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Calculus of Variations

Gradient estimates and Parabolic frequency under the Laplacian G_2 flow

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Abstract

In this paper, we consider the Laplacian G_2 flow on a closed seven-dimensional manifold M with a closed G_2-structure. We first obtain the gradient estimates for positive solutions of the heat equation under the Laplacian G_2 flow and then we get the Harnack inequality on spacetime. As an application, we prove the monotonicity of parabolic frequency for positive solutions of the heat equation with bounded Ricci curvature, and get the integral-type Harnack inequality. Besides, we prove the monotonicity of parabolic frequency for solutions of the linear heat equation with bounded Bakry-Émery Ricci curvature, and then obtain the backward uniqueness.

Mathematics Subject Classification Primary 53E99 · 58J35

1 Introduction

1.1 Gradient estimates under the Laplacian G_2 flow

In [16, 32], P. Li, S.-T. Yau and Hamilton obtained the following gradient estimates for positive solutions of the heat equation on a closed Riemannian manifold with Ricci curvature bounded below.

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Theorem A (Li-Yau, [32]) *Let (M, g) be a closed n-dimensional manifold with nonnegative Ricci curvature, and $u = u(x, t)$ be a positive solution of the heat equation on $M \times (0, \infty)$. Then the following estimate*

$$\frac{|\nabla u|^2}{u^2} - \frac{\partial_t u}{u} \leq \frac{n}{2t}$$

holds on $M \times (0, \infty)$.

Theorem B (Hamilton, [16]) *Let (M, g) be a closed n-dimensional manifold with $\mathrm{Ric} \geq -Kg$ for some $K \geq 0$, and $u = u(x, t)$ be a positive solution of the heat equation with $u(x, t) \leq A$ for all $(x, t) \in M \times (0, \infty)$, where A is a positive constant. Then the following estimate*

$$\frac{|\nabla u|^2}{u^2} \leq \left(\frac{1}{t} + 2K\right) \ln \frac{A}{u}$$

holds on $M \times (0, \infty)$.

These two estimates provide a versatile tool for studying the analytical, topological, and geometrical properties of manifolds.

In 2010, Băileşteanu-Cao-Pulemotov [3] obtained the Li-Yau estimate for positive solutions of the heat equation when the metrics $g(t)$ are evolved by the Ricci flow

$$\partial_t g(t) = -2\mathrm{Ric}(g(t)). \tag{1.1}$$

The Ricci flow was introduced by Hamilton in [17] to study the compact three-manifolds with positive Ricci curvature, which is a special case of the Poincaré conjecture finally proved by Perelman in [43, 44]. Hamilton [17] obtained the short-time existence and uniqueness of the Ricci flow on compact manifolds, and Shi [46] obtained a short-time solution of the Ricci flow on a complete noncompact manifold and the uniqueness with bounded Riemann curvature was proved by Chen-Zhu in [7]. After that, many people began to study the gradient estimate for the positive solutions of the heat equation when the metrics are evolved by geometric flows see [2, 38, 47].

In this paper, we first study gradient estimates for positive solutions of the heat equation under the Laplacian G_2 flow for closed G_2-structure:

$$\begin{cases} \partial_t \varphi(t) = \Delta_{\varphi(t)} \varphi(t), \\ \varphi(0) = \varphi, \end{cases} \tag{1.2}$$

which was introduced by Bryant [6] on a smooth 7-manifold M admitting closed G_2-structure, where $\Delta_{\varphi(t)}\varphi(t) = dd^*_{\varphi(t)}\varphi(t) + d^*_{\varphi(t)} d\varphi(t)$ is the Hodge Laplacian of $g(t)$ and φ is an initial closed G_2-structure. Here $g(t)$ is the associated Riemannian metric of $\varphi(t)$. Since for a closed G_2-structure φ, $\Delta_\varphi \varphi = dd^*_\varphi \varphi$, we see that the closedness of $\varphi(t)$ is preserved along the Laplacian G_2 flow (1.2). The existence for the solution of the Laplacian G_2 flow can be found in [6, 13, 29, 35, 41].

We first consider the following Li-Yau type gradient estimate of the heat equation

$$\partial_t u(t) = \Delta_{g(t)} u(t) \tag{1.3}$$

under the Laplacian G_2 flow (1.2), where $\Delta_{g(t)} = \mathrm{tr}_{g(t)}\left(\nabla^2_{g(t)}\right)$ is the trace Laplacian induced by $g(t)$.

Theorem 1.1 *Let $(M, \varphi(t))_{t\in(0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T < +\infty$ and $-Kg(t) \leq \mathrm{Ric}(g(t)) \leq 0$, where $g(t)$ is the*

Riemannian metric associated with $\varphi(t)$ and K is a positive constant. If $u(t)$ is a positive solution of the heat equation (1.3), then on $M \times (0, T]$, the following estimate

$$\frac{|\nabla_{g(t)} u(t)|^2_{g(t)}}{u^2(t)} - \alpha \frac{\partial_t u(t)}{u(t)} \le \frac{7\alpha}{2at} + \left(\frac{49\alpha}{3a} + \frac{105\alpha^2 - 98\alpha}{2a(\alpha - 1)} + \frac{7\sqrt{29}\alpha}{2\sqrt{ab}} \right) K \tag{1.4}$$

holds for any $\alpha > 1$ and $a, b > 0$ with $a + 2b = \frac{1}{\alpha}$.

As an application, we can get the following Harnack inequality on spacetime.

Corollary 1.2 *Let $(M, \varphi(t))_{t\in(0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T < +\infty$ and $-Kg(t) \le \mathrm{Ric}(g(t)) \le 0$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$ and K is a positive constant. If $u(t)$ is a positive solution of the heat equation (1.3), then for $(x, t_1), (y, t_2) \in M \times (0, T]$ with $t_1 < t_2$, we have*

$$u(x, t_1) \le u(y, t_2) \left(\frac{t_2}{t_1}\right)^{\frac{7}{2a}} \exp\left\{ \int_0^1 \left[\frac{\alpha |\gamma'(s)|^2_{\sigma(s)}}{4(t_2 - t_1)} + (t_2 - t_1) C_{a,b,\alpha} K \right] ds \right\},$$

where $\alpha > 1$,

$$C_{a,b,\alpha} = \frac{49}{3a} + \frac{105\alpha - 98}{2a(\alpha - 1)} + \frac{7\sqrt{29}}{2\sqrt{ab}},$$

$a, b > 0$ with $a + 2b = \frac{1}{\alpha}$, $\gamma(s)$ is a geodesic curve connecting x and y with $\gamma(0) = y$ and $\gamma(1) = x$, and $|\gamma'(s)|_{\sigma(s)}$ is the length of the vector $\gamma'(s)$ at $\sigma(s) = (1 - s)t_2 + st_1$.

For the Hamilton type gradient estimate of the heat equation under the Laplacian G_2 flow (1.2), we have

Theorem 1.3 *Let $(M, \varphi(t))_{t\in(0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T < +\infty$ and $-Kg(t) \le \mathrm{Ric}(g(t)) \le 0$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$ and K is a positive constant. If $u(t)$ is a positive solution of the heat equation, then on $M \times (0, T]$, the following estimate*

$$|\nabla_{g(t)} u(t)|^2_{g(t)} \le \frac{u(t)}{t} \left[u(t) \ln \frac{A}{u(t)} + \lambda A^2 - \lambda \eta^2 \right] \tag{1.5}$$

holds, where $\eta = \min_M u(0)$, $A = \max_M u(0)$ and λ is a constant depending on K, η and T.

1.2 Parabolic frequency under the Laplacian G_2 flow

In 1979, the (elliptic) frequency functional for a harmonic function $u(x)$ on $\mathbb{R}^n$ was introduced by Almgren in [1], which was defined by

$$N(r) = \frac{r \displaystyle\int_{B(r,p)} |\nabla u(x)|^2 dx}{\displaystyle\int_{\partial B(r,p)} u^2(x) d\sigma},$$

where $d\sigma$ is the induced $(n-1)$-dimensional Hausdorff measure on $\partial B(r, p)$, $B(r, p)$ is the ball in $\mathbb{R}^n$ and p is a fixed point in $\mathbb{R}^n$. Almgren obtained that $N(r)$ is monotone nondecreasing for r, and he used this property to investigate the local regularity of harmonic functions and minimal surfaces. Next, Garofalo and Lin [14, 15] considered the monotonicity of frequency functional on Riemannian manifolds to study the unique continuation for elliptic operators. The frequency functional was also used to estimate the size of nodal sets in [39, 40]. For more applications, see [9, 18, 19, 36, 49].

The parabolic frequency for the solution of the heat equation on $\mathbb{R}^n$ was introduced by Poon in [45], and Ni [42] considered the case when $u(t)$ is a holomorphic function, both of them showed that the parabolic frequency is nondecreasing. Besides, on Riemannian manifolds, the monotonicity of the parabolic frequency was obtained by Colding and Minicozzi [10] through the drift Laplacian operator. Using the matrix Harnack's inequality in [16], Li and Wang [33] investigated the parabolic frequency on compact Riemannian manifolds and the 2-dimensional Ricci flow.

In [5], Baldauf-Kim defined the following parabolic frequency for a solution $u(t)$ of the heat equation

$$U(t) = -\frac{\tau(t)\|\nabla_{g(t)}u(t)\|^2_{L^2(d\nu)}}{\|u(t)\|^2_{L^2(d\nu)}} \cdot \exp\left\{-\int_{t_0}^{t} \frac{1-\kappa(s)}{\tau(s)} ds\right\},$$

where $t \in [t_0, t_1] \subset (0, T)$, $\tau(t)$ is the backwards time, $\kappa(t)$ is the time-dependent function and $d\nu$ is the weighted measure. They proved that parabolic frequency $U(t)$ for the solution of the heat equation is monotone increasing along the Ricci flow with the bounded Bakry-Émery Ricci curvature and obtained the backward uniqueness. Baldauf, Ho and Lee derived analogous result to the mean curvature flow in [4].

Recently, Liu and Xu studied the monotonicity of parabolic frequency for the weighted p-Laplacian heat equation on Riemannian manifolds in [37], and they obtained a theorem of Hardy-Pólya-Szegö on Kähler manifolds under the Kähler-Ricci flow. In [34], Li and Zhang derived the matrix Li-Yau-Hamilton estimates for positive solutions to the heat equation and the backward conjugate heat equation under the Ricci flow, and then applied these estimates to study the monotonicity of the parabolic frequency.

In [30], the authors studied the monotonicity of parabolic frequency under Ricci flow and the Ricci-harmonic flow on manifolds. They considered two cases: one is the monotonicity of parabolic frequency for the solution of the linear heat equation with bounded Bakry-Émery Ricci curvature, and the other case is the monotonicity of parabolic frequency for the solution of the heat equation with bounded Ricci curvature.

