respect to moduli of the theory (up to codim 1 walls of marginal stability).

- There are various ways to compute $\Omega(\gamma)$:
  - Quiver mutation [Alim-Cecotti-Cordova-Espahbodi-Rastogi-Vafa],[Del Monte-Longhi]
  - Spectral (exponential) network [Gaiotto-Moore-Neitzke]
  - Coulomb branch formula [Manschot-Pioline-Sen]
  - GW-DT correspondence [Maulik-Nekrasov-Okounkov-Pandharipande],[Bridgeland]

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- A new approach for $\Omega(\gamma)$: Resurgence analysis of the perturbation series $\varphi(g_s)$ of
  - either the supersymmetric theory itself;
  - or a dual theory which might not be supersymmetric.

Resurgence theory

How to make sense of an asymptotic series

A typical Gevrey-1 asymptotic series in physics

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad a_n \sim \frac{n!}{A^n}.$$

- How do we "sum" the asymptotic series?
- Is it possible to relate the series to its (path) integral and the series from other saddles?

Borel resummation

$$\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$$

Borel transform

$$\widehat{\varphi}(\zeta) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \zeta^n$$

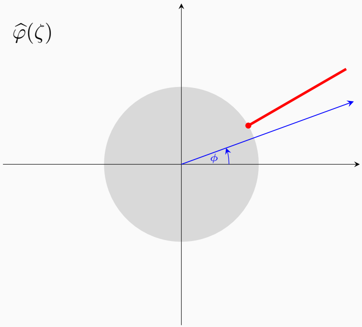
Laplace transform

$$s(\varphi)(z) = \int_0^{\infty} \mathrm{e}^{-\zeta} \widehat{\varphi}(z\zeta) \mathrm{d}\zeta$$

Borel resummation

The Borel resummation $s(\varphi)(z)$ reproduces the series $\varphi(z)$ in small z expansion

Borel resummation

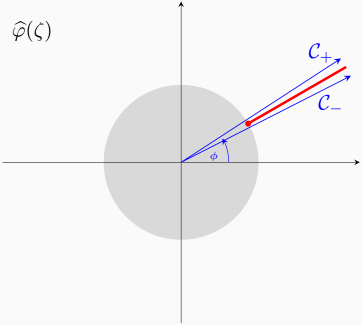


If there is no obstruction along $\phi = \arg z$ in the ζ-plane (Borel plane),

$$s(\varphi)(z) = \int_0^\infty e^{-\zeta}\widehat{\varphi}(e^{i\phi}|z|\zeta)d\zeta,$$

is a well defined integral.

Lateral Borel resummation



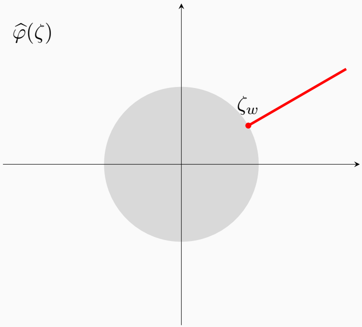
If there is obstruction along $\phi = \arg z$ (Stokes ray), one defines the lateral Borel resummations

$$s_{\pm}(\varphi)(z) = \int_0^{\mathrm{e}^{\mathrm{i}0^{\pm}}\infty} \mathrm{e}^{-\zeta}\widehat{\varphi}(z\zeta)\mathrm{d}\zeta,$$

and Stokes discontinuity

$$\mathrm{disc}(\varphi)(z) = s_{+}(\varphi)(z) - s_{-}(\varphi)(z).$$

Resurgent functions



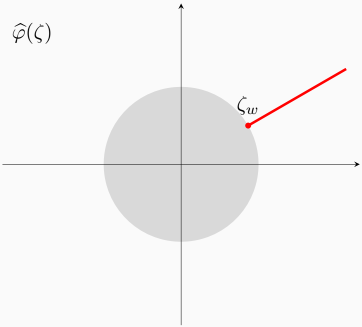
Expansion near ζ_w

$$\widehat{\varphi}(\zeta_w + \xi) = -\mathsf{S}_w \frac{\log(\xi)}{2\pi} \widehat{\varphi}_w(\xi) + \widehat{r}_w(\xi)$$

