Lecture 2. Algebraic Bethe ansatz

Yunfeng Jiang

1 Introduction

In this lecture, we will solve the XXX spin chain in a very different way. In the previous lecture, we have discussed how to construct the eigenvectors of the model by the coordinate Bethe ansatz. There we are guided by physical intuitions which lead to a number of natural educated guesses (or ansatz). The method we are going to introduce in this lecture is rather different in nature. It relies heavily on the underlying algebraic structure of the model and the results can be proven rigorously. This method is called Quantum Inverse Scattering Method (QISM) or Algebraic Bethe Ansatz (ABA) in the literature, which was mainly developed by the Leningrad (now Saint-Petersburg) School in the mid 1970s. The name QISM emphasis its origin, because it can be seen as a quantum version of the inverse scattering method, which is an important method to solve classical integrable differential equations such as the KdV equation. On the other hand, the name ABA highlights the algebraic nature of the method. We will see that it plays a central role in the solution of the model.

For most people who learn about algebraic Bethe ansatz for the first time, it almost look like a magic. You have no idea why one should define such and such an operator, but at the end of the day, it works ! Of course all the quantities defined below have their origins deeply rooted in the classical inverse scattering method, but it would take at least one whole lecture to explain all these. A practical attitude is to first accept the fact that we can define these quantities and see how it works. Once you get used to these notions, they become very natural and powerful.

2 Lax matrix

The starting point of ABA is a quantity called Lax matrix. It is a matrix, as the name suggests, defined at each site of the spin chain. For our case, it is a 2×2 matrix whose matrix elements are basically local spin operators. More precisely, it takes the following form

$$L_{an}(u) = \begin{pmatrix} u + iS_n^z & iS_n^- \\ iS_n^+ & u - iS_n^z \end{pmatrix}_a \tag{2.1}$$

Let us give some more explanation to this definition. First of all, the matrix depends on a parameter u, this is called the *spectral parameter*. Second, we can consider the matrix to be defined in certain 2-dimensional linear space, which we call *auxiliary space* and denote by an abstract index a. The Lax matrix can be regarded as an operator defined on space $\mathbb{C}_a^2 \otimes V_n$ where $\mathbb{C}_a^2$ is the two-dimensional auxiliary space and V_n is the local Hilbert space at site n. Finally S_n^α are the local spin operators which act on the site-n and satisfy the standard commutation relation

$$[S_n^\alpha, S_m^\beta] = i\epsilon^{\alpha\beta\gamma} S^\gamma \delta_{m,n} \tag{2.2}$$

where the $\delta_{m,n}$ indicates that spin operators at different sites commute. The indices $\alpha, \beta = x, y, z$ and

$$S_n^\pm = S_n^x \pm i S_n^y \,. \tag{2.3}$$

We therefore have the following commutation relations

$$[S_n^z, S_n^\pm] = \pm S_n^\pm, \qquad [S_n^+, S_n^-] = 2S_n^z \,. \tag{2.4}$$

The RLL-relation The most important property of the Lax matrix defined in (2.1) is that it satisfies the so-called RLL-relation, given by

$$R_{ab}(u-v)L_{an}(u)L_{bn}(v) = L_{bn}(v)L_{an}(u)R_{ab}(u-v)\,. \tag{2.5}$$

Here 'a' and 'b' are two distinct two-dimensional auxiliary spaces. $R_{ab}(u-v)$ is a 4×4 matrix in the tensor product space $\mathbb{C}_a^2 \otimes \mathbb{C}_b^2$ given by

$$R_{ab}(u-v) = \begin{pmatrix} u-v+i & 0 & 0 & 0 \\ 0 & u-v & i & 0 \\ 0 & i & u-v & 0 \\ 0 & 0 & 0 & u-v+i \end{pmatrix}_{ab} \tag{2.6}$$

The operators $L_{an}(u)$ is a 2×2 matrix in auxiliary space $\mathbb{C}_a^2$, and it is a 4×4 matrix in the space $\mathbb{C}_a^2 \otimes \mathbb{C}_b^2$ by taking a tensor product with the identity matrix in auxiliary space $\mathbb{C}_b$, *i.e.*

$$L_{an}(u) = L_{an}(u) \otimes \mathbb{I}_b = \begin{pmatrix} u+iS_n^z & 0 & iS_n^- & 0 \\ 0 & u+iS_n^z & 0 & iS_n^- \\ iS_n^+ & 0 & u-iS_n^z & 0 \\ 0 & iS_n^+ & 0 & u-iS_n^z \end{pmatrix}_{ab} \tag{2.7}$$

Similarly, $L_{bn}(v)$ can be understood as the following 4×4 matrix

$$L_{bn}(v) = \mathbb{I}_a \otimes L_{bn}(v) = \begin{pmatrix} u + iS_n^z & iS_n^- & 0 & 0 \\ iS_n^+ & u - iS_n^z & 0 & 0 \\ 0 & 0 & u + iS_n^z & iS_n^- \\ 0 & 0 & iS_n^+ & u - iS_n^z \end{pmatrix}_{ab} \tag{2.8}$$

Now all the operators in (2.6), (2.7) and (2.8) are 4×4 matrices in the space $\mathbb{C}_a^2 \otimes \mathbb{C}_b^2$, we can plug them back in the RLL-relation (2.5) and compute both side by the usual matrix multiplication. Notice that since the matrix elements are *local spin operators*, we need to pay attention to the order of these operators. Having this in mind, it is then a straightforward exercise to prove (2.5) by showing that each of the 16 matrix elements on both sides of the equation match. Some of the matrix elements are identical trivially, while to see the identification of others matrix elements, we need to use the SU(2) algebra (2.4). For example, let us consider the matrix elements on the 2nd row and 1st column on both sides. The results from the left hand side and right hand side read

$$\begin{aligned} \texttt{lhs}_{2,1} &= i(u^2 - uv + iv)S_n^+ - iS_n^+ S_n^z - (u - v)S_n^z S_n^+ \,, \\ \texttt{rhs}_{2,1} &= iu(u - v + i)S_n^+ - (u - v + i)S_n^+ S_n^z \end{aligned} \tag{2.9}$$

These two expressions do not look the same naively, but by using the fact $S_n^z S_n^+ = S_n^+ S_n^z + S_n^+$ in the first line, we find that they are indeed identical. The proof for other matrix elements can be done in a similar way. Therefore, we see that the RLL-relation is a non-trivial relation for the Lax matrix which relies on the underlying SU(2) symmetry algebra.