Inspired by [30], we first study the parabolic frequency for the solution of the heat equation (1.3) under the Laplacian G_2 flow (1.2) with bounded Ricci curvature. The parabolic frequency for the positive solution of the heat equation (1.3) is defined by

$$U(t) = \exp\left\{-\int_{t_0}^{t}\left[\frac{h'(s)}{h(s)} - \frac{2}{3}R_0 + \frac{28 + C_1(A,\eta)}{s} + cK + \frac{7}{2}C(s)\right]ds\right\} \cdot \frac{h(t)\int_M |\nabla_{g(t)}u(t)|^2_{g(t)}d\mu_{g(t)}}{\int_M u^2(t)\,d\mu_{g(t)}}$$

where $h(t)$ is a time-dependent function, K and c are both positive constants,

$$R_0 = \min_{M\times[t_0,t_1]} R(t), \quad C_1(A,\eta) = \ln\frac{A}{\eta} + \lambda\frac{A^2}{\eta}, \quad C(t) = \frac{C_1(A,\eta)}{t}$$

and λ is the constant in Theorem 1.3,

$$\eta = \min_M u(0), \quad A = \max_M u(0).$$

Observe that, A and η are both positive constants. Using Theorem 1.1 and Theorem 1.3 as an application, we have

Theorem 1.4 *Let $(M, \varphi(t))_{t\in[0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T < +\infty$ and $-Kg(t) \leq \mathrm{Ric}(g(t)) \leq 0$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$ and K is a positive constant. If $u(t)$ is a positive solution of the heat equation (1.3) with $\eta \leq u(0) \leq A$, then the following holds.*

(i) *If $h(t)$ is a negative time-dependent function, then the parabolic frequency $U(t)$ is monotone increasing along the Laplacian G_2 flow.*
(ii) *If $h(t)$ is a positive time-dependent function, then the parabolic frequency $U(t)$ is monotone decreasing along the Laplacian G_2 flow.*

Besides, we consider the parabolic frequency for the solution of the linear heat equation

$$(\partial_t - \Delta_{g(t)})u(t) = a(t)u(t) \tag{1.6}$$

under the Laplacian G_2 flow (1.2) with bounded Bakry-Émery Ricci curvature, where $a(t)$ is a time-dependent smooth function. The parabolic frequency is defined by

$$U(t) = \exp\left\{-\int_{t_0}^{t}\left[-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right]ds\right\}\frac{h(t)\displaystyle\int_M |\nabla_{g(t)}u(t)|^2_{g(t)}d\mu_{g(t)}}{\displaystyle\int_M u^2(t)\,d\mu_{g(t)}},$$

where $R_0 = \min_{M\times[t_0,t_1]} R(t)$, $h(t)$ and $\kappa(t)$ are both time-dependent smooth functions. Then we get the following theorem, where $\mathrm{Ric}_{f(t)}$ is given in (2.10).

Theorem 1.5 *Let $(M, \varphi(t))_{t\in[0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T < +\infty$ and $Ric_{f(t)} \leq \dfrac{\kappa(t)}{2h(t)}g(t)$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$. Then the following holds.*

(i) *If $h(t)$ is a negative time-dependent function, then the parabolic frequency $U(t)$ is monotone increasing along the Laplacian G_2 flow.*
(ii) *If $h(t)$ is a positive time-dependent function, then the parabolic frequency $U(t)$ is monotone decreasing along the Laplacian G_2 flow.*

The backward uniqueness of solutions to parabolic equations has been the object of consistent study for at least half a century. There are already many results for heat operators concerning it in various domains, such as the exterior domain [11], the half-space [12] and some cones [31, 48]. For the heat equation on manifolds, Colding and Minicozzi [10] obtained the backward uniqueness result. Kotschwar showed a backward uniqueness result to Ricci flow in [25] and gave a general backward uniqueness theorem in [26]. For more backward uniqueness results of geometric flows, see [20–22, 28, 50].

As an application of Theorem 1.5, we get the following backward uniqueness.

Corollary 1.6 *Let $(M, \varphi(t))_{t\in[0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T < +\infty$ and $Ric_{f(t)} \leq \dfrac{\kappa(t)}{2h(t)}g(t)$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$. If $u(t_1) = 0$, then $u(t) \equiv 0$ for any $t \in [t_0, t_1] \subset (0, T)$.*

For the general parabolic equation, we have

Theorem 1.7 *Let $(M, \varphi(t))_{t\in[0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T < +\infty$ and $Ric_{f(t)} \leq \dfrac{\kappa(t)}{2h(t)}g(t)$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$ and $h(t)$ is a negative time-dependent function. Suppose $u(t) : M \times [t_0, t_1] \to \mathbb{R}$ satisfies*

$$\left|\left(\partial_t - \Delta_{g(t)}\right) u(t)\right| \leq C(t)\left[|\nabla_{g(t)} u(t)|_{g(t)} + |u(t)|\right]$$

along the Laplacian G_2 flow (1.2). If $u(t_1) = 0$, then $u(t) \equiv 0$ for all $t \in [t_0, t_1] \subset (0, T)$.

We give an outline of this paper. We review the basic theory in Sect. 2 about G_2-structures, G_2-decompositions of 2-forms and 3-forms, and the torsion tensors of G_2-structures. We also calculate the conjugate heat equation under the Laplacian G_2 flow (1.2). Section 3 proves the Li-Yau type gradient estimate and Hamilton type gradient estimate under the Laplacian G_2 flow (1.2) with bounded Ricci curvature, and as an application, we get the Harnack inequality on spacetime. In Sect. 4, using the Li-Yau type gradient estimate and Hamilton type gradient estimate, we prove the monotonicity of parabolic frequency for the solution of the linear equation (1.6) under the Laplacian G_2 flow (1.2) with bounded Ricci curvature, then we get the integral-type Harnack inequality. In Sect. 5, we consider the monotonicity of parabolic frequency for the heat equation and the general parabolic equation under the Laplacian G_2 flow (1.2) with bounded Bakry-Émery Ricci curvature, and obtain the backward uniqueness.

2 G_2-structure, notations and definitions

In this section, we introduce the G_2-structure on manifolds, G_2-decompositions, the torsion tensor, some notations, and definitions.

2.1 G_2-structure on smooth manifolds

Let $\mathbb{O}$ be the octonions (exceptional division algebra), from the vector cross product "$\times$" on Im $\mathbb{O}$, we can define the 3-form by

$$\phi(a, b, c) := \frac{1}{2}\langle a, [b, c]\rangle = \langle a \times b, c\rangle \qquad \text{for } a, b, c \in \text{Im }\mathbb{O}.$$

Let $\{e_1, e_2, \cdots, e_7\}$ denote the standard basis of $\mathbb{R}^7$ and $\{e^1, e^2, \cdots, e^7\}$ be its dual basis. Using the octonion multiplication table, one can show that

$$\phi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where $e^{ijk} := e^i \wedge e^j \wedge e^k$. When we fix ϕ, the subgroup of $\text{GL}(7, \mathbb{R})$ is the exceptional Lie group G_2, which is a compact, connected, simple 14-dimensional Lie subgroup of $\text{SO}(7)$. In fact, G_2 acts irreducibly on $\mathbb{R}^7$ and preserves the metric and orientation for which $\{e_1, e_2, \cdots, e_7\}$ is an oriented orthonormal basis. Note that G_2 also preserves the 4-form

$$*_\phi\phi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247},$$

where $*_\phi$ is the Hodge star operator determined by the metric and orientation.

Remark 2.1 The vector cross product "×" is an algebraic structure defined in a normed division algebra. Therefore, the G_2-structure can only be defined in the 7-dimensional manifold. For more details, see [24].

For a smooth 7-manifold M and a point $x \in M$, we define as in [35, 41]

$$\wedge^3_+(T^*_x M) := \left\{ \varphi_x \in \wedge^3(T^*_x M) \;\middle|\; h^*\phi = \varphi_x, \text{ for invertible } h \in \mathrm{Hom}_{\mathbb{R}}(T^*_x M, \mathbb{R}^7) \right\}$$

and the bundle

$$\wedge^3_+(T^*M) := \bigcup_{x \in M} \wedge^3_+(T^*_x M).$$

We call a section φ of $\wedge^3_+(T^*M)$ a *positive* 3*-form* on M or a G_2*-structure* on M, and denote the space of positive 3-form by $\Omega^3_+(M)$. The existence of G_2-structure is equivalent to the property that M is oriented and spin, which is equivalent to the vanishing of the first two Stiefel-Whitney classes $\omega_1(TM)$ and $\omega_2(TM)$. For more details, see Theorem 10.6 in [27].

For a 3-form φ, we define a $\Omega^7(M)$-valued bilinear form B_φ by

$$\mathrm{B}_\varphi(u, v) = \frac{1}{6}(u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi,$$

where u, v are tangent vectors on M and "$\lrcorner$" is the interior multiplication operator (*Here we use the orientation in* [6]). Then we can see that any $\varphi \in \Omega^3_+(M)$ determines a Riemannian metric g_φ and an orientation dV_φ, hence the Hodge star operator $*_\varphi$ and the associated 4-form

$$\psi := *_\varphi \varphi$$

can also be uniquely determined by φ.

The group G_2 acts irreducibly on $\mathbb{R}^7$ (and hence on $\wedge^1(\mathbb{R}^7)^*$ and $\wedge^6(\mathbb{R}^7)^*$), but it acts reducibly on $\wedge^k(\mathbb{R}^7)^*$ for $2 \le k \le 5$. Hence a G_2 structure φ induces splittings of the bundles $\wedge^k(T^*M)(2 \le k \le 5)$ into direct summands, which we denote by $\wedge^k_l(T^*M, \varphi)$ with l being the rank of the bundle. We let the space of sections of $\wedge^k_l(T^*M, \varphi)$ be $\Omega^k_l(M)$. Define the natural projections

$$\pi^k_l : \Omega^k(M) \longrightarrow \Omega^k_l(M), \;\; \alpha \longmapsto \pi^k_l(\alpha).$$

Then we have

$$\Omega^2(M) = \Omega^2_7(M) \oplus \Omega^2_{14}(M),$$
$$\Omega^3(M) = \Omega^3_1(M) \oplus \Omega^3_7(M) \oplus \Omega^3_{27}(M).$$

where each component is determined by

$$\Omega^2_7(M) = \{X \lrcorner \varphi : X \in C^\infty(TM)\} = \{\beta \in \Omega^2(M) : *_\varphi(\varphi \wedge \beta) = 2\beta\},$$
$$\Omega^2_{14}(M) = \{\beta \in \Omega^2(M) : \psi \wedge \beta = 0\} = \{\beta \in \Omega^2(M) : *_\varphi(\varphi \wedge \beta) = -\beta\},$$

and

$$\Omega^3_1(M) = \{f\varphi : f \in C^\infty(M)\},$$
$$\Omega^3_7(M) = \{*_\varphi(\varphi \wedge \alpha) : \alpha \in \Omega^1(M)\} = \{X \lrcorner \psi : X \in C^\infty(TM)\},$$
$$\Omega^3_{27}(M) = \{\eta \in \Omega^3(M) : \eta \wedge \varphi = \eta \wedge \psi = 0\}.$$

Remark 2.2 Ω^4 and Ω^5 have the corresponding decompositions by Hodge duality. The more details for G_2-decompositions see [6, 24].

By the definition of G_2 decompositions, we can find unique differential forms $\tau_0 \in \Omega^0(M)$, $\tau_1, \widetilde{\tau}_1 \in \Omega^1(M)$, $\tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$ such that (see [6])

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *_\varphi \tau_3, \tag{2.1}$$

$$d\psi = 4\widetilde{\tau}_1 \wedge \psi + \tau_2 \wedge \varphi. \tag{2.2}$$

In fact, Karigiannis [23] proved that $\tau_1 = \widetilde{\tau}_1$. We call τ_0 the *scalar torsion*, τ_1 the *vector torsion*, τ_2 the *Lie algebra torsion*, and τ_3 the *symmetric traceless torsion*. We also call $\tau_\varphi := \{\tau_0, \tau_1, \tau_2, \tau_3\}$ the intrinsic torsion forms of the G_2-structure φ.