with regular functions $\widehat{r}_w(\xi)$ and

$$\widehat{\varphi}_w(\xi) = \sum_{n \geq 0} a_{n,w} \xi^n,$$

Resurgent functions



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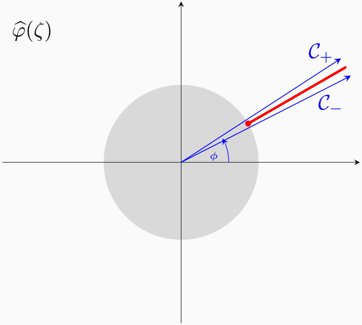
$$\widehat{\varphi}_w(\xi) = \sum_{n \geq 0} a_{n,w} \xi^n,$$

which is regarded as Borel transform of a resurgent series

$$\varphi_w(z) = \sum_{n \geq 0} a_{n,w} z^n, \quad \widehat{a}_{n,w} = \frac{a_{n,w}}{n!}.$$

Resurgent functions and Stokes discontinuity



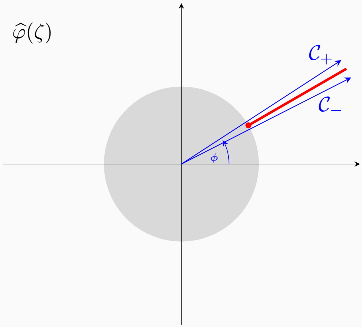
Resurgence at ζ_w

$$\widehat{\varphi}(\zeta_w + \xi) = -\mathsf{S}_w \frac{\log(\xi)}{2\pi \mathrm{i}} \widehat{\varphi}_w(\xi) + \widehat{r}_w(\xi)$$

implies Stokes discontinuity

$$\mathrm{disc}_\phi \varphi(z) = \mathsf{S}_w \, \mathrm{e}^{-\zeta_w/z} s_-(\varphi_w)(z)$$

with Stokes constant S_w.



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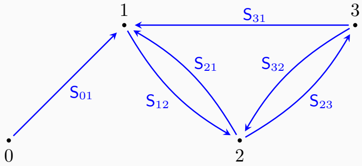
with Stokes constant S_w.

new saddle: $A_w - A_0 = \zeta_w$

Resurgent structure

Starting from one asymptotic series, one finds recursively resurgent asymptotic series, which form a resurgent structure:

$$\varphi_0(z) \to \{\varphi_w(z)\} \to \{\mathsf{S}_{ww'}\}$$



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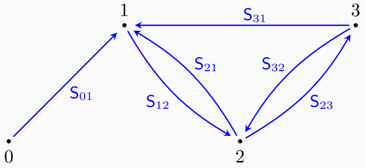
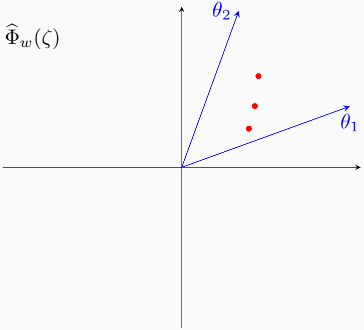


- $\{\mathsf{S}_{ww'}\}$ are new invariants, which are *non-perturbative* in nature.
- Sometimes $\mathsf{S}_{ww'}$ can be interpreted as counting of BPS states.

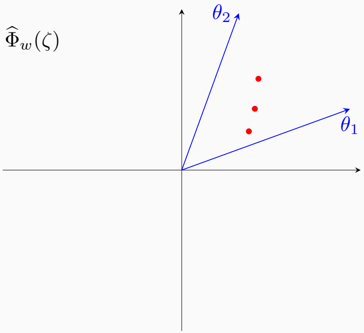
Stokes automorphism



(Local) Stokes automorphism $\mathfrak{S}_\theta$ at angle θ acting on trans-series $\Phi_w(z) = \mathrm{e}^{-A_w/z}\varphi_w(z)$

$$\mathfrak{S}_\theta \Phi_w = \Phi_w + \sum_{\arg(A_{w'}-A_w)=\theta} \mathsf{S}_{ww'}\Phi_{w'}.$$

Stokes automorphism



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Global Stokes automorphism between two angles

$$\mathfrak{S}_{\theta_1,\theta_2} = \prod_{\theta_1<\theta<\theta_2}^{\leftarrow} \mathfrak{S}_\theta.$$

- Ordered product;
- Unique factorisation.

Comparison with Wall-Crossing formula

Let us recall the Wall-Crossing formula of Kontsevich-Soibelman for BPS invariants.

- Let Γ be lattice of elec./mag. charges with pairing $\langle , \rangle$, functions $\mathcal{X}_\gamma : \mathcal{M} \to \mathbb{C}^*$.