3 Monodromy matrix

After defining a Lax matrix at each site of the spin chain, we can multiply them together in order, which leads to the central object of algebraic Bethe ansatz — the monodromy matrix. Let us define

$$M_a(u) = L_{a1}(u)L_{a2}(u)\dots L_{aL}(u) \tag{3.1}$$

where $L_{an}(u)$ is the Lax matrix at the n-th site of the spin chain and all the Lax matrices share the same auxiliary space. This means, the Lax matrices are all 2×2 matrices in the *same* auxiliary space a. The difference is that the matrix elements are local spin operators at different sites. Therefore, $M_a(u)$ can be written formally as a 2×2 matrix in the auxiliary space as

$$M_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} \tag{3.2}$$

where $A(u), B(u), C(u), D(u)$ are some complicated operators which are constructed from local spin operators.

An example To have an idea of what A, B, C, D operators look like, let us consider a length-2 spin chain. We have

$$M_a(u) = L_{a1}(u)L_{a2}(u) = \begin{pmatrix} u + iS_1^z & iS_1^- \\ iS_1^+ & u - iS_1^z \end{pmatrix} \begin{pmatrix} u + iS_2^z & iS_2^- \\ iS_2^+ & u - iS_2^z \end{pmatrix} \tag{3.3}$$

The four operators take the following form

$$\begin{aligned} A(u) &= (u + iS_1^z)(u + iS_2^z) - S_1^- S_2^+ \,, \\ B(u) &= i(u + iS_1^z)S_2^- + iS_1^-(u - iS_2^z) \,, \\ C(u) &= iS_1^+(u + iS_2^z) + i(u - iS_1^z)S_2^+ \,, \\ D(u) &= (u - iS_1^z)(u - iS_2^z) - S_1^+ S_2^- \,. \end{aligned} \tag{3.4}$$

It is straightforward to write down the explicit expressions of A, B, C, D for longer spin chains, but of course the expressions would become rather bulky very quickly.

The operators A, B, C, D are non-local operators which act on all sites of the spin chain. As an example, let us consider how they act on the state $|\uparrow\uparrow\rangle$. Recall that at each site we have

$$\begin{aligned} &S_n^+|\uparrow\rangle_n = 0, \qquad S_n^-|\uparrow\rangle_n = |\downarrow\rangle_n, \qquad S_n^z|\uparrow\rangle_n = \frac{1}{2}|\uparrow\rangle_n \,, \\ &S_n^+|\downarrow\rangle_n = |\uparrow\rangle_n, \qquad S_n^-|\downarrow\rangle_n = 0, \qquad S_n^z|\downarrow\rangle_n = -\frac{1}{2}|\downarrow\rangle_n \,, \end{aligned} \tag{3.5}$$

we find that

$$\begin{aligned} A(u)|\uparrow\uparrow\rangle &= (u + \tfrac{i}{2})^2|\uparrow\uparrow\rangle \,, \\ B(u)|\uparrow\uparrow\rangle &= i(u + \tfrac{i}{2})|\uparrow\downarrow\rangle + i(u - \tfrac{i}{2})|\downarrow\uparrow\rangle \,, \\ C(u)|\uparrow\uparrow\rangle &= 0 \,, \\ D(u)|\uparrow\uparrow\rangle &= (u - \tfrac{i}{2})^2|\uparrow\uparrow\rangle \,. \end{aligned} \tag{3.6}$$

Actions of these operators on other states can be computed similarly.

The key point, however, is that for most of our purpose, we do not need to use the explicit form of the A, B, C, D operators. The crucial thing is the *algebra* that these operators satisfy. Let us now move to this point.

The algebra The algebra between A, B, C, D is a consequence of the RLL-relation proved in the previous section. It takes a very similar form

$$R_{ab}(u-v)M_a(u)M_b(v) = M_b(v)M_a(u)R_{ab}(u-v)\,. \tag{3.7}$$

We first prove this relation. From the definition of the monodromy matrix, the left hand side reads (we omit the spectral parameters for the moment)

$$R_{ab}(L_{a1}L_{a2}\dots L_{aL})(L_{b1}L_{b2}\dots L_{bL}) = (R_{ab}L_{a1}L_{b1})(L_{a2}\dots L_{aL})(L_{b2}\dots L_{bL}). \tag{3.8}$$

Notice that because L_{1b} does not share any common indices with $L_{a2},\dots,L_{aL}$, which means they act in different spaces, it commutes with them and we can move L_{1b} towards right all the way up to L_{1a} where they share one common index "1". Now in the first bracket on the right hand side, we can use the RLL-relation

$$R_{ab}L_{a1}L_{b1} = L_{b1}L_{a1}R_{ab} \tag{3.9}$$

and obtain

$$R_{ab}(L_{a1}L_{a2}\dots L_{aL})(L_{b1}L_{b2}\dots L_{bL}) = (L_{b1}L_{a1})R_{ab}(L_{a2}\dots L_{aL})(L_{b2}\dots L_{bL}) \tag{3.10}$$

Now we can make the same move for L_{a2} and L_{b2} as follows

$$\cdots R_{ab}(L_{a2}\dots L_{aL})(L_{b2}\dots L_{bL}) = \cdots L_{b2}L_{a2}R_{ab}(L_{a3}\dots L_{aL})(L_{b3}\dots L_{bL}) \tag{3.11}$$

Repeating this for all L_{an}, L_{bn} until we move all the Lax matrices to the left of R_{ab}. We obtain

$$\begin{aligned} R_{ab}(L_{a1}L_{a2}\dots L_{aL})(L_{b1}L_{b2}\dots L_{bL}) &= (L_{1b}L_{1a})(L_{2b}L_{2a})\dots(L_{bL}L_{aL})R_{ab} \\ &= (L_{1b}L_{2b}\dots L_{bL})(L_{a1}L_{a2}\dots L_{aL})R_{ab}\,, \end{aligned} \tag{3.12}$$

which is the relation that we want to prove.

To extract the algebra between A, B, C, D, we can write down the RMM-relation (3.7) explicitly and compare the components of the two sides of the relation. Similar to the RLL-relation, we can write

$$M_a(u) = M_a(u)\otimes \mathbb{I}_b = \begin{pmatrix} A(u) & 0 & B(u) & 0 \\ 0 & A(u) & 0 & B(u) \\ C(u) & 0 & D(u) & 0 \\ 0 & C(u) & 0 & D(u) \end{pmatrix}_{a,b} \tag{3.13}$$

and

$$M_b(v) = \mathbb{I}_a \otimes M_b(v) = \begin{pmatrix} A(v) & B(v) & 0 & 0 \\ C(v) & D(v) & 0 & 0 \\ 0 & 0 & A(v) & B(v) \\ 0 & 0 & C(v) & D(v) \end{pmatrix}_{a,b} . \tag{3.14}$$

Plugging (3.13), (3.14) and (2.6) into the RMM-relation, both sides are 4×4 matrices. We therefore obtain 16 relations between the operators A, B, C, D. We leave this computation as an exercise and only present the three relations that are useful for later derivations

$$\begin{aligned} B(u)B(v) &= B(v)B(u) \,, \\ A(u)B(v) &= f(v-u)B(v)A(u) + g(u-v)B(u)A(v) \,, \\ D(u)B(v) &= f(u-v)B(v)D(u) + g(v-u)B(u)D(v) \end{aligned} \tag{3.15}$$

where

$$f(u) = \frac{u+i}{u}, \qquad g(u) = \frac{i}{u} \,. \tag{3.16}$$

4 Transfer matrix

Finally we define the last important quantity for our construction, this is the transfer matrix. It is defined as the trace of the monodromy matrix in the auxiliary space, *i.e.*

$$T(u) = \mathrm{tr}_a M_a(u) = A(u) + D(u) \,. \tag{4.1}$$

Now using the commutations of A, B, C, D, we can prove that $T(u)$ and $T(v)$ actually commute

$$[T(u), T(v)] = 0 \,. \tag{4.2}$$

We leave the proof as an exercise. Although the definition of the transfer matrix looks somewhat simple, it is crucial for the spin chain. We will see later that it plays the role of generating function of the conserved charges. The fact that these charges are conserved is guaranteed by the commutativity relation (4.2).