If φ is closed, which means $d\varphi = 0$, then τ_0, τ_1, τ_3 are all zero, so the only nonzero torsion form is

$$\tau \equiv \tau_2 = \frac{1}{2}(\tau_2)_{ij} dx^i \otimes dx^j = \frac{1}{2}\tau_{ij} dx^i \otimes dx^j.$$

Then from [23, 24], the full torsion tensor $\mathbf{T} = \mathbf{T}_{ij} dx^i \otimes dx^j$ satisfies the followings

$$\mathbf{T}_{ij} = -\mathbf{T}_{ji} = -\frac{1}{2}(\tau_2)_{ij} \quad \text{or equivalently} \quad \mathbf{T} = -\frac{1}{2}\tau,$$

so that $\mathbf{T}$ is a skew-symmetric 2-tensor or a 2-form.

2.2 The Laplacian G_2 flow and some notations

In this subsection, we introduce the Laplacian G_2 flow, some notations, and definitions which will be used in the sequel. We use the notations in Hamilton's paper [17], ∇_g is the Levi-Civita connection induced by g, $\mathrm{Ric}(g)$, R_g, dV_g are Ricci curvature, scalar curvature, and volume form, respectively. The Laplacian of the smooth time-dependent function $f(t)$ with respect to a family of Riemannian metrics $g(t)$ is

$$\Delta_{g(t)} f(t) = g^{ij}(t)\left[\partial_i\partial_j f(t) - \Gamma^k_{ij}(t)\partial_k f(t)\right],$$

where $\Gamma^k_{ij}(t)$ is the Christoffel symbol of $g(t)$ and $\partial_i = \dfrac{\partial}{\partial x^i}$.

In [6], Bryant introduced the following Laplacian G_2 flow on a smooth 7-manifold M admitting closed G_2-structures

$$\begin{cases} \partial_t \varphi(t) = \Delta_{\varphi(t)}\varphi(t), \\ \varphi(0) = \varphi, \end{cases}$$

under the Laplacian G_2 flow. From [41], we see that the associated metric tensor $g(t)$ evolves by

$$\partial_t g(t) = -2\,\mathrm{Sic}(g(t)), \tag{2.3}$$

where

$$\mathrm{Sic}(g(t)) = \mathrm{Ric}(g(t)) + \frac{1}{3}|\mathbf{T}(t)|^2_{g(t)} g(t) + 2\widehat{\mathbf{T}}(t)$$

is the symmetric 2-tensor and its components are given by

$$S_{ij} = R_{ij} + \frac{1}{3}|\mathbf{T}(t)|^2_{g(t)} g_{ij} + 2\widehat{\mathbf{T}}_{ij}, \tag{2.4}$$

and $\widehat{\mathbf{T}}_{ij} = \mathbf{T}_i{}^k\mathbf{T}_{kj}$. In [35, 41], we see that $R_{g(t)} = -|\mathbf{T}(t)|^2_{g(t)}$ and $\mathbf{T}_{ij}$ is skew-symmetric, then we have that

$$\begin{aligned}\mathrm{tr}_{g(t)}\Big(\mathrm{Sic}(g(t))\Big) &= R_{g(t)} + \frac{7}{3}|\mathbf{T}(t)|^2_{g(t)} - 2|\mathbf{T}(t)|^2_{g(t)} \\ &= -|\mathbf{T}(t)|^2_{g(t)} + \frac{7}{3}|\mathbf{T}(t)|^2_{g(t)} - 2|\mathbf{T}(t)|^2_{g(t)} = \frac{2}{3}R_{g(t)}.\end{aligned}$$

Under the Laplacian G_2 flow (1.2), for any smooth functions $u(t), v(t)$ with

$$\int_M u(T)v(T)dV_{g(T)} = \int_M u(0)v(0)dV_{g(0)},$$

we have that

$$\begin{aligned}&\int_0^T \int_M v(t)\left(\partial_t - \Delta_{g(t)}\right)u(t)dV_{g(t)}dt \\ &= \int_0^T \int_M \left[-u(t)\partial_t v(t) + \frac{2}{3}v(t)u(t)R_{g(t)} - u(t)\Delta_{g(t)}v(t)\right]dV_{g(t)}dt \\ &= \int_0^T \int_M u(t)\left(\frac{2}{3}R_{g(t)} - \partial_t - \Delta_{g(t)}\right)v(t)dV_{g(t)}dt.\end{aligned}$$

Let $\tau(t) = T - t$ be the backward time. For any time-dependent smooth function $f(t)$ on M, we denote

$$\mathbf{K}(t) = (4\pi\tau(t))^{-\frac{7}{2}}e^{-f(t)}$$

to be the positive solution of the conjugate heat equation

$$\partial_t\mathbf{K}(t) = -\Delta_{g(t)}\mathbf{K}(t) + \frac{2}{3}R_{g(t)}\mathbf{K}(t). \tag{2.5}$$

From the definition of $\mathbf{K}(t)$, we can calculate the smooth function $f(t)$ satisfies the following equation

$$\partial_t f(t) = -\Delta_{g(t)}f(t) - \frac{2}{3}R_{g(t)} + |\nabla_{g(t)}f(t)|^2_{g(t)} + \frac{7}{2\tau(t)}. \tag{2.6}$$

We can define the weighted measure

$$d\mu_{g(t)} := \mathbf{K}(t)dV_{g(t)} = (4\pi\tau(t))^{-\frac{7}{2}}e^{-f(t)}dV_{g(t)}, \quad \int_M d\mu_{g(t)} = 1. \tag{2.7}$$

And the volume form $dV_{g(t)}$ satisfies

$$\partial_t(dV_{g(t)}) = -\frac{2}{3}R_{g(t)}dV_{g(t)},$$

thus, the conjugate heat kernel measure $d\mu_{g(t)}$ is evolved by

$$\partial_t(d\mu_{g(t)}) = -(\Delta_{g(t)}\mathbf{K}(t))dV_{g(t)} = -\frac{\Delta_{g(t)}\mathbf{K}(t)}{\mathbf{K}(t)}d\mu_{g(t)}. \tag{2.8}$$

On the weighted Riemannian manifold $(M^n, g(t), d\mu_{g(t)})$, the weighted Bochner formula for any smooth function u is as follow

$$\begin{aligned}\Delta_{g(t),f(t)}\left(|\nabla_{g(t)}u|^2_{g(t)}\right) &= 2\left|\nabla^2_{g(t)}u\right|^2_{g(t)} + 2\left\langle\nabla_{g(t)}u, \nabla_{g(t)}\Delta_{g(t),f(t)}u\right\rangle_{g(t)} \\ &\quad + 2\mathrm{Ric}_{f(t)}\left(\nabla_{g(t)}u, \nabla_{g(t)}u\right),\end{aligned} \tag{2.9}$$

where

$$\mathrm{Ric}_{f(t)} := \mathrm{Ric}(g(t)) + \nabla^2_{g(t)} f(t) \tag{2.10}$$

is the Bakry-Émery Ricci tensor introduced in [8], and

$$\Delta_{g(t),f(t)} u := e^{f(t)} \mathrm{div}_{g(t)} \left(e^{-f(t)} \nabla_{g(t)} u \right) = \Delta_{g(t)} u - \left\langle \nabla_{g(t)} f(t), \nabla_{g(t)} u \right\rangle_{g(t)} \tag{2.11}$$

is the drift Laplacian operator for any smooth function u.

3 Gradient estimates under Laplacian G_2 flow

In this section, we consider the Li-Yau type gradient estimate and Hamilton type gradient estimate of the heat equation(1.3) under the Laplacian G_2 flow (1.2). Since $R_{g(t)} = -|\mathbf{T}(t)|^2_{g(t)}$, which implies the scalar curvature is non-positive here, some methods of gradient estimate require the non-negative Ricci curvature condition, which can't hold in this circumstance. Inspired by Liu in [38], we weaken the curvature constraints and obtain the gradient estimate for the solution of the heat equation when the metric is evolved by the Laplacian G_2 flow (1.2).

Theorem 3.1 *Let $(M, \varphi(t))_{t\in(0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with with $T < +\infty$ and $-Kg(t) \le \mathrm{Ric}(g(t)) \le 0$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$ and K is a positive constant. If $u(t)$ is a positive solution of the heat equation, then on $M \times (0, T]$ the following estimate*

$$\frac{|\nabla_{g(t)} u(t)|^2_{g(t)}}{u(t)^2} - \alpha \frac{\partial_t u(t)}{u(t)} \le \frac{7\alpha}{2at} + \left(\frac{49\alpha}{3a} + \frac{105\alpha^2 - 98\alpha}{2a(\alpha-1)} + \frac{7\sqrt{29}\alpha}{2\sqrt{ab}} \right) K \tag{3.1}$$

holds for any $\alpha > 1$ and $a, b > 0$ with $a + 2b = \dfrac{1}{\alpha}$.

Proof We first set $f = \ln u$ and

$$F = t\left(|\nabla f|^2 - \alpha \partial_t f \right).$$

Observe that, (3.1) is true when $F < 0$, hence we always assume that $F \ge 0$ in the sequel. Some computations show that

$$\Delta\left(|\nabla f|^2\right) = \sum_{1\le i,j\le 7} \left(2f_{ij}^2 + 2f_i f_{jji} + 2R_{ij} f_i f_j \right),$$

$$\partial_t(\Delta f) = \partial_t\left(g^{ij} \nabla_i \nabla_j f \right) = \sum_{1\le i,j\le 7} \left(2R_{ij} f_{ij} + 4\widehat{\mathbf{T}}_{ij} f_{ij} \right) + \frac{2}{3} |\mathbf{T}|^2 \Delta f + \Delta(\partial_t f).$$

Combining these two equations we have that

$$\begin{aligned} \Delta F &= t\Big(\Delta\left(|\nabla f|^2\right) - \alpha \Delta(\partial_t f) \Big) \\ &= t\Bigg(2\sum_{1\le i,j\le 7} f_{ij}^2 + 2\sum_{1\le i,j\le 7} f_i f_{jji} + 2\sum_{1\le i,j\le 7} R_{ij} f_i f_j - \alpha \partial_t(\Delta f) \\ &\quad + 2\alpha \sum_{1\le i,j\le 7} R_{ij} f_{ij} + \frac{2}{3}\alpha |\mathbf{T}|^2 \Delta f + 4\alpha \sum_{1\le i,j\le 7} \widehat{\mathbf{T}}_{ij} f_{ij} \Bigg). \end{aligned}$$