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- Define symplectomorphism [Kontsevich,Soibelman][Gaiotto,Moore,Neitzke]

$$\mathfrak{S}(\theta) = \prod_{\gamma_{\mathrm{BPS}:\arg(-Z_{\gamma_{\mathrm{BPS}}})=\theta}} \mathcal{K}_{\gamma_{\mathrm{BPS}}}$$

where $\mathcal{K}_{\gamma_{\mathrm{BPS}}}$ acts by

$$\mathcal{K}_{\gamma_{\mathrm{BPS}}} : \mathcal{X}_\gamma \to \mathcal{X}_\gamma (1 - \sigma(\gamma_{\mathrm{BPS}}) \mathcal{X}_{\gamma_{\mathrm{BPS}}})^{\Omega(\gamma_{\mathrm{BPS}})\langle \gamma, \gamma_{\mathrm{BPS}} \rangle}$$

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- Global symplectomorphism (spectrum generator)

$$\mathfrak{S}(\theta_1, \theta_2) = \overset{\leftarrow}{\prod_{\theta_1 < \theta < \theta_2}} \mathfrak{S}(\theta).$$

  - Ordered product;
  - Unique factorisation.

Stokes constants vs BPS invariants

Stokes constants (if integers!)	BPS invariants
Stokes automorphism	KS symplectomorphism

Messages

- One could combine saddle action and saddle point expansion into trans-series.
- Saddle point trans-series are related to each other by Stokes automorphisms.
- Stokes automorphisms (constants) may be identified with KS symplectomorphism (BPS invariants).
- To compute Stokes constants, many terms in asymptotic series are required.

Example 1: Seiberg-Witten theory

Seiberg-Witten theory and its BPS spectrum

4d $\mathcal{N} = 2$ pure $SU(2)$ theory has moduli space identified with family of spectral curves
[Seiberg,Witten]

$$p^2 + 2\Lambda^2 \cosh x = 2u$$

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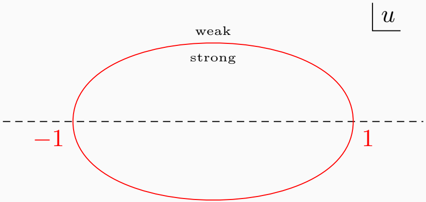
Wall of marginal stability



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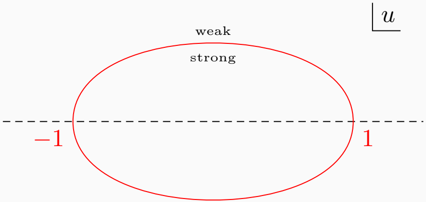


BPS spectrum

- $|u| < 1$: Strong coupling

$$\pm(0,1), \quad \pm(1,1)$$

- $|u| > 1$: Weak coupling

$$\pm(1,0), \quad \pm(\ell,1), \;\; \ell \in \mathbb{Z}$$

Quantum periods

Quantum spectral curve

$$-\hbar^2\psi''(x) + 2\Lambda^2 \cosh(x)\psi(x) = E\psi(x)$$

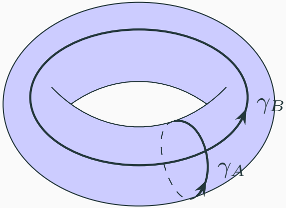
has WKB solutions

$$\psi(x, E) = \exp\left(\frac{\mathsf{i}}{\hbar}\int^x p(x, E; \hbar)\mathrm{d}x\right)$$

Quantum periods

Classical spectral curve

$H_1(\Sigma)$ gives lattice $\Gamma = \mathbb{Z}^2$ with pairing $\langle , \rangle$



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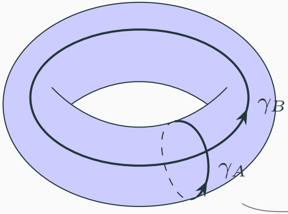

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Quantum periods: $\Pi_\gamma(E;\hbar) = \oint_\gamma p(x,E;\hbar)\mathrm{d}x = \sum_{n=0} \Pi_\gamma^{(n)}(E)\hbar^{2n}$

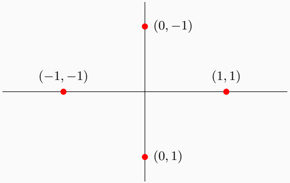
Voros symbols: $\Phi_\gamma(E;\hbar) = \mathrm{e}^{\frac{1}{\hbar}\Pi_\gamma(E;\hbar)} = \mathrm{e}^{\frac{1}{\hbar}\Pi_\gamma^{(0)}(E)} \exp\sum_{n\geq 1}\Pi_\gamma^{(n)}(E)\hbar^{2n-1}$

As solutions to an ODE, $p(x;\hbar)$ and thus $\Pi_\gamma(E;\hbar)$ can be computed efficiently to many terms.