5 Yang-Baxter equation

In our discussions so far, the matrix R_{ab} looks rather arbitrary. We can imagine taking a different kind of Lax-operator (say with a different u dependence) and try to write down certain RLL-relation using a different R-matrix. Repeating the same procedure as before, we could obtain another algebra. However, it turns out that the R-matrix is highly constrained and cannot be taken arbitrarily if we want our algebra to be compatible.

In order to see this, let us consider three monodromy matrices M_a, M_b, M_c acting on three different auxiliary spaces. Consider the product $M_a(u_1)M_b(u_2)M_c(u_3)$. Now using RMM-relation, we can re-order them to the opposite order $M_c(u_3)M_b(u_2)M_a(u_1)$. This can be done in two different ways. One is first moving M_a to the rightmost and then swap the order of M_b and M_c, *i.e.*

$$M_a M_b M_c \to M_b M_a M_c \to M_b M_c M_a \to M_c M_b M_a \,. \tag{5.1}$$

The other is first moving M_c to the leftmost and then swap the order of M_a and M_b, *i.e.*

$$M_a M_b M_c \to M_a M_c M_b \to M_c M_a M_b \to M_c M_b M_a \,. \tag{5.2}$$

When we exchange the order of two monodromy matrices, we can use the RMM-relation. For example

$$R_{ab}(u_1, u_2)M_a(u_1)M_b(u_2) = M_b(u_2)M_a(u_1)R_{ab}(u_1, u_2) \,. \tag{5.3}$$

Or equivalently

$$M_b(u_2)M_a(u_1) = R_{ab}(u_1, u_2)M_a(u_1)M_b(u_2)R_{ab}^{-1}(u_1, u_2) \,. \tag{5.4}$$

Following the first way of reordering, we obtain

$$\begin{aligned} M_c M_b M_a &= R_{ab} \left(M_c M_a M_b\right) R_{ab}^{-1} \\ &= R_{ab} R_{ac} \left(M_a M_c M_b\right) R_{ac}^{-1} R_{ab}^{-1} \\ &= R_{ab} R_{ac} R_{bc} \left(M_a M_b M_c\right) R_{bc}^{-1} R_{ac}^{-1} R_{ab}^{-1} \,. \end{aligned} \tag{5.5}$$

while following the second way of ordering, we obtain

$$\begin{aligned} M_c M_b M_a &= R_{bc} \left(M_b M_c M_a\right) R_{bc}^{-1} \\ &= R_{bc} R_{ac} \left(M_b M_a M_c\right) R_{ac}^{-1} R_{bc}^{-1} \\ &= R_{bc} R_{ac} R_{ab} \left(M_a M_b M_c\right) R_{ab}^{-1} R_{ac}^{-1} R_{bc}^{-1} \,. \end{aligned} \tag{5.6}$$

These two ways of reordering must be consistent. Therefore, comparing (5.5) and (5.6), we find

$$R_{ab}(u_1, u_2)R_{ac}(u_1, u_3)R_{bc}(u_2, u_3) = R_{bc}(u_2, u_3)R_{ac}(u_1, u_3)R_{ab}(u_1, u_2). \tag{5.7}$$

This is the Yang-Baxter equation. This equation was discovered in different contexts in the study of integrable models. It was proposed by J.B. McGuire [1] in 1964 and C.N. Yang in 1967 [2] while studying multi-component Lieb-Liniger model. In statistical mechanics, it goes back to Onsager's solution of 2d Ising model in 1944 [3] culminated in the work of R. Baxter in 1972 [4] while studying integrable lattice models, which he called star-triangle relation.

An analogy with Lie algebra Let us draw an analogy with Lie algebra. We define the algebra by commutation relations of the generators

$$[e_i, e_j] = c_{ij}^k \, e_k \tag{5.8}$$

where c_{ij}^k are the structure constants. In our case, the generators are the components of the monodromy matrix, the algebra is given by the RMM-relation where the R-matrix play the role of structure constants. Now in the theory of Lie algebra, the structure constants cannot be chosen arbitrarily and are constrained by the Jacobi identity

$$[e_i, [e_j, e_k]] + [e_j, [e_k, e_i]] + [e_k, [e_i, e_j]] = 0\,, \tag{5.9}$$

This is a consistency relation of the algebra. Similarly, here we also have consistency relations for the RMM-relation, which is nothing but the Yang-Baxter relation.

Solving YBE Finding solutions of Yang-Baxter relation is a highly non-trivial and deep mathematical problem. If we assume the R-matrix takes the form

$$R(u_1, u_2) = R(u_1 - u_2) \tag{5.10}$$

and certain non-degeneracy conditions, it can be shown that there are three types of solutions: rational, trigonometric and elliptic meaning $R(u)$ is a rational, trigonometric and elliptic functions of u respectively. More precisely, in a paper by Belavin and Drinfeld in 1983, they studied the *classical* Yang-Baxter equation and classified the solutions into these three categories. The quantum R-matrix can be understood as the deformation of classical R-matrices, therefore we also have three types of solutions. Interestingly, the R-matrix of the XXX, XXZ, XYZ model belong to these three types respectively.

There are on-going progress in the field of solving YBE even after so many years. Here we mention two examples. Firstly, the classification of the solution of the YBE can be understood nicely from a 4D Chern-Simons theory point of view following the recent work of K. Costello, E. Witten and M. Yamazaki. This series of works give a new perspective for integrable models. Secondly, one finds more solutions for the R-matrices which does not satisfy the assumption (5.10). Notably, the Beisert's S-matrix in integrability in AdS/CFT correspondence belongs to this type.

For an algebra, a natural question to ask is what are the representations. We will see that different representations of the RMM-relation correspond to different integrable models. To see this more clearly, let us now see how the Heisenberg spin chain emerges from our algebraic construction.