Since $\Delta f = \partial_t f - |\nabla f|^2$, it follows that

$$\alpha\partial_t(\Delta f) = \alpha f_{tt} - \alpha\Big(\sum_{1\le i,j\le 7}\big(2R_{ij}f_if_j + 4\widehat{\mathbf{T}}_{ij}f_if_j\big) + \frac{2}{3}|\mathbf{T}|^2|\nabla f|^2 + 2\nabla f\cdot\nabla(\partial_t f)\Big),$$

and

$$\begin{aligned}\Delta F = t\Big(&2\sum_{1\le i,j\le 7} f_{ij}^2 + 2\nabla f\cdot\nabla\Delta f + 2\sum_{1\le i,j\le 7}R_{ij}f_if_j - \alpha f_{tt} + 2\alpha\sum_{1\le i,j\le 7}R_{ij}f_if_j\\ &+\frac{2}{3}\alpha|\mathbf{T}|^2|\nabla f|^2 + 4\alpha\sum_{1\le i,j\le 7}\widehat{\mathbf{T}}_{ij}f_if_j + 2\alpha\nabla f\cdot\nabla(\partial_t f)\\ &+2\alpha\sum_{1\le i,j\le 7}R_{ij}f_{ij} + \frac{2}{3}\alpha|\mathbf{T}|^2\Delta f + 4\alpha\sum_{1\le i,j\le 7}\widehat{\mathbf{T}}_{ij}f_{ij}\Big).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\partial_t F = |\nabla f|^2 - \alpha\partial_t f + t\Big(&2\sum_{1\le i,j\le 7}R_{ij}f_if_j + \frac{2}{3}|\mathbf{T}|^2|\nabla f|^2\\ &+4\sum_{1\le i,j\le 7}\widehat{\mathbf{T}}_{ij}f_if_j + 2\nabla f\cdot\nabla(\partial_t f)\Big) - t\alpha f_{tt}.\end{aligned}$$

Now we obtain

$$\begin{aligned}(\Delta-\partial_t)\,F = {}& t\Big(2\nabla f\cdot\nabla\Delta f + 2\alpha\nabla f\cdot\nabla(\partial_t f) - 2\nabla f\cdot\nabla(\partial_t f)\Big)\\ &+t\Big(2\sum_{1\le i,j\le 7}f_{ij}^2 + 2\alpha\sum_{1\le i,j\le 7}R_{ij}f_{ij} + 4\alpha\sum_{1\le i,j\le 7}\widehat{\mathbf{T}}_{ij}f_{ij}\Big)\\ &+t\Big(\frac{2}{3}\alpha|\mathbf{T}|^2|\nabla f|^2 + \frac{2}{3}\alpha|\mathbf{T}|^2\Delta f - \frac{2}{3}|\mathbf{T}|^2|\nabla f|^2\Big)\\ &+t\Big(2\alpha\sum_{1\le i,j\le 7}R_{ij}f_if_j + 4\alpha\sum_{1\le i,j\le 7}\widehat{\mathbf{T}}_{ij}f_if_j - 4\sum_{1\le i,j\le 7}\widehat{\mathbf{T}}_{ij}f_if_j\Big)\\ &-\Big(|\nabla f|^2 - \alpha\partial_t f\Big).\end{aligned}\tag{3.2}$$

Again using $\Delta f = \partial_t f - |\nabla f|^2$, the first and third line in the right side of (3.2) become

$$\begin{aligned}&t\Big(2\nabla f\cdot\nabla\Delta f + 2\alpha\nabla f\cdot\nabla(\partial_t f) - 2\nabla f\cdot\nabla(\partial_t f)\Big)\\ &\quad= 2t\nabla f\cdot\nabla\Big(\partial_t f - |\nabla f|^2 + \alpha\partial_t f - \partial_t f\Big) = -2\nabla f\cdot\nabla F,\\ &t\Big(\frac{2}{3}\alpha|\mathbf{T}|^2|\nabla f|^2 + \frac{2}{3}\alpha|\mathbf{T}|^2\Delta f - \frac{2}{3}|\mathbf{T}|^2|\nabla f|^2\Big)\\ &\quad= \frac{2}{3}t|\mathbf{T}|^2\Big(\alpha|\nabla f|^2 + \alpha\Delta f - |\nabla f|^2\Big) = -\frac{2}{3}|\mathbf{T}|^2F.\end{aligned}$$

For the second line, using the trick in [2, 3], we get

$$\sum_{1\le i,j\le 7}\Big(f_{ij}^2 + \alpha R_{ij}f_{ij} + 2\alpha\widehat{\mathbf{T}}_{ij}f_{ij}\Big)$$

$$
\begin{aligned}
&= \sum_{1\le i,j\le 7}\Big((a\alpha+2b\alpha)f_{ij}^2+\alpha R_{ij}f_{ij}+2\alpha\widehat{\mathbf{T}}_{ij}f_{ij}\Big)\\
&= \sum_{1\le i,j\le 7}\left(a\alpha f_{ij}^2+\alpha\left|\sqrt{b}f_{ij}+\frac{R_{ij}}{2\sqrt{b}}\right|^2-\frac{\alpha}{4b}|\mathrm{Ric}|^2+\alpha\left|\sqrt{b}f_{ij}+\frac{\widehat{\mathbf{T}}_{ij}}{\sqrt{b}}\right|^2-\frac{\alpha}{b}|\widehat{\mathbf{T}}|^2\right)\\
&\ge a\alpha\sum_{1\le i,j\le 7}f_{ij}^2-\frac{\alpha}{4b}|\mathrm{Ric}|^2-\frac{\alpha}{b}|\widehat{\mathbf{T}}|^2,
\end{aligned}
$$

where a, b are constants satisfying $a+2b=\dfrac{1}{\alpha}$. Noting that

$$
\sum_{1\le i,j\le 7}f_{ij}^2\ge\frac{1}{7}\Big(\sum_{1\le i\le 7}f_{ii}\Big)^2=\frac{(\Delta f)^2}{7},\quad |\mathrm{Ric}|^2\le 7K^2,\quad |\widehat{\mathbf{T}}|^2\le 49K^2,
$$

so it becomes

$$
\sum_{1\le i,j\le 7}\Big(f_{ij}^2+\alpha R_{ij}f_{ij}+2\alpha\widehat{\mathbf{T}}_{ij}f_{ij}\Big)\ge\frac{a\alpha(\Delta f)^2}{7}-\left(\frac{7\alpha}{4b}+\frac{49\alpha}{b}\right)K^2.
$$

For the fourth line, since $\mathbf{T}_{ij}$ is skew-symmetric, we obtain

$$
\begin{aligned}
\sum_{1\le i,j\le 7}\widehat{\mathbf{T}}_{ij}f_if_j &= \sum_{1\le i,j,k\le 7}\mathbf{T}_i{}^k\mathbf{T}_{kj}f_if_j\\
&= -\sum_{1\le i,j,k\le 7}(\mathbf{T}_{ik}f_i)(\mathbf{T}_{jk}f_j) \;=\; -\Big|\sum_{1\le i\le 7}\mathbf{T}_{ik}f_i\Big|^2.
\end{aligned}
$$

Together with the Cauchy inequality, we have

$$
\begin{aligned}
\sum_{1\le i,j\le 7}\widehat{\mathbf{T}}_{ij}f_if_j &= -\Big|\sum_{1\le i\le 7}\mathbf{T}_{ik}f_i\Big|^2\\
&= -\sum_{1\le k\le 7}\Big(\sum_{1\le i\le 7}\mathbf{T}_{ik}f_i\Big)^2\\
&\ge -\sum_{1\le k\le 7}\Big(\sum_{1\le i\le 7}\mathbf{T}_{ik}^2|\nabla f|^2\Big)\\
&= -|\mathbf{T}|^2|\nabla f|^2=R|\nabla f|^2\ge -7K|\nabla f|^2.
\end{aligned}
$$

Now, the fourth line becomes

$$
\begin{aligned}
&t\sum_{1\le i,j\le 7}\big(2\alpha R_{ij}f_if_j+4\alpha\widehat{\mathbf{T}}_{ij}f_if_j-4\widehat{\mathbf{T}}_{ij}f_if_j\big)\\
&\quad\ge -2\alpha tK|\nabla f|^2-28t(\alpha-1)K|\nabla f|^2=-\Big(2\alpha+28(\alpha-1)\Big)tK|\nabla f|^2.
\end{aligned}
$$

Substituting all these terms into (3.2), we now obtain

$$
\begin{aligned}
(\Delta-\partial_t)\,F\ge{}&-2\nabla f\cdot\nabla F+\frac{2a\alpha t}{7}\Big(|\nabla f|^2-\partial_t f\Big)^2-\Big(2\alpha+28(\alpha-1)\Big)tK|\nabla f|^2\\
&-\frac{2}{3}|\mathbf{T}|^2F-\Big(|\nabla f|^2-\alpha\partial_t f\Big)-\left(\frac{7\alpha}{2b}+\frac{98\alpha}{b}\right)tK^2.
\end{aligned}\tag{3.3}
$$

Following the trick in [3, 38], set $y = |\nabla f|^2$, $z = \partial_t f$. Observe that

$$(y-z)^2 = \frac{1}{\alpha^2}(y-\alpha z)^2 + \left(\frac{\alpha-1}{\alpha}\right)^2 y^2 + \frac{2(\alpha-1)}{\alpha^2} y(y-\alpha z),$$

and $mx^2 - nx \geq -\frac{n^2}{4m}$ for any $m, n > 0$. Now we have

$$\begin{aligned}
&\frac{2a\alpha t}{7}\left(\left(|\nabla f|^2 - f_t\right)^2 - \frac{7\alpha+98(\alpha-1)}{a\alpha}K|\nabla f|^2\right)\\
&= \frac{2a\alpha t}{7}\left((y-z)^2 - \frac{7\alpha+98(\alpha-1)}{a\alpha}Ky\right)\\
&= \frac{2a\alpha t}{7}\left[\frac{1}{\alpha^2}(y-\alpha z)^2 + \left(\frac{\alpha-1}{\alpha}\right)^2 y^2 + \frac{2(\alpha-1)}{\alpha^2}y(y-\alpha z) - \frac{7\alpha+98(\alpha-1)}{a\alpha}Ky\right]\\
&\geq \frac{2a\alpha t}{7}\left(\frac{1}{\alpha^2}(y-\alpha z)^2 + \frac{2(\alpha-1)}{\alpha^2}y(y-\alpha z) - \frac{49K^2[\alpha+14(\alpha-1)]^2}{4(\alpha-1)^2a^2}\right)\\
&= \frac{2a}{7\alpha}\frac{F^2}{t} + \frac{2(\alpha-1)}{\alpha^2}|\nabla f|^2\frac{F}{t}\frac{2a\alpha t}{7} - \frac{7\alpha K^2[\alpha+14(\alpha-1)]^2}{2(\alpha-1)^2a}t\\
&\geq \frac{2a}{7\alpha}\frac{F^2}{t} - \frac{7\alpha K^2[\alpha+14(\alpha-1)]^2}{2(\alpha-1)^2a}t.
\end{aligned}$$

Taking this term into (3.3), we finally arrive at

$$\begin{aligned}
(\Delta - \partial_t)F \geq& -2\nabla f\cdot\nabla F + \frac{2a}{7\alpha}\frac{F^2}{t} - \frac{7\alpha[\alpha+14(\alpha-1)]^2}{2(\alpha-1)^2a}tK^2\\
&+\frac{2}{3}RF - \frac{F}{t} - \left(\frac{7\alpha}{2b}+\frac{98\alpha}{b}\right)tK^2.
\end{aligned} \tag{3.4}$$

We assume that $F(x,t)$ takes its maximum at (x_0,t_0), which means

$$\nabla F(x_0,t_0) = 0, \quad \partial_t F(x_0,t_0) \geq 0, \quad \triangle F(x_0,t_0) \leq 0.$$

Thus, at (x_0,t_0), we have

$$\frac{2a}{7\alpha}F^2 - \left(1-\frac{2}{3}Rt\right)F - \frac{7\alpha[\alpha+14(\alpha-1)]^2}{2(\alpha-1)^2a}t^2K^2 - \left(\frac{7\alpha}{2b}+\frac{98\alpha}{b}\right)t^2K^2 \leq 0. \tag{3.5}$$