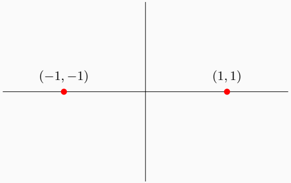
Stokes automorphism

Borel singularities of quantum periods

- $u = 0$



$\Pi_A(\hbar)$



$\Pi_B(\hbar)$

Stokes automorphism

Borel singularities of quantum periods

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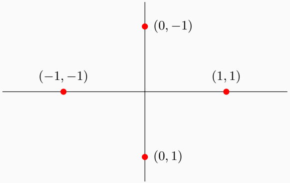

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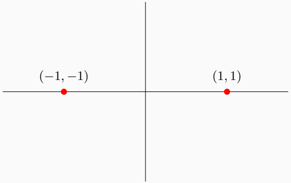
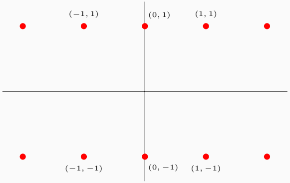


$\Pi_B(\hbar)$

- $u = E/2 = 4$



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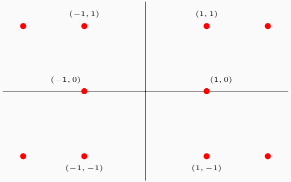

$\Pi_B(\hbar)$

Identification

A,B cycles	elec., mag. charges
Saddle points	BPS states
Classical period $\Pi_\gamma^{(0)}$	Central charge Z_γ

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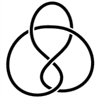
Saddle points	BPS states
Classical period $\Pi_\gamma^{(0)}$	Central charge Z_γ
Voros symbol Φ_γ	function $\mathcal{X}_\gamma$
Stokes automorphism	KS symplectomorphism
$\frac{1}{\hbar}\Pi_\gamma \to \frac{1}{\hbar}\Pi_\gamma + \mathsf{S}_{\gamma\gamma'} \log(1 - \sigma_{\gamma'} \mathrm{e}^{\frac{1}{\hbar}\Pi'_\gamma})$	$\mathcal{X}_\gamma \to \mathcal{X}_\gamma \left(1 - \sigma_{\gamma_{\mathrm{BPS}}} \mathcal{X}_{\gamma_{\mathrm{BPS}}}\right)^{\Omega_{\gamma_{\mathrm{BPS}}} \langle \gamma, \gamma_{\mathrm{BPS}} \rangle}$
Stokes constants $\mathsf{S}_{\gamma\gamma'}$	BPS invariants $\Omega_{\gamma_{\mathrm{BPS}}} \langle \gamma, \gamma_{\mathrm{BPS}} \rangle$

Example 2: Complex Chern-Simons theory

Chern-Simons

$SL(n,\mathbb{C})$ CS on

$M = S^3 \backslash K$



Quantum Topology

HOMFLY-PT (Jones, Alexander) polynomial

HOMFLY-PT (Khovanov, knot Floer) homology

phys. real.

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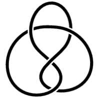

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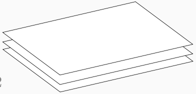
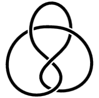
3d-3d correspondence

3d SCFT

$T_n[M]$ on $S^1 \times \mathbb{R}^2$





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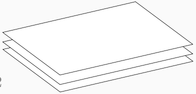
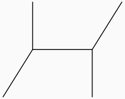


String Theory

Open GW invariants on resolved

conifold $O(-1) \oplus O(-1) \to \mathbb{CP}^1$



Action and saddle points

- Consider (complex) Chern-Simons theory with gauge algebra $\mathfrak{g}$ on a 3d manifold M with the action [Witten][Gukov]

$$S = \frac{t}{8\pi}\int_M \mathrm{Tr}\left(A\wedge \mathrm{d}A + \frac{2}{3}A\wedge A\wedge A\right) + \frac{\tilde{t}}{8\pi}\int_M \mathrm{Tr}\left(\overline{A}\wedge \mathrm{d}\overline{A} + \frac{2}{3}\overline{A}\wedge \overline{A}\wedge \overline{A}\right)$$