6 From transfer matrix to Hamiltonian

So far we have introduced a number of quantities and showed that they have certain nice properties. At this point, one might start to wonder: Well, all these are nice, but what does it have to do with solving the Heisenberg spin chain ? This is, of course a legitimate concern, which we will address now. The relation between our construction so far and the Heisenberg XXX spin chain is based on the following two 'magics' :

1. **Magic 1.** The Hamiltonian of the Heisenberg spin chain H_{XXX} is hidden inside the transfer matrix $T(u)$. More precisely

$$H_{\text{XXX}} \sim \frac{\mathrm{d}}{\mathrm{d}u} \log T(u)\bigg|_{u=i/2} . \tag{6.1}$$

   we will make this relation more precise in this section.

2. **Magic 2.** The eigenstate of the Hamiltonian H_{XXX} (also the transfer matrix $T(u)$) can be constructed by

$$B(u_1)B(u_2)\ldots B(u_N)|\uparrow\uparrow\ldots\uparrow\rangle \tag{6.2}$$

   where B is one of the components in the monodromy matrix. We will discuss this construction in detail in the next section.

We will first discuss 'magic 1' in this section and show how to derive the Hamiltonian from the transfer matrix.

Permutation operators To facilitate our discussions below, it is useful to introduce the permutation operator, together with some of their properties. To this end, consider the following operator which acts on the space $\mathbb{C}^2 \otimes \mathbb{C}^2$

$$\mathrm{P}_{ab} = \frac{1}{2}\Big(\mathrm{I}_a \otimes \mathrm{I}_b + \sum_{\alpha} \sigma_a^{\alpha} \otimes \sigma_b^{\alpha}\Big), \qquad \alpha = x, y, z\,. \tag{6.3}$$

In terms of matrix, it is given by

$$\mathrm{P}_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{ab} \tag{6.4}$$

One can check explicitly that for any vector $|x\rangle_a, |y\rangle_b$ in the two spaces, we have

$$\mathrm{P}_{ab}\left(|x\rangle_a \otimes |y\rangle_b\right) = |y\rangle_a \otimes |x\rangle_b\,. \tag{6.5}$$

For this reason, P_{ab} is called the permutation operator. It is not hard to prove the following identities of the permutation operators (*e.g.* act them on proper states)

$$\mathrm{P}_{n,a}\mathrm{P}_{n,b} = \mathrm{P}_{a,b}\mathrm{P}_{n,a} = \mathrm{P}_{n,b}\mathrm{P}_{b,a}, \qquad \mathrm{P}_{a,b} = \mathrm{P}_{b,a}\,. \tag{6.6}$$

The point of introducing the permutation operator is that both the Hamiltonian of the Heisenberg spin chain and the Lax operator can be written in terms of these operators. Recall that the Hamiltonian of the XXX spin chain can be written as

$$H_{\mathrm{XXX}} = \sum_{n=1}^{L} \vec{S}_n \cdot \vec{S}_{n+1} = \frac{1}{4}\sum_{n=1}^{L} \vec{\sigma}_n \cdot \vec{\sigma}_{n+1}\,. \tag{6.7}$$

Using the definition (6.3), we have

$$\vec{\sigma}_n \cdot \vec{\sigma}_{n+1} = 2\mathrm{P}_{n,n+1} - \mathrm{I}_{n,n+1} \tag{6.8}$$

and

$$H_{\mathrm{XXX}} = \frac{1}{2}\sum_{n=1}^{L} \mathrm{P}_{n,n+1} - \frac{L}{4}\,. \tag{6.9}$$

So we see that up to a constant shift, the Hamiltonian of XXX spin chain is essentially a sum of permutation operators.

So far we haven't specify the representation of the local spin operator S_n^α. In order to write down the Hamiltonian, we take each quantum space to be the same as the auxiliary space, namely $V_n = \mathbb{C}^2$. In this representation, the Lax operator can be written in terms of the permutation operator as

$$L_{an}(u) = \left(u - \tfrac{i}{2}\right)\mathrm{I}_{a,n} + i\mathrm{P}_{a,n}\,. \tag{6.10}$$

From this expression, we have two very simple but extremely useful observations

- At $u = i/2$, the Lax operator becomes essentially the permutation operator

$$L_{an}(i/2) = i\mathrm{P}_{a,n} \tag{6.11}$$

- Taking derivative with respect to u, we obtain the identity operator,

$$\frac{\mathrm{d}}{\mathrm{d}u} L_{an}(u) = \mathrm{I}_{a,n} \,. \tag{6.12}$$

Now we are ready to extract useful quantities from the transfer matrix.

The shift operator The first useful quantity is the so-called the *shift operator*, we will see where this name comes from soon. It is simply given by

$$\begin{aligned} U =& i^{-L}\, T(i/2) = i^{-L}\, \mathrm{tr}_a M_a(i/2) \\ =& \mathrm{tr}_a \mathrm{P}_{a,1}\mathrm{P}_{a,2}\ldots\mathrm{P}_{a,L} = (\mathrm{tr}_a \mathrm{P}_{a,L})\, \mathrm{P}_{L,L-1}\ldots\mathrm{P}_{2,3}\mathrm{P}_{1,2} \\ =& \mathrm{P}_{L,L-1}\ldots\mathrm{P}_{2,3}\mathrm{P}_{1,2} \end{aligned} \tag{6.13}$$

where we have used the fact

$$\mathrm{tr}_a \mathrm{P}_{a,n} = \mathrm{I}_n. \tag{6.14}$$

Using the relation

$$\mathrm{P}_{n,m} X_m \mathrm{P}_{n,m} = X_n \,, \tag{6.15}$$

it is straightforward to show that

$$U^{-1} X_n U = X_{n+1} \,. \tag{6.16}$$

When acting this operator on any spin chain state, it shifts all spins one site towards the right.

The Hamiltonian Now let us expand the transfer matrix around $u = i/2$. First we have

$$\frac{\mathrm{d}}{\mathrm{d}u} M_a(u)\bigg|_{u=i/2} = i^{L-1} \sum_{n=1}^{L} \mathrm{P}_{a,1}\ldots\widehat{\mathrm{P}}_{a,n}\ldots\mathrm{P}_{a,L} \tag{6.17}$$

where $\widehat{\mathrm{P}}_{a,n}$ means the permutation operator $\mathrm{P}_{a,n}$ is missing from the string of operators. Now to obtain the derivative with respect to the transfer matrix, we need to take the trace in the auxiliary space. We can use a similar trick to the case when deriving the shift operator. This leads to

$$\frac{\mathrm{d}}{\mathrm{d}u}T(u)\bigg|_{u=i/2} = i^{L-1}\sum_{n} \mathrm{P}_{L,L-1}\dots\mathrm{P}_{n-1,n+1}\dots\mathrm{P}_{1,2} \tag{6.18}$$