According to the quadratic formula

$$\begin{aligned}
F \leq \frac{7\alpha}{4a}\cdot\Bigg(&\sqrt{\left(1-\frac{2}{3}Rt\right)^2 + \frac{8a}{7\alpha}\left(\frac{7\alpha[\alpha+14(\alpha-1)]^2}{2(\alpha-1)^2a} + \left(\frac{7\alpha}{2b}+\frac{98\alpha}{b}\right)\right)t^2K^2}\\
&+1-\frac{2}{3}Rt\Bigg),
\end{aligned}$$

which implies

$$F \leq \frac{7\alpha}{4a}\left(2 - \frac{4}{3}Rt + \frac{2[\alpha+14(\alpha-1)]}{\alpha-1}tK + \sqrt{8a\left(\frac{1}{2b}+\frac{14}{b}\right)}tK\right).$$

Since F takes its maximum at (x_0, t_0), for all $(x, t) \in M \times (0, T]$,

$$F(x,t) \le F(x_0,t_0) \le \frac{7\alpha}{2a} + \left(\frac{49\alpha}{3a} + \frac{7\alpha[\alpha + 14(\alpha - 1)]}{2a(\alpha - 1)} + \frac{7\alpha}{4a}\sqrt{8a\left(\frac{1}{2b} + \frac{14}{b}\right)} \right) tK.$$

According to the definition of $F(x, t)$, we obtain the desired result

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \le \frac{7\alpha}{2at} + \left(\frac{49\alpha}{3a} + \frac{105\alpha^2 - 98\alpha}{2a(\alpha - 1)} + \frac{7\sqrt{29}\alpha}{2\sqrt{ab}} \right) K, \tag{3.6}$$

where $\alpha > 1$, $a + 2b = \dfrac{1}{\alpha}$. □

Remark 3.2 For example, if we take $a = 2b = \dfrac{1}{2\alpha}$, then the estimate becomes

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \le \frac{7\alpha^2}{t} + \left[\left(\frac{98}{3} + 7\sqrt{58} \right)\alpha^2 + \frac{105\alpha^3 - 98\alpha^2}{\alpha - 1} \right] K,$$

where $\alpha > 1$.

Corollary 3.3 *Let $(M, \varphi(t))_{t\in(0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T < +\infty$ and $-Kg(t) \le \mathrm{Ric}(g(t)) \le 0$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$ and K is a positive constant. If $u(t)$ is a positive solution of the heat equation, then on $M \times (0, T]$ such that $t_1 < t_2$, we have*

$$u(x,t_1) \le u(y,t_2)\left(\frac{t_2}{t_1}\right)^{\frac{7}{2a}} \exp\left\{ \int_0^1 \left[\frac{\alpha|\gamma'(s)|^2_{\sigma(s)}}{4(t_2 - t_1)} + (t_2 - t_1)C_{a,b,\alpha}K \right] ds \right\},$$

where $\alpha > 1$,

$$C_{a,b,\alpha} = \frac{49}{3a} + \frac{105\alpha - 98}{2a(\alpha - 1)} + \frac{7\sqrt{29}}{2\sqrt{ab}},$$

$a, b > 0$, $a + 2b = \dfrac{1}{\alpha}$, $\gamma(s)$ is a geodesics curve connecting x and y with $\gamma(0) = y$ and $\gamma(1) = x$, and $|\gamma'(s)|_{\sigma(s)}$ is the length of the vector $\gamma'(s)$ at $\sigma(s) = (1 - s)t_2 + st_1$.

Proof We can write Li-Yau type gradient estimate in Theorem 3.1 as follows:

$$\frac{|\nabla u|^2}{u^2} - \alpha\frac{\partial_t u}{u} \le \frac{C_\alpha}{t} + C_{\alpha,a,b}K,$$

where

$$C_\alpha = \frac{7\alpha}{2a}, \quad C_{\alpha,a,b} = \frac{49\alpha}{3a} + \frac{105\alpha^2 - 98\alpha}{2a(\alpha - 1)} + \frac{7\sqrt{29}\alpha}{2\sqrt{ab}}.$$

Choosing a geodesics curve $\gamma(s)$ connects x and y with $\gamma(0) = y$ and $\gamma(1) = x$. We define $l(s) = \ln u(\gamma(s), (1-s)t_2 + st_1)$ and $\sigma(s) = (1-s)t_2 + st_1$, then we have $l(0) = \ln u(y, t_2)$ and $l(1) = \ln u(x, t_1)$. By calculating, we have

$$\frac{\partial l(s)}{\partial s} = (t_2 - t_1)\left(\frac{\nabla u}{u}\frac{\gamma'(s)}{(t_2 - t_1)} - \frac{u_t}{u} \right)$$

$$
\begin{aligned}
&\leq (t_2 - t_1)\left(\frac{\alpha|\gamma'(s)|^2_{\sigma(s)}}{4(t_2-t_1)^2} + \frac{|\nabla u|^2}{\alpha u^2} - \frac{u_t}{u}\right)\\
&\leq \frac{\alpha|\gamma'(s)|^2_{\sigma(s)}}{4(t_2-t_1)} + \frac{t_2-t_1}{\alpha}\left(\frac{C_\alpha}{\sigma(s)} + C_{\alpha,a,b}K\right),
\end{aligned}
\tag{3.7}
$$

where $|\gamma'(s)|_{\sigma(s)}$ is the length of the vector $\gamma'(s)$ at $\sigma(s)$. Integrating (3.7) over $\gamma(s)$, we get

$$
\begin{aligned}
\ln\frac{u(x,t_1)}{u(y,t_2)} &= \int_0^1 \frac{\partial l(s)}{\partial s}ds\\
&\leq \int_0^1 \left[\frac{\alpha|\gamma'(s)|^2_{\sigma(s)}}{4(t_2-t_1)} + \frac{t_2-t_1}{\alpha}\left(\frac{C_\alpha}{\sigma(s)} + C_{\alpha,a,b}K\right)\right]ds\\
&= \int_0^1 \left[\frac{\alpha|\gamma'(s)|^2_{\sigma(s)}}{4(t_2-t_1)} + \frac{t_2-t_1}{\alpha}C_{\alpha,a,b}K\right]ds + \frac{7}{2a}\ln\frac{t_2}{t_1}.
\end{aligned}
$$

Thus, we get the desired result. □

Theorem 3.4 *Let $(M,\varphi(t))_{t\in(0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T<+\infty$ and $-Kg(t)\leq \mathrm{Ric}(g(t))\leq 0$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$ and K is a positive constant. If $u(t)$ is a positive solution of the heat equation, then on $M\times(0,T]$ the following estimate*

$$
|\nabla_{g(t)}u(t)|^2_{g(t)} \leq \frac{u(t)}{t}\left(u(t)\ln\frac{A}{u(t)} + \lambda A^2 - \lambda\eta^2\right)
\tag{3.8}
$$

holds, where $\eta = \min_M u(0)$, $A=\max_M u(0)$ and λ is a constant depending on K, η and T.

Proof First, using the Bochner technique with the flow equation, we have that

$$
(\partial_t - \Delta)|\nabla u|^2 = -2|\nabla^2 u|^2 + \frac{2}{3}|\mathbf{T}|^2|\nabla u|^2 - 4\sum_{1\leq k\leq 7}\left|\sum_{1\leq i\leq 7}\mathbf{T}_{ik}\nabla_i u\right|^2.
\tag{3.9}
$$

Let λ be a constant to be fixed later. Setting

$$
P = t\frac{|\nabla u|^2}{u} - u\ln\frac{A}{u} + \lambda u^2,
$$

some calculations show that

$$
\begin{aligned}
\partial_t\left(\frac{|\nabla u|^2}{u}\right) &= \frac{\partial_t|\nabla u|^2}{u} - \frac{|\nabla u|^2}{u^2}\partial_t u,\\
\Delta\left(\frac{|\nabla u|^2}{u}\right) &= \frac{\Delta|\nabla u|^2}{u} + \Delta\left(\frac{1}{u}\right)|\nabla u|^2 + 2\nabla\left(\frac{1}{u}\right)\cdot\nabla(|\nabla u|^2)\\
&= \frac{\Delta|\nabla u|^2}{u} - \left(\frac{\Delta u}{u^2} - 2\frac{|\nabla u|^2}{u^3}\right)|\nabla u|^2 - \frac{4}{u^2}\sum_{1\leq i,j\leq 7}u_{ij}u_iu_j,\\
\partial_t\left(u\ln\frac{A}{u}\right) &= \Delta\left(u\ln\frac{A}{u}\right) + \frac{|\nabla u|^2}{u},\\
\partial_t\left(u^2\right) &= \Delta\left(u^2\right) - 2|\nabla u|^2.
\end{aligned}
$$

Combining these equations we obtain

$$
\begin{aligned}
(\partial_t - \Delta)\, P &= t\left(\partial_t \frac{|\nabla u|^2}{u} - \Delta \frac{|\nabla u|^2}{u}\right) - 2\lambda |\nabla u|^2 \\
&= t\Bigg(\frac{(\partial_t - \Delta)|\nabla u|^2}{u} - 2\frac{|\nabla u|^4}{u^3} + \frac{4}{u^2}\sum_{1\le i,j\le 7} u_{ij}u_i u_j\Bigg) - 2\lambda |\nabla u|^2 \\
&= \frac{t}{u}\Bigg(- 2|\nabla^2 u|^2 + \frac{2}{3}|\mathbf{T}|^2|\nabla u|^2 - 4\sum_{1\le k\le 7}\Bigg|\sum_{1\le i\le 7} \mathbf{T}_{ik}u_i\Bigg|^2\Bigg) \\
&\quad - \frac{2t}{u}\Bigg(\frac{|\nabla u|^4}{u^2} - \frac{2}{u}\sum_{1\le i,j\le 7} u_{ij}u_i u_j\Bigg) - 2\lambda |\nabla u|^2 \\
&= -\frac{2t}{u}\sum_{1\le i,j\le 7}\left|u_{ij} - \frac{u_i u_j}{u}\right|^2 + \left(\frac{2t}{3u}|\mathbf{T}|^2 - 2\lambda\right)|\nabla u|^2 \\
&\quad - \frac{4t}{u}\sum_{1\le k\le 7}\Bigg|\sum_{1\le i\le 7}\mathbf{T}_{ik}u_i\Bigg|^2.
\end{aligned}
$$

Since $\eta \le u(0) \le A$ and $|\mathbf{T}|^2 = -R$, taking $\lambda \ge \dfrac{7KT}{3\eta}$, we obtain $(\partial_t - \Delta)\, P \le 0$.

According to the maximum principle, we obtain

$$
P(t) \le \max_M P(0) = \lambda A^2,
$$

which means

$$
|\nabla u|^2 \le \frac{u}{t}\left(u \ln \frac{A}{u} + \lambda A^2 - \lambda u^2\right), \tag{3.10}
$$

where λ is a constant depending on K, η and T. Thus we get the desired result. □

4 Parabolic frequency on Laplacian G_2 flow with bounded Ricci curvature

In this section, using the Li-Yau type gradient estimate and Hamilton type gradient estimate, we study the parabolic frequency for the solution of the heat equation (1.3) under the Laplacian G_2 flow (1.2) with bounded Ricci curvature.