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- Saddle points are $\mathfrak{g}$ flat connections on M

$$\mathrm{d}A + A\wedge A = 0, \qquad A\in\mathfrak{g},$$

classified via holonomies

$$\sigma : H_1(M)\to\mathbb{C}.$$

Saddle expansion of Chern-Simons

- Saddle point expansion around the flat connection σ [Dimofte-Gukov-Lenells-Zagier]

$$Z^{(\sigma)}(M,\hbar) \sim \exp\left(\frac{1}{\hbar}S_0^{(\sigma)} - \frac{1}{2}\delta^{(\sigma)}\log\hbar + \sum_{n=0}^{\infty} S_{n+1}^{(\sigma)}\hbar^n\right), \quad \hbar = 2\pi/t.$$

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- Let $\mathfrak{g} = SL(2,\mathbb{C})$ and $M = S^3\backslash K$ ($\cong$ solid torus):
  - Trivial (Abelian) flat connection $S_0^{(\sigma_0)} = 0$.
  - Geometric flat connection (Volume Conjecture) $S_0^{(\sigma_1)} = \mathrm{Vol}(M) + \mathsf{i}\,\mathrm{CS}(M)$.

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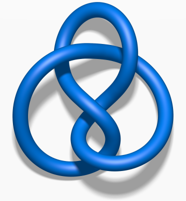
- Let $\mathfrak{g} = SL(2,\mathbb{C})$ and $M = S^3\backslash K$ ($\cong$ solid torus):
  - Trivial (Abelian) flat connection $S_0^{(\sigma_0)} = 0$.
  - Geometric flat connection (Volume Conjecture) $S_0^{(\sigma_1)} = \mathrm{Vol}(M) + \mathsf{i}\,\mathrm{CS}(M)$.
- To work out the resurgent structure of $Z^{(\sigma)}(M,\hbar)$, we need to compute the trans-series efficiently.

Non-trivial flat connections: state integrals



- Perturbation series at non-Abelian flat connections are encoded in the state integral [Dimofte-Gukov-Lenells-Zagier]. For figure eight knot ($\mathbf{4}_1$) [Hikami][Andersen,Kashaev]

$$Z_{\mathbf{4}_1}(\mathsf{b}) = \int_{\mathbb{R}+\mathrm{i}0} \Phi_{\mathsf{b}}(v)^2 \mathrm{e}^{-\pi \mathrm{i} v^2} \mathrm{d}v, \quad \hbar = 2\pi \mathsf{b}^2$$

whose main ingredient is Faddeev's quantum dilogarithm $\Phi_{\mathsf{b}}(v)$.

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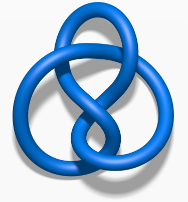

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- It has two saddle points for geom. and conj. flat connections

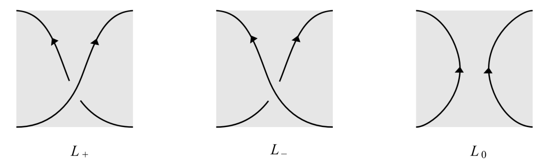
$$\begin{aligned} Z^{(\sigma_1)}(\hbar) =& \mathrm{e}^{\mathcal{V}/\hbar}\left(1 + \frac{11\hbar}{72\sqrt{3}} + \frac{697\hbar^2}{2(72\sqrt{3})^2} + \ldots\right), \\ Z^{(\sigma_2)}(\hbar) =& \mathrm{i}\, Z^{(\sigma_1)}(-\hbar) \end{aligned}$$

with $\mathcal{V} = \mathrm{Vol}(S^3 \backslash \mathbf{4}_1)$.

- Can be computed efficiently with Gaussian expansion up to ~ 300 terms.