This is also a string of products of permutation operators. It might look a bit bulky, but most of the permutations can be cancelled neatly by U^{-1}. It is straightforward to show that

$$\left(\frac{\mathrm{d}}{\mathrm{d}u}T(u)\right)T(u)^{-1}\bigg|_{u=i/2} = \frac{1}{i}\sum_{n=1}^{L}\mathrm{P}_{n,n+1}\,. \tag{6.19}$$

Formally we can write the left hand side as the logarithm derivative

$$\left(\frac{\mathrm{d}}{\mathrm{d}u}T(u)\right)T(u)^{-1} = \frac{\mathrm{d}}{\mathrm{d}u}\log T(u)\,. \tag{6.20}$$

Recall that the Hamiltonian takes the form

$$H = \frac{1}{2}\sum_{n=1}^{L}\mathrm{P}_{n,n+1} - \frac{L}{4}\,, \tag{6.21}$$

we thus found that

$$H_{\mathrm{XXX}} = \frac{i}{2}\frac{\mathrm{d}}{\mathrm{d}u}\log T(u)\bigg|_{u=i/2} - \frac{L}{4}\,. \tag{6.22}$$

7 Construction of eigenvectors

In this section, we consider the second 'magic', namely it turns out that the eigenstates can be constructed by acting B-operators on the pseudovacuum $|\uparrow^L\rangle$. Since we have shown that the transfer matrix is a generating function of conserved charges, our strategy is diagonalizing the transfer matrix $T(u) = A(u) + D(u)$ instead. This might seem to be an even harder problem, but as we will see, it can be achieved nicely by exploiting the algebra which we derived in the previous section.

Pseudovacuum As a first step, we want to show that the pseudovacuum state $|\uparrow^L\rangle$ diagonalizes $A(u)$ and $D(u)$ and is annihilated by $C(u)$. Let us consider the action of the monodromy matrix on the pseudovacuum

$$\begin{aligned} M_a(u)|\uparrow^L\rangle &= L_{a1}(u)L_{a2}(u)\dots L_{aL}(u)|\uparrow^L\rangle \\ &= (L_{a1}(u)|\uparrow\rangle_1)\otimes\dots\otimes(L_{aL}(u)|\uparrow\rangle_L) \end{aligned} \tag{7.1}$$

On each site, we have

$$L_{an}(u)|\uparrow\rangle_n = \begin{pmatrix} u+iS_n^z & iS_n^- \\ iS_n^+ & u-iS_n^z \end{pmatrix}|\uparrow\rangle_n = \begin{pmatrix} u+\frac{i}{2} & iS_n^- \\ 0 & u-\frac{i}{2} \end{pmatrix}|\uparrow\rangle_n \,. \tag{7.2}$$

This implies that at each site we have an upper triangular matrix. The multiplication of upper triangular matrices is still an upper triangular matrix. We therefore have

$$\begin{aligned} M_a(u)|\uparrow^L\rangle &= \begin{pmatrix} u+\frac{i}{2} & iS_1^- \\ 0 & u-\frac{i}{2} \end{pmatrix}\begin{pmatrix} u+\frac{i}{2} & iS_2^- \\ 0 & u-\frac{i}{2} \end{pmatrix}\ldots\begin{pmatrix} u+\frac{i}{2} & iS_L^- \\ 0 & u-\frac{i}{2} \end{pmatrix}|\uparrow^L\rangle \\ &= \begin{pmatrix} (u+\frac{i}{2})^L & \star \\ 0 & (u-\frac{i}{2})^L \end{pmatrix}|\uparrow^L\rangle \end{aligned} \tag{7.3}$$

Comparing with

$$M_a(u)|\uparrow^L\rangle = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}|\uparrow^L\rangle, \tag{7.4}$$

we find

$$A(u)|\uparrow^L\rangle = a(u)|\uparrow^L\rangle, \qquad D(u)|\uparrow^L\rangle = d(u)|\uparrow^L\rangle, \qquad C(u)|\uparrow^L\rangle = 0\,, \tag{7.5}$$

where

$$a(u) = \left(u+\tfrac{i}{2}\right)^L, \qquad d(u) = \left(u-\tfrac{i}{2}\right)^L \,. \tag{7.6}$$

The first two equations in (7.5) shows that $|\uparrow^L\rangle$ is indeed an eigenvector of $A(u)$ and $D(u)$; the last equation shows that it is annihilated by $C(u)$. In the literature, such a state is also called a highest weight state or reference state. The existence of such a state is a non-trivial property of the model and is a necessary condition that the model can be solved by Bethe ansatz. Some integrable models do not have a reference state, for example the XYZ spin chain and the Toda chain. These models have to be solved by other methods such as Sklyanin's separation of variables.

The N-magnon state Now that we have discussed the action of A, C, D on the pseudovacuum state. What about $B(u)$? The action of $B(u)$ on $|\uparrow^L\rangle$ gives something complicated. The action of each $B(u)$ on a given state flips down a spin. The flipped spin can be located at any site of the spin chain which need to be summed over with different weights. We have computed $B(u)|\uparrow^L\rangle$ for $L=2$ in (3.6). Now we shall show that the following state

$$|\mathbf{u}_N\rangle = B(u_1)B(u_2)\ldots B(u_N)|\uparrow^L\rangle \tag{7.7}$$

is an eigenstate of $T(u)$ if $\mathbf{u}_N = \{u_1, \ldots, u_N\}$ satisfy certain conditions. To this end, we need to know the action of $A(u)$ and $D(u)$ on $|\mathbf{u}_N\rangle$.

Let us first see how $A(u)$ acts on $|\mathbf{u}_N\rangle$. We have

$$A(u)|\mathbf{u}_N\rangle = A(u)B(u_1)\ldots B(u_N)|\uparrow^L\rangle\,. \tag{7.8}$$

We want to move the A-operator through the string of B-operators and finally hit $|\uparrow^L\rangle$ which diagonalize the A-operator. From the algebra, when we move A-operator through an B-operator, their spectral parameters might be swapped. Therefore we conclude that the structure of the final result takes the following form

$$\begin{aligned} A(u)B(u_1)\ldots B(u_N)|\uparrow^L\rangle =& a(u)\prod_{k=1}^{N} f(u_k - u)B(u_1)\ldots B(u_N)|\uparrow^L\rangle \\ &+ \sum_{k=1}^{N} M_k(u|\mathbf{u}_N)B(u_1)\ldots \widehat{B}(u_k)\ldots B(u_N)B(u)|\uparrow^L\rangle\,. \end{aligned} \tag{7.9}$$

where $\widehat{B}(u_k)$ indicates that the operator is absent from the string of B-operators. The first term on the right hand side takes the form of an eigenstate and is called the 'wanted term'. To obtain this term, we only use the first term on the right hand side of the algebra. The rest terms on the right hand side are called 'un-wanted terms'. The coefficients $M_k(u|\mathbf{u})$ might seem a bit more complicated. It can be determined in the following way. First we notice that $M_1(u|\mathbf{u})$ can be obtained straightforwardly: we use the second term on the right hand side of the algebra once, and then use the first term for the rest of the commutation relations. In this way, we obtain

$$M_1(u|\mathbf{u}_N) = g(u - u_1)a(u_1)\prod_{k=2}^{N} f(u_k - u_1)\,. \tag{7.10}$$

Secondly, notice that since all the B-operators commute, we can obtain $M_k(u|\mathbf{u})$ simply from $M_1(u|\mathbf{u})$ by a substitution $u_1 \leftrightarrow u_k$. This leads to

$$M_j(u|\mathbf{u}_N) = g(u - u_j)a(u_j)\prod_{k\neq j}^{N} f(u_k - u_j)\,. \tag{7.11}$$

One can check these relations explicitly using simple examples.