For a time-dependent function $u = u(t) : M \times [t_0, t_1] \to \mathbb{R}^+$ with $u(t), \partial_t u(t) \in W_0^{2,2}(d\mu_{g(t)})$ and for all $t \in [t_0, t_1] \subset (0, T)$, we define

$$
\begin{aligned}
I(t) &= \int_M u^2(t)\, d\mu_{g(t)}, \\
D(t) &= h(t)\int_M |\nabla_{g(t)} u(t)|^2_{g(t)}\, d\mu_{g(t)} = -h(t)\int_M \langle u(t), \Delta_{g(t),f(t)} u(t)\rangle_{g(t)}\, d\mu_{g(t)}, \\
U(t) &= \exp\left\{-\int_{t_0}^{t}\left(\frac{h'(s)}{h(s)} - \frac{2}{3}R_0 + \frac{28 + C_1(A,\eta)}{s} + cK + \frac{7}{2}C(s)\right) ds\right\}\frac{D(t)}{I(t)}
\end{aligned}
$$

where $h(t)$ is a time-dependent function, K and c are both positive constants,

$$
R_0 = \min_{M\times[t_0,t_1]} R(t), \quad C_1(A,\eta) = \ln \frac{A}{\eta} + \lambda \frac{A^2}{\eta}, \quad C(t) = \frac{C_1(A,\eta)}{t},
$$

and λ is the constant in Theorem 3.4,

$$\eta = \min_M u(0), \quad A = \max_M u(0).$$

Observe that, A and η are both positive constants.

Lemma 4.1 *Under the Laplacian G_2 flow (1.2), the norm of the gradient of any smooth function $u(t)$ satisfies the following equation*

$$(\partial_t - \Delta)|\nabla u|^2 = -2|\nabla^2 u|^2 + 2\langle \nabla u, \nabla(\partial_t - \Delta)u\rangle + \frac{2}{3}|\mathbf{T}|^2|\nabla u|^2 - 4\sum_{1\le k\le 7}\left|\sum_{1\le i\le 7}\mathbf{T}_{ik}\nabla_i u\right|^2. \tag{4.1}$$

Proof At first, note that

$$\partial_t|\nabla u|^2 = 2\mathrm{Ric}(\nabla u, \nabla u) + \frac{2}{3}|\mathbf{T}|^2|\nabla u|^2 + 4\sum_{1\le i,j\le 7}\widehat{\mathbf{T}}_{ij}\nabla_i u\nabla_j u + 2\langle \nabla u, \nabla\partial_t u\rangle \tag{4.2}$$

Since $\mathbf{T}_{ij}$ is anti-symmetric, we obtain

$$\begin{aligned}\sum_{1\le i,j\le 7}\widehat{\mathbf{T}}_{ij}\nabla_i u\nabla_j u &= \sum_{1\le i,j,m\le 7}\mathbf{T}_{im}\mathbf{T}_j^{\ m}\nabla_i u\nabla_j u\\ &= -\sum_{1\le i,j,m\le 7}(\mathbf{T}_{im}\nabla_i u)(\mathbf{T}_m^{\ j}\nabla_j u)\\ &= -\sum_{1\le k\le 7}\left|\sum_{1\le i\le 7}\mathbf{T}_{ik}\nabla_i u\right|^2.\end{aligned} \tag{4.3}$$

Together with the Bochner formula, we obtain the desired result. □

Theorem 4.2 *Let $(M, \varphi(t))_{t\in(0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T < +\infty$ and $-Kg(t) \le \mathrm{Ric}(g(t)) \le 0$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$ and K is a positive constant. If $u(t)$ is a positive solution of the heat equation (1.3) with $\eta \le u(0) \le A$, then the following holds.*

(i) *If $h(t)$ is a negative time-dependent function, then the parabolic frequency $U(t)$ is monotone increasing along the Laplacian G_2 flow.*
(ii) *If $h(t)$ is a positive time-dependent function, then the parabolic frequency $U(t)$ is monotone decreasing along the Laplacian G_2 flow.*

Proof Before discussing the monotonicity of $U(t)$, we need to calculate the derivative of $I(t)$ and $D(t)$. Taking $\alpha = 2$, $a = \frac{1}{4}$, $b = \frac{1}{8}$ in Theorem 3.1, we obtain

$$\frac{|\nabla u|^2}{u^2} - 2\frac{\partial_t u}{u} \le \frac{28}{t} + cK, \tag{4.4}$$

where $c = \frac{392}{3} + 448 + 28\sqrt{58}$. Then we can get the derivative of $I(t)$,

$$I'(t) = \frac{d}{dt}\left(\int_M u^2 d\mu\right)$$

$$
\begin{aligned}
&= 2\int_M \left(u\cdot\partial_t u - |\nabla u|^2\right) d\mu - 2\int_M u\Delta u\, d\mu \\
&= 2\int_M \left(u\cdot\partial_t u - \frac{1}{2}|\nabla u|^2\right) d\mu - 2\int_M u\Delta u\, d\mu - \int_M |\nabla u|^2 d\mu \\
&\geq -\left(\frac{28}{t} + cK\right) I(t) - 2\int_M u\Delta u\, d\mu - \int_M |\nabla u|^2 d\mu \\
&\geq -\left(\frac{28 + C_1(A,\eta)}{t} + cK + \frac{7C(t)}{2}\right) I(t) - \frac{2}{7C(t)} \int_M |\Delta u|^2 d\mu. \qquad (4.5)
\end{aligned}
$$

where $C_1(A,\eta) = \ln\dfrac{A}{\eta} + \lambda\dfrac{A^2}{\eta}$, and we use Young's inequality and Theorem 3.4 in the last line.

For the derivative of $D(t)$, according to Lemma 4.1, we have

$$
\begin{aligned}
D'(t) &= h'(t)\int_M |\nabla u|^2 d\mu + h(t)\frac{d}{dt}\left(\int_M |\nabla u|^2 d\mu\right) \\
&= h'(t)\int_M |\nabla u|^2 d\mu + h(t)\int_M (\partial_t - \Delta)|\nabla u|^2 d\mu \\
&= h'(t)\int_M |\nabla u|^2 d\mu - 2h(t)\int_M |\nabla^2 u|^2 d\mu \\
&\quad + \frac{2}{3}h(t)\int_M |\mathbf{T}|^2|\nabla u|^2 d\mu - 4h(t)\int_M |\mathbf{T}_{ik}\nabla^i u|^2 d\mu. \qquad (4.6)
\end{aligned}
$$

If $h(t) < 0$, then by (4.6),

$$
D'(t) \geq \left(h' - \frac{2}{3}hR_0\right)\int_M |\nabla u|^2 d\mu - 2h\int_M |\nabla^2 u|^2 d\mu,
$$

together with (4.5) and Theorem 3.4, yields

$$
\begin{aligned}
I^2(t)U'(t) &\geq \exp\left\{-\int_{t_0}^t \left(\frac{h'(s)}{h(s)} - \frac{2}{3}R_0 + \frac{28 + C_1(A,\eta)}{s} + cK + \frac{7C(s)}{2}\right) ds\right\} \\
&\quad \cdot \left[-2hI(t)\left(\int_M |\nabla^2 u|^2 d\mu\right) + \frac{2h}{7C(t)}\left(\int_M |\Delta u|^2 d\mu\right)\left(\int_M |\nabla u|^2 d\mu\right)\right] \\
&\geq \exp\left\{-\int_{t_0}^t \left(\frac{h'(s)}{h(s)} - \frac{2}{3}R_0 + \frac{28 + C_1(A,\eta)}{s} + cK + \frac{7C(s)}{2}\right) ds\right\} \\
&\quad \cdot \left[-\frac{2h}{7}I(t)\left(\int_M |\Delta u|^2 d\mu\right) + \frac{2h}{7C(t)}\left(\int_M |\Delta u|^2 d\mu\right)\left(\int_M |\nabla u|^2 d\mu\right)\right] \\
&\geq \exp\left\{-\int_{t_0}^t \left(\frac{h'(s)}{h(s)} - \frac{2}{3}R_0 + \frac{28 + C_1(A,\eta)}{s} + cK + \frac{7C(s)}{2}\right) ds\right\} \\
&\quad \cdot \left[-\frac{2h}{7}I(t)\left(\int_M |\Delta u|^2 d\mu\right) + \frac{2h}{7C(t)}\cdot\frac{C_1(A,\eta)}{t}I(t)\left(\int_M |\Delta u|^2 d\mu\right)\right] \\
&= 0
\end{aligned}
$$

where we take trace over $|\nabla^2 u|^2$ and let $C(t) = \dfrac{C_1(A,\eta)}{t}$.

On the other hand, if $h(t) > 0$, similarly, we have

$$
I^2(t)U'(t) \leq \exp\left\{-\int_{t_0}^t \left(\frac{h'(s)}{h(s)} - \frac{2}{3}R_0 + \frac{28 + C_1(A,\eta)}{s} + cK + \frac{7C(s)}{2}\right) ds\right\}
$$

$$
\begin{aligned}
&\cdot\left[-2hI(t)\left(\int_M|\nabla^2u|^2d\mu\right)+\frac{2h}{7C(t)}\left(\int_M|\Delta u|^2d\mu\right)\left(\int_M|\nabla u|^2d\mu\right)\right]\\
&\le \exp\left\{-\int_{t_0}^t\left(\frac{h'(s)}{h(s)}-\frac{2}{3}R_0+\frac{28+C_1(A,\eta)}{s}+cK+\frac{7C(s)}{2}\right)ds\right\}\\
&\cdot\left[-\frac{2h}{7}I(t)\left(\int_M|\Delta u|^2d\mu\right)+\frac{2h}{7C(t)}\cdot\frac{C_1(A,\eta)}{t}I(t)\left(\int_M|\Delta u|^2d\mu\right)\right]\\
&=0.
\end{aligned}
$$

Thus we get the desired result. □

We define the first nonzero eigenvalue of the Laplacian G_2 flow $(M,\varphi(t))_{t\in(0,T]}$ with the weighted measure $d\mu_{g(t)}$ by

$$
\lambda_M(t)=\inf\left\{\frac{\int_M|\nabla_{g(t)}u|^2_{g(t)}d\mu_{g(t)}}{\int_M u^2d\mu_{g(t)}}\middle|\, 0<u\in C^\infty(M)\setminus\{0\}\right\}.
$$

Then we have the following corollary by Theorem 4.2.