Jones polynomial

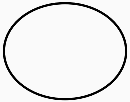
- Using skein relation:

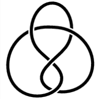


with

$$q^{-1}J_{L_+}(q) - qJ_{L_-}(q) = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})J_{L_0}(q)$$

to compute





Unknot: $J^{\text{unknot}}(q) = 1$ Figure eight: $J^{\mathbf{4}_1}(q) = q^2 - q + 1 - q^{-1} + q^{-2}$

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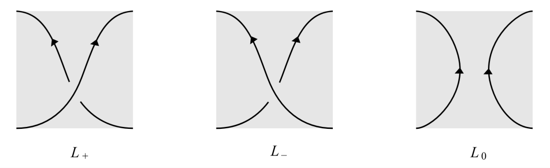

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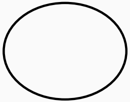

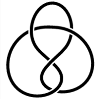

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- Promotion to Khovanov homology:

$$J^K(q) = \sum_{i,j} (-1)^i q^j \dim Kh_{i,j}(K)$$

Trivial flat connection: Jones polynomial

- Colored Jones polynomial

$$J_n^{\mathbf{4}_1}(q) = \sum_{k=0}^{n-1} (-1)^k q^{-k(k+1)/2} \prod_{j=1}^{k} (1 - q^{j+n})(1 - q^{j-n}).$$

is the vev of Wilson loop $\langle \mathbf{4}_1 \rangle_n$ along $\mathbf{4}_1$ with repr. n of $SL(2, \mathbb{C})$ and $q = \exp \frac{2\pi \mathrm{i}}{t}$ [Witten].

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- $J_n^{\mathbf{4}_1}(q)$ allows loop expansion

$$J_n^{\mathbf{4}_1}(e^h) = \sum_{i=0}^{\infty} \sum_{j=0}^{i} a_{i,j} n^j h^i \in \mathbb{Q}[[n,h]]$$

and the perturbative series $Z^{(\sigma_0)}(\hbar)$ for trivial flat connections is

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$$Z^{(\sigma_0)}(\hbar) = \sum_{i=0}^{\infty} a_{i,0} \hbar^i.$$

- $J_n^{\mathbf{4}_1}(q)$ (and thus $Z^{(\sigma_0)}(\hbar)$) can be computed efficiently by recursion relations in n.

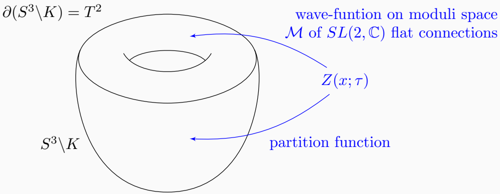
Another way of computation: Quantum A-polynomial

- Turn on holonomy on the boundary T^2: $Z^{(\sigma)}(\hbar) \to Z^{(\sigma)}(x, \hbar)$.
- $Z^{(\sigma)}(x, \hbar)$ satisfy the difference equation (quantum A-polynomial)

$$\widehat{A}(\hat{x}, \hat{y}) Z(x, \hbar) = 0,$$

with

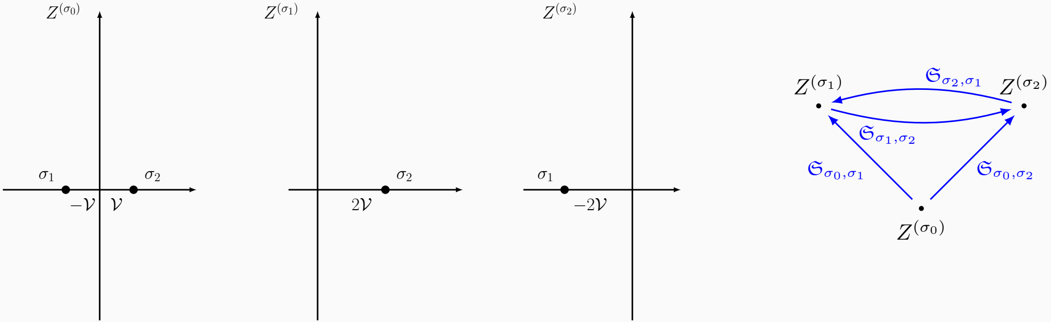
$$\hat{x} Z(x, \hbar) = x Z(x, \hbar), \quad \hat{y} Z(x, \hbar) = Z(qx, \hbar).$$



$\widehat{A}$ is the Schrödinger equation on $\mathcal{M}$. [Gukov]

Borel singularities

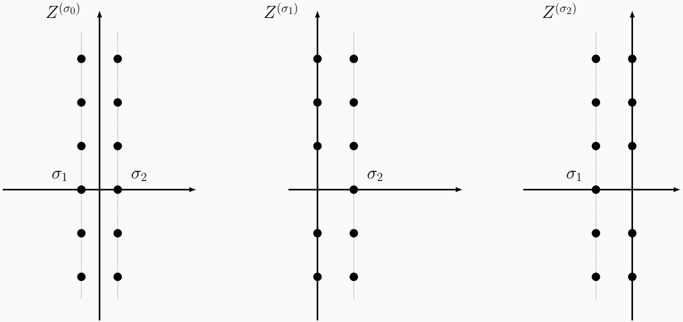
"Classical" Borel singularities [Gukov-Marino-Putrov][Gang-Hatsuda][Garoufalidis-Zagier]