Similarly, one can derive that

$$\begin{aligned} D(u)A(u_1)\ldots B(u_N)|\Omega\rangle =& d(u)\prod_{k=1}^{N} f(u - u_k)B(u_1)\ldots B(u_N)|\uparrow^L\rangle \\ &+ \sum_{j=1}^{N} N_j(u|\mathbf{u}_N)B(u_1)\ldots \widehat{B}(u_k)\ldots B(u_N)B(u)|\uparrow^L\rangle\,. \end{aligned} \tag{7.12}$$

where

$$N_j(u|\mathbf{u}_N) = g(u_j - u)d(u_j)\prod_{k\neq j}^{N} f(u_j - u_k) \tag{7.13}$$

Notice that $g(u - u_j) = -g(u_j - u)$, we see that there is chance that we can actually cancel the unwanted terms by taking the sum of $A(u) + D(u)$ acting on $|\mathbf{u}\rangle$. We get that

$$(A(u) + D(u))|\mathbf{u}_N\rangle = \tau(u|\mathbf{u}_N)|\mathbf{u}_N\rangle \tag{7.14}$$

where $\tau(u|\mathbf{u}_N)$ is the eigenvalue of the transfer matrix

$$\tau(u|\mathbf{u}_N) = a(u)\prod_{j=1}^{N}\frac{u - u_j - i}{u - u_j} + d(u)\prod_{j=1}^{N}\frac{u - u_j + i}{u - u_j}\,. \tag{7.15}$$

The unwanted terms cancel under the condition

$$a(u_j)\prod_{k\neq j}^{N} f(u_j - u_k) = d(u_j)\prod_{k\neq j}^{N} h(u_j - u_k)\,, \qquad j = 1, 2, \ldots, N\,. \tag{7.16}$$

Or written more explicitly

$$\left(\frac{u_j + i/2}{u_j - i/2}\right)^L \prod_{k\neq j}^{N}\frac{u_j - u_k - i}{u_j - u_k + i} = 1, \qquad j = 1, 2, \ldots, N\,. \tag{7.17}$$

We see that this is precisely the Bethe ansatz equation which we derived in Lecture 1 by coordinate Bethe ansatz.

To summarize, we have proven that the state $B(u_1)\ldots B(u_N)|\Omega\rangle$ is indeed an eigenstate of the transfer matrix $T(u)$ provided $\{u_1, \ldots, u_N\}$ satisfy the Bethe equation (7.17). The corresponding eigenvalue is $\tau(u|\mathbf{u}_N)$ given in (7.15).

8 SU(2) symmetry

The XXX model enjoys the SU(2) symmetry. This can be seen easily by the fact

$$[H, S^\alpha] = 0 \tag{8.1}$$

where

$$S^\alpha = \sum_{n=1}^{L} S_n^\alpha, \qquad \alpha = \pm, z. \tag{8.2}$$

Let us discuss how to understand this symmetry in the algebraic construction. There are at least two manifestations of this fact. Firstly, the SU(2) algebra emerges in the large-u limit of the RMM-relation. Secondly, the Hilbert space is organized in SU(2) multiplets.

8.1 Large-u asymptotics and SU(2) algebra

We will see that the spin operators (8.2) appear in the $u \to \infty$ limit. Recall that the monodromy matrix can be written as

$$M_a(u) = \left[u\,\mathbb{I}_a + i \begin{pmatrix} S_1^z & S_1^- \\ S_1^+ & -S_1^z \end{pmatrix} \right] \cdots \left[u\,\mathbb{I}_a + i \begin{pmatrix} S_L^z & S_L^- \\ S_L^+ & -S_L^z \end{pmatrix} \right] \tag{8.3}$$

The full expansion in u is complicated, but the first two leading terms in u are rather easy to extract, we have

$$\begin{aligned} M_a(u) &= u^L\,\mathbb{I}_a + iu^{L-1} \sum_{n=1}^{L} \begin{pmatrix} S_n^z & S_n^- \\ S_n^+ & -S_n^z \end{pmatrix} + \cdots \\ &= u^L\,\mathbb{I}_a + iu^{L-1} \begin{pmatrix} S^z & S^- \\ S^+ & -S^z \end{pmatrix} + \cdots \end{aligned} \tag{8.4}$$

where the ellipsis denote lower order terms in u. Form (8.4) we can extract the large u behavior of the A, B, C, D operators. Keeping up to order $1/u$, we have

$$\begin{aligned} \lim_{u\to\infty} \frac{A(u)}{u^L} &= 1 + \frac{i}{u} S^z\,, \\ \lim_{u\to\infty} \frac{B(u)}{u^L} &= \frac{i}{u} S^-\,, \\ \lim_{u\to\infty} \frac{C(u)}{u^L} &= \frac{i}{u} S^+\,, \\ \lim_{u\to\infty} \frac{D(u)}{u^L} &= 1 - \frac{i}{u} S^z\,. \end{aligned} \tag{8.5}$$

Therefore we see that indeed we can extract the global spin operators from the large u asymptotics of the monodromy matrix.