Corollary 4.3 *Let $(M,\varphi(t))_{t\in(0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T<+\infty$ and $-Kg(t)\le \mathrm{Ric}(g(t))\le 0$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$ and K is a positive constant. If $u(t)$ is a positive solution of the heat equation (1.3) with $\eta\le u(0)\le A$, then for any $t\in[t_0,t_1]\subset(0,T)$, the following holds.*

(i) *If $h(t)$ is a negative time-dependent function, then $\beta(t)h(t)\lambda_M(t)$ is a monotone increasing function.*
(ii) *If $h(t)$ is a positive time-dependent function, then $\beta(t)h(t)\lambda_M(t)$ is a monotone decreasing function.*

where

$$
\beta(t)=\exp\left\{-\int_{t_0}^t\left(\frac{h'(s)}{h(s)}-\frac{2}{3}R_0+\frac{28+C_1(A,\eta)}{s}+cK+\frac{7C(s)}{2}\right)ds\right\}.
$$

Corollary 4.4 *Let $(M,\varphi(t))_{t\in(0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T<+\infty$ and $-Kg(t)\le \mathrm{Ric}(g(t))\le 0$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$ and K is a positive constant. If $u(t)$ is a positive solution of the heat equation (1.3) with $\eta\le u(0)\le A$, then for any $t\in[t_0,t_1]\subset(0,T)$,*

$$
I(t_1)\ge\exp\left\{2U(t_0)\int_{t_0}^{t_1}\frac{dt}{-h(t)\beta(t)}\right\}I(t_0).
$$

Proof We give the proof of case $h(t)<0$ (The case $h(t)>0$ is similar to it). According to the definition of $U(t)$, yields

$$
\frac{d}{dt}\ln(I(t))=\frac{I'(t)}{I(t)}=-\frac{2D(t)}{h(t)I(t)}=\frac{2U(t)}{-h(t)\beta(t)}. \tag{4.7}
$$

By Theorem 4.2, integrating (4.7) from t_0 to t_1, we get

$$
\ln I(t_1)-\ln I(t_0)=2\int_{t_0}^{t_1}\frac{U(t)}{-h(t)\beta(t)}dt\ge 2U(t_0)\int_{t_0}^{t_1}\frac{dt}{-h(t)\beta(t)}.
$$

From the boundedness of time-dependent function $h(t)$, we have

$$I(t_1) \geq \exp\left\{2U(t_0)\int_{t_0}^{t_1} \frac{dt}{-h(t)\beta(t)}\right\} I(t_0).$$

We prove this corollary. □

5 Parabolic frequency on Laplacian G_2 flow with bounded Bakry-Émery Ricci curvature

In this section, we study the parabolic frequency for the solution of the linear equation (5.1) and the more general equations under the Laplacian G_2 flow (1.2) with bounded Bakry-Émery Ricci curvature.

For a time-dependent function $u = u(t) : M \times [t_0, t_1] \to \mathbb{R}$ with $u(t), \partial_t u(t) \in W_0^{2,2}(d\mu_{g(t)})$ for all $t \in [t_0, t_1] \subset (0, T)$, we denote by

$$\begin{aligned} I(t) &= \int_M u^2(t) d\mu_{g(t)}, \\ D(t) &= h(t)\int_M |\nabla_{g(t)} u(t)|^2_{g(t)} d\mu_{g(t)} \\ &= -h(t)\int_M \langle u(t), \Delta_{g(t),f(t)} u(t)\rangle_{g(t)} d\mu_{g(t)}, \\ U(t) &= \exp\left\{-\int_{t_0}^{t}\left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right)ds\right\}\frac{D(t)}{I(t)}, \end{aligned}$$

where $R_0 = \min\limits_{M\times[t_0,t_1]} R(t)$, $h(t)$ and $\kappa(t)$ are both time-dependent smooth functions.

5.1 Parabolic frequency for the linear heat equation under Laplacian G_2 flow

In this section, we consider the parabolic frequency $U(t)$ for the solution of the linear heat equation

$$(\partial_t - \Delta_{g(t)})u(t) = a(t)u(t) \tag{5.1}$$

under the Laplacian G_2 flow (1.2), where $a(t)$ is a time-dependent smooth function. At first, we give some lemmas.

Lemma 5.1 *For any $u \in W_0^{2,2}(d\mu_{g(t)})$, we have*

$$\int_M |\nabla^2_{g(t)} u|^2_{g(t)} d\mu_{g(t)} = \int_M \left(|\Delta_{g(t),f(t)} u|^2_{g(t)} - Ric_{f(t)}(\nabla_{g(t)} u, \nabla_{g(t)} u)\right) d\mu_{g(t)}.$$

Proof This result has been proved in Lemma 1.13 of [5]. □

Theorem 5.2 *Let $(M, \varphi(t))_{t\in[0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T < +\infty$ and $Ric_{f(t)} \leq \dfrac{\kappa(t)}{2h(t)} g(t)$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$.*

(i) *If $h(t)$ is a negative time-dependent function, then the parabolic frequency $U(t)$ is monotone increasing along the Laplacian G_2 flow.*

(ii) *If $h(t)$ is a positive time-dependent function, then the parabolic frequency $U(t)$ is monotone decreasing along the Laplacian G_2 flow.*

Proof We only give the proof of the first case (The second case is similar to it). Our main purpose is to compute the $I'(t)$ and $D'(t)$. Under the Laplacian G_2 flow (1.2), combining with the linear heat equation (5.1) and Lemma 4.1, we can obtain

$$\begin{aligned}(\partial_t - \Delta)|\nabla u|^2 &= 2\langle \nabla u, \nabla(\partial_t - \Delta)u\rangle - 2|\nabla^2 u|^2 + \frac{2}{3}|\mathbf{T}|^2|\nabla u|^2 \\ &\quad - 4\sum_{1\le k\le 7}\Big|\sum_{1\le i\le 7}\mathbf{T}_{ik}\nabla_i u\Big|^2 \\ &= 2a(t)|\nabla u|^2 - 2|\nabla^2 u|^2 + \frac{2}{3}|\mathbf{T}|^2|\nabla u|^2 - 4\sum_{1\le k\le 7}\Big|\sum_{1\le i\le 7}\mathbf{T}_{ik}\nabla_i u\Big|^2. \quad (5.2)\end{aligned}$$

From (2.8) and integration by parts, we get the derivative of $I(t)$ as follow

$$\begin{aligned}I'(t) &= \int_M \left(2u\partial_t u - u^2\frac{\Delta \mathbf{K}}{\mathbf{K}}\right)d\mu \\ &= \int_M \left(2u\partial_t u - \Delta(u^2)\right)d\mu \\ &= \int_M \left(2u\partial_t u - 2u\Delta u - 2|\nabla u|^2\right)d\mu \\ &= -\frac{2}{h}D(t) + 2a(t)I(t). \quad (5.3)\end{aligned}$$

If we write

$$\hat{I}(t) = \exp\left\{-\int_{t_0}^t 2a(s)ds\right\} I(t),$$

then we can easily find

$$\hat{I}'(t) = -\frac{2}{h}\exp\left\{-\int_{t_0}^t 2a(s)ds\right\} D(t). \quad (5.4)$$

Next, it turns to compute the derivative of $D(t)$. Using (2.8), (5.2) and the assumption of the Bakry-Émery Ricci curvature, we obtain

$$\begin{aligned}D'(t) &= h'\int_M |\nabla u|^2 d\mu + h\int_M \left(\partial_t|\nabla u|^2 - |\nabla u|^2\frac{\Delta\mathbf{K}}{\mathbf{K}}\right)d\mu \\ &= h'\int_M |\nabla u|^2 d\mu + h\int_M (\partial_t - \Delta)|\nabla u|^2 d\mu \\ &= (2ah + h')\int_M |\nabla u|^2 d\mu - 2h\int_M |\nabla^2 u|^2 d\mu + \frac{2}{3}h\int_M |\mathbf{T}|^2|\nabla u|^2 d\mu \\ &\quad - 4h\int_M |\mathbf{T}_{ik}\nabla^i u|^2 d\mu \\ &\ge (2ah + h' - \frac{2}{3}hR_0)\int_M |\nabla u|^2 d\mu - 2h\int_M |\nabla^2 u|^2 d\mu \\ &= (2ah + h' - \frac{2}{3}hR_0)\int_M |\nabla u|^2 d\mu - 2h\int_M \left[|\Delta_f u|^2 - \mathrm{Ric}_f(\nabla u, \nabla u)\right]d\mu\end{aligned}$$

$$
\begin{aligned}
&\geq (\kappa + 2ah + h' - \frac{2}{3}hR_0)\int_M |\nabla u|^2 d\mu - 2h\int_M |\Delta_f u|^2 d\mu \\
&= \left(2a - \frac{2}{3}R_0 + \frac{h'+\kappa}{h}\right) D(t) - 2h\int_M |\Delta_f u|^2 d\mu.
\end{aligned}
$$

where we write $R_0 = \min\limits_{M\times[t_0,t_1]} R(t)$. Similarly, if we write

$$
\widehat{D}(t) = \exp\left\{-\int_{t_0}^{t}\left[2a(s) - \frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right]ds\right\} D(t),
$$

then we get

$$
\widehat{D}'(t) \geq -2h\exp\left\{-\int_{t_0}^{t}\left[2a(s) - \frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right]ds\right\}\int_M |\Delta_f u|^2 d\mu. \tag{5.5}
$$

Finally, the parabolic frequency $U(t)$ can be written as $U(t) = \dfrac{\widehat{D}(t)}{\hat{I}(t)}$. By (5.4) and (5.5), we can compute the derivative of $U(t)$

$$
\begin{aligned}
\hat{I}^2(t)U'(t) &= \widehat{D}'(t)\hat{I}(t) - \hat{I}'(t)\widehat{D}(t) \\
&\geq -2h\exp\left\{-\int_{t_0}^{t}\left[4a(s) - \frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right]ds\right\} \\
&\quad\cdot\left[\left(\int_M |\Delta_f u|^2 d\mu\right)\cdot\left(\int_M |u|^2 d\mu\right) - \left(\int_M |\nabla u|^2 d\mu\right)^2\right] \\
&\geq -2h\exp\left\{-\int_{t_0}^{t}\left[4a(s) - \frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right]ds\right\} \\
&\quad\cdot\left[\left(\int_M \langle u(t), \Delta_f u\rangle d\mu\right)^2 - \left(\int_M |\nabla u|^2 d\mu\right)^2\right] \\
&= 0.
\end{aligned} \tag{5.6}
$$

The last inequality is directly obtained by the definition of $D(t)$ and the Cauchy-Schwarz inequality. □

Then we have the following

Corollary 5.3 *Let $(M, \varphi(t))_{t\in[0,T]}$ be the solution of the Laplacian G_2 flow (1.2) on a closed 7-dimensional manifold M with $T < +\infty$ and $Ric_{f(t)} \leq \dfrac{\kappa(t)}{2h(t)} g(t)$, where $g(t)$ is the Riemannian metric associated with $\varphi(t)$. If $u(t_1) = 0$, then $u(t) \equiv 0$ for any $t \in [t_0, t_1] \subset (0, T)$.*

Proof We give the proof of case $h(t) < 0$ (The case $h(t) > 0$ is similar to it). Recalling the definition of $U(t)$, we get

$$
\begin{aligned}
\frac{d}{dt}\ln(I(t)) = \frac{I'(t)}{I(t)} &= -\frac{2D(t)}{h(t)I(t)} + 2a(t) \\
&= -\frac{2}{h(t)}\exp\left\{\int_{t_0}^{t}\left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right)ds\right\} U(t) + 2a(t).
\end{aligned} \tag{5.7}
$$

According to Theorem 5.2 and integrating (5.7) from t' to t_1 for any $t' \in [t_0, t_1]$, yields

$$\begin{aligned}
&\ln I(t_1) - \ln I(t') \\
&= -2\int_{t'}^{t_1} \exp\left\{\int_{t_0}^{t}\left(\frac{h'(s)+\kappa(s)}{h(s)} - \frac{2}{3}R_0\right)ds\right\}\frac{U(t)}{h(t)}dt + 2\int_{t'}^{t_1} a(t)dt \\
&\geq -2U(t_0)\int_{t'}^{t_1} \exp\left\{\int_{t_0}^{t}\left(\frac{h'(s)+\kappa(s)}{h(s)} - \frac{2}{3}R_0\right)ds\right\}\frac{dt}{h(t)} + 2\int_{t'}^{t_1} a(t)dt.
\end{aligned}$$

Since $a(t), h(t)$ are finite, it follows from the last inequality that

$$\begin{aligned}
\frac{I(t_1)}{I(t')} \geq \exp\Bigg\{&-2U(t_0)\int_{t'}^{t_1} \exp\left\{\int_{t_0}^{t}\left(\frac{h'(s)+\kappa(s)}{h(s)} - \frac{2}{3}R_0\right)ds\right\}\frac{dt}{h(t)} \\
&+2\int_{t'}^{t_1} a(t)dt\Bigg\},
\end{aligned}$$

which implies Corollary 5.3. □

Remark 5.4 If we let

$$-\frac{2}{h(t)}\exp\left\{\int_{t_0}^{t}\left(\frac{h'(s)+\kappa(s)}{h(s)} - \frac{2}{3}R_0\right)ds\right\} \equiv C_3,$$

and $a'(t) \geq 0$, where C_3 is a constant, then we get $\ln I(t)$ is convex, which is a parabolic version of the classical Hadamard's three-circle theorem for holomorphic functions. For example, if we let

$$h \equiv C_4, \quad \kappa = \frac{2}{3}R_0C_4,$$

where C_4 is any constant, then we get the classical Hadamard's three-circle theorem.