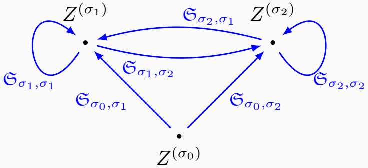


Borel singularities

More singularities due to multivaluedness of CS action and the state integral potential

[Garoufalidis][Witten][Gukov-Marino-Putrov]



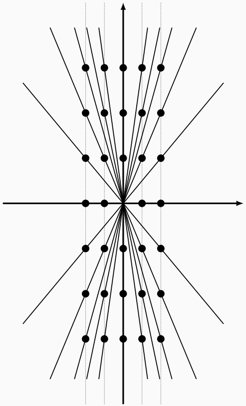


A family of trans-series but with the *same* power series

$$Z_n^{(\sigma_j)}(\hbar) = Z^{(\sigma_j)}(\hbar)\mathrm{e}^{-n\frac{4\pi^2\mathrm{i}}{\hbar}}, \quad n \in \mathbb{Z}.$$

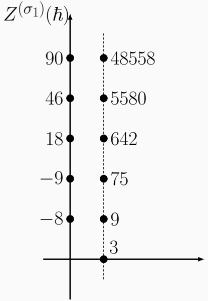
Peacock pattern of Stokes rays

- Stokes rays in the Borel plane for the vector $(Z^{(\sigma_0)}(\hbar), Z^{(\sigma_1)}(\hbar), Z^{(\sigma_2)}(\hbar))^T$.



Stokes constants are non-trivial integers

The Stokes constants are non-trivial *integers*!



- Complete set of Stokes constants can be solved!
- The Stokes q-series

$$\mathsf{S}^{\pm}_{\sigma\sigma'}(q) = 1 + \sum_{n=1}^{\infty} \mathsf{S}_{\sigma\sigma';\pm n} q^{\pm n}, \quad \mathsf{S}_{\sigma\sigma';\pm n} \in \mathbb{Z}$$

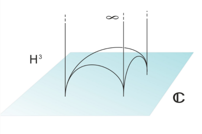
are given by bilinear expressions in fundamental solutions of the equation

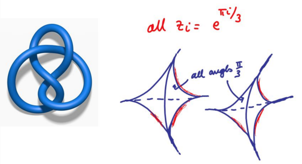
$$y_{m+1}(q) + y_{m-1}(q) - (2 - q^m) y_m(q) = 0$$

3d-3d correspondence

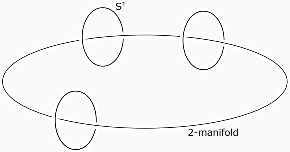
Wrap n M5 branes on $M \times \Lambda$, with topological twist on M

- M is a 3d manifold that allows hyperbolic metric: tetrahedron, $S^3 \backslash K$.



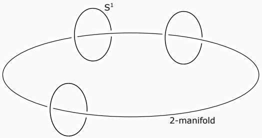


- Λ is a 3d Seifert manifold that has S^1 fibration: $\mathbb{R}^2 \times_q S^1$, $S^2 \times_q S^1$, $S^3_b/\mathbb{Z}_k$.



3d-3d correspondence

There are two possibities



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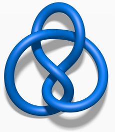
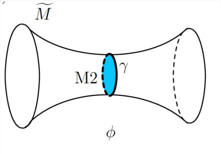


- Shrinking M leads to 3d $N = 2$ Chern-Simons-matter theory that flows to SCFT $T_n[M]$ on Λ in IR.
- Shrinking Λ leads to 3d $SL(n, \mathbb{C})$ Chern-Simons theory on M (supersymmetry is broken).

SUSY ground states of $T_n[M]$ = $SL(n, \mathbb{C})$ flat connections on M

3d-3d correspondence

- BPS states in $T_n[M]$ arise from M2 branes ending on M5 branes.