Commutation relations Using these relations, we can obtain the commutation relations between the SU(2) generators and the matrix elements of the monodromy matrix. Let us see how it works by considering the following example

$$A(u)B(v) = f(v-u)B(v)A(u) + g(u-v)B(u)A(v)\,. \tag{8.6}$$

We divide both sides by u^L and take the $u \to \infty$ limit. In this limit, the function $f(v-u)$ and $g(u-v)$ becomes

$$\begin{aligned} \lim_{u\to\infty} f(v-u) &= \lim_{u\to\infty} \frac{v-u+i}{v-u} = 1 - \frac{i}{u} + \mathcal{O}(u^{-2})\,, \\ \lim_{u\to\infty} g(u-v) &= \lim_{u\to\infty} \frac{i}{u-v} = \frac{i}{u} + \mathcal{O}(u^{-2})\,. \end{aligned} \tag{8.7}$$

The left hand side of (8.6) becomes

$$\lim_{u\to\infty}\frac{A(u)}{u^L}B(v)=\left(1+\frac{i}{u}S^z\right)B(v)+\cdots \qquad (8.8)$$
$$=B(v)+\frac{i}{u}\,S^zB(v)+\mathcal{O}(u^{-2})$$

The right hand side of (8.6) reads

$$\lim_{u\to\infty}\left(f(v-u)B(v)\frac{A(u)}{u^L}+g(u-v)\frac{B(u)}{u^L}A(v)\right) \qquad (8.9)$$
$$=\left(1-\frac{i}{u}\right)B(v)\left(1+\frac{i}{u}S^z\right)+\cdots$$
$$=B(v)+\frac{i}{u}\left(-B(v)+B(v)S^z\right)+\mathcal{O}(u^{-2})$$

Now comparing (8.8) and (8.9), we see that the leading term is identical. The $\mathcal{O}(u^{-1})$ term leads to the following non-trivial relation

$$S^z\,B(v)=-B(v)+B(v)\,S^z\,. \qquad (8.10)$$

Or written in a slightly nicer form

$$[S^z,B(v)]=-B(v)\,. \qquad (8.11)$$

Similarly, we can derive other relations such as

$$[S^+,B(u)]=A(u)-D(u)\,. \qquad (8.12)$$

Of course, taking both u and v in the RMM relations to infinity, we recover the SU(2) Lie algebra. For example, taking $v\to\infty$ in (8.11), we obtain $[S^z,S^-]=-S^z$. Taking $u\to\infty$ in (8.12) we obtain $[S^+,S^-]=2S^z$.

8.2 SU(2) multiplets

Since the theory is SU(2) invariant, the spectrum must be organized in terms of SU(2) multiplets. Let us see how this works. The pseudovacuum state $|\Omega\rangle=|\uparrow^L\rangle$ satisfy

$$S^+|\Omega\rangle=0,\qquad S^z|\Omega\rangle=\frac{L}{2}|\Omega\rangle \qquad (8.13)$$

In the Lie algebra terminology, such a state is called a *highest weight state* of SU(2).

We will show that all the on-shell Bethe states whose Bethe roots are finite are highest weight states. Here we emphasis that roots are finite because roots at infinity corresponds to acting S^-'s as we have seen. The action of S^z on an N-magnon state is easy

$$S^z|\mathbf{u}_N\rangle = \left(\frac{L}{2} - N\right)|\mathbf{u}_N\rangle. \tag{8.14}$$

Let us now show that it is also a highest weight state. Using the commutation relation (8.12), we find

$$\begin{aligned} S^+|\mathbf{u}_N\rangle &= \sum_{j=1}^{N} B(u_1)\ldots B(u_{j-1})\,[A(u_j) - D(u_j)]\,B(u_{j+1})\ldots B(u_N)|\Omega\rangle \\ &= \sum_{j=1}^{N} O_k(\mathbf{u}_N)\,B(u_1)\ldots \widehat{B}(u_k)\ldots B(u_N)|\Omega\rangle \end{aligned} \tag{8.15}$$

The calculation of $O_k(\mathbf{u}_k)$ is the same as the derivation of the unwanted terms when computing $A(u)|\mathbf{u}_N\rangle$. Namely, we first determine $O_1(\mathbf{u}_N)$ and then use the argument that the result is symmetric with respect to all $\{u_k\}$ to determine the rest of $O_k(\mathbf{u}_N)$. This leads to

$$O_k(\mathbf{u}_N) = a(u_k)\prod_{j\neq k}^{N} f(u_j - u_k) - d(u_k)\prod_{j\neq k}^{N} f(u_k - u_j) \tag{8.16}$$

It is then also clear that when BAE is satisfied, we have $O_k(\mathbf{u}_N) = 0$. Therefore we conclude that

$$S^+|\mathbf{u}_N\rangle = 0 \tag{8.17}$$

which means on-shell Bethe states are highest weight states. They are also called *primary states*. Suppose we have a primary state which satisfies

$$H_{\text{XXX}}|\mathbf{u}_N\rangle = E(\mathbf{u}_N)|\mathbf{u}_N\rangle \tag{8.18}$$

We can act the spin operator S^- on the primary state. The action of each S^- flip down one spin on the state. Consider the state $(S^-)^n|\mathbf{u}_N\rangle$. Due to SU(2) invariance of the spin chain, we have $[H_{\text{XXX}}, S^-] = 0$ and

$$H_{\text{XXX}}(S^-)^n|\mathbf{u}_N\rangle = (S^-)^n\,H_{\text{XXX}}|\mathbf{u}_N\rangle = E(\mathbf{u}_N)\,(S^-)^n\,|\mathbf{u}_N\rangle \tag{8.19}$$

This indicates that the states $(S^-)^n|\mathbf{u}_N\rangle$ have the same energy with $|\mathbf{u}_N\rangle$ and belongs to the same multiplet. Such states are called descendant states of $|\mathbf{u}_N\rangle$. Recall that the highest weight representation of SU(2) algebra of spin-j is spanned by the states $|j,m\rangle$ with $m=-j,-j+1,\dots,j$. Since $|\mathbf{u}_N\rangle$ is the highest weight state of spin-$(\frac{L}{2}-N)$ representation, we have $N\le\frac{L}{2}$ and $n=1,2,\dots,L-2N$[1]. Therefore, we see that eigenstates of the Hamiltonian can indeed be organized as SU(2) multiplets.

Since each finite solution of Bethe ansatz equation corresponds to a primary state. In order to see whether the whole Hilbert space can be covered by Bethe states (taking into account both primary and descendant states), we need to count the number of solutions of Bethe ansatz equations for given L and N. This is a non-trivial and important question, which will be addressed in detail in Lecture 3.

8.3 Momentum and energy

Using algebraic Bethe ansatz, we can obtain the eigenvalue of the transfer matrix $\tau(u|\mathbf{u}_N)$ for any u. From this eigenvalue, we can obtain the eigenvalue of all conserved charges including the momentum, energy and higher conserved charges.

Momentum Recall that the shift operator is related to the transfer matrix as

$$U=i^{-L}\,T(i/2) \tag{8.20}$$

Its eigenvalue on a Bethe state $|\mathbf{u}_N\rangle$ is thus given by

$$i^{-L}\tau(i/2|\mathbf{u}_N)=\prod_{j=1}^{N}\frac{u_j+\frac{i}{2}}{u_j-\frac{i}{2}} \tag{8.21}$$

Since the shift operator can be seen as $U=e^{i\hat{P}}$ where $\hat{P}$ is the momentum, we find that

$$\hat{P}|\mathbf{u}_N\rangle=\sum_{k=1}^{N}p(u_k)|\mathbf{u}_N\rangle \tag{8.22}$$

where

$$p(u_k)=\frac{1}{i}\log\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}}\,. \tag{8.23}$$

Recall that in Lecture 1, we introduce the rapidity by

$$e^{ip_k}=\frac{u_k+\frac{i}{2}}{u_k-\frac{i}{2}} \tag{8.24}$$

where p_k is the momentum of the magnon.