5.2 Parabolic frequency for the more general parabolic equations under Laplacian G_2 flow

This section considers the parabolic frequency for more general parabolic equations. We use the definition of parabolic frequency in Sect. 5.1, **here we assume $h(t)$ is the negative smooth function.**

Theorem 5.5 *Suppose that $u(t)$ satisfies*

$$|(\partial_t - \Delta_{g(t)})u(t)| \leq C(t)\left(|\nabla_{g(t)}u(t)|_{g(t)} + |u(t)|\right)$$

along the Laplacian G_2 flow (1.2). Then

$$\begin{aligned}
(\ln I(t))' &\geq -\big(2 + C(t)\big)\exp\left\{\int_{t_0}^{t}\left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right)ds\right\}\frac{U(t)}{h(t)} - 3C(t), \\
U'(t) &\geq C^2(t)\left(U(t) + h(t)\exp\left\{\int_{t_0}^{t}\left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right)ds\right\}\right).
\end{aligned}$$

Proof At first, we calculate

$$I'(t) = \int_M \left(2u\partial_t u - 2u\Delta u - 2|\nabla u|^2\right)d\mu$$

$$
\begin{aligned}
&= -\frac{2}{h}D(t) + 2\int_M u(\partial_t - \Delta)u d\mu \\
&\geq -\frac{2}{h}D(t) - 2C\int_M |u|(|\nabla u| + |u|)d\mu \\
&= -\frac{2}{h}D(t) - 2CI(t) - 2C\int_M |\nabla u||u|d\mu \\
&\geq -\frac{2}{h}D(t) - 3CI(t) - \frac{C}{h}D(t) \\
&= -\frac{2+C}{h}D(t) - 3CI(t) \\
&= -(2+C)\exp\left\{\int_{t_0}^t \left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right) ds\right\} I(t)\frac{U(t)}{h} - 3CI(t).
\end{aligned}
$$

Then we get the first inequality.

For the second inequality, we write

$$
D(t) = -h\int_M u\left(\Delta_f + \frac{1}{2}(\partial_t - \Delta)\right) u d\mu + \frac{h}{2}\int_M (\partial_t - \Delta)u d\mu.
$$

Then from (5.3), we get

$$
\begin{aligned}
I'(t) &= \int_M \left(2u\partial_t u - 2u\Delta u - 2|\nabla u|^2\right) d\mu \\
&= 2\int_M u(\Delta_f + \partial_t - \Delta)u d\mu \\
&= 2\int_M u\left(\Delta f + \frac{1}{2}(\partial_t - \Delta)\right) u d\mu + \int_M (\partial_t - \Delta)u d\mu.
\end{aligned}
$$

From the above two equalities, we get

$$
I'(t)D(t) = -2h\left(\int_M u\left(\Delta_f + \frac{1}{2}(\partial_t - \Delta)\right) u d\mu\right)^2 + \frac{h}{2}\left(\int_M (\partial_t - \Delta)u d\mu\right)^2.
$$

According to Lemma 4.1 and Lemma 5.1, we can calculate

$$
\begin{aligned}
D'(t) &= h'\int_M |\nabla u|^2 d\mu + h\int_M (\partial_t - \Delta)|\nabla u|^2 d\mu \\
&= h\int_M \Big(\frac{h'}{h}|\nabla u|^2 - 2|\nabla^2 u|^2 + 2\langle \nabla u, \nabla(\partial_t - \Delta)u\rangle \\
&\quad + \frac{2}{3}|\mathbf{T}|^2|\nabla u|^2 - 4|\mathbf{T}_{ik}\nabla_i u|^2\Big) d\mu \\
&= h\int_M \Big(\frac{h'}{h}|\nabla u|^2 - 2|\Delta_f u|^2 + 2\mathrm{Ric}_f(\nabla u, \nabla u) - 2\Delta_f u\cdot(\partial_t - \Delta)u \\
&\quad + \frac{2}{3}|\mathbf{T}|^2|\nabla u|^2 - 4|\mathbf{T}_{ik}\nabla_i u|^2\Big) d\mu \\
&= 2h\int_M \left(\mathrm{Ric}_f(\nabla u, \nabla u) + \frac{h'}{2h}|\nabla u|^2 + \frac{1}{3}|\mathbf{T}|^2|\nabla u|^2 - 2|\mathbf{T}_{ik}\nabla_i u|^2\right) d\mu \\
&\quad - 2h\int_M \left(|\Delta_f u|^2 + \Delta_f u\cdot(\partial_t - \Delta)u\right) d\mu
\end{aligned}
$$

$$
\begin{aligned}
&\geq \int_M \left(\kappa + h' - \frac{2}{3}hR_0\right)|\nabla u|^2 d\mu - 2h\int_M \left(|\Delta_f u|^2 + \Delta_f u\cdot(\partial_t - \Delta)u\right)d\mu \\
&= -2h\int_M \left(\left|\left(\Delta_f + \frac{1}{2}(\partial_t - \Delta)\right)u\right|^2 - \frac{1}{4}|(\partial_t - \Delta)u|^2\right)d\mu \\
&\quad + \left(\kappa + h' - \frac{2}{3}hR_0\right)\frac{D(t)}{h}.
\end{aligned}
$$

where $R_0 = \min\limits_{M\times[t_0,t_1]} R(t)$.

Together with the above, using Cauchy-Schwarz inequality, yields

$$
\begin{aligned}
I^2(t)U'(t) &= \exp\left\{-\int_{t_0}^t \left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right)ds\right\} \\
&\quad \cdot\left[\left(\frac{2}{3}R_0 - \frac{h'(s)+\kappa(s)}{h(s)}\right)I(t)D(t) + I(t)D'(t) - I'(t)D(t)\right] \\
&\geq \exp\left\{-\int_{t_0}^t \left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right)ds\right\} \\
&\quad \cdot\left\{-2hI(t)\int_M \left(\left|\left(\Delta_f + \frac{1}{2}(\partial_t - \Delta)\right)u\right|^2 - \frac{1}{4}|(\partial_t - \Delta)u|^2\right)d\mu\right. \\
&\quad \left. + 2h\left(\int_M u\left(\Delta_f + \frac{1}{2}(\partial_t - \Delta)\right)u d\mu\right)^2 - \frac{h}{2}\left(\int_M (\partial_t - \Delta)u d\mu\right)^2\right\} \\
&\geq \frac{h}{2}I(t)\exp\left\{-\int_{t_0}^t \left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right)ds\right\}\int_M |(\partial_t - \Delta)u|^2 d\mu \\
&\geq \frac{h}{2}I(t)C^2\exp\left\{-\int_{t_0}^t \left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right)ds\right\}\int_M (|\nabla u| + |u|)^2 d\mu \\
&\geq C^2\exp\left\{-\int_{t_0}^t \left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right)ds\right\} I(t)\big(D(t) + hI(t)\big).
\end{aligned}
$$

Then we prove this Theorem. □

Corollary 5.6 *Suppose that $u(t) : M \times [t_0, t_1] \to \mathbb{R}$ satisfies*

$$
|(\partial_t - \Delta_{g(t)})u(t)| \leq C(t)\left(|\nabla_{g(t)}u(t)|_{g(t)} + |u(t)|\right)
$$

along the Laplacian G_2 flow (1.2). Then

$$
\begin{aligned}
I(t_1) \geq I(t_0)\exp&\left\{\int_{t_0}^{t_1} -\big(2 + \sup_{[t_0,t_1]} C(t)\big)\exp\left\{\int_{t_0}^t \left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right)ds\right\}\right. \\
&\cdot\frac{1}{h(t)}\left[\exp\left\{-\int_{t_0}^t C^2(s)ds\right\}U(t_0) + \exp\left\{-\int_{t_0}^t C^2(s)ds\right\}\right. \\
&\cdot\left.\int_{t_0}^{t_1}\exp\left\{\int_{t_0}^{\tau}\left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)} - C^2(s)\right)ds\right\}d\tau\right]dt \\
&\left. -3\int_{t_0}^{t_1}\sup_{[t_0,t_1]} C(t)dt\right\}.
\end{aligned}
$$

In particular, if $u(t_1) = 0$, then $u \equiv 0$ for all $t \in [t_0, t_1]$.

Proof By the first inequality in Theorem 5.5, we get

$$\begin{aligned}\ln(I(t_1)) - \ln(I(t_0)) &\geq \int_{t_0}^{t_1} -(2+C(t)) \exp\left\{\int_{t_0}^{t}\left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)}\right)ds\right\}\frac{U(t)}{h(t)}dt \\ &\quad - \int_{t_0}^{t_1} 3C(t)dt. \end{aligned} \tag{5.8}$$

If we write $\widehat{U}(t)$ as $\widehat{U}(t) = \exp\{-\int_{t_0}^{t} C^2(s)ds\}U(t)$, then from the second inequality in Theorem 5.5, we have

$$\widehat{U}(t)' \geq h(t)C^2(t)\exp\left\{\int_{t_0}^{t}\left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)} - C^2(s)\right)ds\right\}. \tag{5.9}$$

Integrating (5.9) from t_0 to t for any $t \in [t_0, t_1]$, yields

$$\widehat{U}(t) \geq \widehat{U}(t_0) + \int_{t_0}^{t} h(\tau)C^2(\tau)\exp\left\{\int_{t_0}^{\tau}\left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)} - C^2(s)\right)ds\right\}d\tau,$$

which means

$$\begin{aligned} U(t) \geq \exp&\left\{-\int_{t_0}^{t} C^2(s)ds\right\}U(t_0) + \exp\left\{-\int_{t_0}^{t} C^2(s)ds\right\} \\ &\cdot \int_{t_0}^{t_1} h(\tau)C^2(\tau)\exp\left\{\int_{t_0}^{\tau}\left(-\frac{2}{3}R_0 + \frac{h'(s)+\kappa(s)}{h(s)} - C^2(s)\right)ds\right\}d\tau. \end{aligned} \tag{5.10}$$

where we use $h(t)$ is the negative smooth function. Submitting (5.10) to (5.8), we get the desired result. □

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Declarations

Conflict of interest No Conflict of interest.

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