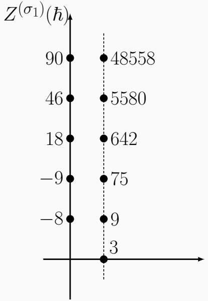
- If $M = S^3 \backslash K$, $T_n[M]$ has a $U(1)$ flavor symmetry. Define supersymmetric index [Dimofte,Gaiotto,Gukov]

$$\mathrm{Ind}(m, \zeta; q) = \mathrm{Tr}_{\mathcal{H}_m} (-1)^F q^{\frac{R}{2} + j_3} \zeta^e .$$

- $\mathrm{Ind}(m, \zeta; q)$ is SUSY partition function of $T_n[M]$ on $S^2 \times_q S^1$.

Integer Stokes constants as BPS counting

The Stokes constants are non-trivial *integers*!



- Generating series of Stokes constants in positive imaginary axis

$$\mathsf{S}^{+}_{\sigma_1\sigma_1}(q) = 1 - 8q - 9q^2 + 18q^3 + 46q^4 + 90q^5 + \ldots, \quad q = \mathrm{e}^{4\pi^2 \mathrm{i}/\hbar}.$$

(Conjecture) It coincides with index $\mathrm{Ind}(0,1;q)$ of dual 3d superconformal field theory!

$$\mathrm{Ind}(m,\zeta;q) = \mathrm{Tr}_{\mathcal{H}_m}(-1)^F q^{\frac{R}{2}+j_3}\zeta^e.$$

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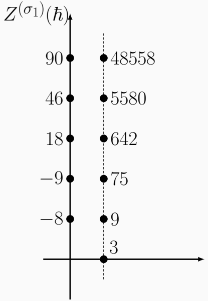

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- The generating series for other Stokes constants $\mathsf{S}^{+}_{\sigma_1\sigma_2}(q), \mathsf{S}^{+}_{\sigma_2\sigma_2}(q)$ are also identified with the index with magnetic flux turned on.

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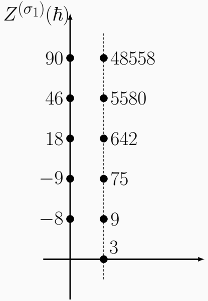

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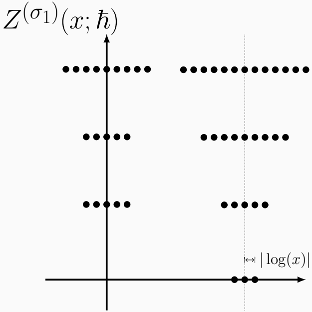
- *Can we turn on flavor fugacity ζ?*

Turning on deformation



- Turning on holonomoy $x = \mathrm{e}^u$ on boundary

$$Z^{(\sigma_{1,2})}(\hbar) \to Z^{(\sigma_{1,2})}(x;\hbar) \sim \int \Phi_{\mathsf{b}}(v)\Phi_{\mathsf{b}}(v+u)\mathrm{e}^{-\pi\mathsf{i}(v^2+4uv)}\mathrm{d}v$$

- Generating series of Stokes constants in vertical towers

$$\begin{aligned}\mathsf{S}^{+}_{\sigma_1\sigma_1}(x;q) =& 1-(2x^{-2}+x^{-1}+2+x+2x^2)q\\ &-(x^{-2}+2x^{-1}+3+2x+x^2)q^2+\mathcal{O}(q^3)\end{aligned}$$

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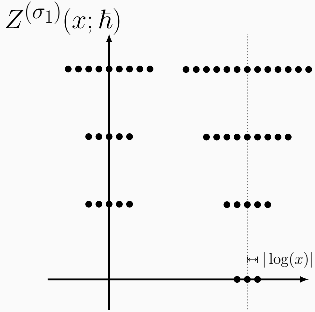

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- They coincide with the index $\mathrm{Ind}(m,x;q)$ with the flavor fugacity turned on.

Conclusions and open questions

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- Stokes constants define new *non-perturbative* invariants.
- In some models (SW theory, complex Chern-Simons, topological string) they are non-trivial integers and are BPS countings,
- and they can be solved *completely*.

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- In some models (SW theory, complex Chern-Simons, topological string) they are non-trivial integers and are BPS countings,
- and they can be solved *completely*.

Open questions

- Proof of BPS interpretation of Stokes constants in complex Chern-Simons? [Gregory Moore, "Number Theory, Strings, and Quantum Physics", Jun-2021]
- BPS interpretation of Stokes constants in $\mathsf{S}^{+}_{\sigma_0,\sigma_1}(q), \mathsf{S}^{+}_{\sigma_0,\sigma_2}(q)$?
- Resurgence in other theories where perturbative coefficients are efficiently computable (integrable models)?

Thank you for your attention!