[1] Acting more S^- on the state will lead to null vector, this can also be checked explicitly.

Energy The energy for N-magnon state is given by

$$E(\mathbf{u}_N) = \frac{i}{2}\frac{\mathrm{d}}{\mathrm{d}u}\log\tau(u|\mathbf{u}_N)\Big|_{u=i/2} - \frac{L}{4} \tag{8.25}$$

It is easy to verify that

$$E_N(\mathbf{u}_N) - E_0 = -\sum_{k=1}^{N}\frac{2}{4u_k^2+1}\,. \tag{8.26}$$

where the energy of each magnon is given by

$$\varepsilon(u_k) = -\frac{2}{4u_k^2+1}\,. \tag{8.27}$$

The eigenvalue of higher conserved charges can be derived in a similar way.

9 Physical models as representations

It is now clear that the algebraic structure under the Heisenberg XXX spin chain is the RMM-relation. Since it is an algebra, it makes sense to talk about its representations. It turns out the representations are 'labeled' by the two functions $a(u)$ and $d(u)$ which are eigenvalues of the operators $A(u)$ and $D(u)$ on the pseudovacuum

$$A(u)|\Omega\rangle = a(u)|\Omega\rangle, \qquad D(u)|\Omega\rangle = d(u)|\Omega\rangle\,. \tag{9.1}$$

For the Heisenberg XXX spin chain, we have

$$a(u) = (u+\tfrac{i}{2})^L, \qquad d(u) = (u-\tfrac{i}{2})^L\,. \tag{9.2}$$

Let us see a few other representations.

Higher spin model We mainly considered the spin operators in the spin-1/2 representation in this lecture. In fact, we can consider higher spin representations of the local spin operators. The Lax operator takes the same form. The only difference is that it can no longer be written as Pauli matrices. We can construct the monodromy and transfer matrix in exactly the same way. Therefore the RMM-relation is the same, which implies that we have the same underlying algebra. However, this time we have

$$A(u)|\Omega_s\rangle = (u+is)^L|\Omega_s\rangle, \qquad D(u) = (u-is)^L|\Omega_s\rangle \tag{9.3}$$

where $|\Omega_s\rangle$ is the pseudovacuum state in the spin-s representation. The Hamiltonian of this model takes more effort to work out and will be discussed in Lecture 4. For example, the spin-1 Hamiltonian takes the following form

$$H = \sum_{n=1}^{L} \vec{S}_n \cdot \vec{S}_{n+1} - (\vec{S}_n \cdot \vec{S}_{n+1})^2 \tag{9.4}$$

Non-linear Schrodinger model Probably a slightly more surprising example is the quantum non-linear Schrodinger model, or Lieb-Liniger model. It describes a one dimensional Bose gas. The Hamiltonian is given by

$$H = \int_0^L \left(\partial_x \Psi^\dagger \partial_x \Psi(x) + c\, \Psi^\dagger \Psi^\dagger \Psi(x)\Psi(x)\right) \mathrm{d}x \tag{9.5}$$

where the bosonic fields satisfy the usual commutation relation

$$[\Psi(x), \Psi^\dagger(y)] = \delta(x-y)\,. \tag{9.6}$$

This model can be solved by coordinate Bethe ansatz, which was done by Lieb and Liniger. Here we solve it by algebraic Bethe ansatz. To define Lax and monodromy operators, we need to discretize the model first. To this end, let us pick N points $x_1, \ldots, x_N$ on the interval $[0, L]$ such that $x_n = \Delta n$ and $x_N = L$. We define

$$\psi_n = \frac{1}{\Delta}\int_{x_{n-1}+0}^{x_n-0} \Psi(x)\mathrm{d}x, \qquad \psi_n^\dagger = \frac{1}{\Delta}\int_{x_{n-1}+0}^{x_n-0} \Psi^\dagger(x)\mathrm{d}x\,. \tag{9.7}$$

It is easy to check that the operator ψ_n and $\psi_n^\dagger$ satisfy the following relation

$$[\psi_n, \psi_m^\dagger] = \frac{\delta_{nm}}{\Delta}\,. \tag{9.8}$$

Now we can define the Lax-operator of the form

$$L_n(u) = \begin{pmatrix} 1 - \frac{iu\Delta}{2} + \frac{c\Delta^2}{2}\psi_n^\dagger\psi_n & -i\Delta\psi_n^\dagger\rho_n^+ \\ i\Delta\rho_n^-\psi_n & 1 + \frac{iu\Delta}{2} + \frac{c\Delta^2}{2}\psi_n^\dagger\psi_n \end{pmatrix} \tag{9.9}$$

The operators $\rho_n^\pm$ satisfy two constraints. One is that it is a combination of $\psi_n^\dagger\psi_n$ only, namely $\rho_n^\pm = \rho_n^\pm(\psi_n^\dagger\psi_n)$ and

$$\rho_n^+\rho_n^- = c + \frac{c^2\Delta^2}{4}\psi_n^\dagger\psi_n\,. \tag{9.10}$$

For example, we can take

$$\rho_n^+ = 1, \qquad \rho_n^- = c + \frac{c^2\Delta^2}{4}\psi_n^\dagger\psi_n\,. \tag{9.11}$$

We can check explicitly that the RLL-relation is satisfied with the same R-matrix[2]. We can define the monodromy matrix as before. It then follows that RMM-relation is satisfied and we have the same algebra. For this model the pseudovacuum is identified with the Fock vacuum, *i.e.* $|\Omega\rangle = |0\rangle$ where

$$\Psi(x)|0\rangle = 0, \qquad \psi_n|0\rangle = 0\,. \tag{9.12}$$

We then have

$$A(u)|\Omega\rangle = a(u)|\Omega\rangle, \qquad D(u)|\Omega\rangle = d(u)|\Omega\rangle \tag{9.13}$$

with

$$a(u) = \left(1 - \frac{iu\Delta}{2}\right)^N, \qquad d(u) = \left(1 + \frac{iu\Delta}{2}\right)^N \tag{9.14}$$

To recover the results of the Lieb-Liniger model, we take the continuous limit $\Delta \to 0$, $N \to \infty$ and $N\Delta = L$. In this limit, we obtain

$$a(u) = e^{-iuL/2}, \qquad d(u) = e^{iuL/2}\,. \tag{9.15}$$

This constitute another representation for the same algebra.

In this sense, algebraic Bethe ansatz is a more universal approach to integrable models. We see that the XXX_s and Lieb-Liniger model share the same underlying algebra.

References

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[2] More precisely, we need to change the functions slightly to $f(u,v) = \frac{u-v+c}{u-v}$ and $g(u,v) = \frac{c}{u-v}$.